2

Essential background

Plurality should not be posited without necessity
William of Ockham (*ca*. 1285–1349).

The purpose of this chapter is to introduce the basic knowledge required by the reader to understand the description of the sonar equations introduced in Chapter 3, and no more than this. The knowledge is sub-divided into four general subject areas: oceanography, acoustics, signal processing, and detection theory. Further details of these four areas, omitted here for simplicity, are described in Chapters 4 and following.

2.1 ESSENTIALS OF SONAR OCEANOGRAPHY

This section describes those basic physical properties of the sea and the air–sea boundary of relevance to the generation, propagation, and scattering of sound at sonar frequencies. The speed of sound and the density of water influence the generation and propagation of sound. These and other related parameters are described in Section 2.1.1, followed by the relevant properties of air in Section 2.1.2. A more comprehensive description of these oceanographic properties is provided in Chapter 4.

Many parameters of relevance to underwater acoustics vary with temperature T, salinity S, and hydrostatic pressure (often parameterized through the depth from the sea surface z). Where "representative" numerical values are quoted, they are evaluated for the following conditions:

$$T = 10\,^{\circ}\text{C},$$

$$S = 35,$$

$$\text{depth } z = 0.$$

By convention the depth co-ordinate z is zero at the sea surface and increases downwards to the seabed.

For simplicity, in Chapters 2 and 3 the ocean is assumed to extend to infinite depth, with uniform properties occupying the entire half-space satisfying $z > 0$. For example, the speed of sound and density are assumed independent of depth and range. The purpose of the quantitative numerical calculations based on this idealization (see *Worked Examples* in Chapter 3) is to illustrate the main principles of sonar performance modeling, rather than to provide realistic estimates of detection performance. More realistic examples are presented at the end of the book, in Chapter 11.

2.1.1 Acoustical properties of seawater

The two most important acoustical properties of seawater are the speed at which sound waves travel (abbreviated as *sound speed*) and the rate at which they decay with distance traveled (the *decay rate*, or *absorption coefficient*). A third parameter that can influence sound propagation, through its effect on interaction with boundaries, is density. The density of seawater, and the speed and absorption of sound in seawater, all depend on salinity, temperature, and pressure. At low frequency the absorption also depends on acidity or *p*H (see Chapter 4).

2.1.1.1 Speed of sound

For the representative conditions described above, the sound speed in seawater, denoted c_{water}, is 1490 m/s. More generally, the parameters S, T, and P all vary with depth and therefore so too does the sound speed, resulting in significant refraction. A discussion of these gradients and their important acoustic effects is deferred to Chapters 4 and 9. Here and in Chapter 3 they are neglected for simplicity. The wavelength λ at frequency f is

$$\lambda = c_{\text{water}}/f. \tag{2.1}$$

2.1.1.2 Density

The density of seawater (ρ_{water}) under the representative conditions introduced above is 1027 kg/m^3. Departures of seawater density from this value are small and for most sonar performance applications may be neglected.

2.1.1.3 Attenuation of sound

Attenuation is the name given to the process of decay in amplitude due to a combination of *absorption* and *scattering* of sound. The term "absorption" implies conversion to some other form of energy, usually heat, whereas "scattering" implies a redistribution in angle away from the original propagation direction, with no overall loss of acoustic energy.

The rate of attenuation of sound in water is less than in air and much less than that of electromagnetic waves in water. Low-frequency sound, of order 1 Hz to 10 Hz, can travel for thousands of kilometers, but high-frequency sound is attenuated more rapidly. The attenuation coefficient α_{water} increases monotonically with frequency by about four orders of magnitude in the frequency range from 30 Hz to 300 kHz, and quadratically with frequency thereafter. For frequencies f exceeding 200 Hz, it can be written (see Chapter 4)

$$\alpha_{\text{water}} = \alpha_1 \frac{f^2}{f^2 + f_1^2} + \alpha_2 \frac{f^2}{f^2 + f_2^2} + \alpha_3 f^2. \tag{2.2}$$

For the specified representative conditions, the three coefficients α_i are[1]

$$\alpha_1 = 1.40 \times 10^{-2} \quad \text{Np km}^{-1},$$

$$\alpha_2 = 5.58 \quad \text{Np km}^{-1}$$

and

$$\alpha_3 = 3.90 \times 10^{-5} \quad \text{Np km}^{-1}\ \text{kHz}^{-2}. \tag{2.3}$$

The frequencies f_1 and f_2, explained further in Chapter 4, are known as *relaxation frequencies* and for the representative conditions are equal, respectively, to 1.15 kHz and 75.6 kHz.

Numerical evaluation of Equation (2.2) gives (to the nearest order of magnitude) $\alpha_{\text{water}} \sim 10^{-3}$, 10^{-1}, and 10^{+1} Np/km at 300 Hz, 10 kHz, and 300 kHz, respectively. A graph of α_{water} vs. frequency, computed using Equation (2.2), is plotted in Figure 2.1. The reciprocal of the attenuation coefficient (i.e., α^{-1}), plotted on the same graph, provides a rough measure of the distance that sound can travel in water if unimpeded by physical obstacles. This quantity is referred to here as "audibility", the acoustical analogue of optical "visibility", and varies between 10^2 m Np^{-1} at 300 kHz and 10^6 m Np^{-1} at 300 Hz. By comparison, the attenuation coefficient of green light[2] is at least 10^{-2} Np m^{-1} for clear seawater, so that the optical visibility in water never exceeds 10^2 m Np^{-1} and is usually less than 10^1 m Np^{-1}. Thus, for acoustic frequencies up to 300 kHz, the audibility of sound exceeds the maximum visibility of light by up to six orders of magnitude. This is the reason why sound waves have

[1] Two sound waves are said to differ in level by 1 Np if their amplitudes are in the ratio 1 : *e*. The neper (Np) and the related unit the decibel (dB) are defined in Appendix B.

[2] The sea is opaque to electromagnetic radiation with the exception of visible light, very low frequency radio waves, and gamma rays. The extinction coefficient is a measure of the decay of light intensity with distance, and in the present notation is equal to $2\alpha_{\text{opt}}$,where α_{opt} is the optical attenuation coefficient in units of nepers per unit distance. For example, the value quoted by (Clarke and James, 1939) of 4 % per meter for the extinction coefficient in the Sargasso Sea means that $\exp(-2\alpha_{\text{opt}}x) = 0.96$ when the distance $x = 1$ m. Taking logarithms gives $2\alpha_{\text{opt}} = 0.04$/m.

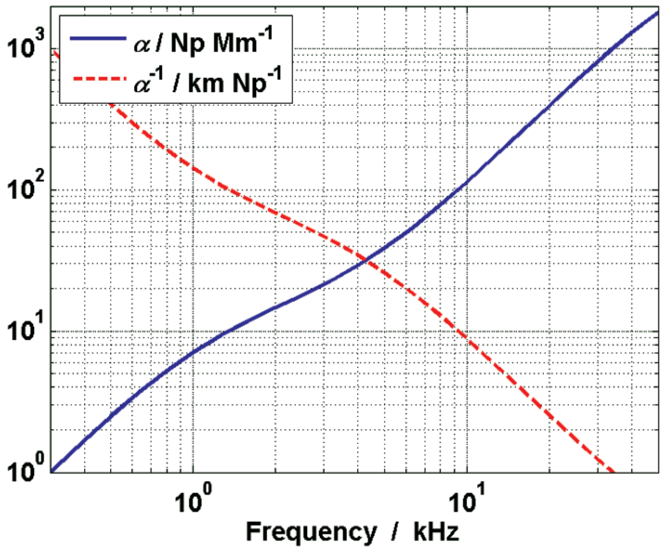

Figure 2.1. Numerical value of attenuation coefficient *vs*. frequency of sound in seawater α (expressed in units of nepers per megameter) and of its reciprocal, α^{-1} (in kilometers per neper), calculated using Equation (2.2) for the specified representative conditions: $S = 35$; $T = 10\,^{\circ}\mathrm{C}$; $z = 0$.

become the most successful means of probing the underwater environment. It is the *raison d'être* of sonar.

2.1.2 Acoustical properties of air

Together with those of seawater, the properties of air determine the reflection coefficient at the air–sea boundary. The sound speed and density of air depend on temperature (T) and pressure (P). For the representative conditions, these are $c_{\mathrm{air}} = 337\,\mathrm{m/s}$ and $\rho_{\mathrm{air}} = 1.25\,\mathrm{kg/m^3}$. Thus, both ρ and c in air are considerably lower than their counterparts in water, which has important implications for the behavior of underwater sound.

2.2 ESSENTIALS OF UNDERWATER ACOUSTICS

2.2.1 What is sound?

Steady-state pressure increases with increasing depth z and is equal to the total weight per unit area of water plus atmosphere supported above that depth. This quantity is called the *static pressure* (or hydrostatic pressure) and can be expressed quantitatively in the form

$$P_{\text{stat}}(z) = P_{\text{atm}} + P_{\text{gauge}}(z), \tag{2.4}$$

where P_{atm} is the atmospheric pressure (approximately 101 kPa); and P_{gauge} is the additional pressure due to the weight of the water above depth z (the gauge pressure)

$$P_{\text{gauge}}(z) = \int_0^z \rho_{\text{water}}(\zeta) g(\zeta)\, \mathrm{d}\zeta. \tag{2.5}$$

Underwater disturbances result in departures δP from this value, for an arbitrary position vector $\mathbf{x}$,

$$\mathrm{P}_{\text{tot}}(\mathbf{x}, \mathrm{t}) = \mathrm{P}_{\text{stat}}(\mathrm{z}) + \delta \mathrm{P}(\mathbf{x}, \mathrm{t}). \tag{2.6}$$

Once created, provided that certain basic conditions are met (Pierce, 1989), a pressure disturbance propagates with the speed of sound, and δP is known as the *acoustic pressure*, henceforth denoted $q(\mathbf{x}, t)$ and assumed small by comparison with static pressure.[3] Such an acoustic disturbance is known as *underwater sound*. The study of this sound is called *underwater acoustics*. The assumption of small q simplifies the mathematics and is generally justified because atmospheric pressure is large compared with typical acoustic pressure fluctuations.

A brief account is given here of radiation and scattering of underwater sound from simple sources and for a simple geometry. First, radiation is considered from a point source in an infinite uniform medium, with and without a perfectly reflecting plane boundary (Section 2.2.2). Then the scattering of plane waves is considered, first from a point object and then from a rough surface (Section 2.2.3). The sea surface is considered as an example of a reflecting surface, a radiating surface, and a scattering boundary. A more complete treatment of these phenomena is presented in Chapters 5 and 8.

2.2.2 Radiation of sound

2.2.2.1 Radiation from a point monopole source

2.2.2.1.1 Spherical spreading

Consider a point monopole[4] source of power W. To generate sound at a given frequency, the source must expand and contract at that frequency. During expansion the source motion causes an increase in density of the surrounding fluid, with a corresponding increase in its pressure. The resulting high-pressure disturbance propagates outwards in the form of a spherical wave, traveling at the speed of sound c_{water}. The same sequence follows a contraction, except with a low-pressure disturbance replacing the high-pressure one.

At any fixed moment in time the radiated field comprises a series of concentric "rings" (actually spherical shells in three dimensions) of alternating high and low

[3] The symbol p, introduced in Section 2.2.2, is reserved for a complex variable representing the acoustic pressure. See footnote 5.

[4] A monopole source is one with a fluctuating volume, such as a pulsating bubble.

pressure. The potential of these rings to do work on the surrounding medium can be expressed in terms of their potential energy density (Pierce, 1989)

$$E_V^{(\mathrm{P})} = \frac{q^2}{2B_{\mathrm{water}}}, \tag{2.7}$$

where E_V denotes energy per unit volume; and B_{water} is the bulk modulus of water, a measure of its opposition to compression or rarefaction, analagous to the stiffness of a spring, and equal to

$$B_{\mathrm{water}} = \rho_{\mathrm{water}} c_{\mathrm{water}}^2. \tag{2.8}$$

The rings also contain kinetic energy, due to the particle velocity $\mathbf{u}$, given by

$$E_V^{(\mathrm{K})} = \frac{\rho_{\mathrm{water}} |\mathbf{u}|^2}{2}. \tag{2.9}$$

The superscripts (P) and (K) in Equations (2.7) and (2.9) denote potential and kinetic energy, respectively. If the pressure and particle velocity are in phase, it can be shown that the kinetic and potential densities are equal (Pierce, 1989), so that the average rate of energy flux (i.e., intensity) is

$$I = c_{\mathrm{water}} \overline{\left(E_V^{(\mathrm{K})} + E_V^{(\mathrm{P})} \right)} = 2c_{\mathrm{water}} \overline{E_V^{(\mathrm{P})}} = \frac{\overline{q^2}}{\rho_{\mathrm{water}} c_{\mathrm{water}}}, \tag{2.10}$$

where the overbar indicates an average in time. Conservation of energy demands that the total radiated power at distance s from the source, $4\pi s^2 I$, be constant, which means that the RMS pressure (i.e., $\sqrt{\overline{q^2}}$) must vary as $1/s$ with distance.

A point monopole source radiates *omni-directionally* (i.e., with equal power in all directions). At a distance s from the source, in the assumed uniform medium the energy is distributed uniformly on a sphere of surface area $4\pi s^2$ (Figure 2.2), so the component of *acoustic intensity* normal to the surface of the sphere at a distance s is

$$I(s) = \frac{W}{4\pi s^2}. \tag{2.11}$$

Also of interest is the acoustic pressure resulting from the point source. Using standard complex variable notation for a diverging harmonic spherical wave of angular frequency ω, the complex pressure field p varies with time t and distance s according to Pierce (1989)[5]

$$p(s,t) = \sqrt{2} p_0 s_0 \frac{\mathrm{e}^{\mathrm{i}(ks - \omega t)}}{s}, \tag{2.12}$$

where p_0 is the RMS pressure at a distance s_0 from the source; and k is the acoustic wave number, so that

$$k = \omega / c_{\mathrm{water}}. \tag{2.13}$$

[5] The real part of the complex variable $p(s,t)$ is the acoustic pressure $q(s,t)$. Unless otherwise stated, an $\exp(-\mathrm{i}\omega t)$ time convention is used for traveling waves throughout.

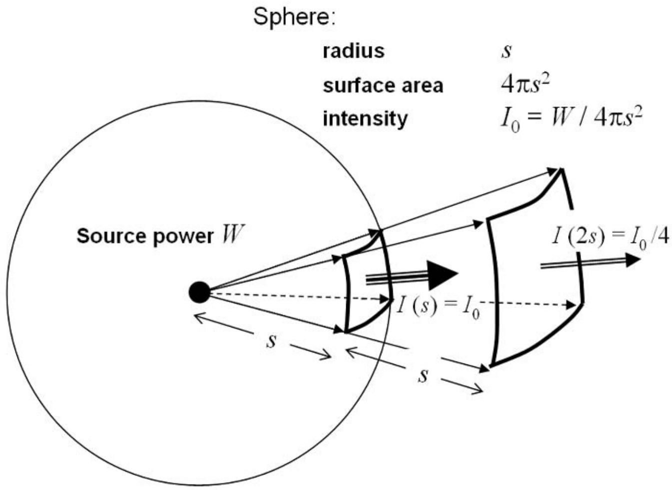

Figure 2.2. Radiation from a point source of power W in free space. The intensity at a distance s is $I_0 = W/(4\pi s^2)$. At distance $2s$ the same power has spread into four times the area, reducing the intensity by a factor of 4.

The true acoustic pressure is obtained by taking the real part of Equation (2.12), so that

$$q(s,t) = \sqrt{2} p_0 s_0 \frac{\cos(ks - \omega t)}{s}. \tag{2.14}$$

From Equation (2.10) it follows that

$$I = \frac{|p|^2}{2\rho_w c_w}, \tag{2.15}$$

where the "water" subscript is abbreviated henceforth as "w". From Equations (2.11), (2.12), and (2.15) it then follows that

$$p_0 s_0 = \left(\rho_w c_w \frac{W}{4\pi}\right)^{1/2}, \tag{2.16}$$

or more generally (for a directional source)

$$p_0 s_0 = (\rho_w c_w W_\Omega)^{1/2}, \tag{2.17}$$

where W_Ω indicates the radiated power per unit solid angle (the *radiant intensity*).

It is convenient to define a steady-state *propagation factor* $F(s)$ in terms of the ratio of the mean square pressure at the receiver to that at a small distance (s_0) from the source, such that

$$F(s) = \frac{\overline{q^2}}{p_0^2 s_0^2} = \frac{|p|^2}{2 p_0^2 s_0^2}. \tag{2.18}$$

Defined in this way, the propagation factor has dimensions [distance]$^{-2}$. For a spherical wave in a medium of uniform impedance it is equal to the ratio of received intensity I to the radiant intensity W_Ω.

It follows by substituting for $p(s,t)$ from Equation (2.12), scaled by $\exp(-\alpha s)$ to account for absorption, that the propagation factor for a point source in a uniform medium is

$$F(s) = \frac{e^{-2\alpha s}}{s^2}, \tag{2.19}$$

where α is the sound attenuation coefficient introduced in Section 2.1.1.3.

The above arguments apply to a steady-state field due to a source of constant radiant intensity. If the power is transmitted for a short time only, it is useful to think in terms of the transient field resulting from the total radiated energy per unit solid angle E_Ω. The appropriate propagation factor under these conditions is obtained by integrating the numerator and denominator of Equation (2.18) over time instead of averaging them:

$$F(s) \equiv \frac{\int q^2 \, dt}{\rho_w c_w E_\Omega}. \tag{2.20}$$

To summarize, the steady-state mean square pressure for a source of radiant intensity W_Ω, from Equation (2.18), is

$$\overline{q^2} = \rho_w c_w W_\Omega F(s) \tag{2.21}$$

and for a transient field, the time-integrated pressure squared, from Equation (2.20), is

$$\int q^2 \, dt = \rho_w w E_\Omega F(s). \tag{2.22}$$

Either way, $F(s)$ is given by Equation (2.19) for an omni-directional point source in an infinite uniform medium. The same equation applies also for a directional source, provided that W_Ω (or E_Ω) is measured in the direction of the receiver.

The behavior described by Equation (2.19), characterized by its s^{-2} dependence due to the spherical nature of the expanding wave front, is known as *spherical spreading*.

2.2.2.1.2 Reflection from the sea surface

Now consider the effect of placing a reflecting boundary, such as the sea surface, close to the point source of Section 2.2.2.1.1. There are two straight-line ray paths connecting the source to any given receiver position: the direct path and a surface reflected one. If the source is at depth z_0 below the surface (see Figure 2.3), the contribution to the pressure field at the receiver due to the direct path is given by Equation (2.12) with a source–receiver separation equal to

$$s_- = \sqrt{r^2 + (z - z_0)^2}. \tag{2.23}$$

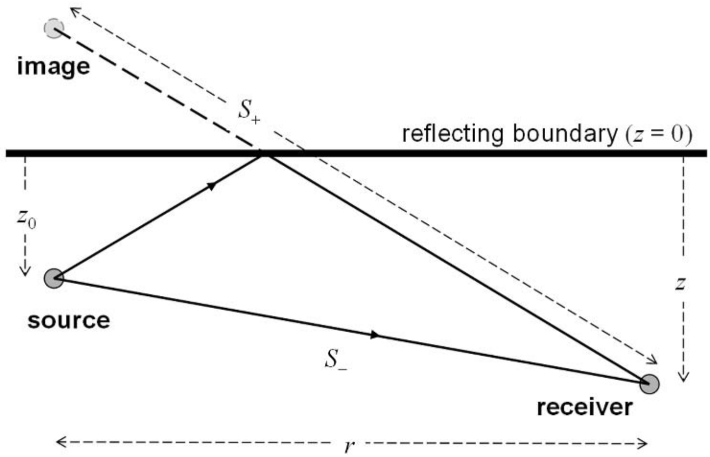

Figure 2.3. Radiation from a point source in the presence of a reflecting boundary.

The reflected path can be thought of as originating from an image source at height z_0 above the boundary, with image–receiver separation of

$$s_+ = \sqrt{r^2 + (z + z_0)^2}. \tag{2.24}$$

Using the method of images, the two contributions from source and image are added coherently to obtain the total pressure at the receiver, scaling the reflected path by the surface reflection coefficient R

$$p = \sqrt{2} s_0 p_0 \left[\frac{\mathrm{e}^{(\mathrm{i}k-\alpha)s_-}}{s_-} + R \frac{\mathrm{e}^{(\mathrm{i}k-\alpha)s_+}}{s_+} \right] \mathrm{e}^{-i\omega t}. \tag{2.25}$$

If R is real (implying a phase change on reflection of 0 or π) it follows that

$$F(s_+, s_-) = \frac{\mathrm{e}^{-2\alpha s_-}}{s_-^2} + \frac{R^2 \mathrm{e}^{-2\alpha s_+}}{s_+^2} + \frac{2R\mathrm{e}^{-\alpha(s_- + s_+)}}{s_- s_+} \cos(2kx), \tag{2.26}$$

where

$$x \equiv \frac{s_+ - s_-}{2} = \frac{2 z_0 z}{s_- + s_+}. \tag{2.27}$$

Equation (2.26) can be interpreted as follows. The first and second terms on the right-hand side are associated with the energy from the direct and surface-reflected ray paths, respectively. The third term is due to interference between these two paths, resulting from the coherent addition of complex pressures. The expression is useful for broadband applications because the third term vanishes after averaging over frequency.

An (equivalent) alternative version, convenient for narrowband applications, is

$$F(s_+, s_-) = \mathrm{e}^{-\alpha(s_- + s_+)} \left[\left(\frac{\mathrm{e}^{\alpha x}}{s_-} + R \frac{\mathrm{e}^{-\alpha x}}{s_+} \right)^2 - \frac{4R}{s_- s_+} \sin^2(kx) \right]. \tag{2.28}$$

It is often the case that the distances s_+ and s_- are approximately equal, such that the product αx is sufficiently small to neglect the αx terms, and $s_+ - s_- \ll s_+ + s_-$. Furthermore, for many applications the sea surface can be treated as a perfect reflector with a π-phase change (i.e., $R = -1$; see Box on p. 37). It then follows that

$$F_{\mathrm{coh}}(s) \approx \frac{4\mathrm{e}^{-2\alpha s}}{s^2} \sin^2\left(\frac{kz_0 z}{s}\right), \tag{2.29}$$

where

$$s = (s_- + s_+)/2. \tag{2.30}$$

The sequence of sinusoidal peaks and troughs predicted by Equation (2.29) is known as a *Lloyd mirror* interference pattern. The "coh" subscript stands for "coherent addition", indicating that the two contributions to the total pressure, from the direct and reflected paths, respectively, are added with regard to their phase, before squaring. This means that the phase difference information is used for the purpose of combining the contributions from the two paths. Specifically, if Equation (2.25) is written in the form

$$p = \sqrt{2} s_0 p_0 (\Phi_+ + \Phi_-), \tag{2.31}$$

where

$$\Phi_- = \frac{\mathrm{e}^{(\mathrm{i}k-\alpha)s_-}}{s_-} \mathrm{e}^{-\mathrm{i}\omega t} \tag{2.32}$$

and

$$\Phi_+ = R \frac{\mathrm{e}^{(\mathrm{i}k-\alpha)s_+}}{s_+} \mathrm{e}^{-\mathrm{i}\omega t}, \tag{2.33}$$

it follows that

$$F_{\mathrm{coh}} = |\Phi_+ + \Phi_-|^2. \tag{2.34}$$

The "incoherent" propagation factor is obtained by discarding the phase terms. In other words

$$F_{\mathrm{inc}} = |\Phi_+|^2 + |\Phi_-|^2. \tag{2.35}$$

Alternatively, averaging F_{coh} over (say) frequency

$$\langle F_{\mathrm{coh}} \rangle = \langle |\Phi_+|^2 \rangle + \langle |\Phi_-|^2 \rangle + \langle 2|\Phi_+||\Phi_-| \cos(2kx) \rangle. \tag{2.36}$$

The first two terms are hardly affected by the average and together approximate to F_{inc}. If the average is over several cycles of the cosine function, the third term can be expected to average out to zero, and hence

$$\langle F_{\mathrm{coh}} \rangle \approx F_{\mathrm{inc}}. \tag{2.37}$$

For this reason, there are many situations in which a *coherent sum* (add and square), followed by an average over frequency, gives the same result as an *incoherent sum* (square and add).

The reflection coefficient at the air–sea boundary

The reflection coefficient at the sea surface is determined by the impedance of air relative to that of water. The *characteristic impedance* of air for the assumed representative conditions is given by

$$Z_{\text{air}} = \rho_{\text{air}} c_{\text{air}} = 420\ \text{kg m}^{-2}\ \text{s}^{-1},$$

more than three orders of magnitude smaller than that of water, which is equal to

$$Z_{\text{water}} = \rho_{\text{water}} c_{\text{water}} = 1.53 \times 10^6\ \text{kg m}^{-2}\ \text{s}^{-1}.$$

The low impedance of air compared with that of water means that the acoustic pressure required to achieve a given acoustic intensity is much smaller in air than in water. From the continuity of pressure across the boundary it follows that the pressure on the boundary itself must also be small, and to first order this can be approximated by the boundary condition $p = 0$ at $z = 0$. The only way this can be achieved for an incident plane wave of finite amplitude in water is for a reflected wave to be generated at the surface of the same amplitude and *opposite phase*. Let the horizontal and vertical wave numbers be κ and γ, respectively, so that the incident wave can be represented by

$$p_{\text{incident}} = e^{i(\kappa x - \gamma z)} e^{-i\omega t}$$

and the reflected wave by

$$p_{\text{reflected}} = R e^{i(\kappa x + \gamma z)} e^{-i\omega t}.$$

By adding these two terms it can be seen that the only way the total pressure $p_{\text{incident}} + p_{\text{reflected}}$ can be zero everywhere on the $z = 0$ boundary is if $R = -1$. This has two important consequences. First, the unit magnitude corresponds to 100 % reflection of energy, so that sound becomes trapped in the sea, potentially traveling very long distances. Second, the negative sign means a π phase change of the reflected wave relative to the incident one, which results in near-perfect cancellation of acoustic pressure close to the sea surface.

In general, the reflection coefficient is a function of frequency, and of the physical properties of the reflecting boundary. For example, the sea surface reflection coefficient depends on the wave height and on the population of near-surface bubbles created by breaking waves (see Chapters 5 and 8.)

2.2.2.2 Radiation from an infinite sheet of uniformly distributed dipoles

One of the factors that limit sonar performance is the presence of background noise in the sea. Much of this background noise originates at the sea surface (e.g., due to breaking waves). Consider an infinitesimal patch of sea surface with surface area δA and radiating power per unit surface area and solid angle $W_{A\Omega}$. The contribution from this patch to the mean square pressure at a receiver situated at a distance s (see

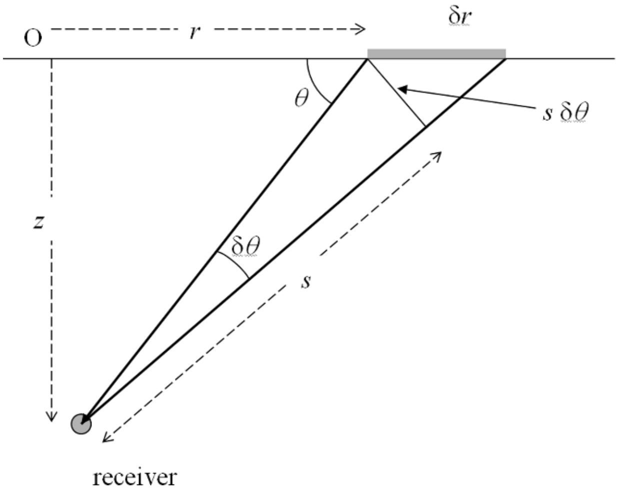

Figure 2.4. Radiation from a sheet source element of width δr.

Figure 2.4) is

$$\delta\overline{q^2} = \rho_w c_w W_{A\Omega}\,\delta A\,\frac{\mathrm{e}^{-2\alpha s}}{s^2}. \tag{2.38}$$

The sea surface behaves like a sheet of dipoles,[6] with a radiation pattern proportional to $\sin^2\theta$, so that[7]

$$W_{A\Omega} = \frac{3}{2\pi} W_A \sin^2\theta, \tag{2.39}$$

where θ is the ray *grazing angle* (the angle between the ray path and the horizontal), so that

$$\delta\overline{q^2} = \frac{3}{2\pi} W_A\,\delta A\,\rho_\mathrm{w} c_\mathrm{w}\,\frac{\mathrm{e}^{-2\alpha s}}{s^2}\sin^2\theta. \tag{2.40}$$

The solid angle $\delta\Omega$ subtended by the surface element δr at the receiver, for an azimuthal increment $\delta\phi$, is

$$\delta\Omega = \cos\theta\,\delta\theta\,\delta\phi \tag{2.41}$$

[6] A dipole source is one made out of two out-of-phase monopole sources, placed an infinitesimal distance apart (in practice, they must be separated by at most a small fraction of a wavelength). An important distinction between a monopole and a dipole is that in the case of the dipole source, there is no net change in volume. See Crocker (1997) for details.

[7] The constant $3/(2\pi)$ ensures that the power radiated per unit area, integrated over all solid angles (into the lower half-space) $\int_{2\pi} W_{A\Omega}\,\mathrm{d}\Omega$ is W_A. This follows from the use of $\mathrm{d}\Omega = \cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ and the result

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathrm{d}\theta\,\sin^2\theta\cos\theta = 2\pi/3,$$

where ϕ is the azimuth angle.

where (Figure 2.4)

$$\delta\theta = \frac{\delta r \sin\theta}{s}. \tag{2.42}$$

The corresponding element of area at the sea surface is

$$\delta A = r\,\delta r\,\delta\phi \tag{2.43}$$

so that

$$\delta A = \frac{s^2\,\delta\Omega}{\sin\theta}, \tag{2.44}$$

and hence (taking the limit of infinitesimal $\delta\Omega$)

$$\frac{\mathrm{d}\overline{q^2}}{\mathrm{d}\Omega} = \frac{3}{2\pi}\rho_{\mathrm{w}} c_{\mathrm{w}} W_A \sin\theta \exp\left(-\frac{2\alpha z}{\sin\theta}\right). \tag{2.45}$$

This is the mean square pressure per unit solid angle, at the receiver position. Now assume that the contribution to the pressure field from each patch of the surface, covering an infinitesimal solid angle $\mathrm{d}\Omega$, is uncorrelated with all other contributions. This means that the energy contributions (mean square pressures) may be added incoherently. The contribution from a concentric ring centered on the origin O is therefore obtained by replacing $\mathrm{d}\phi$ with 2π, so that $\mathrm{d}\Omega$ becomes $2\pi\cos\theta\,\mathrm{d}\theta$. The total mean square pressure is then found by integrating over θ

$$\overline{q^2} = 2\pi\int_0^{\pi/2}\frac{\mathrm{d}\overline{q^2}}{\mathrm{d}\Omega}\cos\theta\,\mathrm{d}\theta = 3\rho_{\mathrm{w}} c_{\mathrm{w}} W_A E_3(2\alpha z), \tag{2.46}$$

where $E_3(x)$ is a third-order exponential integral (see Appendix A). For small arguments, the limiting form may be used

$$\lim_{x\to 0} E_3(x) = \tfrac{1}{2} \tag{2.47}$$

so that, in the case of negligible attenuation ($\alpha z \ll 1$), Equation (2.46) becomes

$$\overline{q^2} \approx \tfrac{3}{2}\rho_{\mathrm{w}} c_{\mathrm{w}} W_A. \tag{2.48}$$

As an example, consider the radiation of sound from the sea surface, already identified as an important source of background noise. The acoustic power radiated by the sea surface due to wind (per unit area and bandwidth) can be written in the form[8] (see Chapter 8 for details)

$$W_{Af} = \frac{2\pi}{3\rho_{\mathrm{w}} c_{\mathrm{w}}} K, \tag{2.49}$$

[8] The subscripts A and f denote derivatives with respect to area and frequency such that

$$W_f \equiv \frac{\mathrm{d}W}{\mathrm{d}f} \qquad W_A \equiv \frac{\mathrm{d}W}{\mathrm{d}A} \qquad W_{Af} \equiv \frac{\mathrm{d}^2 W}{\mathrm{d}A\,\mathrm{d}f}.$$

This is a generalization of the notation introduced previously for power and energy per unit solid angle

$$W_\Omega \equiv \frac{\mathrm{d}W}{\mathrm{d}\Omega} \qquad E_\Omega \equiv \frac{\mathrm{d}E}{\mathrm{d}\Omega}.$$

where the parameter K varies with frequency f and wind speed v. An empirical expression for K, based on measurements over a wide range of frequencies is

$$K = 1.32 \times 10^4 \frac{\hat{v}^{2.24}}{1.5 + F^{1.59}} \quad \mu\text{Pa}^2\,\text{Hz}^{-1} \tag{2.50}$$

where F is the frequency in units of kilohertz

$$F \equiv \frac{f}{1\,\text{kHz}} \tag{2.51}$$

and $\hat{v}$ is the wind speed in meters per second

$$\hat{v} \equiv \frac{v}{1\,\text{m/s}}. \tag{2.52}$$

Throughout this book, standard SI units and prefixes are used so that 1 μPa (one micropascal) is equal to 10^{-6} Pa. See Appendix B for a complete list of SI prefixes.

2.2.3 Scattering of sound

2.2.3.1 Scattering from a small object

The likelihood that an echo from a distant object is detected depends on how much sound is reflected (i.e., scattered) in the direction of the receiving system. The ability of a small underwater object to scatter sound is quantified by its *scattering cross-section*, defined as the ratio of total scattered power W to incoming intensity I (from a specified grazing angle θ_{in}), of an incident plane wave

$$\sigma(\theta_{\text{in}}) = \frac{W}{I(\theta_{\text{in}})}. \tag{2.53}$$

Thus, σ is the scattered power per unit incident intensity and has dimensions of area. A related quantity is the *differential scattering cross-section*, proportional to the power per unit solid angle scattered in a specified direction[9] (elevation θ_{out}, and bearing ϕ relative to that of the incident plane wave):

$$\sigma_\Omega(\theta_{\text{in}}; \theta_{\text{out}}, \phi) = \frac{W_\Omega(\theta_{\text{out}}, \phi)}{I(\theta_{\text{in}})} \tag{2.54}$$

[9] That is, the radiant intensity of the scattered sound.

so that

$$\sigma(\theta_{\text{in}}) = \int \sigma_\Omega(\theta_{\text{in}}, \Omega_{\text{out}})\, \mathrm{d}\Omega_{\text{out}}, \tag{2.55}$$

where the shorthand Ω_{out} denotes the direction $(\theta_{\text{out}}, \phi)$ and $\mathrm{d}\Omega_{\text{out}}$ is an element of solid angle such that

$$\mathrm{d}\Omega_{\text{out}} = \cos\theta_{\text{out}}\, \mathrm{d}\theta_{\text{out}}\, \mathrm{d}\phi. \tag{2.56}$$

The *backscattering cross-section* is defined as the differential cross-section evaluated in the backscattering direction, multiplied by 4π. In equation form[10] (Pierce, 1989; Morfey, 2001):[11]

$$\sigma^{\text{back}}(\theta) \equiv 4\pi\sigma_\Omega(\theta; \theta, \pi). \tag{2.57}$$

The backscattering cross-section of a rigid sphere of radius a at high frequency $(ka \gg 1)$, is (see Chapter 5)

$$\sigma^{\text{back}} = \pi a^2. \tag{2.58}$$

2.2.3.2 Scattering from a rough surface

The echo from an underwater object can be masked by sound that happens to arrive at the same time, after it has been scattered from a rough boundary. For a rough surface, the scattered power is proportional to the area ensonified by the incident wave. In this situation it makes sense to define a (dimensionless) scattering coefficient as the differential scattering cross-section per unit scattering area; that is,

$$\sigma_{\Omega A}(\theta_{\text{in}}; \Omega_{\text{out}}) \equiv \frac{W_{\Omega A}(\Omega_{\text{out}})}{I(\theta_{\text{in}})}. \tag{2.59}$$

The parameter $\sigma_{\Omega A}$ is also known as the *scattering coefficient* of a rough surface.

Provided that the wind speed is low enough to neglect the influence of near-surface bubbles, the scattering coefficient for the sea surface is approximately (except in directions close to that of specular reflection) (see Chapter 8)

$$\sigma_{\Omega A}(\theta_{\text{in}}; \theta_{\text{out}}, \phi) \approx \frac{C_{\text{PM}}}{16\pi} \tan^2\theta_{\text{in}} \tan^2\theta_{\text{out}}, \tag{2.60}$$

where C_{PM} is a constant equal to 0.0081. Often the scattering coefficient is written as a function of a single angle θ, in which case the backscattering direction is implied;

[10] By "in the backscattering direction" is meant that the propagation direction of the scattered wave is taken to be the same as that of the incident wave, except that its sense is reversed.
[11] An alternative definition of backscattering cross-section as the differential scattering cross-section in the backscattering direction (i.e., omitting the factor 4π from Equation 2.57) is sometimes used. In this book the definition of Equation (2.57) is used throughout. This point is discussed further in Chapter 5.

that is,

$$\sigma_{\Omega A}(\theta) \equiv \sigma_{\Omega A}(\theta; \theta, \pi) \approx \frac{C_{\mathrm{PM}}}{16\pi} \tan^4 \theta. \tag{2.61}$$

Substituting for the numerical value of C_{PM} gives

$$\sigma_{\Omega A}(\theta) \approx 1.61 \times 10^{-4} \tan^4 \theta. \tag{2.62}$$

2.3 ESSENTIALS OF SONAR SIGNAL PROCESSING

When a sound wave reaches a sonar receiver, the acoustic pressure is first converted to an electrical voltage by an underwater microphone, or *hydrophone*. This voltage could be displayed on an oscilloscope and monitored for evidence of something other than background noise, such as the voltage exceeding some pre-established threshold. Such a simple system might work in practice for a strong signal, while a weak one would first need to be enhanced by signal processing. For example, the signal-to-noise ratio can be increased by filtering out sound at unwanted frequencies, or from unwanted directions or ranges, or a combination of these. Before any such enhancement begins, the electrical signal is passed through an anti-alias filter and then digitized.[12] The details of subsequent processing depend on the characteristics of the expected signal, but almost all sonar systems use either a temporal filter to remove noise at unwanted frequencies or a spatial filter to remove noise from unwanted directions or both. No distinction is made in the following between *acoustic* and *electrical* signals. The justification for this is that the waveform, once digitized, can be rescaled by an arbitrary constant factor to represent either the voltage or the original pressure.

A time domain filter operation involves sampling a waveform in time and combining successive samples in such a way as to remove any unwanted sound, whereas a spatial filter (or *beamformer*) samples in space instead of time. Both are described below, with the purpose of introducing some basic relationships between time duration and frequency bandwidth (Section 2.3.1) and between spatial aperture and beamwidth (Section 2.3.2). A more complete treatment of signal processing is presented in Chapter 6.

2.3.1 Temporal filter

Imagine a receiving system that passes frequencies between $f_{\min}$ and $f_{\max}$ and blocks frequencies outside this range. Such a system is called a *passband filter* of *bandwidth* $\Delta f \equiv f_{\max} - f_{\min}$. If $f_{\min}$ is zero it is a *low-pass* filter; if $f_{\max}$ is infinite it is a *high-pass* filter. General filter theory is beyond the present scope, but it is useful to introduce

[12] An anti-alias filter is one that removes high-frequency signals above a threshold that depends on the sampling rate of the subsequent digital sampler. According to the *Nyquist–Shannon sampling theorem*, the maximum acoustic frequency that can be correctly sampled is half of the sampling rate. This maximum permissible frequency is known as the *Nyquist frequency*.

some basic concepts. A special kind of filter of particular interest is a discrete Fourier transform (DFT),[13] the basic properties of which are outlined below.

Let $F(t)$ denote the time domain waveform of interest, sampled at discrete times t_n at fixed intervals δt. The DFT of $F(t)$ is the spectrum $G(\omega)$ (see Appendix A for details)

$$G(\omega) \equiv \sum_{n=0}^{N-1} F(t_n) \exp(-\mathrm{i}\omega t_n), \qquad t_n = t_0 + n\,\delta t. \tag{2.63}$$

The inverse transform is the operation that recovers the original function $F(t)$ from the spectrum at discrete frequencies ω_m:

$$F(t) = \frac{1}{N}\sum_{m=0}^{N-1} G(\omega_m) \exp(+\mathrm{i}\omega_m t), \qquad \omega_m = \frac{2\pi}{N\,\delta t} m. \tag{2.64}$$

For the special case of simple harmonic time dependence of angular frequency ω

$$F(t) = \mathrm{e}^{\mathrm{i}\omega t}, \tag{2.65}$$

it follows that

$$G(\omega_m) = \sum_{n=0}^{N-1} \exp[\mathrm{i}(\omega - \omega_m) t_n] = \frac{\sin\left[(\omega - \omega_m)\dfrac{\Delta t}{2}\right]}{\sin\left[(\omega - \omega_m)\dfrac{\delta t}{2}\right]}, \tag{2.66}$$

where the time origin is chosen for convenience to be at the center of the sequence of time samples (such that $t_0 + t_{N-1} = 0$) and Δt is given by[14]

$$\Delta t = N\,\delta t, \tag{2.67}$$

approximately equal to the signal duration.

If the signal is well sampled in time (such that $|(\omega - \omega_m)\,\delta t| \ll 1$), the denominator of Equation (2.66) may be approximated by the argument of the sine function. The spectrum is then given by

$$G(\omega_m) \approx N\,\mathrm{sinc}(y), \tag{2.68}$$

where

$$y = (\omega - \omega_m)\frac{\Delta t}{2} \tag{2.69}$$

and $\mathrm{sinc}(y)$ is the sine cardinal function, defined (see Appendix A) as

$$\mathrm{sinc}(y) \equiv \frac{\sin y}{y}. \tag{2.70}$$

Written in this form it can be seen that the DFT operation, with output $G(\omega_m)$, is a passband filter, centered on ω. The parameter Δt (the total time duration) determines

[13] When the number of points in a DFT is a power of 2, a particularly efficient implementation is possible. This efficient implementation is also known as a fast Fourier transform (FFT).

[14] The time between first and last samples is equal to $\Delta t - \delta t$, which is approximately equal to Δt if δt is assumed small.

the frequency resolution of the filter through the argument of the sinc function. Specifically, the full width at half-maximum (fwhm), i.e., the spectral width between half-power points, is given by

$$\delta\omega_{\text{fwhm}} = \frac{4}{\Delta t}\operatorname{sinc}^{-1}\left(\frac{1}{\sqrt{2}}\right). \tag{2.71}$$

Evaluation of the inverse sinc function gives

$$\operatorname{sinc}^{-1}(1/\sqrt{2}) \approx 1.3916, \tag{2.72}$$

so the frequency resolution, as defined by

$$\delta f_{\text{fwhm}} \equiv \frac{\delta\omega_{\text{fwhm}}}{2\pi}, \tag{2.73}$$

is approximately equal to the reciprocal of the total time duration Δt

$$\delta f_{\text{fwhm}} = \frac{2\operatorname{sinc}^{-1}(1/\sqrt{2})}{\pi}\frac{1}{\Delta t} \approx \frac{0.886}{\Delta t}. \tag{2.74}$$

2.3.2 Spatial filter (beamformer)

Mathematically, there is no difference between spatial and temporal filtering, except that spatial filtering can be carried out in more than one dimension. For simplicity, the scope is limited here to a single dimension, so the expression for a spatial DFT can be obtained from Equations (2.63) and (2.64) by replacing the time variable (t) with the spatial one (x). It is also customary to represent the spatial "frequency" (the wave number) variable by the symbol k. Thus, the forward and inverse transforms are, respectively,

$$G(k) \equiv \sum_{n=0}^{N-1} F(x_n)\exp(-\mathrm{i}kx_n), \qquad x_n = x_0 + n\,\delta x, \tag{2.75}$$

and

$$F(x) = \frac{1}{N}\sum_{m=0}^{N-1} G(k_m)\exp(+\mathrm{i}k_m x), \qquad k_m = \frac{2\pi}{N\,\delta x}m. \tag{2.76}$$

A collection of hydrophones whose output is combined to carry out spatial filtering is known as a hydrophone *array* or, if it extends in only one dimension, a *line array*. Consider a pressure field whose spatial distribution along such an array is of the form

$$F(x) = \mathrm{e}^{\mathrm{i}kx}. \tag{2.77}$$

By an exact analogy with the time domain filter, if the origin is at the geometrical center of the array it follows that

$$G(k_m) = \frac{\sin\left[(k - k_m)\dfrac{\Delta x}{2}\right]}{\sin\left[(k - k_m)\dfrac{\delta x}{2}\right]}, \tag{2.78}$$

or (for sufficiently small hydrophone spacing δx)

$$G(k_m) \approx N \operatorname{sinc}(y), \tag{2.79}$$

where

$$y = (k - k_m)\frac{\Delta x}{2}, \tag{2.80}$$

and the parameter Δx is

$$\Delta x = N\,\delta x. \tag{2.81}$$

Here, the distance Δx plays the role of Δt in the time domain filter, and determines the resolution of the spatial filter in the wavenumber domain. It is approximately equal to the length of the array.

The approximation Equation (2.79) requires the signal to be well sampled in space such that

$$|k - k_m|\,\delta x \ll 1. \tag{2.82}$$

The fwhm in wave number, by analogy with Equation (2.71), is

$$\delta k_{\text{fwhm}} = \frac{4}{\Delta x}\operatorname{sinc}^{-1}\left(\frac{1}{\sqrt{2}}\right). \tag{2.83}$$

The output wavenumber spectrum of the hydrophone array is referred to as the *array response*. The squared magnitude of the normalized array response for an incident plane wave, known as the *beam pattern* of the array, is

$$B \equiv \left|\frac{G(k_m)}{G_{\max}}\right|^2 = \frac{\sin^2 y}{N^2 \sin^2 \dfrac{y}{N}} \approx (\operatorname{sinc} y)^2. \tag{2.84}$$

Because of its ability to amplify selectively acoustic waves arriving from a narrow range of angles (a "beam"), the spatial filter is called a *beamformer*. To see how the angle selection process works, consider an array aligned along the x-axis ($y = z = 0$) and an acoustic plane wave traveling in a direction parallel to the x–y plane. The field of this plane wave as a function of space and time can be written

$$p(\mathbf{x}, t) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{e}^{-\mathrm{i}\omega t} = \exp(\mathrm{i}k_x x)\mathrm{e}^{\mathrm{i}(k_y y - \omega t)}. \tag{2.85}$$

Defining α as the angle between the wavenumber vector and the plane normal to the array axis, the along-axis wavenumber component is[15]

$$k_x = \frac{2\pi}{\lambda}\sin\alpha, \tag{2.86}$$

where λ is the acoustic wavelength. Specializing to the field along the array ($y = 0$), Equation (2.85) becomes

$$p(x, t) = \exp(\mathrm{i}k_x x)e^{-i\omega t}. \tag{2.87}$$

Now consider the field at an arbitrary instant in time (say $t = 0$) and use this field as

[15] Similarly, $k_y = (2\pi/\lambda)\cos\alpha$.

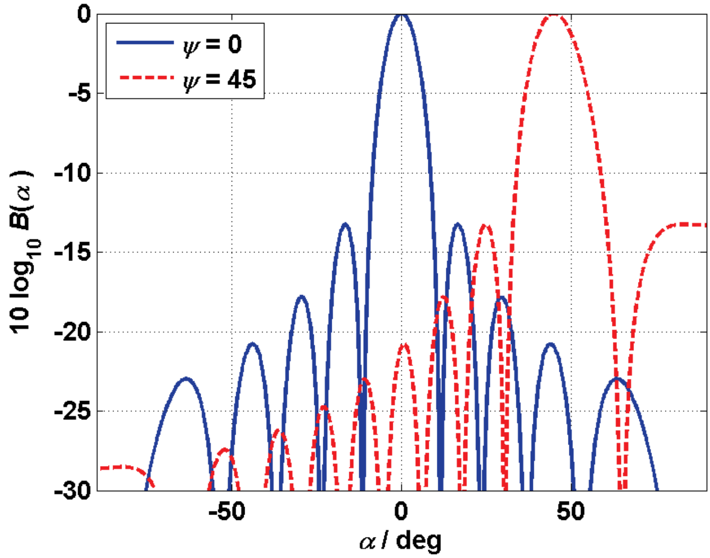

Figure 2.5. Beam patterns for $L/\lambda = 5$ and steering angles 0, 45 deg as indicated.

input to the beamformer, so that

$$F(x) \equiv p(x,0) = \exp(\mathrm{i}k_x x). \tag{2.88}$$

The response is Equation (2.79), with

$$y = \left(\frac{2\pi}{\lambda}\sin\alpha - k_m\right)\frac{\Delta x}{2}. \tag{2.89}$$

If the magnitude of k_m does not exceed $2\pi/\lambda$, there exists an arrival angle α at which the beamformer output is maximized, corresponding to $y = 0$. This value of α is given by

$$\alpha_m = \arcsin\frac{k_m}{2\pi/\lambda} \tag{2.90}$$

and is known as the beam *steering angle*. It is measured from the direction perpendicular to the array axis, known as the *broadside* direction. Beam patterns for two different steering angles are shown in Figure 2.5.

The angular width of the beam varies with steering angle as follows. Taking a finite difference of Equation (2.86)

$$\delta k_x \approx \frac{2\pi}{\lambda}\cos\alpha\,\delta\alpha, \tag{2.91}$$

the fwhm beamwidth is obtained by equating the right-hand sides of Equations (2.83) and (2.91):

$$\delta\alpha_{\text{fwhm}} \approx \frac{2\,\text{sinc}^{-1}(1/\sqrt{2})}{\pi}\frac{\lambda}{\Delta x \cos\alpha_m}. \tag{2.92}$$

In radians this is

$$\delta\alpha_{\text{fwhm}} \approx 0.886\frac{\lambda}{\Delta x \cos\alpha_m} \quad \text{rad} \tag{2.93}$$

and in degrees

$$\delta\alpha_{\text{fwhm}} \approx 50.8\frac{\lambda}{\Delta x \cos\alpha_m} \quad \text{deg.} \tag{2.94}$$

This approximation for the beamwidth works best at angles close to the broadside direction. For the case of Figure 2.5, the predicted and observed half-power widths are about 10 deg at broadside ($\psi = 0$) and 14 deg at $\psi = 45$ deg. The approximation breaks down at angles close to ± 90 deg from broadside (i.e., parallel to the array axis, known as the *endfire* direction), due to the singularity in the derivative $\mathrm{d}\alpha/\mathrm{d}k_x$ in that direction. The equation for wavenumber width (Equation 2.83) is valid at any angle (see Chapter 6).

2.4 ESSENTIALS OF DETECTION THEORY

The calculation of detection probability is the whole point of sonar performance modeling and the ultimate goal of this book. Hence, considerable attention is paid to its calculation. The end result of the processing, after all filtering, is presented to a sonar operator, whose job it is to report the detection or not of a (potential) sonar target, based on the information provided by the sonar. Depending on the signal-to-noise ratio, the probability of making a detection might be high or low, but it is never certain. The objective of statistical detection theory is to quantify this probability.

2.4.1 Gaussian distribution

In this section, expressions are derived for the probability of detection (denoted p_{d}) for a simple case involving a constant signal in Gaussian noise.[16] The signal is represented by the constant x_{S} and noise by the variable $x_{\text{N}}(t)$. The signal, if present, is always accompanied by a noise background, and the combination of both is represented by $x_{\text{S+N}}(t)$. The parameter x can be the amplitude or energy of an acoustic wave, depending on the processing, and is referred to below as the "observable".

The two possibilities "signal present" and "signal absent" are represented by the total observable x_{tot} given by either

$$x_{\text{tot}}(t) = x_{\text{S+N}}(t) \quad \text{(signal present)} \tag{2.95}$$

[16] The choice of constant signal and Gaussian noise is for mathematical convenience and does not necessarily represent a realistic situation for a sonar system. More realistic distributions are considered in Section 2.4.2.

Table 2.1. Detection truth table; p_d is the probability of deciding correctly that a signal is present ("detection probability") and p_{fa} is the probability of declaring a detection when there is no signal.

	Threshold exceeded $x_{tot} > x_T$	*Threshold not exceeded* $x_{tot} < x_T$
Signal present $x_{tot} = x_{S+N}$	Correct decision (probability p_d)	Incorrect decision (probability $1 - p_d$)
Signal absent $x_{tot} = x_N$	Incorrect decision (probability p_{fa})	Correct decision (probability $1 - p_{fa}$)

or

$$x_{tot}(t) = x_N(t) \quad \text{(signal absent).} \tag{2.96}$$

A decision-maker (the sonar operator) presented with the data sequence $x_{tot}(t)$ deems a signal to be present whenever the value of x_{tot} exceeds some threshold x_T. To avoid too many false alarms it is desirable for the threshold x_T to exceed the noise, in some average sense, but how should the noise be averaged, and by how much must this average be exceeded? The answer depends, among other things, on the rate of false alarms considered acceptable. The larger the threshold, the fewer false alarms will result, at the expense of a reduced probability of detection.

Assuming that the operator must always choose between the two decisions "signal present" and "signal absent", irrespective of the chosen threshold there are always four possible outcomes, according to Table 2.1.

Because of the statistical fluctuations in the noise there is always a chance that the threshold is exceeded when there is no signal, and conversely there is also a chance that the threshold is not exceeded even when the target is present. Both situations lead to an incorrect decision, indicated in the table by gray shading. The probability of making a correct "signal present" decision is known as the detection probability and denoted p_d. The false alarm probability p_{fa} is the probability of making an *incorrect* "signal present" decision.

One's objective is to make the correct decision as often as possible. In other words, to maximize p_d, implying a low threshold, while at the same time minimizing p_{fa}, which requires a high threshold. These conflicting requirements are resolved in practice by deciding in advance on a highest acceptable false alarm rate, and then determining the threshold consistent with this rate. Thus, the values of p_d and p_{fa} depend on the choice of x_T as well as on the statistical fluctuations of noise and of signal + noise.

The following calculations assume a randomly fluctuating noise observable $x_N(t)$ with Gaussian statistics, and a non-fluctuating signal so that the signal-plus-noise (x_{S+N}) has the same statistics (the same Gaussian distribution with the same standard deviation) as noise alone (x_N). Expressions for p_{fa} and p_d are derived below for these assumptions.

2.4.1.1 Noise only

Let the probability density function (pdf) of the noise observable be $f_N(x)$, so that the mean and variance of the distribution are

$$\overline{x_N} = \int_{-\infty}^{+\infty} x f_N(x)\,dx \tag{2.97}$$

and

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \overline{x_N})^2 f_N(x)\,dx, \tag{2.98}$$

respectively. The Gaussian distribution with these properties (see Figure 2.6, *upper graph*) is

$$f_N(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \overline{x_N})^2}{2\sigma^2}\right]. \tag{2.99}$$

2.4.1.2 Signal plus noise

Similarly, if the non-fluctuating signal is added

$$f_{S+N}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \overline{x_{S+N}})^2}{2\sigma^2}\right], \tag{2.100}$$

illustrated by the *lower graph* of Figure 2.6. Assuming that the observable terms add linearly, if the signal is constant, the signal-plus-noise can be written

$$x_{S+N}(t) = x_S + x_N(t) \tag{2.101}$$

and therefore

$$\overline{x_{S+N}} \equiv \int_{-\infty}^{+\infty} x f_{S+N}(x)\,dx = x_S + \overline{x_N}. \tag{2.102}$$

Thus, the signal-plus-noise distribution has the same pdf as the noise alone but with a higher mean value.

Suppose that a detection is declared by the operator whenever the threshold x_T is exceeded. The probability of this occurring as the result of a single observation is equal to the area under the pdf curve to the right of the threshold. This area, depending on whether in reality a signal is absent or present, is either p_{fa} or p_d. In other words, respectively,

$$p_{fa} = \int_{x_T}^{\infty} f_N(x)\,dx = \frac{1}{2}\text{erfc}\left(\frac{x_T - \overline{x_N}}{\sqrt{2}\sigma}\right) \tag{2.103}$$

where erfc(x) is the complementary error function (Appendix A), or

$$p_d = \int_{x_T}^{\infty} f_{S+N}(x)\,dx = \frac{1}{2}\text{erfc}\left(\frac{x_T - \overline{x_{S+N}}}{\sqrt{2}\sigma}\right). \tag{2.104}$$

In this treatment, a large negative result is arbitrarily not considered a threshold crossing. This choice might be justified if the observable is a positive definite quantity, such that the negative tail of the Gaussian has no physical meaning. If large negative

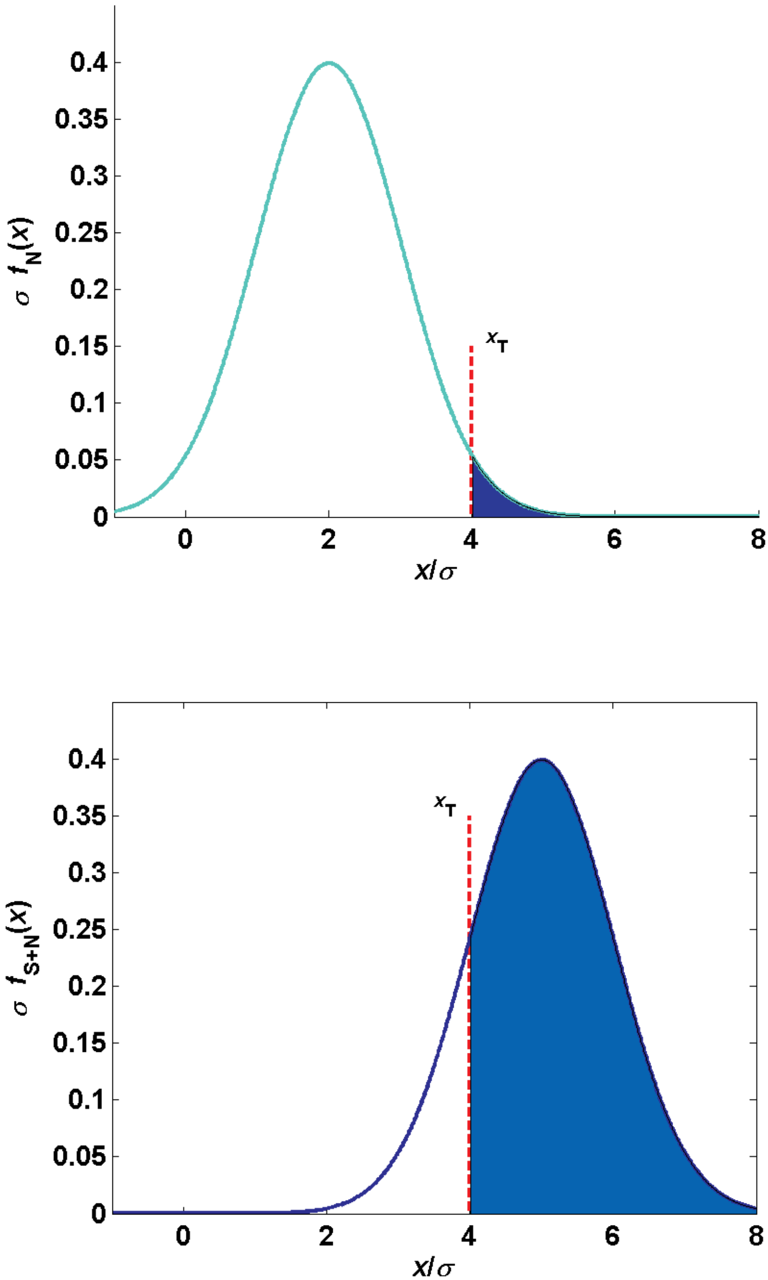

Figure 2.6. Probability density functions of noise (*upper graph*) and signal-plus-noise (*lower*) observables. The threshold for declaring a detection, x_T, is shown as a vertical dashed line. The shaded areas are the probabilities of false alarm (*upper graph*) and detection (*lower*). The example shown is for the case $\overline{x_N} = 2\sigma$, $\overline{x_{S+N}} = 5\sigma$, and $x_T = 4\sigma$.

values were considered to be threshold crossings, the expressions for both p_{fa} and p_d would then need to include contributions from values of x between $-\infty$ and $-x_T$.

Both p_{fa} and p_d vary between 0 and 1. It is convenient to replace x_T in Equation (2.104) by expressing it as a function of p_{fa} (from Equation 2.103). The result is

$$p_d = \frac{1}{2}\operatorname{erfc}\left[\operatorname{erfc}^{-1}(2p_{fa}) - \frac{x_S}{\sqrt{2}\sigma}\right]. \tag{2.105}$$

2.4.2 Other distributions

The analysis of sonar detection problems requires the consideration of more complicated distributions than that of a constant signal in Gaussian background noise. A preview of some important results from Chapter 7 is presented below. In each case, expressions are quoted for the false alarm probability p_{fa} and detection probability p_d as a function of the signal-to-noise ratio (SNR).

2.4.2.1 Coherent processing (Rayleigh statistics)

Coherent processing for Gaussian noise results in a Rayleigh distribution for the noise amplitude A. For an amplitude threshold A_T, and assuming a Rayleigh distribution for the *signal* as well as for the noise, the false alarm and detection probabilities are

$$p_{fa} = \exp\left(-\frac{A_T^2}{2\sigma^2}\right) \tag{2.106}$$

and

$$p_d = p_{fa}^{1/(1+R)}, \tag{2.107}$$

where R is the SNR

$$R = \frac{\overline{A^2}}{2\sigma^2}. \tag{2.108}$$

2.4.2.2 Incoherent processing (chi-squared statistics with many samples)

Incoherent addition of a number of Rayleigh-distributed samples results in a chi-squared (or "χ^2") distribution for the total energy. The detection and false alarm probabilities can be expressed for this distribution in terms of special functions as described in Chapter 7. If the number of samples M is sufficiently large ($M > 100$), these expressions simplify to those presented below. For an energy threshold E_T, the false alarm probability becomes

$$p_{fa} \approx \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{M}{2}}\left(\frac{E_T}{2M\sigma^2} - 1\right)\right], \tag{2.109}$$

where σ is the standard deviation of the noise samples before any averaging. The detection probability simplifies to

$$p_{\mathrm{d}} \approx \frac{1}{2}\mathrm{erfc}\left[\frac{\mathrm{erfc}^{-1}(2p_{\mathrm{fa}}) - \sqrt{M/2}R}{1+R}\right], \tag{2.110}$$

where R is the power signal-to-noise ratio. Equation (2.110) can be rearranged for R:

$$R = \frac{\mathrm{erfc}^{-1}(2p_{\mathrm{fa}}) - \mathrm{erfc}^{-1}(2p_{\mathrm{d}})}{\sqrt{M/2} + \mathrm{erfc}^{-1}(2p_{\mathrm{d}})}. \tag{2.111}$$

If $M^{1/2}$ is large compared with $\mathrm{erfc}^{-1}(2p_{\mathrm{d}})$, this simplifies further to

$$R \approx \frac{\mathrm{erfc}^{-1}(2p_{\mathrm{fa}}) - \mathrm{erfc}^{-1}(2p_{\mathrm{d}})}{\sqrt{M/2}}. \tag{2.112}$$

The condition on M makes Equation (2.112) mainly relevant to situations involving a low SNR. If R and M are both large, there is usually no need for a detailed analysis, because in this situation the detection probability is always close to unity.

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