Preface

Establishing a new concept of local Lyapunov exponents, two separate theories are brought together, namely Lyapunov exponents and the theory of large deviations.

Specifically, for the stochastic differential system

$$\begin{aligned} dZ_t^\varepsilon &= \mathbf{A}\left(X_t^\varepsilon\right)\, Z_t^\varepsilon\, dt \\ dX_t^\varepsilon &= b\left(X_t^\varepsilon\right)\, dt + \sqrt{\varepsilon}\, \sigma\left(X_t^\varepsilon\right)\, dW_t \end{aligned}$$

the new concept is introduced. Due to stationarity, the Lyapunov exponents of Z_t^ε (which by Oseledets' Multiplicative Ergodic Theorem describe the exponential growth rates of Z_t^ε) do not depend on the initial position x of X^ε. Now the goal of this work is to provide a Lyapunov-type number for each regime of the drift b. As this characteristic number shall depend on the domain in which X^ε, a dynamical system perturbed by additive white noise, is starting, it yields a concept of locality for the Lyapunov exponents of Z_t^ε. Furthermore, the locality of such local Lyapunov exponents is to be understood as reflecting the quasi-deterministic behavior of X^ε which asserts that in the limit of small noise, $\varepsilon \to 0$, the process X^ε has *metastable states* depending on its initial value as well as on the time scale chosen (Freidlin-Wentzell theory).

Up to now local Lyapunov exponents have been defined as finite time versions of Lyapunov exponents by several authors, but here we target at investigating the large time asymptotics $t \to \infty$. So the goal is to connect the large parameters t and ε^{-1} in the customary definition of the Lyapunov exponents in order to approach the *sublimiting distributions* (Freidlin) which are supported by the metastable states of X^ε. The *local Lyapunov exponent* is then understood to be the exponential growth rate of Z^ε on the time scale chosen, subject to convergence in probability as $\varepsilon \to 0$. Notably, the system itself changes in the sense that the noise intensity converges to zero with the time horizon depending on the noise intensity parameter. In contrast to this new concept the Lyapunov exponents as obtained by the Multiplicative Ergodic Theorem reflect the information of *limit distributions*, i.e. of invariant measures, as time increases to infinity for the system parameter $\varepsilon > 0$ being fixed.

As a result we prove that the local Lyapunov exponent is bounded from above by the largest real part of the spectrum of the matrix $\mathbf{A}$ evaluated at the metastable state corresponding to the time scale; the respective bound from below holds true with the smallest real part of an eigenvalue of $\mathbf{A}$ at the corresponding metastable state.

Assuming that $\mathbf{A}$ takes its values in the diagonal matrices, it is shown that its eigenvalues at the respective metastable state are precisely the possible local Lyapunov exponents. Moreover, in a "strongly" hypoelliptic situation it can be proved that only the largest eigenvalue is observed under convergence in probability. The latter result is regarded as sublimiting Furstenberg-Khasminskii formula, since the resulting limit is obtained as a (trivial) integral which produces the top eigenvalue.

For the above tasks the prerequisites which fundamentally consist of knowing the exit probabilities of all the stochastic systems involved will be provided in detail: For this purpose, an integrated account of the theory for non-degenerate stochastic differential systems (Freidlin and Wentzell) and of the exit probabilities for degenerate stochastic systems (Hernández-Lerma) is given in chapters 2 and 3. The subsequent final chapter is the heart of the book. Here, all the results are proven and discussed.

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