Chapter 17
Hom-Lie Admissible Hom-Coalgebras and Hom-Hopf Algebras

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Abstract The aim of this paper is to develop the coalgebra counterpart of the notions introduced by the authors in a previous paper, we introduce the notions of Hom-coalgebra, Hom-coassociative coalgebra and G-Hom-coalgebra for any subgroup G of permutation group $\mathscr{S}_3$. Also we extend the concept of Lie-admissible coalgebra by Goze and Remm to Hom-coalgebras and show that G-Hom-coalgebras are Hom-Lie admissible Hom-coalgebras, and also establish duality correspondence between classes of G-Hom-coalgebras and G-Hom-algebras. In another hand, we provide relevant definitions and basic properties of Hom-Hopf algebras generalizing the classical Hopf algebras and define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra.

17.1 Introduction

In [4, 7, 8], the class of quasi-Lie algebras and subclasses of quasi-hom-Lie algebras and Hom-Lie algebras have been introduced. These classes of algebras are tailored in a way suitable for simultaneous treatment of the Lie algebras, Lie superalgebras, the color Lie algebras and the deformations arising in connection with twisted, discretized or deformed derivatives [5] and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. It has been shown in [4, 7–9] that the class of quasi-Hom-Lie algebras contains as a subclass on the one hand the color Lie algebras and in particular Lie superalgebras and Lie algebras, and

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on the another hand various known and new single and multi-parameter families of algebras obtained using twisted derivations and constituting deformations and quasi-deformations of universal enveloping algebras of Lie and color Lie algebras and of algebras of vector-fields. The main feature of quasi-Lie algebras, quasi-Hom-Lie algebras and Hom-Lie algebras is that the skew-symmetry and the Jacobi identity are twisted by several deforming twisting maps and also in quasi-Lie and quasi-Hom-Lie algebras the Jacobi identity in general contains six twisted triple bracket terms.

In the paper [12], we provided a different way for constructing Hom-Lie algebras by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. To this end we defined the notion of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted, and showed that the commutator product defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. We introduced also Hom-Lie-admissible algebras and more general G-Hom-associative algebras with subclasses of Hom-Vinberg and pre-Hom-Lie algebras, generalizing to the twisted situation Lie-admissible algebras, G-associative algebras, Vinberg and pre-Lie algebras respectively, and show that for these classes of algebras the operation of taking commutator leads to Hom-Lie algebras as well. We constructed also all the twistings so that the brackets $[X_1,X_2] = 2X_2,\ [X_1,X_3] = -2X_3,\ [X_2,X_3] = X_1$ determine a three-dimensional Hom-Lie algebra. Finally, we provided for a subclass of twistings, the list of all three-dimensional Hom-Lie algebras. This list contains all three-dimensional Lie algebras for some values of structure constants. The families of Hom-Lie algebras in these list can be viewed as deformations of Lie algebras into a class of Hom-Lie algebras. The notion, constructions and properties of the enveloping algebras of Hom-Lie algebras are yet to be properly studied in full generality. An important progress in this direction has been made in the recent work by D. Yau [14].

In the present paper we develop the coalgebra counterpart of the notions and results of [12], extending in particular in the framework of Hom-associative and Hom-Lie algebras and Hom-coalgebras, the notions and results on associative and Lie admissible coalgebras obtained in [2]. In Sect. 17.2 we summarize the relevant definitions of Hom-associative algebra, Hom-Lie algebra, Hom-Leibniz algebra, and define the notions of Hom-coalgebras and Hom-coassociative coalgebras. In Sect. 17.3, we introduce the concept of Hom-Lie admissible Hom-coalgebra, describe some useful relations between coproduct, opposite coproduct, the cocommutator defined as their difference, and their β-twisted coassociators and β-twisted co-Jacobi sums. We also introduce the notion of G-Hom-coalgebra for any subgroup G of permutation group S_3. We show that G-Hom-coalgebras are Hom-Lie admissible Hom-coalgebras, and also establish duality correspondence between classes of G-Hom-coalgebras and G-Hom-algebras. Section 17.4 is dedicated to relevant definitions and basic properties of the Hom-Hopf algebra which generalize the classical Hopf algebra structure. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra.

17.2 Hom-Algebra and Hom-Coalgebra Structures

A Hom-algebra structure is a multiplication on a vector space where the structure is twisted by a homomorphism. The structure of Hom-Lie algebra was introduced by Hartwig et al. [4]. In the following we summarize the definitions of Hom-associative, Hom-Leibniz, and Hom-Lie-admissible algebraic structures introduced in [12] and generalizing the well known associative, Leibniz and Lie-admissible algebras. By dualization of Hom-associative algebra we define the Hom-coassociative coalgebra structure.

17.2.1 Hom-Algebra Structures

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and V be a linear space over $\mathbb{K}$.

Definition 17.1. A *Hom-associative algebra* is a triple (V,μ,α) consisting of a linear space V, a linear map $\mu : V\otimes V \to V$ and a homomorphism α satisfying

$$\mu(\alpha(x)\otimes\mu(y\otimes z)) = \mu(\mu(x\otimes y)\otimes\alpha(z)). \tag{17.1}$$

The Hom-associativity condition (17.1) may be expressed by the following commutative diagram.

$$\begin{array}{ccc} V\otimes V\otimes V & \stackrel{\mu\otimes\alpha}{\longrightarrow} & V\otimes V \\ \downarrow{\scriptstyle \alpha\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ V\otimes V & \stackrel{\mu}{\longrightarrow} & V \end{array}$$

The Hom-associative algebra is unital if there exists a homomorphism $\eta : \mathbb{K}\to V$ such that the following diagrams are commutative

$$\begin{array}{ccccc} \mathbb{K}\otimes V & \stackrel{\eta\otimes id}{\longrightarrow} & V\otimes V & \stackrel{id\otimes\eta}{\longleftarrow} & V\otimes\mathbb{K} \\ & \searrow^{\cong} & \downarrow{\scriptstyle \mu} & \swarrow^{\cong} & \\ & & V & & \end{array}$$

In the language of Hopf algebra, a Hom-associative algebra $\mathscr{A}$ is a quadruple (V,μ,α,η) where V is the vector space, μ is the Hom-associative multiplication, α is the twisting homomorphism and η is the unit.

Let (V,μ,α,η) and (V',μ',α',η') be two Hom-associative algebras. A linear map $f : V\to V'$ is a morphism of Hom- associative algebras if

$$\mu'\circ(f\otimes f) = f\circ\mu \quad , \quad f\circ\eta = \eta' \quad \text{and} \quad f\circ\alpha = \alpha'\circ f.$$

In particular, (V,μ,α,η) and (V,μ',α',η') are isomorphic if there exists a bijective linear map f such that

$$\mu = f^{-1}\circ\mu'\circ(f\otimes f) \quad , \quad \eta = f^{-1}\circ\eta' \quad \text{and} \quad \alpha = f^{-1}\circ\alpha'\circ f.$$

The tensor product of two Hom-associative algebras $(V_1, \mu_1, \alpha_1, \eta_1)$ and $(V_2, \mu_2, \alpha_2, \eta_2)$ is defined in an obvious way as the Hom-associative algebra $(V_1 \otimes V_2, \mu_1 \otimes \mu_2, \alpha_1 \otimes \alpha_2, \eta_1 \otimes \eta_2)$.

The Hom-Lie algebras were initially introduced in [4] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields.

Definition 17.2. A *Hom-Lie algebra* is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \to V$ and a linear space homomorphism $\alpha : V \to V$ satisfying

$$[x,y] = -[y,x] \quad \text{(skew-symmetry)}$$
$$\circlearrowleft_{x,y,z} [\alpha(x), [y,z]] = 0 \quad \text{(Hom-Jacobi condition)}$$

for all x, y, z from V, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

In a similar way we have the following definition of Hom-Leibniz algebra.

Definition 17.3. A *Hom-Leibniz algebra* is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$[[x,y], \alpha(z)] = [[x,z], \alpha(y)] + [\alpha(x), [y,z]]. \tag{17.2}$$

Note that if a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

17.2.2 Hom-Coalgebra Structures

Definition 17.4. A *Hom-coassociative coalgebra* is a quadruple $(V, \Delta, \beta, \varepsilon)$ where V is a $\mathbb{K}$-vector space and

$$\Delta : V \to V \otimes V, \quad \beta : V \to V \text{ and } \varepsilon : V \to \mathbb{K}$$

are linear maps satisfying the following conditions:

(C1) $(\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta$
(C2) $(id \otimes \varepsilon) \circ \Delta = id$ and $(\varepsilon \otimes id) \circ \Delta = id$.

The condition (C1) expresses the Hom-coassociativity of the comultiplication Δ. Also, it is equivalent to the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \beta \otimes \Delta} \\ V \otimes V & \xrightarrow{\Delta \otimes \beta} & V \otimes V \otimes V \end{array}$$

The condition (C2) expresses that ε is the counit which is also equivalent to the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{K}\otimes V & \stackrel{\varepsilon\otimes id_V}{\longleftarrow} & V\otimes V & \stackrel{id\otimes\varepsilon}{\longrightarrow} & V\otimes\mathbb{K} \\ & \nwarrow^{\cong} & \uparrow^{\Delta} & \nearrow^{\cong} & \\ & & V & & \end{array}$$

Let $(V,\Delta,\beta,\varepsilon)$ and $(V',\Delta',\beta',\varepsilon')$ be two Hom-coassociative coalgebras. A linear map $f\ :V\rightarrow V'$ is a morphism of Hom-coassociative coalgebras if

$$(f\otimes f)\circ\Delta=\Delta'\circ f \quad , \quad \varepsilon=\varepsilon'\circ f \quad \text{and} \quad f\circ\beta=\beta'\circ f.$$

If $V=V'$, then the previous Hom-coassociative coalgebras are isomorphic if there exists a bijective linear map $f\ :V\rightarrow V$ such that

$$\Delta'=(f\otimes f)\circ\Delta\circ f^{-1} \quad , \quad \varepsilon'=\varepsilon\circ f^{-1} \quad \text{and} \quad \beta=f^{-1}\circ\beta'\circ f.$$

In the sequel, we call *Hom-coalgebra* a triple (V,Δ,β) where V is a $\mathbb{K}$-vector space, Δ is a comultiplication not necessarily coassociative or Hom-coassociative, that is a linear map $\Delta:V\rightarrow V\otimes V$, and β is a linear map $\beta:V\rightarrow V$.

17.3 Hom-Lie Admissible Hom-Coalgebras

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and V be a vector space over $\mathbb{K}$. Let (V,Δ,β) be a Hom-coalgebra where $\Delta:V\rightarrow V\otimes V$ and $\beta:V\rightarrow V$ are linear maps and Δ is not necessarily coassociative or Hom-coassociative.

By a β*-coassociator* of Δ we call a linear map $\mathbf{c}_\beta(\Delta)$ defined by

$$\mathbf{c}_\beta(\Delta):=(\Delta\otimes\beta)\circ\Delta-(\beta\otimes\Delta)\circ\Delta.$$

Let $\mathscr{S}_3$ be the symmetric group of order 3. Given $\sigma\in\mathscr{S}_3$, we define a linear map

$$\Phi_\sigma\ :V^{\otimes 3}\longrightarrow V^{\otimes 3}$$

by

$$\Phi_\sigma(x_1\otimes x_2\otimes x_3)=x_{\sigma^{-1}(1)}\otimes x_{\sigma^{-1}(2)}\otimes x_{\sigma^{-1}(3)}.$$

Recall that $\Delta^{op}=\tau\circ\Delta$ where τ is the usual flip that is $\tau(x\otimes y)=y\otimes x$.

Definition 17.5. A triple (V,Δ,β) is a *Hom-Lie admissible Hom-coalgebra* if the linear map

$$\Delta_L:V\longrightarrow V\otimes V$$

defined by $\Delta_L=\Delta-\Delta^{op}$, is a Hom-Lie coalgebra multiplication, that is the following condition is satisfied

$$\mathbf{c}_\beta(\Delta_L)+\Phi_{(213)}\circ\mathbf{c}_\beta(\Delta_L)+\Phi_{(231)}\circ\mathbf{c}_\beta(\Delta_L)=0 \tag{17.3}$$

where (213) and (231) are the two cyclic permutations of order 3 in $\mathscr{S}_3$.

Remark 17.1. Since $\Delta_L = \Delta - \Delta^{op}$, the equality $\Delta_L^{op} = -\Delta_L$ holds.

Lemma 17.1. *Let (V,Δ,β) be a Hom-coalgebra where $\Delta : V \to V\otimes V$ and $\beta : V \to V$ are linear maps and Δ is not necessarily coassociative or Hom-coassociative, then the following relations are true*

$$\begin{aligned}
\boldsymbol{c}_\beta(\Delta^{op}) &= -\Phi_{(13)}\circ \boldsymbol{c}_\beta(\Delta) && (17.4)\\
(\beta\otimes\Delta^{op})\circ\Delta &= \Phi_{(13)}\circ(\Delta\otimes\beta)\circ\Delta^{op} && (17.5)\\
(\beta\otimes\Delta)\circ\Delta^{op} &= \Phi_{(13)}\circ(\Delta^{op}\otimes\beta)\circ\Delta && (17.6)\\
(\Delta\otimes\beta)\circ\Delta^{op} &= \Phi_{(213)}\circ(\beta\otimes\Delta)\circ\Delta && (17.7)\\
(\Delta^{op}\otimes\beta)\circ\Delta &= \Phi_{(12)}\circ(\Delta\otimes\beta)\circ\Delta. && (17.8)
\end{aligned}$$

Lemma 17.2. *The β-coassociator of Δ_L is expressed using Δ and Δ^{op} as follows:*

$$\begin{aligned}
\boldsymbol{c}_\beta(\Delta_L) &= \boldsymbol{c}_\beta(\Delta)+\boldsymbol{c}_\beta(\Delta^{op}) && (17.9)\\
&\quad -(\Delta\otimes\beta)\circ\Delta^{op}-(\Delta^{op}\otimes\beta)\circ\Delta+\\
&\quad \Phi_{(13)}\circ(\Delta\otimes\beta)\circ\Delta^{op}+\Phi_{(13)}\circ(\Delta^{op}\otimes\beta)\circ\Delta\\
&= \boldsymbol{c}_\beta(\Delta)-\Phi_{(13)}\circ\boldsymbol{c}_\beta(\Delta) && (17.10)\\
&\quad -\Phi_{(213)}\circ(\beta\otimes\Delta)\circ\Delta-\Phi_{(12)}\circ(\Delta\otimes\beta)\circ\Delta\\
&\quad +\Phi_{(23)}\circ(\beta\otimes\Delta)\circ\Delta+\Phi_{(231)}\circ(\Delta\otimes\beta)\circ\Delta.
\end{aligned}$$

Proposition 17.1. *Let (V,Δ,β) be a Hom-coalgebra. Then one has*

$$\boldsymbol{c}_\beta(\Delta_L)+\Phi_{(213)}\circ\boldsymbol{c}_\beta(\Delta_L)+\Phi_{(231)}\circ\boldsymbol{c}_\beta(\Delta_L)=2\sum_{\sigma\in\mathscr{S}_3}(-1)^{\varepsilon(\sigma)}\Phi_\sigma\circ\boldsymbol{c}_\beta(\Delta) \quad (17.11)$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ.

Proof. By (17.10) and multiplication rules in the group $\mathscr{S}_3$, it follows that

$$\begin{aligned}
\Phi_{(213)}\circ\mathbf{c}_\beta(\Delta_L) &= \Phi_{(213)}\circ\mathbf{c}_\beta(\Delta)-\Phi_{(213)}\circ\Phi_{(13)}\circ\mathbf{c}_\beta(\Delta)\\
&\quad -\Phi_{(213)}\circ\Phi_{(213)}\circ(\beta\otimes\Delta)\circ\Delta-\Phi_{(213)}\circ\Phi_{(12)}\circ(\Delta\otimes\beta)\circ\Delta\\
&\quad +\Phi_{(213)}\circ\Phi_{(23)}\circ(\beta\otimes\Delta)\circ\Delta+\Phi_{(213)}\circ\Phi_{(231)}\circ(\Delta\otimes\beta)\circ\Delta\\
&= \Phi_{(213)}\circ\mathbf{c}_\beta(\Delta)-\Phi_{(12)}\circ\mathbf{c}_\beta(\Delta) && (17.12)\\
&\quad -\Phi_{(231)}\circ(\beta\otimes\Delta)\circ\Delta-\Phi_{(23)}\circ(\Delta\otimes\beta)\circ\Delta\\
&\quad +\Phi_{(13)}\circ(\beta\otimes\Delta)\circ\Delta+(\Delta\otimes\beta)\circ\Delta,\\
\Phi_{(231)}\circ\mathbf{c}_\beta(\Delta_L) &= \Phi_{(231)}\circ\mathbf{c}_\beta(\Delta)-\Phi_{(231)}\circ\Phi_{(13)}\circ\mathbf{c}_\beta(\Delta)\\
&\quad -\Phi_{(231)}\circ\Phi_{(213)}\circ(\beta\otimes\Delta)\circ\Delta-\Phi_{(231)}\circ\Phi_{(12)}\circ(\Delta\otimes\beta)\circ\Delta\\
&\quad +\Phi_{(231)}\circ\Phi_{(23)}\circ(\beta\otimes\Delta)\circ\Delta+\Phi_{(231)}\circ\Phi_{(231)}\circ(\Delta\otimes\beta)\circ\Delta\\
&= \Phi_{(231)}\circ\mathbf{c}_\beta(\Delta)-\Phi_{(23)}\circ\mathbf{c}_\beta(\Delta) && (17.13)\\
&\quad -(\beta\otimes\Delta)\circ\Delta-\Phi_{(13)}\circ(\Delta\otimes\beta)\circ\Delta\\
&\quad +\Phi_{(12)}\circ(\beta\otimes\Delta)\circ\Delta+\Phi_{(213)}\circ(\Delta\otimes\beta)\circ\Delta.
\end{aligned}$$

After summing up the equalities (17.10), (17.12) and (17.13) the terms on the right hand sides may be pairwise combined into the terms of the form $(-1)^{\varepsilon(\sigma)}\Phi_\sigma \circ \mathbf{c}_\beta(\Delta)$ with each one being present in the sum twice for all $\sigma \in \mathscr{S}_3$.

Definition 17.5 together with (17.11) yields the following corollary.

Corollary 17.1. *A triple (V,Δ,β) is a Hom-Lie admissible Hom-coalgebra if and only if*

$$\sum_{\sigma\in\mathscr{S}_3}(-1)^{\varepsilon(\sigma)}\Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ.

17.3.1 G-Hom-Coalgebra Structures

In this section we introduce, as in the multiplication case, the notion of *G-Hom-coalgebra* where G is a subgroup of the symmetric group $\mathscr{S}_3$.

Definition 17.6. Let G be a subgroup of the symmetric group $\mathscr{S}_3$, A Hom-coalgebra (V,Δ,β) is called *G-Hom-coalgebra* if

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0 \tag{17.14}$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ.

Proposition 17.2. *Let G be a subgroup of the permutations group $\mathscr{S}_3$. Then any G-Hom-Coalgebra (V,Δ,β) is a Hom-Lie admissible Hom-coalgebra.*

Proof. The skew-symmetry follows straightaway from the definition. Take the set of conjugacy classes $\{gG\}_{g\in I}$ where $I \subseteq G$, and for any $\sigma_1,\sigma_2 \in I, \sigma_1 \neq \sigma_2 \Rightarrow \sigma_1 G \bigcap \sigma_1 G = \emptyset$. Then

$$\sum_{\sigma\in\mathscr{S}_3}(-1)^{\varepsilon(\sigma)}\Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = \sum_{\sigma_1\in I}\sum_{\sigma_2\in\sigma_1 G}(-1)^{\varepsilon(\sigma)}\Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0.$$

The subgroups of $\mathscr{S}_3$ are

$$G_1 = \{Id\},\ G_2 = \{Id,\tau_{12}\},\ G_3 = \{Id,\tau_{23}\},$$

$$G_4 = \{Id,\tau_{13}\},\ G_5 = A_3,\ G_6 = \mathscr{S}_3,$$

where A_3 is the alternating group and where τ_{ij} is the transposition between i and j.

We obtain the following type of Hom-Lie-admissible Hom-coalgebras:

- The G_1-Hom-coalgebras are the Hom-associative coalgebras defined above.
- The G_2-Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(12)}\mathbf{c}_\beta(\Delta) = 0.$$

- The G_3-Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta)+\Phi_{(23)}\mathbf{c}_\beta(\Delta)=0.$$

- The G_4-Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta)+\Phi_{(13)}\mathbf{c}_\beta(\Delta)=0.$$

- The G_5-Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta)+\Phi_{(213)}\mathbf{c}_\beta(\Delta)+\Phi_{(231)}\mathbf{c}_\beta(\Delta)=0.$$

  If the product μ is skewsymmetric then the previous condition is exactly the Hom-Jacobi identity.
- The G_6-Hom-coalgebras are the Hom-Lie-admissible coalgebras.

The G_2-Hom-coalgebras may be called Vinberg-Hom-coalgebras and G_3-Hom-coalgebras may be called preLie-Hom-coalgebras. The two classes define in fact the same class.

Definition 17.7. A *Vinberg-Hom-coalgebra* is a triple (V,Δ,β) consisting of a linear space V, a linear map $\mu : V \to V\times V$ and a homomorphism β satisfying

$$\mathbf{c}_\beta(\Delta)+\Phi_{(12)}\mathbf{c}_\beta(\Delta)=0.$$

Definition 17.8. A *preLie-Hom-coalgebra* is a triple (V,Δ,β) consisting of a linear space V, a linear map $\mu : V \to V\times V$ and a homomorphism β satisfying

$$\mathbf{c}_\beta(\Delta)+\Phi_{(23)}\mathbf{c}_\beta(\Delta)=0.$$

More generally, by dualization we have a correspondence between G-Hom-associative algebras introduced in [12] and G-Hom-coalgebras for a subgroup G of $\mathscr{S}_3$.

Let G be a subgroup of $\mathscr{S}_3$ and (V,μ,α) be a G-Hom-associative algebra that is $\mu : V\otimes V \to V$ and $\alpha : V\to V$ are linear maps and the following condition is satisfied

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}a_{\alpha,\mu}\circ\Phi_\sigma=0. \tag{17.15}$$

where $a_{\alpha,\mu}$ is the α-associator that is $a_{\alpha,\mu}=\mu\circ(\mu\otimes\alpha)-\mu\circ(\alpha\otimes\mu)$.

Setting

$$(\mu\otimes\alpha)_G=\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}(\mu\otimes\alpha)\circ\Phi_\sigma \text{ and } (\alpha\otimes\mu)_G=\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}(\alpha\otimes\mu)\circ\Phi_\sigma$$

the condition (17.15) is equivalent to the following commutative diagram

$$\begin{array}{ccc} V\otimes V\otimes V & \xrightarrow{(\mu\otimes\alpha)_G} & V\otimes V \\ \Big\downarrow{\scriptstyle (\alpha\otimes\mu)_G} & & \downarrow{\scriptstyle \mu} \\ V\otimes V & \xrightarrow{\mu} & V \end{array}$$

By the dualization of the square one may obtain the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V\otimes V \\ \downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle (\beta\otimes\Delta)_G} \\ V\otimes V & \xrightarrow{(\Delta\otimes\beta)_G} & V\otimes V\otimes V \end{array}$$

where

$$(\beta\otimes\Delta)_G=\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\Phi_\sigma\circ(\beta\otimes\Delta) \text{ and } (\Delta\otimes\beta)_G=\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\Phi_\sigma\circ(\Delta\otimes\beta).$$

The previous commutative diagram expresses that (V,Δ,β) is a G-Hom-coalgebra. More precisely we have the following connection between G-Hom-coalgebras and G-Hom-associative algebras.

Proposition 17.3. *Let (V,Δ,β) be a G-Hom-coalgebra where G is a subgroup of $\mathscr{S}_3$. Its dual vector space $V^\star$ is provided with a G-Hom-associative algebra $(V^\star,\Delta^\star,\beta^\star)$ where $\Delta^\star,\beta^\star$ are the transpose maps.*

Proof. Let (V,Δ,β) be a G-Hom-coalgebra. Let $V^\star$ be the dual space of V ($V^\star = Hom(V,\mathbb{K})$).

Consider the map

$$\lambda_n : (V^\star)^{\otimes n} \longrightarrow (V^\star)^{\otimes n}$$
$$f_1\otimes\cdots\otimes f_n \longrightarrow \lambda_n(f_1\otimes\cdots\otimes f_n)$$

such that for $v_1\otimes\cdots\otimes v_n\in V^{\otimes n}$

$$\lambda_n(f_1\otimes\cdots\otimes f_n)(v_1\otimes\cdots\otimes v_n)=f_1(v_1)\otimes\cdots\otimes f_n(v_n)$$

and set

$$\mu := \Delta^\star\circ\lambda_2 \qquad \alpha := \beta^*$$

where the star $\star$ denotes the transpose linear map. Then, the quadruple $(V^\star,\mu,\eta,\alpha)$ is a G-Hom-associative algebra. Indeed, $\mu(f_1,f_2)=\mu_{\mathbb{K}}\circ\lambda_2(f_1\otimes f_2)\circ\Delta$ where $\mu_{\mathbb{K}}$ is the multiplication of $\mathbb{K}$ and $f_1,f_2\in V^*$. One has

$$\begin{aligned} \mu\circ(\mu\otimes\alpha)(f_1\otimes f_2\otimes f_3) &= \mu(\mu(f_1\otimes f_2)\otimes\alpha(f_3)) \\ &= \mu_{\mathbb{K}}\circ\lambda_2(\mu(f_1\otimes f_2)\otimes\alpha(f_3))\circ\Delta \\ &= \mu_{\mathbb{K}}\circ\lambda_2(\lambda_2((f_1\otimes f_2)\circ\Delta)\otimes\alpha(f_3))\circ\Delta \\ &= \mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes id)\circ\lambda_3(f_1\otimes f_2\otimes f_3)\circ(\Delta\otimes\beta)\circ\Delta. \end{aligned}$$

Similarly

$$\mu \circ (\alpha \otimes \mu)(f_1 \otimes f_2 \otimes f_3) = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\beta \otimes \Delta) \circ \Delta.$$

Using the associativity and the commutativity of $\mu_{\mathbb{K}}$, the α-associator may be written as

$$a_{\alpha,\mu} = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ ((\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta).$$

Then we have the following connection between the α-associator and β-coassociator

$$a_{\alpha,\mu} = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ \mathbf{c}_\beta(\Delta).$$

Therefore if (V, Δ, β) is a G-Hom-coalgebra, then the $(V^\star, \Delta^\star, \beta^\star)$ is a G-Hom-associative algebra.

Proposition 17.4. *Let (V, μ, α) be a finite-dimensional G-Hom-associative algebra where G is a subgroup of $\mathscr{S}_3$. Its dual vector space $V^\star$ is provided with a G-Hom-coalgebra $(V^\star, \mu^\star, \alpha^\star)$, where $\mu^\star, \alpha^\star$ are the transpose maps.*

Proof. Let $\mathscr{A} = (V, \mu, \alpha)$ be a n-dimensional Hom-associative algebra (n finite). Let $\{e_1, \cdots, e_n\}$ be a basis of V and $\{e_1^*, \cdots, e_n^*\}$ be the dual basis. Then $\{e_i^* \otimes e_j^*\}_{i,j}$ is a basis of $\mathscr{A}^\star \otimes \mathscr{A}^\star$. The comultiplication $\Delta = \mu^\star$ on $\mathscr{A}^*$ is defined for $f \in \mathscr{A}^\star$ by

$$\Delta(f) = \sum_{i,j=1}^{n} f(\mu(e_i \otimes e_j))\; e_i^* \otimes e_j^*.$$

Set $\mu(e_i \otimes e_j) = \sum_{k=1}^n C_{ij}^k e_k$ and $\alpha(e_i) = \sum_{k=1}^n \alpha_i^k e_k$. Then $\Delta(e_k^*) = \sum_{i,j=1}^n C_{ij}^k\; e_i^* \otimes e_j^*$ and $\beta(e_i) = \alpha^\star(e_i) = \sum_{k=1}^n \alpha_k^i e_k$.

The condition (17.14) of G-Hom-coassociativity of Δ, applied to any element e_k^* of the basis, is equivalent to

$$\sum_{p,q,s=1}^{n} \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\sum_{i,j=1}^{n} \alpha_s^j C_{ij}^k C_{pq}^i - \alpha_p^i C_{ij}^k C_{qs}^j) e_{\sigma^{-1}(p)}^* \otimes e_{\sigma^{-1}(q)}^* \otimes e_{\sigma-1(s)}^* = 0.$$

Therefore Δ is G-Hom-coassociative if for any $p, q, s, k \in \{1, \cdots, n\}$ one has

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\sum_{i,j=1}^{n} \alpha_s^j C_{ij}^k C_{pq}^i - \alpha_p^i C_{ij}^k C_{qs}^j) = 0.$$

The previous system is exactly the condition (17.15) of G-Hom-associativity of μ, written on $e_{p'} \otimes e_{q'} \otimes e_{s'}$ and setting $p = \sigma(p'),\ q = \sigma(q'),\ s = \sigma(s')$.

Corollary 17.2. *The dual vector space of a Hom-coassociative coalgebra $(V, \Delta, \beta, \varepsilon)$ is a Hom-associative algebra $(V^\star, \Delta^\star, \beta^\star, \varepsilon^\star)$, where $V^\star$ is the dual vector space and the star for the linear maps denotes the transpose map. The dual vector space of finite-dimensional Hom-associative algebra is a Hom-coassociative coalgebra.*

Proof. It is a particular case of the previous Propositions ($G = G_1$).

17.4 Hom-Hopf Algebras

In this section, we introduce a generalization of Hopf algebras and show some relevant properties of the new structure. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra. For classical Hopf algebras theory, we refer to [1, 3, 6, 10, 11, 13]. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and V be a vector space over $\mathbb{K}$.

Definition 17.9. A *Hom-bialgebra* is a quintuple $(V,\mu,\alpha,\eta,\Delta,\beta,\varepsilon)$ where

(B1) (V,μ,α,η) is a Hom-associative algebra
(B2) $(V,\Delta,\beta,\varepsilon)$ is a Hom-coassociative coalgebra
(B3) The linear maps Δ and ε are morphisms of algebras (V,μ,α,η).

Remark 17.2. The condition (B3) could be expressed by the following system:

$$\begin{cases} \Delta\left(e_1\right)=e_1\otimes e_1 \qquad \text{where } e_1=\eta\left(1\right) \\ \Delta\left(\mu(x\otimes y)\right)=\Delta\left(x\right)\bullet\Delta\left(y\right)=\sum_{(x)(y)}\mu(x^{(1)}\otimes y^{(1)})\otimes\mu(x^{(2)}\otimes y^{(2)}) \\ \varepsilon\left(e_1\right)=1 \\ \varepsilon\left(\mu(x\otimes y)\right)=\varepsilon\left(x\right)\varepsilon\left(y\right) \end{cases}$$

where the bullet $\bullet$ denotes the multiplication on tensor product and by using the Sweedler's notation $\Delta\left(x\right)=\sum_{(x)}x^{(1)}\otimes x^{(2)}$. If there is no ambiguity we denote the multiplication by a dot.

Remark 17.3. One can consider a more restrictive definition where linear maps Δ and ε are morphisms of Hom-associative algebras that is the condition (B3) becomes equivalent to

$$\begin{cases} \Delta\left(e_1\right)=e_1\otimes e_1 \qquad \text{where } e_1=\eta\left(1\right) \\ \Delta\left(\mu(x\otimes y)\right)=\Delta\left(x\right)\bullet\Delta\left(y\right)=\sum_{(x)(y)}\mu(x^{(1)}\otimes y^{(1)})\otimes\mu(x^{(2)}\otimes y^{(2)}) \\ \varepsilon\left(e_1\right)=1 \\ \varepsilon\left(\mu(x\otimes y)\right)=\varepsilon\left(x\right)\varepsilon\left(y\right) \\ \Delta\left(\alpha(x)\right)=\sum_{(x)}\alpha(x^{(1)})\otimes\alpha(x^{(2)}) \\ \varepsilon\circ\alpha\left(x\right)=\varepsilon\left(x\right) \end{cases}$$

Given a Hom-bialgebra $(V,\mu,\alpha,\eta,\Delta,\beta,\varepsilon)$, we show that the vector space $Hom\,(V,V)$ with the multiplication given by the convolution product carries a structure of Hom-algebra.

Proposition 17.5. *Let $(V,\mu,\alpha,\eta,\Delta,\beta,\varepsilon)$ be a Hom-bialgebra. Then the algebra $Hom(V,V)$ with the multiplication given by the convolution product defined by*

$$f\star g=\mu\circ(f\otimes g)\circ\Delta$$

and the unit being $\eta\circ\varepsilon$ is a Hom-associative algebra with the homomorphism map defined by $\gamma(f)=\alpha\circ f\circ\beta$.

Proof. Let $f,g,h \in Hom(V,V)$. Then

$$\begin{aligned}\gamma(f)*(g*h)) &= \mu\circ(\gamma(f)\otimes(g*h))\Delta\\ &= \mu\circ(\gamma(f)\otimes(\mu\circ(g\otimes h)\circ\Delta))\Delta\\ &= \mu\circ(\alpha\otimes\mu)\circ(f\otimes g\otimes h)\circ(\beta\otimes\Delta))\Delta.\end{aligned}$$

Similarly

$$(f*g)*\gamma(h) = \mu\circ(\mu\otimes\alpha)\circ(f\otimes g\otimes h)\circ(\Delta\otimes\beta))\Delta.$$

Then, the Hom-associativity of μ and the Hom-coassociativity of Δ lead to the Hom-associativity of the convolution product. The unitality is as usual.

Definition 17.10. An endomorphism S of V is said to be an *antipode* if it is the inverse of the identity over V for the Hom-algebra $Hom(V,V)$ with the multiplication given by the convolution product defined by

$$f\star g = \mu\circ(f\otimes g)\Delta$$

and the unit being $\eta\circ\varepsilon$.

The condition being antipode may be expressed by the condition:

$$\mu\circ S\otimes Id\circ\Delta = \mu\circ Id\otimes S\circ\Delta = \eta\circ\varepsilon.$$

Definition 17.11. A *Hom-Hopf algebra* is a Hom-bialgebra with an antipode.

Then, a Hom-Hopf algebra over a $\mathbb{K}$-vector space V is given by

$$\mathscr{H} = (V,\mu,\alpha,\eta,\Delta,\beta,\varepsilon,S)$$

where the following homomorphisms

$$\mu : V\otimes V\rightarrow, \quad \eta:\mathbb{K}\rightarrow V, \quad \alpha: V\rightarrow V$$

$$\Delta: V\rightarrow V\otimes V, \quad \varepsilon: V\rightarrow\mathbb{K}, \quad \beta: V\rightarrow V$$

$$S: V\rightarrow\mathbb{K}$$

satisfy the following conditions:

1. (V,μ,α,η) is a unital Hom-associative algebra.
2. $(V,\Delta,\beta,\varepsilon)$ is a counital Hom-coalgebra.
3. Δ and ε are morphisms of algebras, which translate to

$$\begin{cases}\Delta(e_1) = e_1\otimes e_1 \qquad \text{where } e_1 = \eta(1)\\ \Delta(x\cdot y) = \Delta(x)\bullet\Delta(y) = \sum_{(x)(y)} x^{(1)}\cdot y^{(1)}\otimes x^{(2)}\cdot y^{(2)}\\ \varepsilon(e_1) = 1\\ \varepsilon(x\cdot y) = \varepsilon(x)\,\varepsilon(y)\end{cases}$$

4. S is the antipode, so

$$\mu \circ S \otimes Id \circ \Delta = \mu \circ Id \otimes S \circ \Delta = \eta \circ \varepsilon.$$

Remark 17.4. Let V be a finite-dimensional $\mathbb{K}$-vector space.
If $\mathscr{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ is a Hom-Hopf algebra, then

$$\mathscr{H}^* = (V^*, \Delta^*, \beta^*, \varepsilon^*, \mu^*, \alpha^*, \eta^*, S^*)$$

is also a Hom-Hopf algebra.

17.4.1 Primitive Elements and Generalized Primitive Elements

In the following, we discuss the properties of primitive elements in a Hom-bialgebra.

Let $\mathscr{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit.

Definition 17.12. An element $x \in \mathscr{H}$ is called primitive if $\Delta(x) = e_1 \otimes x + x \otimes e_1$.

Let $x \in \mathscr{H}$ be a primitive element. The coassociativity of Δ implies

$$(\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)$$

where τ_{13} is a permutation in the symmetric group $\mathscr{S}_3$.

Lemma 17.3. *Let x be a primitive element in $\mathscr{H}$, then $\varepsilon(x) = 0$.*

Proof. By counity property, we have $x = (id \otimes \varepsilon) \circ \Delta(x)$. If $\Delta(x) = e_1 \otimes x + x \otimes e_1$, then $x = \varepsilon(x)e_1 + \varepsilon(e_1)x$, and since $\varepsilon(e_1) = 1$ it implies $\varepsilon(x) = 0$.

Proposition 17.6. *Let $\mathscr{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If x and y are two primitive elements in $\mathscr{H}$. Then we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a primitive element.*

The set of all primitive elements of $\mathscr{H}$, denoted by $Prim(\mathscr{H})$, has a structure of Hom-Lie algebra.

Proof. By a direct calculation one has

$$\begin{aligned}
\Delta([x,y]) &= \Delta(\mu(x \otimes y) - \mu(y \otimes x)) \\
&= \Delta(x) \bullet \Delta(y) - \Delta(y) \bullet \Delta(x) \\
&= (e_1 \otimes x + x \otimes e_1) \bullet (e_1 \otimes y + y \otimes e_1) - (e_1 \otimes y + y \otimes e_1) \bullet (e_1 \otimes x + x \otimes e_1) \\
&= e_1 \otimes \mu(x \otimes y) + y \otimes x + x \otimes y + \mu(x \otimes y) \otimes e_1 \\
&\quad - e_1 \otimes \mu(y \otimes x) - x \otimes y - y \otimes x - \mu(y \otimes x) \otimes e_1 \\
&= e_1 \otimes (\mu(x \otimes y) - \mu(y \otimes x)) + (\mu(x \otimes y) - \mu(y \otimes x)) \otimes e_1 \\
&= e_1 \otimes [x,y] + [x,y] \otimes e_1
\end{aligned}$$

which means that $Prim(\mathscr{H})$ is closed under the bracket multiplication $[\cdot,\cdot]$.

We have seen in [12] that there is a natural map from the Hom-associative algebras to Hom-Lie algebras. The bracket $[x,y] = \mu(x\otimes y) - \mu(y\otimes x)$ is obviously skewsymmetric and one checks that the Hom-Jacobi condition is satisfied:

$$\begin{gathered}[\alpha(x),[y,z]] - [[x,y],\alpha(z)] - [\alpha(y),[x,z]] = \\ \mu(\alpha(x)\otimes\mu(y\otimes z)) - \mu(\alpha(x)\otimes\mu(z\otimes y)) - \mu(\mu(y\otimes z)\otimes\alpha(x)) \\ +\mu(\mu(z\otimes y)\otimes\alpha(x)) - \mu(\mu(x\otimes y)\otimes\alpha(z)) + \mu(\mu(y\otimes x)\otimes\alpha(z)) \\ +\mu(\alpha(z)\otimes\mu(x\otimes y)) - \mu(\alpha(z)\otimes\mu(y\otimes x)) - \mu(\alpha(y)\otimes\mu(x\otimes z)) \\ +\mu(\alpha(y)\otimes\mu(z\otimes x)) + \mu(\mu(x\otimes z)\otimes\alpha(y)) - \mu(\mu(z\otimes x)\otimes\alpha(y)) = 0\end{gathered}$$

We introduce now a notion of generalized primitive element.

Definition 17.13. An element $x\in\mathscr{H}$ is called generalized primitive element if it satisfies the conditions

$$(\beta\otimes\Delta)\circ\Delta(x) = \tau_{13}\circ(\Delta\otimes\beta)\circ\Delta(x) \tag{17.16}$$

$$\Delta^{op}(x) = \Delta(x) \tag{17.17}$$

where τ_{13} is a permutation in the symmetric group $\mathscr{S}_3$.

Remark 17.5.
1. In particular, a primitive element in $\mathscr{H}$ is a generalized primitive element.
2. The condition (17.16) may be written

$$(\Delta\otimes\beta)\circ\Delta(x) = \tau_{13}\circ(\beta\otimes\Delta)\circ\Delta(x).$$

Proposition 17.7. *Let $\mathscr{H} = (V,\mu,\alpha,\eta,\Delta,\beta,\varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If x and y are two generalized primitive elements in $\mathscr{H}$. Then, we have $\varepsilon(x) = 0$ and the commutator $[x,y] = \mu(x\otimes y) - \mu(y\otimes x)$ is also a generalized primitive element.*

The set of all generalized primitive elements of $\mathscr{H}$, denoted by GPrim($\mathscr{H}$), has a structure of Hom-Lie algebra.

Proof. Let x and y be two generalized primitive elements in $\mathscr{H}$. In the following the multiplication μ is denoted by a dot. The following equalities hold:

$$\begin{aligned}(\Delta\otimes\beta)\circ\Delta(x\cdot y - y\cdot x) &= (\Delta\otimes\beta)\circ\Delta(x\cdot y) - (\Delta\otimes\beta)\circ\Delta(y\cdot x) \\ &= (\Delta\otimes\beta)(\Delta(x)\bullet\Delta(y)) - (\Delta\otimes\beta)(\Delta(y)\bullet\Delta(x)) \\ &= \Delta(x^{(1)}\cdot y^{(1)})\otimes\beta(x^{(2)}\cdot y^{(2)}) - \Delta(y^{(1)}\cdot x^{(1)})\otimes\beta(y^{(2)}\cdot x^{(2)}) \\ &= (x^{(1)(1)}\cdot y^{(1)(1)})\otimes(x^{(1)(2)}\cdot y^{(1)(2)})\otimes\beta(x^{(2)}\cdot y^{(2)}) \\ &\quad -(y^{(1)(1)}\cdot x^{(1)(1)})\otimes(y^{(1)(2)}\cdot x^{(1)(2)})\otimes\beta(y^{(2)}\cdot x^{(2)}).\end{aligned}$$

Then, using the fact that $\Delta^{op} = \Delta$ for generalized primitive elements one has:

$$\begin{aligned}\tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) &= \beta(x^{(2)} \cdot y^{(2)}) \otimes (x^{(1)(2)} \cdot y^{(1)(2)}) \otimes (x^{(1)(1)} \cdot y^{(1)(1)}) \\ &\quad - \beta(y^{(2)} \cdot x^{(2)}) \otimes (y^{(1)(2)} \cdot x^{(1)(2)}) \otimes (y^{(1)(1)} \cdot x^{(1)(1)}) \\ &= (\beta \otimes \Delta) \circ \Delta(x \cdot y - y \cdot x).\end{aligned}$$

The structure of Hom-Lie algebra follows from the same argument as in the primitive elements case.

17.4.2 Antipode's Properties

Let $\mathscr{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ be a Hom-Hopf algebra.

For any element $x \in V$, using the counity and Sweedler notation, one may write

$$x = \sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes x^{(2)}. \tag{17.18}$$

Then, for any $f \in End_{\mathbb{K}}(V)$, we have

$$f(x) = \sum_{(x)} f(x^{(1)}) \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes f(x^{(2)}). \tag{17.19}$$

Let $f \star g = \mu \circ (f \otimes g) \Delta$ be the convolution product of $f, g \in End_{\mathbb{K}}(V)$. One may write

$$(f \star g)(x) = \sum_{(x)} \mu(f(x^{(1)}) \otimes g(x^{(2)})). \tag{17.20}$$

Since the antipode S is the inverse of the identity for the convolution product then S satisfies

$$\varepsilon(x)\eta(1) = \sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)}) = \sum_{(x)} \mu(x^{(1)} \otimes S(x^{(2)})). \tag{17.21}$$

Proposition 17.8. *The antipode S is unique and we have*

- $S(\eta(1)) = \eta(1)$.
- $\varepsilon \circ S = \varepsilon$.

Proof. (1) We have $S \star id = id \star S = \eta \circ \varepsilon$. Thus, $(S \star id) \star S = S \star (id \star S) = S$. If S' is another antipode of $\mathscr{H}$ then

$$S' = S' \star id \star S' = S' \star id \star S = S \star id \star S = S.$$

Therefore the antipode when it exists is unique.

(2) Setting $e_1 = \eta(1)$ and since $\Delta(e_1) = e_1 \otimes e_1$ one has

$$(S \star id)(e_1) = \mu(S(e_1) \otimes e_1) = S(e_1) = \eta(\varepsilon(e_1)) = e_1.$$

(3) Applying (17.19) to S, we obtain $S(x) = \sum_{(x)} S(x^{(1)}) \varepsilon(x^{(2)})$.

Applying ε to (17.21), we obtain

$$\varepsilon(x) = \varepsilon(\sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)})).$$

Since ε is a Hom-algebra morphism, one has

$$\varepsilon(x) = \sum_{(x)} \varepsilon(S(x^{(1)}))\varepsilon(x^{(2)}) = \varepsilon(\sum_{(x)} S(x^{(1)})\varepsilon(x^{(2)})) = \varepsilon(S(x)).$$

Thus $\varepsilon \circ S = \varepsilon$.

17.4.3 Modules and Comodules

We introduce in the following the structure of module and comodule over Hom-associative algebras.

Let $\mathscr{A} = (V, \mu, \alpha)$ be a Hom-associative $\mathbb{K}$-algebra, an $\mathscr{A}$-module (left) is a triple (M, f, γ) where M is $\mathbb{K}$-vector space and f, γ are $\mathbb{K}$-linear maps, $f : M \to M$ and $\gamma : V \otimes M \to M$, such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes V \otimes M & \xrightarrow{\mu \otimes f} & V \otimes M \\ \downarrow{\scriptstyle \alpha \otimes \gamma} & & \downarrow{\scriptstyle \gamma} \\ V \otimes M & \xrightarrow{\gamma} & M \end{array}$$

The dualization leads to comodule definition over a Hom-coassociative coalgebra.

Let $C = (V, \Delta, \beta)$ be a Hom-coassociative coalgebra. A C-comodule (right) is a triple (M, g, ρ) where M is a $\mathbb{K}$-vector space and g, ρ are $\mathbb{K}$-linear maps, $g : M \to M$ and $\rho : M \to M \otimes V$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes V \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle g \otimes \Delta} \\ M \otimes V & \xrightarrow{\rho \otimes \beta} & M \otimes V \otimes V \end{array}$$

Remark 17.6. A Hom-associative $\mathbb{K}$-algebra $\mathscr{A} = (V, \mu, \alpha)$ is a left $\mathscr{A}$-module with $M = V$, $f = \alpha$ and $\gamma = \mu$. Also, a Hom-coassociative coalgebra $C = (V, \Delta, \beta)$ is a right C-comodule with $M = V$, $g = \beta$ and $\rho = \Delta$. The properties of modules and comodules over Hom-associative algebras or Hom-coassociative algebras will be discussed in a forthcoming paper.

17.4.4 Examples

The classification of two-dimensional Hom-associative algebras, up to isomorphism, yields the following two classes. Let $B = \{e_1, e_2\}$ be a basis where $\eta(1) = e_1$ is the unit.

1. The multiplication μ_1 is defined by $\mu_1(e_1 \otimes e_i) = \mu_1(e_i \otimes e_1) = e_i$ for $i = 1,2$ and $\mu_1(e_2 \otimes e_2) = e_2$, and the homomorphism α_1 is defined, with respect to the basis B, by $\begin{pmatrix} a_1 & 0 \\ a_2 - a_1 & a_2 \end{pmatrix}$.
2. The multiplication μ_2 is defined by $\mu_2(e_1 \otimes e_i) = \mu_2(e_i \otimes e_1) = e_i$ for $i = 1,2$ and $\mu_2(e_2 \otimes e_2) = 0$, and the homomorphism α_2 is defined, with respect to the basis B, by $\begin{pmatrix} a_1 & 0 \\ a_2 & a_1 \end{pmatrix}$.

The Hom-bialgebras corresponding to the Hom-associative algebra defined by μ_1 and α_1 are given in the following table

	Comultiplication	Co-unit	homomorphism
1	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 1$	$\begin{pmatrix} b_1 & 0 \\ b_3 & b_2 \end{pmatrix}$
2	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 0$	$\begin{pmatrix} b_1 & \frac{b_1 - b_3}{2} \\ b_2 & b_3 \end{pmatrix}$
3	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 0$	$\begin{pmatrix} b_1 & b_1 - b_3 \\ b_2 & b_3 \end{pmatrix}$

Only Hom-bialgebra (2) carries a structure of Hom-Hopf algebra with an antipode defined, with respect to a basis B, by the identity matrix.

Remark 17.7. There is no Hom-bialgebra associated to the Hom-associative algebra defined by the multiplication μ_2 and any homomorphism α_2.

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