Chapter 2
Feynman Path Integral Formulation

2.1 The Path Integral

So far the discussion of quantum gravity has focused almost entirely on perturbative scenarios, where the gravitational coupling G is assumed to be weak, and the weak field expansion based on $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ can be performed with some degree of reliability. At every order in the loop expansion the problem then reduces to the systematic evaluation of an increasingly complex sequence of Gaussian integrals over the small quantum fluctuation $h_{\mu\nu}$.

But there are reasons to expect that non-perturbative effects play an important role in quantum gravity. Then an improved formulation of the quantum theory is required, which does not rely exclusively on the framework of a perturbative expansion. Indeed already classically a black hole solution can hardly be considered as a small perturbation of flat space. Furthermore, the fluctuating metric field $g_{\mu\nu}$ is dimensionless and carries therefore no natural scale. For the simpler cases of a scalar field and non-Abelian gauge theories a consistent non-perturbative formulation based on the Feynman path integral has been known for some time and is by now well developed. Combined with the lattice approach, it provides an effective and powerful tool for systematically investigating non-trivial strong coupling behavior, such as confinement and chiral symmetry breaking. These phenomena are known to be generally inaccessible in weak coupling perturbation theory. Furthermore, the Feynman path integral approach provides a manifestly covariant formulation of the quantum theory, without the need for an artificial $3+1$ split required by the more traditional canonical approach, and the ambiguities that may follow from it. In fact, as will be seen later, in its non-perturbative lattice formulation no gauge fixing of any type is required.

In a nutshell, the Feynman path integral formulation for pure quantum gravitation can be expressed in the functional integral formula

$$Z = \int_{\text{geometries}} e^{\frac{i}{\hbar} I_{\text{geometry}}} \,, \tag{2.1}$$

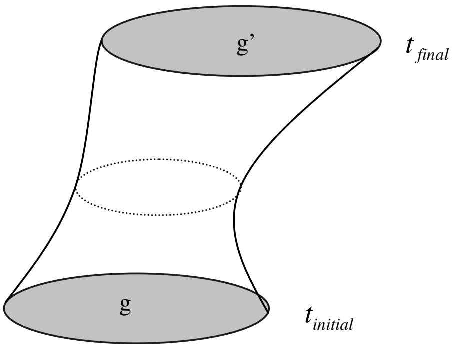

Fig. 2.1 Quantum mechanical amplitude of transitioning from an initial three-geometry described by g at time $t_{initial}$ to a final three-geometry described by g' at a later time t_{final}. The full amplitude is a sum over all intervening metrics connecting the two bounding three-surfaces, weighted by $\exp(iI/\hbar)$ where I is a suitably defined gravitational action.

(for an illustration see Fig. 2.1), just like the Feynman path integral for a non-relativistic quantum mechanical particle (Feynman, 1948; 1950; Feynman and Hibbs, 1965) expresses quantum-mechanical amplitudes in terms of sums over paths

$$A(i \to f) = \int_{\text{paths}} e^{\frac{i}{\hbar} I_{\text{path}}} \quad . \tag{2.2}$$

What is the precise meaning of the expression in Eq. (2.1)? The remainder of this section will be devoted to discussing attempts at a proper definition of the gravitational path integral of Eq. (2.1). A modern rigorous discussion of path integrals in quantum mechanics and (Euclidean) quantum field theory can be found, for example, in (Albeverio and Hoegh-Krohn, 1976), (Glimm and Jaffe, 1981) and (Zinn-Justin, 2002).

2.2 Sum over Paths

Already for a non-relativistic particle the path integral needs to be defined quite carefully, by discretizing the time coordinate and introducing a short distance cutoff. The standard procedure starts from the quantum-mechanical transition amplitude

$$A(q_i, t_i \to q_f, t_f) \; = < q_f \, | \, e^{-\frac{i}{\hbar} H(t_f - t_i)} \, | \, q_i > \;\; , \tag{2.3}$$

and subdivides the time interval into $n+1$ segments of size ε with $t_f = (n+1)\varepsilon + t_i$. Using completeness of the coordinate basis $|q_j>$ at all intermediate times, one obtains the textbook result, here for a non-relativistic particle described by a Hamiltonian $H(p,q) = p^2/(2m) + V(q)$,

$$A(q_i,t_i \to q_f,t_f) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{dq_j}{\sqrt{2\pi i\hbar\varepsilon/m}}$$
$$\times \exp\left\{ \frac{i}{\hbar} \sum_{j=1}^{n+1} \varepsilon \left[\tfrac{1}{2} m \left(\frac{q_j - q_{j-1}}{\varepsilon} \right)^2 - V\left(\frac{q_j + q_{j-1}}{2} \right) \right] \right\} \; . \tag{2.4}$$

The expression in the exponent is easily recognized as a discretized form of the classical action. The above quantum-mechanical amplitude A is then usually written in shorthand as

$$A(q_i,t_i \to q_f,t_f) = \int_{q_i(t_i)}^{q_f(t_f)} [dq] \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt\, L(q,\dot{q}) \right\} \; , \tag{2.5}$$

with $L = \frac{1}{2} m\dot{q}^2 - V(q)$ the Lagrangian for the particle. What appears therefore in the exponent is the classical action

$$I = \int_{t_i}^{t_f} dt\, L(q,\dot{q}) \; , \tag{2.6}$$

associated with a given trajectory $q(t)$, connecting the initial coordinate $q_i(t_i)$ with the final one $q_f(t_f)$. Then the quantity $[dq]$ is the functional measure over paths $q(t)$, as spelled out explicitly in the precise lattice definition of Eq. (2.4). One advantage associated with having the classical action appear in the quantum mechanical amplitude is that all the symmetries of the theory are manifest in the Lagrangian form. The symmetries of the Lagrangian then have direct implications for the study of quantum mechanical amplitudes. A stationary phase approximation to the path integral, valid in the limit $\hbar \to 0$, leads to the least action principle of classical mechanics

$$\delta I = 0 \; . \tag{2.7}$$

In the above derivation it is not necessary to use a uniform lattice spacing ε; one could have used as well a non-uniform spacing $\varepsilon_i = t_i - t_{i-1}$ but the result would have been the same in the limit $n \to \infty$ (in analogy with the definition of the Riemann sum for ordinary integrals). Since quantum mechanical paths have a zig-zag nature and are nowhere differentiable, the mathematically correct definition should be taken from the finite sum in Eq. (2.4). In fact it can be shown that differentiable paths have zero measure in the Feynman path integral: already for the non-relativistic particle most of the contributions to the path integral come from paths that are far from smooth on all scales (Feynman and Hibbs, 1965), the so-called Wiener paths, in turn related to Brownian motion. In particular, the derivative $\dot{q}(t)$ is not always defined, and the correct definition for the path integral is the one given in Eq. (2.4). A very complete and contemporary reference to the many applications of path integrals to non-relativistic quantum systems and statistical physics can be found in two recent monographs (Zinn-Justin, 2005; Kleinert, 2006).

As a next step, one can generalized the Feynman path integral construction to N particles with coordinates $q_i(t)$ ($i = 1, N$), and finally to the limiting case of continuous fields $\phi(x)$. If the field theory is defined from the start on a lattice, then the quantum fields are defined on suitable lattice points as ϕ_i.

2.3 Eulidean Rotation

In the case of quantum fields, one is generally interested in the vacuum-to-vacuum amplitude, which requires $t_i \to -\infty$ and $t_f \to +\infty$. Then the functional integral with sources is of the form

$$Z[J] = \int [d\phi] \exp\left\{ i \int d^4x [\mathscr{L}(x) + J(x)\phi(x)] \right\} \;, \tag{2.8}$$

where $[d\phi] = \prod_x d\phi(x)$, and $\mathscr{L}$ the usual Lagrangian density for the scalar field,

$$\mathscr{L} = -\tfrac{1}{2} [(\partial_\mu \phi)^2 - \mu^2 \phi^2 - i\varepsilon \phi^2] - V(\phi) \;. \tag{2.9}$$

However even with an underlying lattice discretization, the integral in Eq. (2.8) is in general ill-defined without a damping factor, due to the i in the exponent (Zinn-Justin, 2003).

Advances in axiomatic field theory (Osterwalder and Schrader, 1972; 1973; 1975; Glimm and Jaffe, 1974; Glimm and Jaffe, 1981) indicate that if one is able to construct a well defined field theory in Euclidean space $x = (\mathbf{x}, \tau)$ obeying certain axioms, then there is a corresponding field theory in Minkowski space $(\mathbf{x}, t)$ with

$$t = -i\tau \;, \tag{2.10}$$

defined as an analytic continuation of the Euclidean theory, such that it obeys the Wightmann axioms (Streater and Wightman, 2000). The latter is known as the *Euclidicity Postulate*, which states that the Minkowski Green's functions are obtained by analytic continuation of the Green's function derived from the Euclidean functional. One of the earliest discussion of the connection between Euclidean and Minkowski filed theory can be found in (Symanzik, 1969). In cases where the Minkowski theory appears pathological, the situation generally does not improve by rotating to Euclidean space. Conversely, if the Euclidean theory is pathological, the problems are generally not removed by considering the Lorentzian case. From a constructive field theory point of view it seems difficult for example to make sense, for either signature, out of one of the simplest cases: a scalar field theory where the kinetic term has the wrong sign (Gallavotti, 1985).

Then the Euclidean functional integral with sources is defined as

$$Z_E[J] = \int [d\phi] \exp\left\{ - \int d^4x [\mathscr{L}_E(x) + J(x)\phi(x)] \right\} \;, \tag{2.11}$$

with $\int \mathscr{L}_E$ the Euclidean action, and

$$\mathscr{L}_E = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}\mu^2\phi^2 + V(\phi) \ , \tag{2.12}$$

with now $(\partial_\mu \phi)^2 = (\nabla\phi)^2 + (\partial\phi/\partial\tau)^2$. If the potential $V(\phi)$ is bounded from below, then the integral in Eq. (2.11) is expected to be convergent. In addition, the Euclidicity Postulate determines the correct boundary conditions to be imposed on the propagator (the Feynman $i\varepsilon$ prescription). Euclidean field theory has a close and deep connection with statistical field theory and critical phenomena, whose foundations are surveyed for example in the comprehensive monographs of (Parisi, 1981) and (Cardy, 1997).

Turning to the case of gravity, it should be clear that to all orders in the weak field expansion there is really no difference of substance between the Lorentzian (or pseudo-Riemannian) and the Euclidean (or Riemannian) formulation. Indeed most, if not all, of the perturbative calculations in the preceding sections could have been carried out with the Riemannian weak field expansion about flat Euclidean space

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \ , \tag{2.13}$$

with signature $++++$, or about some suitable classical Riemannian background manifold, without any change of substance in the results. The structure of the divergences would have been identical, and the renormalization group properties of the coupling the same (up to the trivial replacement of say the Minkowski momentum q^2 by its Euclidean expression $q^2 = q_0^2 + \mathbf{q}^2$ etc.). Starting from the Euclidean result, the analytic continuation of results such as Eq. (1.161) to the pseudo-Riemannian case would have been trivial.

2.4 Gravitational Functional Measure

It is still true in function space that one needs a metric before one can define a volume element. Therefore, following DeWitt (DeWitt, 1962; 1964), one needs first to define an invariant norm for metric deformations

$$\|\delta g\|^2 = \int d^d x \, \delta g_{\mu\nu}(x) \, G^{\mu\nu,\alpha\beta}\big(g(x)\big) \, \delta g_{\alpha\beta}(x) \ , \tag{2.14}$$

with the supermetric G given by the ultra-local expression

$$G^{\mu\nu,\alpha\beta}\big(g(x)\big) = \tfrac{1}{2}\sqrt{g(x)}\left[g^{\mu\alpha}(x)g^{\nu\beta}(x) + g^{\mu\beta}(x)g^{\nu\alpha}(x) + \lambda\, g^{\mu\nu}(x)g^{\alpha\beta}(x)\right] \ , \tag{2.15}$$

with λ a real parameter, $\lambda \neq -2/d$. The DeWitt supermetric then defines a suitable volume element $\sqrt{G}$ in function space, such that the functional measure over the $g_{\mu\nu}$'s taken on the form

$$\int [d\, g_{\mu\nu}] \equiv \int \prod_x \left[\det G[g(x)] \right]^{1/2} \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \ . \tag{2.16}$$

The assumed locality of the supermetric $G^{\mu\nu,\alpha\beta}[g(x)]$ implies that its determinant is a local function of x as well. By a scaling argument given below one finds that, up to an inessential multiplicative constant, the determinant of the supermetric is given by

$$\det G[g(x)] \propto (1 + \tfrac{1}{2} d\, \lambda) \left[g(x) \right]^{(d-4)(d+1)/4} \ , \tag{2.17}$$

which shows that one needs to impose the condition $\lambda \neq -2/d$ in order to avoid the vanishing of $\det G$. Thus the local measure for the Feynman path integral for pure gravity is given by

$$\int \prod_x \left[g(x) \right]^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \ . \tag{2.18}$$

In four dimensions this becomes simply

$$\int [d\, g_{\mu\nu}] = \int \prod_x \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \ . \tag{2.19}$$

However it is not obvious that the above construction is unique. One could have defined, instead of Eq. (2.15), G to be almost the same, but without the $\sqrt{g}$ factor in front,

$$G^{\mu\nu,\alpha\beta}\left[g(x) \right] = \tfrac{1}{2} \left[g^{\mu\alpha}(x) g^{\nu\beta}(x) + g^{\mu\beta}(x) g^{\nu\alpha}(x) + \lambda\, g^{\mu\nu}(x) g^{\alpha\beta}(x) \right] \ . \tag{2.20}$$

Then one would have obtained

$$\det G[g(x)] \propto (1 + \tfrac{1}{2} d\, \lambda) \left[g(x) \right]^{-(d+1)} \ , \tag{2.21}$$

and the local measure for the path integral for gravity would have been given now by

$$\int \prod_x \left[g(x) \right]^{-(d+1)/2} \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \ . \tag{2.22}$$

In four dimensions this becomes

$$\int [d\, g_{\mu\nu}] = \int \prod_x \left[g(x) \right]^{-5/2} \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \ , \tag{2.23}$$

which was originally suggested in (Misner, 1957).

One can find in the original reference an argument suggesting that the last measure is unique, provided the product $\prod_x$ is interpreted over "physical" points, and invariance is imposed at one and the same "physical" point. Furthermore since there are $d(d+1)/2$ independent components of the metric in d dimensions, the Misner measure is seen to be invariant under a re-scaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ of the metric for any d, but as a result is also found to be singular at small g. Indeed the DeWitt measure of Eq. (2.18) and the Misner scale invariant measure of Eqs. (2.22) and (2.23) could be just as well regarded as two special cases of a slightly more general supermetric G with prefactor $\sqrt{g}^{(1-\omega)}$, with $\omega = 0$ and $\omega = 1$ corresponding to the original DeWitt and Misner measures, respectively.

The power in Eqs. (2.17) and (2.18) can be found for example as follows. In the Misner case, Eq. (2.22), the scale invariance of the functional measure follows directly from the original form of the supermetric $G(g)$ in Eq. (2.20), and the fact that the metric $g_{\mu\nu}$ has $\frac{1}{2}d(d+1)$ independent components in d dimensions. In the DeWitt case one rescales the matrix $G(g)$ by a factor $\sqrt{g}$. Since $G(g)$ is a $\frac{1}{2}d(d+1) \times \frac{1}{2}d(d+1)$ matrix, its determinant is modified by an overall factor of $g^{d(d+1)/4}$. So the required power in the functional measure is $-\frac{1}{2}(d+1)+\frac{1}{8}d(d+1) = \frac{1}{8}(d-4)(d+1)$, in agreement with Eq. (2.18).

Furthermore, one can show that if one introduces an n-component scalar field $\phi(x)$ in the functional integral, it leads to further changes in the gravitational measure. First, in complete analogy to the gravitational case, one has for the scalar field deformation

$$\|\delta\phi\|^2 = \int d^d x \sqrt{g(x)} \left(\delta\phi(x)\right)^2 \ , \tag{2.24}$$

and therefore for the functional measure over ϕ one has the expression

$$\int [d\phi] = \int \prod_x \left[\sqrt{g(x)}\right]^{n/2} \prod_x d\phi(x) \ . \tag{2.25}$$

The first factor clearly represents an additional contribution to the gravitational measure. One can indeed verify that one just followed the correct procedure, by evaluating for example the scalar functional integral in the large mass limit,

$$\int \prod_x \left[\sqrt{g(x)}\right]^{n/2} \prod_x d\phi(x) \exp\left(-\tfrac{1}{2}m^2 \int \sqrt{g}\,\phi^2\right) = \left(\frac{2\pi}{m^2}\right)^{nV/2} = \text{const.} \tag{2.26}$$

so that, as expected, for a large scalar mass m the field ϕ completely decouples, leaving the dynamics of pure gravity unaffected.

These arguments would lead one to suspect that the volume factor $g^{\sigma/2}$, when included in a slightly more general gravitational functional measure of the form

$$\int [d\, g_{\mu\nu}] = \prod_x [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \ , \tag{2.27}$$

perhaps does not play much of a role after all, at least as far as physical properties are concerned. Furthermore, in d dimensions the $\sqrt{g}$ volume factors are entirely absent ($\sigma = 0$) if one chooses $\omega = 1 - 4/d$, which would certainly seem the simplest choice from a practical point of view.

When considering a Hamiltonian approach to quantum gravity, one finds a rather different form for the functional measure (Leutwyler, 1964), which now includes non-covariant terms. This is not entirely surprising, as the introduction of a Hamiltonian requires the definition of a time variable and therefore a preferred direction, and a specific choice of gauge. The full invariance properties of the original action are no longer manifest in this approach, which is further reflected in the use of a rigid lattice to properly define and regulate the Hamiltonian path integral, allowing subsequent formal manipulations to have a well defined meaning. In the covariant approach one can regard formally the measure contribution as effectively a modification of the Lagrangian, leading to an L_{eff}. The additional terms, if treated consistently will result in a modification of the Hamiltonian, which therefore in general will not be of the form one would have naively guessed from the canonical rules (Abers, 2004). One can see therefore that the possible original measure ambiguity found in the covariant approach is still present in the canonical formulation. One new aspect of the Hamiltonian approach is though that conservation of probability, which implies the unitarity of the scattering matrix, can further restrict the form of the measure, if such a requirement is pushed down all the way to the cutoff scale (in a simplicial lattice context, the latter would be equivalent to the requirement of Osterwalder-Schrader reflection positivity at the cutoff scale). Whether such a requirement is physical and meaningful in a geometry that is strongly fluctuating at short distances, and for which a notion of time and orthogonal space-like hypersurfaces is not necessarily well defined, remains an open question, and perhaps mainly an academic one. When an ultraviolet cutoff is introduced (without which the theory would not be well defined), one is after all concerned in the end only with distance scales which are much larger than this short distance cutoff.

Along these lines, the following argument supporting the possible irrelevance of the measure parameter σ can be given (Faddeev and Popov, 1973; Fradkin and Vilkovisky, 1973). Namely, one can show that the gravitational functional measure of Eq. (2.27) is invariant under infinitesimal general coordinate transformations, irrespective of the value of σ. Under an infinitesimal change of coordinates $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$ one has

$$\prod_x [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \rightarrow \prod_x \left(\det \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \right)^{\gamma} [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \ , \tag{2.28}$$

with γ a power that depends on σ and the dimension. But for an infinitesimal coordinate transformations the additional factor is equal to one,

$$\prod_x \left(\det \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \right)^{\gamma} = \prod_x \left[\det (\delta_{\alpha}{}^{\beta} + \partial_{\alpha} \varepsilon^{\beta}) \right]^{\gamma} = \exp \left\{ \gamma \delta^d(0) \int d^d x \, \partial_{\alpha} \varepsilon^{\alpha} \right\} = 1 \ , \tag{2.29}$$

and we have used

$$a^d \sum_x \rightarrow \int d^d x \ , \tag{2.30}$$

with lattice spacing $a = \pi / \Lambda$ and momentum cutoff Λ [see Eq. (1.98)]. So in some respects it appears that σ can be compared to a gauge parameter.

In conclusion, there is no clear a priori way of deciding between the various choices for σ, and the evidence so far suggests that it may very well turn out to be an irrelevant parameter. The only constraint seems that the regularized gravitational path integral should be well defined, which would seem to rule out singular measures, which need additional regularizations at small volumes. It is noteworthy though that the $g^{\sigma/2}$ volume term in the measure is completely local and contains no derivatives. Thus in perturbation theory it cannot affect the propagation properties of gravitons, and only contributes ultralocal $\delta^d(0)$ terms to the effective action, as can be seen from

$$\prod_x [g(x)]^{\sigma/2} = \exp\left\{ \tfrac{1}{2}\,\sigma\,\delta^d(0) \int d^d x \ln g(x) \right\} \tag{2.31}$$

with

$$\ln g(x) = \tfrac{1}{2}\, h_\mu^{\ \mu} - \tfrac{1}{4}\, h_{\mu\nu} h^{\mu\nu} + O(h^3) \ , \tag{2.32}$$

which follows from the general formal expansion formula for an operator $\mathbf{M} \equiv 1 + \mathbf{K}$

$$\operatorname{tr}\ln(1+\mathbf{K}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{tr}\mathbf{K}^n \ , \tag{2.33}$$

which is valid provided the traces of all powers of $\mathbf{K}$ exist. On a spacetime lattice one can interpret the delta function as an ultraviolet cutoff term, $\delta^d(0) \approx \Lambda^d$. Then the first term shifts the vacuum solution and the second one modifies the bare cosmological constant. To some extent these type of contributions can be regarded as similar to the effects arising from a renormalization of the cosmological constant, ultimately affecting only the distribution of local volumes. So far numerical studies of the lattice models to be discussed later show no evidence of any sensitivity of the critical exponents to the measure parameter σ.

Later in this review (Sect. 6.9) we will again return to the issue of the functional measure for gravity in possibly the only context where it can be posed, and to some extent answered, satisfactorily: in a lattice regularized version of quantum gravity, going back to the spirit of the original definition of Eq. (2.4).

In conclusion, the Euclidean Feynman path integral for pure Einstein gravity with a cosmological constant term is given by

$$Z_{cont} \ = \ \int [d\, g_{\mu\nu}] \ \exp\left\{ -\lambda_0 \int dx \sqrt{g} + \frac{1}{16\pi G} \int dx \sqrt{g} R \right\} \ . \tag{2.34}$$

It involves a functional integration over all metrics, with measure given by a suitably regularized form of

$$\int [d\, g_{\mu\nu}] \ = \ \int \prod_x \, [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \ , \tag{2.35}$$

as in Eqs. (2.18), (2.22) and (2.27). For geometries with boundaries, further terms should be added to the action, representing the effects of those boundaries. Then the path integral will depend in general on some specified initial and final three-geometry (Hartle and Hawking, 1977; Hawking, 1979).

2.5 Conformal Instability

Euclidean quantum gravity suffers potentially from a disastrous problem associated with the conformal instability: the presence of kinetic contributions to the linearized action entering with the wrong sign.

As was discussed previously in Sect. 1.7, the action for linearized gravity without a cosmological constant term, Eq. (1.7), can be conveniently written using the three spin projection operators $P^{(0)}, P^{(1)}, P^{(2)}$ as

$$I_{\text{lin}} = \frac{k}{4} \int dx \, h^{\mu\nu} \, [P^{(2)} - 2P^{(0)}]_{\mu\nu\alpha\beta} \, \partial^2 \, h^{\alpha\beta} \quad , \tag{2.36}$$

so that the spin-zero mode enters with the wrong sign, or what is normally referred to as a ghost contribution. Actually to this order it can be removed by a suitable choice of gauge, in which the trace mode is made to vanish, as can be seen, for example, in Eq. (1.13). Still, if one were to integrate in the functional integral over the spin-zero mode, one would have to distort the integration contour to complex values, so as to render the functional integral convergent.

The problem is not removed by introducing higher derivative terms, as can be seen from the action for the linearized theory of Eq. (1.150),

$$\begin{aligned} I_{\text{lin}} = \tfrac{1}{2} \int dx \; \{ \; & h^{\mu\nu} \, [\tfrac{1}{2} k + \tfrac{1}{2} a \, (-\partial^2)] (-\partial^2) \, P^{(2)}_{\mu\nu\rho\sigma} \, h^{\rho\sigma} \\ & + h^{\mu\nu} \, [-k - 2 b \, (-\partial^2)] (-\partial^2) \, P^{(0)}_{\mu\nu\rho\sigma} \, h^{\rho\sigma} \, \} \quad , \end{aligned} \tag{2.37}$$

as the instability reappears for small momenta, where the higher derivative terms can be ignored [see for example Eq. (1.152)]. There is a slight improvement, as the instability is cured for large momenta, but it is not for small ones. If the perturbative quantum calculations can be used as a guide, then at the fixed points one has $b < 0$, corresponding to a tachyon pole in the spin-zero sector, which would indicate further perturbative instabilities. Of course in perturbation theory there never is a real problem, with or without higher derivatives, as one can just define Gaussian integrals by a suitable analytic continuation.

But the instability seen in the weak field limit is not an artifact of the weak field expansion. If one attempts to write down a path integral for pure gravity of the form

$$Z = \int [d \, g_{\mu\nu}] \, e^{-I_E} \quad , \tag{2.38}$$

with an Euclidean action

$$I_E = \lambda_0 \int dx \sqrt{g} - \frac{1}{16\pi G} \int dx \sqrt{g} R \ , \tag{2.39}$$

one realizes that it too appears ill defined due to the fact that the scalar curvature can become arbitrarily positive, or negative. In turn this can be seen as a direct consequence of the fact that while gravitational radiation has positive energy, gravitational potential energy is negative because gravity is attractive. To see more clearly that the gravitational action can be made arbitrarily negative consider the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where Ω is some positive function. Then the Einstein action transforms into

$$I_E(\tilde{g}) = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \left(\Omega^2 R + 6\, g^{\mu\nu} \partial_\mu \Omega \, \partial_\nu \Omega \right) \ , \tag{2.40}$$

which can be made arbitrarily negative by choosing a rapidly varying conformal factor Ω. Indeed in the simplest case of a metric $g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$ one has

$$\sqrt{g}\,(R - 2\lambda) = 6\, g^{\mu\nu} \partial_\mu \Omega \, \partial_\nu \Omega \ - \ 2\lambda \Omega^4 \ , \tag{2.41}$$

which looks like a $\lambda \phi^4$ theory but with the wrong sign for the kinetic term. The problem is referred to as the conformal instability of the classical Euclidean gravitational action (Hawking, 1977). The gravitational action is unbounded from below, and the functional integral is possibly divergent, depending on the detailed nature of the gravitational measure contribution $[dg_{\mu\nu}]$, more specifically its behavior in the regime of strong fields and rapidly varying conformal factors.

A possible solution to the unboundedness problem has been described by Hawking, who suggests performing the integration over all metrics by first integrating over conformal factors by distorting the integration contour in the complex plane to avoid the unboundedness problem, followed by an integration over conformal equivalence classes of metrics (Gibbons and Hawking, 1977; Hawking, 1978a,b; Gibbons, Hawking and Perry, 1978; Gibbons and Perry, 1978). Explicit examples have been given where manifestly convergent Euclidean functional integrals have been formulated in terms of physical (transverse-traceless) degrees of freedom, where the weighting can be shown to arise from a manifestly positive action (Schleich, 1985; Schleich, 1987). A similar convergent procedure seem obtainable for some so-called minisuperspace models, where the full functional integration over the fluctuating metric is replace by a finite dimensional integral over a set of parameters characterizing the reduced subspace of the metric in question, see for example (Barvinsky, 2007). But it is unclear how this procedure can be applied outside perturbation theory, where it not obvious how such a split for the metric should be performed.

An alternate possibility is that the unboundedness of the classical Euclidean gravitational action (which in the general case is certainly physical, and cannot therefore be simply removed by a suitable choice of gauge) is not necessarily an obstacle to defining the quantum theory. The quantum mechanical attractive Coulomb well problem has, for zero orbital angular momentum or in the one-dimensional case, a similar type of instability, since the action there is also unbounded from below. The way the quantum mechanical treatment ultimately evades the problem is that the particle has a vanishingly small probability amplitude to fall into the infinitely deep well. In other words, the effect of quantum mechanical fluctuations in the paths (their zig-zag motion) is just as important as the fact that the action is unbounded. Not unexpectedly, the Feynman path integral solution of the Coulomb problem requires again first the introduction of a lattice, and then a very careful treatment of the behavior close to the singularity (Kleinert, 2006). For this particular problem one is of course aided by the fact that the exact solution is known from the Schrödinger theory.

In quantum gravity the question regarding the conformal instability can then be rephrased in a similar way: Will the quantum fluctuations in the metric be strong enough so that physical excitations will not fall into the conformal well? Phrased differently, what is the role of a non-trivial gravitational measure, giving rise to a density of states $n(E)$

$$Z \propto \int_0^\infty dE \, n(E) \, e^{-E} \, , \tag{2.42}$$

regarding the issue of ultimate convergence (or divergence) of the Euclidean path integral. Of course to answer such questions satisfactorily one needs a formulation which is not restricted to small fluctuations and to the weak field limit. Ultimately in the lattice theory the answer is yes, for sufficiently strong coupling G (Hamber and Williams, 1984; Berg, 1985).

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