

Contents

Preface	vii
Acknowledgements	xi
Common Notations and Terminology	xvii
1 Introduction	1
2 Unwinding proofs ('Proof Mining')	13
2.1 Introductory remark	13
2.2 Informal treatment of ineffective proofs	13
2.3 Herbrand's theorem and the no-counterexample interpretation	22
2.4 Exercises, historical comments and suggested further reading	38
3 Intuitionistic and classical arithmetic in all finite types	41
3.1 Intuitionistic and classical predicate logic	41
3.2 Intuitionistic ('Heyting') arithmetic HA and Peano arithmetic PA .	44
3.3 Extensional intuitionistic ('Heyting') and classical ('Peano')	
arithmetic in all finite types	46
3.4 Fragments of (W)E-HA ^ω and (W)E-PA ^ω	52
3.5 Fragments corresponding to the Grzegorzcyk hierarchy	54
3.6 Models of E-PA ^ω	67
3.7 Exercises, historical comments and suggested further reading	73
4 Representation of Polish metric spaces	77
4.1 Representation of real numbers	77
4.2 Representation of complete separable metric ('Polish') spaces	81
4.3 Special representation of compact metric spaces	88
4.4 Fragments, exercises, historical comments and suggested further	
reading	94

5	Modified realizability	97
5.1	The soundness and program extraction theorems	97
5.2	Remarks on fragments of E-HA ^ω	105
5.3	Exercises, historical comments and suggested further reading	107
6	Majorizability and the fan rule	109
6.1	A syntactic treatment of majorization and the fan rule	109
6.2	Exercises, historical comments and suggested further reading	114
7	Semi-intuitionistic systems and monotone modified realizability	115
7.1	The soundness and bound extraction theorems	115
7.2	Fragments, exercises, historical comments and suggested further reading	123
8	Gödel’s functional (‘Dialectica’) interpretation	125
8.1	Introduction	125
8.2	The soundness and program extraction theorems	129
8.3	Fragments, exercises, historical comments and suggested further reading	138
9	Semi-intuitionistic systems and monotone functional interpretation . .	141
9.1	The soundness and bound extraction theorems	141
9.2	Applications of monotone functional interpretation	146
9.3	Examples of axioms Δ : Weak König’s lemma WKL	149
9.4	WKL as a universal sentence Δ	156
9.5	Fragments, exercises, historical comments and suggested further reading	160
10	Systems based on classical logic and functional interpretation	163
10.1	The negative translation	163
10.2	Combination of negative translation and functional interpretation .	165
10.3	Application: Uniform weak König’s lemma UWKL	178
10.4	Elimination of extensionality	180
10.5	Fragments of (W)E-PA ^ω	188
10.6	The computational strength of full extensionality	191
10.7	Exercises, historical comments and suggested further reading	195
11	Functional interpretation of full classical analysis	199
11.1	Functional interpretation of full comprehension	199
11.2	Functional interpretation of dependent choice	206
11.3	Functional interpretation of arithmetical comprehension	209
11.4	Functional interpretation of (IPP) by finite bar recursion	213
11.5	Models of bar recursion	214
11.6	Exercises, historical comments and suggested further reading	219

12	A non-standard principle of uniform boundedness	223
12.1	The Σ_1^0 -boundedness principle	223
12.2	Applications of Σ_1^0 -boundedness	232
12.3	Remarks on the fragments $E\text{-G}_nA^\omega$	238
12.4	Exercises, historical comments and suggested further reading	241
13	Elimination of monotone Skolem functions	243
13.1	Skolem functions of type degree 1 in fragments of finite type arithmetic	243
13.2	Elimination of Skolem functions for monotone formulas	247
13.3	The principle of convergence for bounded monotone sequences of real numbers (PCM)	262
13.4	Π_1^0 -CA and Π_1^0 -AC	265
13.5	The Bolzano-Weierstraß property for bounded sequences in \mathbb{R}^d ...	269
13.6	Exercises, historical comments and suggested further reading	272
14	The Friedman A-translation	273
14.1	The A-translation	273
14.2	Historical comments and suggested further reading	277
15	Applications to analysis: general metatheorems I	279
15.1	A general metatheorem for Polish spaces	279
15.2	Applications to uniqueness proofs	284
15.3	Applications to monotone convergence theorems	291
15.4	Applications to proofs of contractivity	292
15.5	Remarks on fragments of \mathcal{T}^ω	293
15.6	Historical comments and suggested further reading	295
16	Case study I: Uniqueness proofs in approximation theory	297
16.1	Uniqueness proofs in best approximation theory	297
16.2	Best Chebycheff approximation I	303
16.3	Best Chebycheff approximation II	328
16.4	Best L_1 -approximation	348
16.5	Exercises, historical comments and suggested further reading	376
17	Applications to analysis: general metatheorems II	377
17.1	Introduction	377
17.2	Main results in the metric and hyperbolic case	391
17.3	The case of normed spaces	410
17.4	Proofs of theorems 17.35, 17.52 and 17.69	420
17.5	Further variations	431
17.6	Treatment of several metric or normed spaces X_1, \dots, X_n simultaneously	435
17.7	A generalized uniform boundedness principle $\exists\text{-UB}^X$	436
17.8	Applications of $\exists\text{-UB}^X$	441
17.9	Fragments of $\mathcal{A}^\omega[\dots]$	450

17.10 Exercises, historical comments and suggested further reading	452
18 Case study II: Applications to the fixed point theory of nonexpansive mappings	455
18.1 General facts	455
18.2 Applications of the metatheorems from chapter 17	461
18.3 Logical analysis of the proof of the Borwein-Reich-Shafir theorem	468
18.4 Asymptotically nonexpansive mappings	496
18.5 Applications of proof mining in ergodic theory	499
18.6 Exercises, historical comments and suggested further reading	501
19 Final comments	503
References	507
List of formal systems and term classes	525
List of axioms and rules	527
Index	529



<http://www.springer.com/978-3-540-77532-4>

Applied Proof Theory: Proof Interpretations and their
Use in Mathematics

Kohlenbach, U.

2008, XX, 536 p., Hardcover

ISBN: 978-3-540-77532-4