1 Maxwell’s Equations

1.1 Introduction

In this book, we describe the principles which govern electric and magnetic or electromagnetic fields and waves. This area of knowledge, frequently referred to as Electromagnetism, has a long history and is associated with many famous names among which Maxwell has a prominent place. Maxwell was the one who, in the nineteenth century, gave electromagnetism its final form, by fixing an inconsistency and summarizing the then voluminous material into few equations, through which everything else can be derived. These equations are called Maxwell’s equations. They form the foundation of the so-called Classical Electromagnetism. The first chapter of this book shall serve to introduce these equations.

We have to emphasize, however, that Classical Electromagnetism, which is mostly expressed through Maxwell’s equations is not really complete. The 20th century brought insights that have caused extensions in two different directions. The first is related to Albert Einstein and leads to the Theory of Relativity. Application of this fundamental idea is intimately related, but not limited to electromagnetism. One could even go as far as stating that Classical Electromagnetism can only be understood, and its full importance recognized, through the perspective of the Theory of Relativity. Later we will discuss, that electromagnetic fields propagate in the form of waves. The thereby created electromagnetic waves manifest themselves in manifold ways: as radio waves, heat radiation, visible light, x-rays, gamma rays, etc. In vacuum the velocity of this propagation is the speed of light in vacuum \( c \approx 3 \cdot 10^8 \text{m/s} \). The Theory of Relativity elevates the speed of light to a quantity that is fundamental for the structure of space and time and thus making it a fundamental constant of nature. Besides this, electromagnetic waves have also brought another important knowledge. Light consists, as we have known since Planck, of individual particles called photons. Together with other fundamental discoveries, which we do not want to discuss here, this has lead to Quantum Electrodynamics. This theory treats electromagnetic fields as what they, according to the current state of knowledge, really are: namely waves and particles simultaneously. That is to say, it describes how they are created, destroyed, how they interact with other matter, etc.

Of these three closely related theories – Classical Electromagnetism, Special Relativity, and quantum-electrodynamics – we will only deal with classical electrodynamics. Nevertheless, occasionally it will be necessary to mention facts that go beyond it, and to clarify a situation may require use of elements from other theories, for example the Theory of Relativity. This restriction is purely of didactical nature and certainly not based on the idea that only classical electrodynamics is of practical value. The opposite would be true. To mention just a few examples: the characteristics and behavior of electrons in metals (band
model), behavior of semiconductors and consequently that of transistors, the processes in photoelectric cells, the achievements of laser technology, effects – equally as strange as important – such as superconductivity, etc., can only be discussed and understood by means of quantum theory.

1.2 Charge and Coulomb’s Law

In the following section we will derive Maxwell’s equations. We will do this in an abbreviated form, but roughly following the historical derivation. We will begin by considering an historically old experience, which most of us have experienced many times. If certain objects are rubbed and then separated, they exert a force on each other. Rubbing changes these objects. They are transformed into a state which we will call electric or electrically charged – whatever that may mean. To learn about those forces, we conduct the following thought experiment.

We start by choosing three different objects (A, B, C), which were electrically charged by rubbing them. There are now the following possibilities:

1. A and B attract each other
2. A and C attract each other

What would be the force between B and C? Is the answer to this question trivial? Can we make a prediction? In any case, the experiment provides us with the answer:

3. B and C repel each other

Is this surprising? Is it by chance? No, it is not chance, but a law of nature. We can repeat this experiment infinitely often and always get the same result: If A attracts both B and C, then B and C repel each other. There are other possibilities:

1. A and B repel each other
2. A and C repel each other
3. B and C repel each other

also:

1. A and B attract each other
2. A and C repel each other
3. B and C attract each other

This result may be so familiar to us that we take it for granted and it may appear trivial, but this is not so. Were we to instead deal with gravitational forces or nuclear forces, our experiment would exhibit different results. Strictly speaking, our result is correct only under the implicit assumption that the electric force is greater than any other kind of potentially superimposed force, like gravitational or nuclear forces. This restriction is very important in natural behavior. The nucleus of an atom consists partly of particles that repel each other. The nucleus would burst apart if there were not attracting forces that more than compensate the repelling electric force. Gravitation, while an attracting force, is too weak to
prevent destruction of the nucleus. One needs to remain conscious of this fact when, in the following, we make statements about electric forces.

We can summarize the experience with electrically charged objects in the following way:

1. There are two sorts of electrical charge, which we term positive and negative charges.

2. Like charges repel, while opposite charges attract each other.

These qualitative statements are, however, insufficient. At the end of the day, we want to formulate physical laws quantitatively. We will utilize the following experimental result: To begin, one can measure the force between charged objects, for example, by utilizing springs. We measure the force of A on B, and A on C. Next, we combine B and C, and then measure the force that A exerts on the combined object B+C. We will find that this force is the sum of the individual forces of the previous experiment.

This is a principal realization, whose consequence is far reaching. For now, we simply want to justify the right to expand on our qualitative statements on charge into a more quantitative one based on the magnitude of charge. We will call the charge quantity $Q$. The question of units with which to express $Q$ shall be left for later. For now, assume we have already defined the unit and found a method to measure the charge quantity $Q$ in this unit. This allows us to measure charges $Q_1$ and $Q_2$ and so the force between the two charges, which in turn, enables us to formulate Coulomb’s law:

1. The force between two charges $Q_1$ and $Q_2$ is proportional to both $Q_1$ and $Q_2$ and also inversely proportional to the square of the distance $r_{12}^2$ between them

$$ F \sim \frac{Q_1 Q_2}{r_{12}^2} \quad (1.1)$$

2. The axis of the force lies on the direct line between the charges; it is repelling for like charges, and attractive for opposite charges.

The fact that $F_{12}$ being proportional to $1/r_{12}^2$ is of great significance i.e. it is an inverse square law. We will come back to discuss the consequences of this law later. This property is shared between electric and gravitational forces.

Forces are vector quantities. An arbitrary force $\mathbf{F}$ is therefore determined by three components, for example in a Cartesian coordinate system:

$$ \mathbf{F} = \langle F_x, F_y, F_z \rangle \quad (1.2)$$

Suppose there is a charge $Q_1$ at point $\mathbf{r}_1$

$$ \mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle, \quad (1.3)$$

and a charge $Q_2$ at point $\mathbf{r}_2$

$$ \mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle \quad (1.4)$$
then we can write Coulomb’s law, combining all those statements, in the following way:

$$
F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}
$$

(1.5)

where $F_{12}$ is the force that $Q_1$ exerts on $Q_2$. Conversely, the force that $Q_2$ exerts on $Q_1$ is:

$$
F_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}
$$

(1.6)

It follows from these relations that

$$
F_{12} + F_{21} = 0
$$

(1.7)

Here, $4\pi\varepsilon_0$ is just an arbitrary proportionality constant. It is arbitrary because we still have to select the units for force, charge, and length. We will later make a selection, which in turn uniquely defines the physical constant $\varepsilon_0$, the so-called permittivity or dielectric constant in free space. Currently we have created a fairly simple world, which consists only of charges in an otherwise empty space (vacuum).

### 1.3 Electric Field Strength $E$ and Displacement Field $D$

A single charge in the otherwise empty space causes that space to change. A second charge brought into this space experiences a force at every point of that space. This force is expressed by Coulomb’s law and varies from point to point. At this stage it will be beneficial to introduce the concept of the electric field. It is the quintessence of all possible effects by such forces at the different locations of this space, which become obvious only after we place a charge at a particular point.

The term field, more generally, refers to a quantity of any kind that is a function of space (and possibly of time). This book will also deal with various kinds of fields.

The electric field strength is described by a vector quantity represented by the symbol $E$. It is defined as the force in the field per unit charge.

$$
E = \frac{F}{Q}
$$

(1.8)

This definition makes sense because the force, according to Coulomb’s law, is proportional to $Q$ and thus $E$ is independent on the (test) charge.

Furthermore, Coulomb’s law states that a charge $Q_1$ at location $\mathbf{r}_1$, at an arbitrary field point $\mathbf{r}$ produces the following electric field:

$$
E(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}
$$

(1.9)
For reasons which we will be able to understand only later, we will now define only for vacuum, the vector quantity of the electric displacement Field $D$ in the following way:

$$D = \varepsilon_0 E$$  \hspace{1cm} (1.10)

### 1.4 Electric Flux

By means of $D$ and Fig. 1.1 we define the electric flux:

$$\Omega = \int_A D \cdot dA = \int_A D_n dA$$  \hspace{1cm} (1.11)

$D_n$ is the component of $D$ that is perpendicular (normal) to the surface element $dA$. The dot indicates a scalar or dot product of two vectors. The vector $dA$ is always perpendicular to the surface element and its magnitude $|dA|$ equals the value of its surface area. That is:

$$|dA| = dA$$  \hspace{1cm} (1.12)

The term electric flux is based on the analogy to a moving fluid, where the velocity is:

$$v(r, t)$$

If the fluid is incompressible, then the amount of fluid that moves through a surface $A$ per unit of time can be expressed as

$$\int_A v \cdot dA$$

This is called flux through the surface. This analogy is frequently used for the definition of all kinds of fluxes. We now want to ponder about the question, how much electric flux passes through an arbitrary closed surface if there are charges somewhere, that is to say, charges can be inside or outside the space enclosed by our surface.

The answer to this question is rather easy if we limit ourselves to the area of a sphere (radius $r_0$), with a charge $Q_I$ is at its center. (For a closed surface let $dA$ be always oriented outwardly, see Fig. 1.2.)
Use was made of the fact that for symmetry reasons \( D_n = D = |\mathbf{D}| \). So, in this case, one finds that the flux is the charge itself. How would this result change if we changed the spherical surface into an arbitrary one? To formally solve the integral of the flux (1.11) could become very difficult. A trick, however, allows to reduce the new problem into the already solved one. We surround the charge simultaneously by an arbitrarily large surface of a sphere \( A_1 \), centered at the location of the charge \( Q_1 \), and an arbitrary surface \( A_2 \) (see Fig. 1.3). As a result, for every small cone we find the following relation:

\[
\mathbf{D}_1 \cdot d\mathbf{A}_1 = \mathbf{D}_2 \cdot d\mathbf{A}_2
\]

This is a consequence of:

\[
\mathbf{D} \cdot d\mathbf{A} = D \, dA_t,
\]

where \( dA_t \) is the component of \( d\mathbf{A} \) parallel to \( \mathbf{D} \) and the fact that although \( D \) decreases with \( 1/r^2 \) on one hand, on the other, \( dA_t \) increases with \( r^2 \), given that \( r \)
is the distance from the charge. This means, that independently of the shape of $A_2$, the same flux passes through it as passes through the surface of the sphere $A_1$.

Let us next study the flux through a closed surface where the charge is outside of it. The same arguments as before shall serve to analyze the situation in Fig. 1.4, and to show that the flux through this closed surface will vanish entirely. Every flux entering the surface will exit it as well. In summary one writes:

$$\Omega = \begin{cases} Q_1 \\ 0 \end{cases} \text{ if } Q_1 \text{ is inside of the closed surface}$$

What happens if there are multiple charges in our space? We start with the statement that in order to determine the total force caused by all charges simultaneously, it is permissible to add the forces exerted by those charges. As forces are vectors, this addition is a vectorial addition. Addition is also permitted for the electric field. This only seemingly trivial fact has received its own term:

Superposition principle, which applies to the electric field

We have made use of this principle before, when we introduced charge. We must emphasize: The superposition principle does not state that it is allowed to add forces as vectors. This fact is a basic principle of mechanics and is the reason for the usefulness of vectors altogether. The crucial point is that the force between charges is independent of the existence of other charges in its vicinity, i.e., is not changed by those other charges. This, however, is highly nontrivial and perhaps not even true under all circumstances (it may not apply, for instance, when we deal with very strong fields).

The superposition principle allows us to write an expression for $n$ charges $Q_i$ at the locations $\mathbf{r}_i$

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n} \mathbf{E}_i = \sum_{i=1}^{n} \frac{Q_i}{4\pi \varepsilon_0} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}$$

The flux $\Omega$ through any arbitrary closed surface is thus
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\[ \Omega = \oint \mathbf{D} \cdot d\mathbf{A} = \oint \sum \mathbf{D}_i \cdot d\mathbf{A} = \sum \oint \mathbf{D}_i \cdot d\mathbf{A} = \sum_{\text{enclosed}} Q_i \]

Finally

\[ \Omega = \sum_{\text{enclosed}} Q_i \]  \hspace{1cm} (1.17)

\( \Omega \) equals the sum of all charges inside this closed surface. Instead of point charges, we are able to study continuously distributed charges in some space. This requires the definition of the volume charge density \( \rho(r, t) \). It is defined as the differential quotient

\[ \rho = \lim_{d\tau \to 0} \frac{dQ}{d\tau} , \]  \hspace{1cm} (1.18)

with \( dQ \) being the charge contained in the volume element \( d\tau \). The total charge in a volume \( V \) is thus

\[ Q = \int_V \rho d\tau . \]  \hspace{1cm} (1.19)

This, on the other hand, equals the electric flux that passes through the surface of this volume, and enables us to write for any volume (see Fig. 1.5)

\[ \oint_A \mathbf{D} \cdot d\mathbf{A} = \int_V \rho d\tau . \]  \hspace{1cm} (1.20)

With this, we found a fundamental relation. It is the integral form of one of the (in total four) Maxwell’s equations. Before discussing it in more detail, we will need to introduce several other terms and concepts.

1.5 Divergence of a Vector Field and Gauss’ Integral Theorem

Equation (1.20) is applicable for any volume, in particular for an infinitesimally small one. This allows one to rewrite

\[ \int_V \rho d\tau = \rho V = \oint_A \mathbf{D} \cdot d\mathbf{A} \]

or

Fig. 1.5

Electric flux for arbitrary volume
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\[ \rho = \lim_{V \to 0} \frac{\oint_{A} \mathbf{D} \cdot d\mathbf{A}}{V} \tag{1.21} \]

This is a very important relation in vector analysis. The divergence \( \text{div} \mathbf{a} \) or \( \nabla \cdot \mathbf{a} \) for an arbitrary vector field \( \mathbf{a}(\mathbf{r}) \) is defined as the limit:

\[ \nabla \cdot \mathbf{a} = \lim_{V \to 0} \frac{\oint_{A} \mathbf{a} \cdot d\mathbf{A}}{V} \tag{1.22} \]

Comparison of (1.21) with (1.22) reveals

\[ \nabla \cdot \mathbf{D} = \rho \tag{1.23} \]

This is the equivalent of (1.20), the differential form of Maxwell’s equation. We will verify that it is in fact a differential equation.

The way we derived this equation also illustrates its significance. We use our previous example of the incompressible fluid where \( \oint_{A} \mathbf{v} \cdot d\mathbf{A} \). Therefore, \( \nabla \cdot \mathbf{v} \) can only be non-zero, if fluid flows out of the volume element (source), or flows into it (sink). To apply this to our field lines \( \mathbf{E} \) or \( \mathbf{D} \), we can say that they can only originate at locations where electric charges are (Fig. 1.6).

Electric charges are sources or sinks of the electric field

Divergence is a mathematical term suited for this fact and is a measure of the strength of the source or sink.

At this point one should be alerted to what our conclusions are based on. They are a consequence of Coulomb’s law, or more precisely of the \( 1/r^2 \) dependency in it. Would this dependency be any different, the relation between \( D, A \), and \( Q \) would not hold: \( \oint_{A} \mathbf{D} \cdot d\mathbf{A} \neq Q \) and \( \nabla \cdot \mathbf{D} \neq \rho \). In view of the streaming fluid and the \( 1/r^2 \) dependency, however, we find our results to be rather trivial. A water fountain idealized as a point source pours water evenly in all directions and produces a purely radial flux field, with \( v_r \propto (1/r^2) \). The flux \( \oint_{A} \mathbf{v} \cdot d\mathbf{A} \) which does not enclose a source has to be zero. On the other hand, we have to note that any, even the slightest deviation from Coulomb’s law, would be significant and would result

![Fig. 1.6](image-url)
in a quantitatively different electrodynamics. For that reason, it was interesting and necessary to verify by measurements, whether any deviation could be found. Up to now, not even the most precise measurements have found any deviation. It cannot be excluded, however, that such deviations might be found in the future when even more precise measurements become available. In such a case, this will require that this theory be modified at least in parts. These are areas of concern, which reach far into the domain of Quantum Mechanics and Relativity. They are related to the question whether the rest mass of photons is actually zero or not. Appendix A.1 will deal with this topic in more detail.

The above definition of the divergence leads to a for us very important theorem. We want to integrate \( \nabla \cdot \mathbf{a} \), the divergence of a vector field \( \mathbf{a} \) over the volume shown in Fig. 1.7. We use the following fact:

This means to separate the macroscopic volume into many microscopic volume elements and then calculate the divergence for each such micro element by taking the limit of \( V \) approaching zero. In this case, all the surface integrals inside cancel because each surface occurs twice, each time with a different sign, as the normal vector for each of those two surface elements has the opposite direction. What remains is the surface integral over the outer surface of the macroscopic volume, i.e.:

This is Gauss’ Integral Theorem.

This equation formally establishes the relation between (1.20) and (1.23). Using (1.24) in (1.20) gives:

\[
\oint_{A} \mathbf{D} \cdot d\mathbf{A} = \int_{V} \rho \, d\tau = \int_{V} \nabla \cdot \mathbf{D} \, d\tau .
\]
Because this has to be true for an arbitrary volume, the integrands have to be equal to each other, *i.e.*

\[ \nabla \cdot \mathbf{D} = \rho. \]

This means that (1.23) follows from (1.20). Conversely, from (1.23) follows

\[ \int_V \nabla \cdot \mathbf{D} d\tau = \int_V \rho d\tau = \oint_A \mathbf{D} \cdot d\mathbf{A} \]

and hence (1.20). In conclusion, we realize that Gauss’ integral theorem provides the rigorous formal proof to our previous plausibility arguments.

The definition of the divergence in (1.22) is didactically advantageous, but impractical for actual computations. Therefore, we will calculate \( \nabla \cdot \mathbf{a} \) in the Cartesian components of \( \mathbf{a} \):

\[
\mathbf{a} = (a_x(x, y, z), a_y(x, y, z), a_z(x, y, z))
\]

(1.25)

We write the related surface integral and take the limit of its volume as it goes to zero (see Fig. 1.8),

\[
\nabla \cdot \mathbf{a} = \lim_{V \to 0} \frac{\oint_{A} \mathbf{a} \cdot d\mathbf{A}}{V}
\]

\[
= \lim_{dx, dy, dz \to 0} \frac{1}{dx dy dz} \left\{ [a_x(x + dx) - a_x(x)] dy dz + [a_y(y + dy) - a_y(y)] dx dz \\
+ [a_z(z + dz) - a_z(z)] dx dy \right\}
\]

\[
= \lim_{dx, dy, dz \to 0} \frac{[a_x(x) + \frac{\partial a_x}{\partial x} dx - a_x(x)] dy dz + \ldots + \ldots \ldots dx dz + \ldots dx dy}{dx dy dz}
\]
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\[
\lim_{dx, dy, dz \to 0} \frac{\partial a_x}{\partial x} dx dy dz + \frac{\partial a_y}{\partial y} dx dy dz + \frac{\partial a_z}{\partial z} dx dy dz
\]

\[
= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}
\]

that is

\[
\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}
\]  

The divergence is a scalar quantity, formally expressed as the scalar product (dot product) of the vector operator \( \nabla \) (Nabla or “del”), for Cartesian coordinates with the vector \( \mathbf{a} \), and using the Cartesian unity vectors \( \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \) as

\[
\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}.
\]  

1.6 Work and the Electric Field

A charge \( Q \) within the reach of an electric field experiences the force \( \mathbf{F} \) and moves, if not held fixed in place. The field performs work on the charge. Conversely, to move the charge against the field requires one to do work.

If we move the charge from the starting point \( P_A \) along the contour \( C_1 \) to an endpoint \( P_E \) (Fig. 1.9), then the total work we have to do is given by

\[
W_1 = - \int_{C_1} \mathbf{F} \cdot d\mathbf{s} = -Q \int_{C_1} \mathbf{E} \cdot d\mathbf{s}.
\]  

This is because \( dW \) for the path element \( d\mathbf{s} \) is

\[
dW_1 = -\mathbf{F} \cdot d\mathbf{s}.
\]  

We could have moved the charge along path \( C_2 \) with the result:

\[
W_2 = - \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = -Q \int_{C_2} \mathbf{E} \cdot d\mathbf{s}.
\]  

Fig. 1.9
Initially, we will only deal with time-independent fields. Suppose the results $W_1$ and $W_2$ would be different. We then could take advantage of this to build a perpetuum mobile (1st kind). Suppose, for instance, $W_2 > W_1$, then it would be possible to move the charge from $P_A$ to $P_B$ via path $C_1$ and then back to $P_A$ via path $C_2$. We would need to invest the work $W_1$, but gain work $W_2$ on the way back. Overall, the work of the closed loop would be $W_2 - W_1 > 0$. Repeating this process would manifest itself as a perpetuum mobile. Of course, we have reasons to assume that this is impossible. The theorem of conservation of energy requires to state that:

$$W_1 = W_2.$$  

(1.31)

or

$$\int_{C_1} \mathbf{E} \cdot ds - \int_{C_2} \mathbf{E} \cdot ds = 0.$$  

(1.32)

Consequently, the work over any closed contour is

$$\oint \mathbf{E} \cdot ds = 0.$$  

(1.33)

This important relation was derived without using the knowledge about electric fields we have gained so far. We need to verify that the electric fields, in fact, meet this requirement. Again, we are currently studying time-independent fields only. Time dependency will come in later, and we will find that (1.33) is not applicable in such a case. Nevertheless, if (1.33) applies to a single point charge, it also applies to an arbitrary distribution of charges at rest. The reason for this is the superposition principle. It is therefore sufficient to prove (1.33) for a point charge.

Before starting our proof, we will investigate some simple properties of line integrals over closed curves. Fig. 1.10 shows a closed curve $C$, which is separated into two closed curves $C_1$ and $C_2$ by inserting a line. We get:

$$\oint_C \mathbf{a} \cdot ds = \int_{C_1} \mathbf{a} \cdot ds + \int_{C_2} \mathbf{a} \cdot ds.$$  

Notice that the two newly added path elements identically cancel. This kind of subdivision can be repeated by individually subdividing $C_1$ and $C_2$, respectively, and so on. If we now study the field of a point charge, we can start with any closed contour. We can reduce the integral $\oint \mathbf{E} \cdot ds$ to integrals over the kind shown in Fig. 1.11. For this case we write:
On the paths (arcs) from $P_2$ to $P_3$ and $P_4$ to $P_1$, $E$ is perpendicular to $ds$.

$$
\oint_{P_2} E \cdot ds + \oint_{P_4} E \cdot ds = 0.
$$

On the other two paths from $P_1$ to $P_2$ and $P_3$ to $P_4$, on the other hand, $E$ and $ds$ are parallel and anti-parallel, respectively. This means that

$$
\oint_{P_1} E \cdot ds = \oint_{P_3} E \cdot ds = \oint_{P_4} E \cdot ds.
$$

This finalizes the proof. Unfortunately the perpetuum mobile is not a feasible option. Relation (1.33) will prove to be far-reaching. First, we need to introduce some terms, which will be done in the next section.

1.7 The Rotation of a Vector Field and Stoke’s Integral Theorem

Consider an arbitrary vector field $\mathbf{a}(\mathbf{r})$. For any closed curve, we can write the line integral $\oint \mathbf{a} \cdot ds$. We may also look at arbitrarily small area elements, and the line integrals corresponding to their boundary. Reducing the size of the area elements more and more will make the line integrals smaller and smaller, such that they will vanish in that limit. However, the ratio of the line integral over its related area element will converge towards a limit. We define a new vector field which we call the circulation or curl of $\mathbf{a}$ ($\text{curl}(\mathbf{a})$ or $\nabla \times \mathbf{a}$) as follows:

We choose three perpendicular, but otherwise arbitrary area elements $dA_1$, $dA_2$, $dA_3$, which share a common center in space. Together they form a right handed system. With this we write the limit.

$$
\lim_{dA_i \to 0} \frac{\oint \mathbf{a} \cdot ds}{dA_i} = r_i \quad (i = 1, 2, 3)
$$

(1.34)

Note that the line integral in the numerator is an infinitesimal loop extending over the boundary of the area element and the orientation is such that the line integral forms a right handed system with the vector $dA_1$ (Fig. 1.12).
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