2

Nilpotent Groups: Explicit Examples

In this chapter we list some of the (now numerous) calculations of zeta functions of $\mathfrak{T}$-groups and Lie rings. The primary emphasis is on bringing into print explicit calculations that have yet to be published. However, we aim this chapter to be more than just a gallery of results. Hence we begin the chapter with some details about how these zeta functions have been calculated.

2.1 Calculating Zeta Functions of Groups

Zeta functions of groups have been calculated using a number of different methods. The first examples counted ideals in $\mathfrak{T}$-groups of class 2 and were calculated by Grunewald, Segal and Smith in [32]. A key part of their work is the formula [32, Lemma 6.1]

$$\zeta^{\triangleleft}_{G,p}(s) = \zeta_{\mathbb{Z}^n,p}(s) \sum_{B \leq A} |A : B|^{n-s} |G : X(B)|^{-s} , \tag{2.1}$$

where $A = \gamma_2(G)$, $G/A \cong \mathbb{Z}^d$ and $X(B)/B = Z(G/B)$. Their calculations are made by evaluating (2.1) for each group in turn. Although there are a few general lemmas proved which help speed matters along, their methods are to some extent tailored to each group individually. Nonetheless, their methods suffice to calculate all but perhaps finitely many of the local factors $\zeta^{\triangleleft}_{G,p}(s)$ for every $\mathfrak{T}$-group G of class 2 and Hirsch length at most 6.

In [60], Voll uses (2.1) and the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$ to compute normal zeta functions of $\mathfrak{T}$-groups whose centres are free abelian of rank 2 or 3. In particular, Voll computes the normal zeta function of all $\mathfrak{T}$-groups whose centre is of rank 2, and confirms the functional equation (1.5). This work is based on the classification of such groups by Grunewald and Segal [31]. For centres of rank 3, the geometry of the associated Pfaffian hypersurface comes into play. Provided the singularities of this hypersurface are in some sense not too severe, Voll gives a formula for the local normal zeta function of L

depending on the number of points on the Pfaffian hypersurface. A highlight of this work is explicit expressions for the rational functions $P_1(X,Y)$ and $P_2(X,Y)$ in the local normal zeta function of the 'elliptic curve example' (1.6).

A more general approach is used by Voll in [61], where he considers the case where the Pfaffian hypersurface has no lines. Indeed this occurs generically if the abelianisation has rank greater than $4r-10$, where r is the dimension of the centre. Provided this Pfaffian is smooth and absolutely irreducible, the functional equation (1.5) holds. Voll also gives in [61] an explicit formula for the normal zeta functions of the class-2 nilpotent groups known as 'Grenham groups', using a combinatorial formula for the number of points on flag varieties. This formula is also employed by Voll in [58], where he gives an explicit formula for the local zeta functions counting all subgroups in the Grenham groups.

One key assumption Voll makes in [61] is that the associated Pfaffian hypersurface has no lines. A forthcoming paper by Paajanen [49] presents the first step in overcoming this obstacle. She considers the normal zeta function of a class-2 nilpotent group $G_{\mathcal{S}}$ which encodes the Segre surface $\mathcal{S} : x_1x_4 - x_2x_3 = 0$. In particular, she calculates that

$$\zeta^{\triangleleft}_{G_{\mathcal{S}},p}(s) = W_0(p,p^{-s}) + (p+1)^2 W_1(p,p^{-s}) + 2(p+1)W_2(p,p^{-s})$$

for explicit rational functions $W_i(p,p^{-s})$, $i=0,1,2$. The coefficients $(p+1)^2$ and $2(p+1)$ arise from the geometry of $\mathcal{S}$ reduced mod p: being isomorphic to $\mathbb{P}^1(\mathbb{F}_p) \times \mathbb{P}^1(\mathbb{F}_p)$ it has $(p+1)^2$ points and $2(p+1)$ lines.

Voll has also used combinatorial methods to yield an explicit expression for the local normal zeta functions of the class-2 free nilpotent groups [62]. One key ingredient is an explicit expression for a sum of certain Hall polynomials. Whilst there seems to be no simple formula for the Hall polynomials themselves, a polynomial expression for the sum has been known for some time.

One approach common to the work of Voll and Paajanen is to decompose the local normal zeta function as a sum of rational functions with coefficients corresponding to invariants of a suitable algebraic variety. They are then able to deduce functional equations by virtue of the fact that each individual rational function with its coefficient satisfies the same functional equation. In particular,

$$\zeta^{\triangleleft}_{G_{\mathcal{S}},p}(s)\big|_{p\to p^{-1}} = p^{28-12s}\zeta^{\triangleleft}_{G_{\mathcal{S}},p}(s)\ ,$$

with the three rational functions above satisfying

$$\begin{aligned} W_0(X^{-1},Y^{-1}) &= X^{28}Y^{12}W_0(X,Y)\ , \\ W_1(X^{-1},Y^{-1}) &= X^{26}Y^{12}W_1(X,Y)\ , \\ W_2(X^{-1},Y^{-1}) &= X^{27}Y^{12}W_2(X,Y)\ . \end{aligned}$$

The 'missing' powers of X are provided by the coefficients $(p+1)^2$ and $2(p+1)$.

2.2 Calculating Zeta Functions of Lie Rings

Most of the zeta functions presented in this chapter have been calculated by the method of Lie rings, p-adic integrals and ad-hoc resolutions of singularities. In particular, the zeta functions calculated in the theses of Taylor [57] and the second author [64] were calculated this way. In particular, we shall work with Lie rings instead of groups, and leave the reader to obtain the corresponding results concerning groups via the Mal'cev correspondence. We shall also make the assumption that our Lie rings are additively isomorphic to either $\mathbb{Z}^d$ or $\mathbb{Z}_p^d$, i.e. (additively) finitely generated and torsion-free.

Recall that $\zeta^*_{L,p}(s) = \zeta^*_{L\otimes\mathbb{Z}_p}(s)$. Given a $\mathbb{Z}_p$-Lie ring L with basis $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_d)$ for L, calculating either of the zeta functions $\zeta^{\leq}_{L,p}$ or $\zeta^{\triangleleft}_{L,p}$ is essentially a four-stage calculation:

1. Constructing the cone integral.
2. Breaking the integral into a sum of monomial integrals.
3. Evaluating the monomial integrals.
4. Summing the resulting rational functions.

2.2.1 Constructing the Cone Integral

Let M be an upper-triangular $d \times d$ matrix $M = (m_{i,j})$ with entries in $\mathbb{Z}_p$. We may consider the rows $\mathbf{m}_1,\dots,\mathbf{m}_d$ of this matrix to be additive generators of a submodule of L. This submodule will be a subring if

$$[\mathbf{m}_i, \mathbf{m}_j] \in \langle \mathbf{m}_1, \dots, \mathbf{m}_d \rangle_{\mathbb{Z}_p} \text{ for all } 1 \leq i < j \leq d \tag{2.2}$$

and an ideal if

$$[\mathbf{e}_i, \mathbf{m}_j] \in \langle \mathbf{m}_1, \dots, \mathbf{m}_d \rangle_{\mathbb{Z}_p} \text{ for all } 1 \leq i, j \leq d\,. \tag{2.3}$$

The following proposition and its proof gives us an explicit description of the *cone conditions*, i.e. the conditions of the form $v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x}))$ for $1 \leq i \leq l$. It is essentially Theorem 5.5 of [17].

Proposition 2.1. *Let L be a $\mathbb{Z}$-Lie ring with basis $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_d)$. Let $V_p^{\triangleleft}$ be the set of all upper-triangular matrices over $\mathbb{Z}_p$ such that $\mathbb{Z}_p^d \cdot M \triangleleft L \otimes \mathbb{Z}_p$, and $V_p^{\leq}$ the set of such matrices such that $\mathbb{Z}_p^d \cdot M \leq L \otimes \mathbb{Z}_p$. Then $V_p^{\triangleleft}$ and $V_p^{\leq}$ are defined by the conjunction of polynomial divisibility conditions $v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x}))$ for $1 \leq i \leq l$. Furthermore, the conditions defining $V_p^{\triangleleft}$ satisfy* $\deg f_i(\mathbf{x}) = \deg g_i(\mathbf{x})$*, and those defining $V_p^{\leq}$ satisfy* $\deg f_i(\mathbf{x}) + 1 = \deg g_i(\mathbf{x})$.

Proof. Let $\mathbf{m}_1, \dots, \mathbf{m}_d$ denote the rows of the matrix M, C_j the matrix whose rows are $\mathbf{c}_i = [\mathbf{e}_i, \mathbf{e}_j]$. Let M' denote the adjoint matrix of M and

$$M^{\natural} = M' \operatorname{diag}(m_{2,2}^{-1} \dots m_{d,d}^{-1}, m_{3,3}^{-1} \dots, m_{d,d}^{-1}, \dots, m_{dd}^{-1}, 1) \; .$$

Since M is upper-triangular, the (i,k) entry of $M^{\natural}$ is a homogeneous polynomial of degree $k-1$ in the variables $m_{r,s}$ with $1 \le r \le s \le k-1$.

The rows of M generate an ideal if we can solve, for each $1 \le i, j \le d$, the equation

$$\mathbf{m}_i C_j = (y_{i,j,1}, \dots, y_{i,j,d}) M$$

for $(y_{i,j,1}, \dots, y_{i,j,d}) \in \mathbb{Z}_p^d$. This rearranges to

$$\mathbf{m}_i C_j M^{\natural} = (m_{1,1} y_{i,j,1}, \dots, m_{1,1} \dots m_{d,d} y_{i,j,d})$$

for $(y_{i,j,1}, \dots, y_{i,j,d}) \in \mathbb{Z}_p^d$. Set $g^{\lhd}_{i,j,k}(\mathbf{x})$ to be the k^{th} entry of the d-tuple $\mathbf{m}_i C_j M^{\natural}$. $g^{\lhd}_{i,j,k}(\mathbf{x})$ is a homogeneous polynomial of degree k in the $m_{r,s}$, and if we set $f_{i,j,k}(\mathbf{x}) = m_{1,1} \dots m_{k,k}$, we obtain the conditions $v(f_{i,j,k}(\mathbf{x})) \le v(g^{\lhd}_{i,j,k}(\mathbf{x}))$ with $\deg(f_{i,j,k}(\mathbf{x})) = \deg(g^{\lhd}_{i,j,k}(\mathbf{x}))$.

Similarly, the rows of M generate a subring if we can solve, for $1 \le i < j \le d$,

$$\mathbf{m}_i \left(\sum_{r=j}^{d} m_{j,r} C_r \right) M^{\natural} = (m_{1,1} y_{i,j,1}, \dots, m_{1,1} \dots m_{d,d} y_{i,j,d})$$

for $(y_{i,j,1}, \dots, y_{i,j,d}) \in \mathbb{Z}_p^d$. Again, we set $g^{\lhd}_{i,j,k}(\mathbf{x})$ to be the k^{th} entry of the d-tuple $\mathbf{m}_i \left(\sum_{r=j}^{d} m_{j,r} C_r \right) M^{\natural}$. However, this time $g^{\lhd}_{i,j,k}(\mathbf{x})$ is a homogeneous polynomial of degree $k+1$, so we obtain conditions $v(f_{i,j,k}(\mathbf{x})) \le v(g^{\lhd}_{i,j,k}(\mathbf{x}))$. Furthermore, $\deg(f_{i,j,k}(\mathbf{x})) + 1 = \deg(g^{\lhd}_{i,j,k}(\mathbf{x}))$. □

Whilst every subring or ideal H has a matrix M whose rows additively generate H, these matrices are by no means unique. Multiplying a row by a p-adic unit or adding a multiple of a row to another row above it may change the matrix but does not alter the subring additively generated by the rows. Each diagonal entry $m_{i,i}$ is unique up to multiplication by p-adic units, hence the measure of values it can take is $(1-p^{-1})|m_{i,i}|_p$. Each off-diagonal entry $m_{i,j}$ is only unique modulo $|m_{j,j}|_p^{-1}$. Hence the measure of upper-triangular matrices generating H is $(1-p^{-1})^d |m_{1,1}|_p |m_{2,2}|_p^2 \dots |m_{d,d}|_p^d$. Note that although $m_{i,i}$ may vary, $|m_{i,i}|_p$ is uniquely determined by H.

Finally, we note that the index of H is $|m_{1,1} m_{2,2} \dots m_{d,d}|_p^{-1}$. Hence we may write

$$\zeta^*_{L,p}(s) = (1-p^{-1})^{-d} \int_{V_p^*} |m_{1,1} \dots m_{d,d}|_p^s |m_{1,1}^1 \dots m_{d,d}^d|_p^{-1} \, \mathrm{d}\mu \; , \tag{2.4}$$

or

$$\zeta^*_{L,p}(s+d) = (1-p^{-1})^{-d} \int_{V_p^*} |m_{1,1} \dots m_{d,d}|_p^s |m_{1,1}^{d-1} \dots m_{d-1,d-1}^1|_p \, \mathrm{d}\mu \, . \quad (2.5)$$

Note that the translation in (2.5) is necessary. Equation (2.4) is not a cone integral since the constant (independent of s) term in the integrand has a negative exponent. We complete the set of cone data by setting $f_0(\mathbf{x}) = m_{1,1} \dots m_{d,d}$, $g_0(\mathbf{x}) = m_{1,1}^{d-1} \dots m_{d-1,d-1}$ and $\mathcal{D} = \{f_0(\mathbf{x}), g_0(\mathbf{x}), \dots, f_l(\mathbf{x}), g_l(\mathbf{x})\}$. We therefore obtain the following result.

Proposition 2.2. *Let L be a Lie ring additively isomorphic to $\mathbb{Z}^d$, $* \in \{\leq, \lhd\}$. There exists a set of cone integral data $\mathcal{D} = \{f_0, g_0, \dots, f_l, g_l$ such that, for all primes p,*

$$\zeta^*_{L,p}(s+d) = (1-p^{-1})^{-d} Z_{\mathcal{D}}(s,p) \, .$$

Furthermore, $\deg f_0 = d$, $\deg g_0 = \binom{d}{2}$.

2.2.2 Resolution

Once we have constructed the cone integral, the next step is to break the integral into a sum of integrals with monomial conditions. As mentioned in the Introduction, resolution of singularities gives us one way of doing this, and more importantly guarantees that this can always be done. Hironaka's proof of resolution of singularities of any singular variety defined over a field of characteristic 0 has been refined by Villamayor, Encinas, Bierstone and Milman, and Hauser amongst others to produce an explicit constructive procedure. In particular, Bodnár and Schicho have implemented a computer program to calculate resolutions. We refer the reader wanting to know more to Hauser's accessible article on resolution [34] and its comprehensive bibliography.

However, we shall not use resolution of singularities, for a number of reasons. Firstly, the computer program of Bodnár and Schicho works best in small dimensions, and we shall typically require resolutions of a polynomial with a large number of variables. Secondly, we shall find that we do not need to resolve all the singularities of the polynomial $F = \prod_{i=0}^{l} f_i(\mathbf{x}) g_i(\mathbf{x})$. Singularities lying outside V_p^* do not need to be resolved. Thirdly, there are 'tricks' that can be applied to simplify the polynomial conditions and speed up the process of decomposing the integral as a sum of monomial integrals. Some of these will take advantage of the fact we are working over $\mathbb{Q}_p$, whereas resolution is a general procedure for arbitrary fields of characteristic 0. A further disadvantage of resolution is the highly technical language it is most rigorously formulated in. We do not wish to alienate readers unfamiliar with this advanced machinery.

Therefore, we resolve singularities in an elementary and 'ad-hoc' manner. A collection of 'tricks' are used to simplify the conditions under the integral, and when the conditions can be simplified no further we bisect the integral. This bisection is achieved by choosing a pair of variables and splitting the domain of integration into two parts depending on which variable has the larger valuation. Further 'tricks' and bisections may then be necessary to reduce the integral into smaller and smaller pieces until all the pieces become monomial.

The idea of bisecting the integral as described above has its origins in the concept of a *blow-up*, an operation fundamental to the process of resolution of singularities. Indeed, we shall refer to our bisections as 'blow-ups'. Furthermore, we can use ideas originating from algebraic geometry to provide motivation for our choices of blow-ups. For example, suppose a non-monomial factor of one of the cone conditions is of the form $Px_j + Qx_k$ for variables x_j and x_k and nonzero polynomials P and Q. Let us also assume x_j and x_k have nontrivial integrand exponent or feature somewhere in a monomial condition. The polynomial F, being the product of all the cone data polynomials, has the factors x_j, x_k and $Px_j + Qx_k$, and therefore has a singularity with non-normal crossings at $x_j = x_k = 0$. A blow-up involving x_j and x_k will then replace this polynomial factor with $x_j(P + Qx'_k)$ (where $x_k = x_j x'_k$) or $x_k(Px'_j + Q)$ (where $x_j = x'_j x_k$) on the two sides of the blow-up. If P and Q are both independent of x_j and x_k, then this trick reduces the sum of the total degrees of the terms of the non-monomial factor. This trick is even more useful when one of x_j and x_k divides the other side of the condition, since the monomial factor x_j or x_k introduced above will cancel out. Algebraic geometry therefore provides inspiration for our method, but we do not totally rely on it.

Initially, the integrand and the left-hand side of each condition $v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x}))$ is monomial, and this is something we preserve. For brevity we also write $f_i(\mathbf{x}) \mid g_i(\mathbf{x})$ instead of $v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x}))$.

Examples of 'Resolution'

To illustrate the concepts in the previous section, we present two example calculation, where we construct the p-adic integral corresponding to a Lie ring and in each case apply some 'tricks' and blow-ups to split it into monomial integrals. The first example will illustrate the basic ideas, with some more unusual and less obvious tricks employed in the second.

For the first example, we shall choose to count all subrings of the Lie ring

$$L = \langle x_1, x_2, x_3, x_4, y_1, y_2 : [x_1, x_2] = y_1, [x_1, x_3] = y_2, [x_2, x_4] = y_2 \rangle \ .$$

In this case, the set $V_p^{\leq}$ is given by

$$V_p^{\leq} = \{ (m_{1,1}, m_{1,2} \dots, m_{6,6}) \in \mathbb{Z}_p^{21} : f_i(\mathbf{x}) \leq g_i(\mathbf{x}) \text{ for } 1 \leq i \leq 6 \} ,$$

where the six[1] conditions $f_i(\mathbf{x}) \mid g_i(\mathbf{x})$ are listed below:

[1] It is mere coincidence that there are six conditions in this case. Generally the number of conditions obtained bears no relation to the rank of the underlying Lie ring.

$$m_{5,5} \mid m_{1,1}m_{2,2} ,$$
$$m_{6,6} \mid m_{1,2}m_{4,4} ,$$
$$m_{6,6} \mid m_{2,2}m_{3,4} ,$$
$$m_{6,6} \mid m_{2,2}m_{4,4} ,$$
$$m_{6,6} \mid m_{1,1}m_{3,3} + m_{1,2}m_{3,4} ,$$
$$m_{5,5}m_{6,6} \mid m_{1,1}m_{2,2}m_{5,6} - m_{1,1}m_{2,3}m_{5,5} - m_{1,2}m_{2,4}m_{5,5} + m_{1,4}m_{2,2}m_{5,5} .$$

These conditions are independent of $m_{1,3}$ and $m_{i,j}$ for $1 \leq i \leq 4$, $5 \leq j \leq 6$. For the sake of clarity, we shall relabel the remaining 12 variables as $a, b, \dots, l$. Thus,

$$\zeta^{\leq}_{L,p}(s) = (1 - p^{-1})^{-6} I ,$$

where

$$I = \int_W |a|_p^{s-1}|d|_p^{s-2}|g|_p^{s-3}|i|_p^{s-4}|j|_p^{s-5}|l|_p^{s-6}\,\mathrm{d}\mu$$

and W is the subset of $(a, b, \dots, l) \in \mathbb{Z}_p^{12}$ defined by the conditions

$$j \mid ad , \quad l \mid bi , \quad l \mid dh , \quad l \mid di , \quad l \mid ag + bh , \quad jl \mid adk - aej - bfj + cdj .$$

We perform a blow-up with l and d to remove the variable c. On one side of the blow-up it disappears altogether, on the other its coefficient dj divides the sum of the other terms of the polynomial:

1. $v(l) \leq v(d)$: set $d = d'l$. The conditions $l \mid dh$ and $l \mid di$ become trivially true, and we can also remove the term $cd'jl$ from the last condition. Thus

$$I_1 = \int_{\substack{j \mid ad'l \\ l \mid bi \\ l \mid ag+bh \\ jl \mid ad'kl-aej-bfj}} |a|_p^{s-1}|d'|_p^{s-2}|g|_p^{s-3}|i|_p^{s-4}|j|_p^{s-5}|l|_p^{2s-7}\,\mathrm{d}\mu .$$

   Note that the exponent of $|l|_p$ is $2s-7$, as opposed to $2s-8 = (s-2)+(s-6)$. The discrepancy is caused by the dilation of the measure that the change $d = d'l$ brings about. By dividing the l out of d, we have allowed d' to take a greater measure of values in $\mathbb{Z}_p$ than d. Hence we introduce a *Jacobean* $|l|_p$ into the integrand to balance out the dilation.
2. $v(l) > v(d)$: set $l = dl'$ with $v(l') \geq 1$. This then implies $l' \mid h$ and $l' \mid i$. To remove these two variable-divides-variable conditions, set $h = h'l'$ and $i = i'l'$.

$$I_2 = \int_{\substack{j \mid ad \\ d \mid bi' \\ dl' \mid ag+bh'l' \\ djl' \mid adk-aej-bfj+cdj \\ v(l') \geq 1}} |a|_p^{s-1}|d|_p^{2s-7}|g|_p^{s-3}|i|_p^{s-4}|j|_p^{s-5}|l'|_p^{s-4}\,\mathrm{d}\mu .$$

The last condition implies

$$dj \mid adk - aej - bfj \tag{2.6}$$

and thus $l \mid c + (adk - aej - bfj)/dj$, so we shall set $c = c' - (adk - aej - bfj)/dj$. After this substitution, the conditions no longer imply (2.6), so to avoid altering the value of the integral, we must explicitly enforce (2.6). We can also set $c' = c''l$ to remove the condition $l \mid c'$. Hence

$$I_2 = \int_{\substack{j|ad \\ d|bi' \\ dl'|ag+bh'l' \\ dj|adk-aej-bfj \\ v(l')\geq 1}} |a|_p^{s-1} |d|_p^{2s-7} |g|_p^{s-3} |i'|_p^{s-4} |j|_p^{s-5} |l'|_p^{2s-7} \,\mathrm{d}\mu \,.$$

In both cases we have removed c or c'' from the conditions and the number of terms in the last condition has dropped from 4 to 3.

We play a similar trick on I_1 and I_2 to remove f. By a stroke of luck it turns out to also eliminate h from I_1 and h' from I_2:

1.1. $v(l) \leq v(b)$: set $b = b'l$. Terms $b'hl$ and $-b'fjl$ disappear from the last two conditions:

$$I_{1.1} = \int_{\substack{j|ad'l \\ l|ag \\ jl|a(d'kl-ej)}} |a|_p^{s-1} |d'|_p^{s-2} |g|_p^{s-3} |i|_p^{s-4} |j|_p^{s-5} |l|_p^{2s-6} \,\mathrm{d}\mu \,.$$

1.2. $v(l) > v(b)$: set $l = bl'$ with $v(l') \geq 1$, and $i = i'l'$. Now $b \mid ag$ and $bj \mid a(bd'kl' - ej)$ are implied by the last two conditions, so we set $h = h'l - ag/b$ and $f = f'l + a(bd'kl' - ej)/bj$. Again, we must introduce explicitly the implied conditions.

$$I_{1.2} = \int_{\substack{j|abd'l' \\ b|ag \\ bj|a(bd'kl'-ej) \\ v(l')\geq 1}} |a|_p^{s-1} |b|_p^{2s-6} |d'|_p^{s-2} |g|_p^{s-3} |i'|_p^{s-4} |j|_p^{s-5} |l'|_p^{3s-8} \,\mathrm{d}\mu \,.$$

2.1. $v(d) \leq v(b)$: set $b = b'd$:

$$I_{2.1} = \int_{\substack{j|ad \\ dl'|ag \\ dj|a(dk-ej) \\ v(l')\geq 1}} |a|_p^{s-1} |d|_p^{2s-6} |g|_p^{s-3} |i'|_p^{s-4} |j|_p^{s-5} |l'|_p^{2s-7} \,\mathrm{d}\mu \,.$$

2.2. $v(d) > v(b)$: set $d = bd'$ with $v(d') \geq 1$, $i' = d'i''$. Also $bl' \mid ag$ and $bj \mid a(bd'k - ej)$, so we can set $h' = d'h' - ag/bl'$ and $f = df' + a(bd'k - ej)/bj$:

$$I_{2.2} = \int_{\substack{j|abd' \\ bl'|ag \\ bj|a(bd'k-ej) \\ v(l')\geq 1 \\ v(d')\geq 1}} |a|_p^{s-1}|b|_p^{2s-6}|d'|_p^{3s-8}|g|_p^{s-3}|i''|_p^{s-4}|j|_p^{s-5}|l'|_p^{2s-7}\,\mathrm{d}\mu\ .$$

All four of these integrals are very similar, and can be reduced to monomials in the same way. For simplicity we shall consider only $I_{1.1}$.

1.1.1. $v(j) \leq v(d'kl)$: in this case, $d'kl/j$ is an integer, so we may set $e = e' + d'kl/j$:

$$I_{1.1.1} = \int_{\substack{j|ad'l \\ l|ag \\ j|d'kl \\ l|ae'}} |a|_p^{s-1}|d'|_p^{s-2}|g|_p^{s-3}|i|_p^{s-4}|j|_p^{s-5}|l|_p^{2s-6}\,\mathrm{d}\mu\ .$$

1.1.2. $v(j) > v(d'kl)$: set $j = j'd'kl$ with $v(j') \geq 1$:

$$I_{1.1.2} = \int_{\substack{j'k|a \\ l|ag \\ j'l|a(1-ej') \\ v(j')\geq 1}} |a|_p^{s-1}|d'|_p^{2s-6}|g|_p^{s-3}|i|_p^{s-4}|j'|_p^{s-5}|k|_p^{s-4}|l|_p^{3s-10}\,\mathrm{d}\mu\ .$$

Since $v(j') \geq 1$, $v(1 - ej') = 0$. Thus

$$I_{1.1.2} = \int_{\substack{j'k|a \\ l|ag \\ j'l|a \\ v(j')\geq 1}} |a|_p^{s-1}|d'|_p^{2s-6}|g|_p^{s-3}|i|_p^{s-4}|j'|_p^{s-5}|k|_p^{s-4}|l|_p^{3s-10}\,\mathrm{d}\mu\ .$$

In this case we can break up the initial integral into eight monomial integrals, however larger examples may need to be broken up into many more integrals. Evaluating these monomial integrals and summing gives us the local zeta function counting all subrings in $\mathfrak{g}_{6,4}$, which can be found below on p. 44.

The second example is more involved, and demonstrates some other tricks which sometimes come in useful. We count ideals in the free class-3 2-generator nilpotent Lie ring $F_{3,2}$. This has presentation

$$\langle x_1, x_2, y, z_1, z_2 : [x_1, x_2] = y, [x_1, y] = z_1, [x_2, y] = z_2 \rangle\ .$$

Now

$$I := \zeta^{\triangleleft}_{F_{3,2},p}(s) = (1-p^{-1})^{-5} \int_W |m_{1,1}|_p^{s-1} \ldots |m_{5,5}|_p^{s-5} \, \mathrm{d}\mu \ ,$$

where W is defined by the conjunction of the following conditions:

$$\begin{gathered} m_{3,3} \mid m_{1,1} \ , \quad m_{3,3} \mid m_{1,2} \ , \quad m_{3,3} \mid m_{2,2} \ , \quad m_{4,4} \mid m_{1,1} \ , \quad m_{4,4} \mid m_{3,3} \ , \\ m_{5,5} \mid m_{2,2} \ , \quad m_{5,5} \mid m_{2,3} \ , \quad m_{5,5} \mid m_{3,3} \ , \quad m_{3,3}m_{4,4} \mid m_{1,1}m_{3,4} \ , \\ m_{4,4}m_{5,5} \mid m_{3,3}m_{4,5} \ , \quad m_{3,3}m_{4,4} \mid m_{1,2}m_{3,4} - m_{1,3}m_{3,3} \ , \\ m_{3,3}m_{4,4} \mid m_{2,2}m_{3,4} - m_{2,3}m_{3,3} \ , \quad m_{4,4}m_{5,5} \mid m_{1,1}m_{4,5} - m_{1,2}m_{4,4} \ , \\ m_{3,3}m_{4,4}m_{5,5} \mid m_{1,2}m_{3,4}m_{4,5} - m_{1,2}m_{3,5}m_{4,4} - m_{1,3}m_{3,3}m_{4,5} \ , \\ m_{3,3}m_{4,4}m_{5,5} \mid m_{2,2}m_{3,4}m_{4,5} - m_{2,2}m_{3,5}m_{4,4} - m_{2,3}m_{3,3}m_{4,5} \ , \\ m_{3,3}m_{4,4}m_{5,5} \mid m_{1,1}m_{3,4}m_{4,5} - m_{1,1}m_{3,5}m_{4,4} - m_{1,3}m_{3,3}m_{4,4} \ . \end{gathered}$$

We start by setting $m_{1,1} = m'_{1,1}m_{3,3}$, $m_{1,2} = m'_{1,2}m_{3,3}$, $m_{2,2} = m'_{2,2}m_{3,3}$, $m_{3,3} = m'_{3,3}m_{4,4}$ and $m_{2,3} = m'_{2,3}m_{5,5}$. Doing so 'uses up' five of the first eight conditions. These conditions, and the changes that eliminate them, are typical when calculating local ideal zeta functions. Variables $m_{1,4}$, $m_{1,5}$, $m_{2,4}$ and $m_{2,5}$ don't feature among the above conditions. Relabelling the remainder from a to k tells us that

$$I = (1-p^{-1})^{-5} \int_{W'} |a|_p^{s-1} |d|_p^{s-2} |f|_p^{3s-3} |i|_p^{4s-6} |k|_p^{s-4} \, \mathrm{d}\mu \ ,$$

where W is the subset of all $(a, \ldots, k) \in \mathbb{Z}_p^{11}$ satisfying

$$\begin{gathered} i \mid ag \ , \quad k \mid fi \ , \quad k \mid fj \ , \quad i \mid bg - c \ , \quad i \mid dg - ek \ , \quad ik \mid agj - ahi - ci \ , \\ ik \mid bgj - bhi - cj \ , \quad ik \mid dgj - dhi - ekj \ . \end{gathered}$$

Our focus is on the conditions and how to perform blow-ups to reduce the conditions to monomials. We shall therefore neglect to track the changes to the integrand.

We started the last calculation by aiming to remove a variable from the integral. We cannot do the same here. Instead, we choose a blow-up between i and j. Note that each term of the right-hand side of each of the last three conditions above contains an i or a j. Where $v(i) \leq v(j)$, we set $i = i'j$ and then $h = h' + gj'$ to obtain that

$$W_1 := \left\{ (a, \ldots, k) \in \mathbb{Z}_p^{11} : \begin{matrix} i \mid ag \ , \quad k \mid fi \ , \quad k \mid dh' \ , \quad i \mid bg - c \ , \\ i \mid dg - ek \ , \quad k \mid ah' + c \ , \quad k \mid bh' + cj' \end{matrix} \right\} .$$

A blow-up with k and c is the thing to do here. Where $v(k) \leq v(c)$, two of the binomial conditions drop to monomial and a blow-up with i and k will suffice to reduce to monomials. However, more interesting things happen when $v(k) > v(c)$. Firstly, let's set $k = ck'$ with $v(k') \geq 1$, and then set $j' = j''k - bh'/c$:

$$W_{1.2} := \left\{ (a,\ldots,k') \in \mathbb{Z}_p^{11} : \begin{array}{c} c \mid bh' \,, \quad i \mid ag \,, \quad ck' \mid fi \,, \quad ck' \mid dh' \,, \\ i \mid bg - c \,, \quad i \mid dg - eck' \,, \quad ck' \mid ah' + c \,, \\ v(k') \geq 1 \end{array} \right\} .$$

Consider the last condition, $ck' \mid ah' + c$. Since $v(k') \geq 1$, $v(ck') > v(c)$. This implies that $v(ah') = v(c)$, so that $ah' \mid c$. Set $c = ac'h'$:

$$W_{1.2} = \left\{ (a,\ldots,k') \in \mathbb{Z}_p^{11} : \begin{array}{c} a \mid b \,, \quad i \mid ag \,, \quad ac'h'k' \mid fi \,, \quad ac'k' \mid d \,, \\ i \mid bg - ac'h' \,, \quad i \mid dg - ac'eh'k' \,, \\ c'k' \mid 1 + c' \,, \quad v(k') \geq 1 \end{array} \right\} .$$

$c'k' \mid 1 + c'$ and $v(k') \geq 1$ imply that $c' \equiv -1 \pmod p$, in particular c' is a unit. We set $c' = c''k - 1$ as well as $b = ab'$ and $d = ad'k'$. After some tidying, we end with the following monomial conditions:

$$W_{1.2} = \left\{ (a,\ldots,k') \in \mathbb{Z}_p^{11} : i \mid ag \,, \quad ah'k' \mid fi \,, \quad i \mid ah' \,, \quad v(k') \geq 1 \right\} .$$

We now return to the second half of the initial blow-up. We have

$$W_2 := \left\{ (a,\ldots,k) \in \mathbb{Z}_p^{11} : \begin{array}{c} k \mid fj \,, \quad i'j \mid ag \,, \quad i'j \mid bg - c \,, \quad j \mid dh - e''k \,, \\ k \mid d(g - hi') \,, \quad i'k \mid bg - bhi' - c \,, \\ i'k \mid ag - ahi' - ci' \,, \quad v(i') \geq 1 \end{array} \right\} .$$

It is best not to do a blow-up at this point. Instead, we do a couple of changes of variable. Firstly, we set $g = g' + hi'$. Note that this change will make two conditions longer. Setting $c = c' + bg'$ and then $c' = c''i'k$ gives us the binomial conditions

$$W_2 = \left\{ (a,\ldots,k) \in \mathbb{Z}_p^{11} : \begin{array}{c} k \mid dg' \,, \quad k \mid fj \,, \quad j \mid bh - c''k \,, \\ j \mid dh - e''k \,, \quad i'j \mid a(g' + hi') \,, \\ i'k \mid g'(a - bi') \,, \quad v(i') \geq 1 \end{array} \right\} .$$

A blow-up between j and k will remove the first two binomial conditions. It is then routine (although not trivial) to split the two parts into monomials. Evaluating the resulting monomial integrals and summing yields $\zeta^{\triangleleft}_{F_{3,2},p}(s)$, on p. 51.

2.2.3 Evaluating Monomial Integrals

A p-adic cone integral with monomial conditions can be expressed as a sum of integral points within a polyhedral cone in $\mathbb{R}^n$, and there are algorithms for evaluating such sums. One such example is the Elliott–MacMahon algorithm described in [54]. However, the second author considered an alternative approach, which appears to be more efficient for the monomial cone integrals arising from zeta functions of Lie rings, but is not guaranteed to terminate.

This approach is to continue applying 'blow-ups' to further decompose the monomial integrals until the conditions become trivial. One strategy for choosing blow-ups is to choose the two variables which appear most frequently on opposite sides of conditions without appearing on the same side. It is not difficult to automate this strategy, and in practice it has worked well, but it is not difficult to construct integrals for which this strategy will fail.

Most of the 'tricks' described in the previous section are aimed at reducing non-monomial conditions to monomials and so cannot be applied. The exception is that any conditions $f_i(\mathbf{x}) \mid g_i(\mathbf{x})$ where $g_i(\mathbf{x})$ is a single variable x_j can be removed by setting $x_j = x'_j f_i(\mathbf{x})$.

2.2.4 Summing the Rational Functions

The final stage is to sum the rational functions resulting from the trivial integrals. Whilst being the most elementary, it can also be the most computationally intensive. Given a perhaps large collection of rational functions in two variables, we must add them up. This sort of summation can easily be performed by a computer algebra system such as Maple or Magma. Indeed this is the approach used by Taylor [57]. However, we can make use of the fact that these rational functions are of the form

$$\frac{P(X,Y)}{\prod_{i=1}^{r}(1 - X^{a_i}Y^{b_i})}$$

for some bivariate polynomial $P(X,Y)$ with $a_i, b_i \in \mathbb{N}$. Typically, many of the factors of the denominator will cancel out once all the terms have been summed. If there are a large number of rational functions, it is advantageous to pick factors we believe will cancel, sum all the rational functions with this factor in the denominator and then hope that the factor cancels in this partial sum. We may then replace the rational functions we summed with the partial sum and continue. With less factors in the denominator, the remaining rational functions should sum more quickly.

2.3 Explicit Examples

For the rest of this chapter we give explicit expressions for the local zeta functions of many Lie rings. We also list the functional equation satisfied by these local zeta functions (where applicable), and the abscissae of convergence of the corresponding global zeta functions. We also give the order of the pole on the abscissa of convergence when it is not a simple pole. Unless we state otherwise, the local zeta functions we present are uniform, i.e. are given by the same rational function in p and p^{-s} for all primes p.

It may be noted that there are more zeta functions counting ideals than all subrings. There are usually more conditions under a p-adic integral counting ideals than under one counting all subrings, but the cone conditions for counting ideals are simpler.

The calculations involved are frequently long and tedious and were often performed with computer assistance. Therefore we shall not provide proofs of the calculations. This contrasts with the approach of Taylor [57], who does provide proofs of his calculations in his thesis. One such proof runs to 40 pages. There are several zeta functions of comparable or greater complexity presented in this chapter, and we simply don't have the space to present the proofs. Nonetheless we believe that all the zeta functions listed below are correct. In particular, there shouldn't have been any errors in transcription since the LaTeX source for each zeta function was generated from the computer calculations.

The advent of computer calculations has also led to zeta functions with the numerator and denominator of large degree. We have confined some of the larger numerator polynomials to Appendix A. However, there are four excessively large polynomials which we have chosen not to include since we do not feel the extra 23 pages they would require would be justified. Further details may be obtained from the authors on request.

Many of the examples will satisfy a functional equation of the form

$$\zeta^*_{L,p}(s)\big|_{p\to p^{-1}} = (-1)^c p^{b-as} \zeta^*_{L,p}(s) \tag{2.7}$$

for all but perhaps finitely many primes p. However, there are a small number that don't. When we say that a local zeta function 'satisfies no functional equation', we mean that it satisfies no functional equation of the form (2.7).

The Lie rings we shall be considering can be presented conveniently by giving a basis and the nontrivial Lie brackets of the basis elements. Most of these Lie brackets will be zero, so we make the convention that, up to antisymmetry, any Lie bracket not listed is zero.

2.4 Free Abelian Lie Rings

Let $L = \mathbb{Z}^d$, the free abelian Lie ring of rank d. Then

$$\zeta^{\triangleleft}_L(s) = \zeta^{\leq}_L(s) = \prod_{i=0}^{d-1} \zeta(s-i) \ ,$$

where $\zeta(s)$ is the Riemann zeta function. Hence this function is meromorphic on the whole of $\mathbb{C}$. In particular, the Tauberian Theorem (Theorem 1.8) mentioned in the Introduction allows us to deduce that if a_n is the number of subgroups of index n in $\mathbb{Z}^2$, then

$$\sum_{i=1}^{n} a_i \sim \frac{\pi^2}{12} n^2 \ ,$$

a result which seems remarkably difficult to obtain without the machinery of zeta functions.

In [22] it is shown that for any finite extension G of the free abelian group $\mathbb{Z}^d$, the zeta functions $\zeta_G^*(s)$ are all meromorphic. This is proved by relating the zeta functions to classical L-functions that arise in the work of Solomon, Bushnell and Reiner. The zeta functions of the 17 plane crystallographic groups, also known as the 'wallpaper groups', were calculated by McDermott and are listed in [22].

We shall see that many of the zeta functions have a factor similar to the local factor of $\zeta_{\mathbb{Z}^d}(s)$. It is therefore convenient to use the notation

$$\zeta_{\mathbb{Z}^n,p}(s) = \prod_{i=0}^{n-1} \zeta_p(s-i) \ , \tag{2.8}$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$ is the p-factor of the Riemann zeta function.

2.5 Heisenberg Lie Ring and Variants

Let $\mathcal{H}$ be the free class two, two generator nilpotent Lie ring. This is the Lie ring of strictly upper-triangular matrices

$$U_3(\mathbb{Z}) = \begin{pmatrix} 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 \end{pmatrix} \ .$$

It is given by the presentation

$$\mathcal{H} = \langle x, y, z : [x,y] = z \rangle \ ,$$

where, as mentioned above, $[x,z] = [y,z] = 0$. For $n \geq 2$, let $\mathcal{H}^n$ denote the direct product of n copies of the Heisenberg Lie ring.

Theorem 2.3 ([32]).

$$\begin{aligned} \zeta^{\triangleleft}_{\mathcal{H},p}(s) &= \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(3s-2) \ , \\ \zeta^{\leq}_{\mathcal{H},p}(s) &= \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(2s-2)\zeta_p(2s-3)\zeta_p(3s-3)^{-1} \ . \end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned} \zeta^{\triangleleft}_{\mathcal{H},p}(s)\Big|_{p\to p^{-1}} &= -p^{3-5s}\zeta^{\triangleleft}_{\mathcal{H},p}(s) \ , \\ \zeta^{\leq}_{\mathcal{H},p}(s)\Big|_{p\to p^{-1}} &= -p^{3-3s}\zeta^{\leq}_{\mathcal{H},p}(s) \ . \end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}} = \alpha^{\leq}_{\mathcal{H}} = 2$*, with* $\zeta^{\leq}_{\mathcal{H}}(s)$ *having a double pole at* $s = 2$*.*

Theorem 2.4 ([32, 57]).

$$\zeta^{\triangleleft}_{\mathcal{H}^2,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(5s-4)^{-1} ,$$
$$\zeta^{\leq}_{\mathcal{H}^2,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-4)^2\zeta_p(2s-5)^2\zeta_p(3s-5)\zeta_p(3s-7)\zeta_p(3s-8) \times W^{\leq}_{\mathcal{H}^2}(p,p^{-s}) ,$$

where $W^{\leq}_{\mathcal{H}^2}(X,Y)$ *is*

$$\begin{aligned}&1 - X^4Y^3 - 3X^5Y^3 - X^7Y^3 + X^5Y^4 - X^9Y^4 - X^8Y^5 + 3X^9Y^5 - 2X^{11}Y^5\\ &+ X^{10}Y^6 + 3X^{11}Y^6 + 3X^{12}Y^6 + 2X^{13}Y^6 + X^{14}Y^6 - X^{14}Y^7 + X^{15}Y^7\\ &- X^{14}Y^8 + X^{15}Y^8 - X^{15}Y^9 - 2X^{16}Y^9 - 3X^{17}Y^9 - 3X^{18}Y^9 - X^{19}Y^9\\ &+ 2X^{18}Y^{10} - 3X^{20}Y^{10} + X^{21}Y^{10} + X^{20}Y^{11} - X^{24}Y^{11} + X^{22}Y^{12}\\ &+ 3X^{24}Y^{12} + X^{25}Y^{12} - X^{29}Y^{15} .\end{aligned}$$

These zeta functions satisfy the functional equations

$$\zeta^{\triangleleft}_{\mathcal{H}^2,p}(s)\Big|_{p\to p^{-1}} = p^{15-10s}\zeta^{\triangleleft}_{\mathcal{H}^2,p}(s) ,$$
$$\zeta^{\leq}_{\mathcal{H}^2,p}(s)\Big|_{p\to p^{-1}} = p^{15-6s}\zeta^{\leq}_{\mathcal{H}^2,p}(s) .$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}^2} = \alpha^{\leq}_{\mathcal{H}^2} = 4$.

Theorem 2.5 ([57]).

$$\zeta^{\triangleleft}_{\mathcal{H}^3,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)^3\zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(8s-14)W^{\triangleleft}_{\mathcal{H}^3}(p,p^{-s}) ,$$

where $W^{\triangleleft}_{\mathcal{H}^3}(X,Y)$ *is*

$$\begin{aligned}&1 - 3X^6Y^5 + 2X^7Y^5 + X^6Y^7 - 2X^7Y^7 + X^{12}Y^8 - 2X^{13}Y^8 + 2X^{13}Y^{12}\\ &- X^{14}Y^{12} + 2X^{19}Y^{13} - X^{20}Y^{13} - 2X^{19}Y^{15} + 3X^{20}Y^{15} - X^{26}Y^{20} .\end{aligned}$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathcal{H}^3,p}(s)\Big|_{p\to p^{-1}} = -p^{36-15s}\zeta^{\triangleleft}_{\mathcal{H}^3,p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}^3} = 6$.

Theorem 2.6 ([64]).

$$\zeta^{\triangleleft}_{\mathcal{H}^4,p}(s) = \zeta_{\mathbb{Z}^8,p}(s)\zeta_p(3s-8)^4\zeta_p(5s-9)\zeta_p(7s-10)\zeta_p(8s-18)\zeta_p(9s-11) \times \zeta_p(10s-20)\zeta_p(11s-27)W^{\triangleleft}_{\mathcal{H}^4}(p,p^{-s}) ,$$

where the polynomial $W^{\triangleleft}_{\mathcal{H}^4}(X,Y)$ *is given in Appendix A on p. 179. This zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{\mathcal{H}^4,p}(s)\Big|_{p\to p^{-1}} = p^{66-20s}\zeta^{\triangleleft}_{\mathcal{H}^4,p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}^4} = 8$.

Theorem 2.7. *Let* $(K:\mathbb{Q}) = 2$, R *be the ring of integers of* K *and* $L = U_3(R)$. *Then*

1. *If* p *is inert (of which there are possibly infinitely many) then*
$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(5s-5)\zeta_p(6s-8)(1+p^{4-5s}) \ .$$
2. *If* p *is ramified (of which there are only finitely many) then*
$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(5s-5) \ .$$
3. *If* p *is split then* $U_3(R\otimes\mathbb{Z}_p) = U_3(\mathbb{Z}_p)\times U_3(\mathbb{Z}_p)$ *and we already have a calculation of this factor from Theorem 2.4 above.*

For all split or inert primes p, *this zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{L,p}(s)\big|_{p\to p^{-1}} = p^{15-10s}\zeta^{\triangleleft}_{L,p}(s) \ ,$$

whereas for p *ramified,*

$$\zeta^{\triangleleft}_{L,p}(s)\big|_{p\to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{L,p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{L} = 4$.

Taking the Euler product of all these factors we can represent the global zeta function in terms of the Riemann zeta function and the Dedekind zeta function $\zeta_K(s)$ of the underlying quadratic number field K (as observed in Corollary 8.2 of [32]):

Corollary 2.8.

$$\zeta^{\triangleleft}_{L}(s) = \zeta_{\mathbb{Z}^4}(s)\zeta(5s-4)\zeta(5s-5)\zeta_K(3s-4)/\zeta_K(5s-4) \ . \qquad (2.9)$$

Theorem 2.9 ([32, 57]). *Let* $L = U_3(R_3)$ *be the Lie ring of* 3×3 *upper triangular matrices over the ring of integers* R_3 *of a algebraic number field* K *of degree* 3 *over* $\mathbb{Q}$.

1. *If* p *is inert in* R_3, *then*
$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(7s-8)\zeta_p(8s-14)\zeta_p(9s-18)W^{\triangleleft}_{L,\mathrm{in}}(p,p^{-s})$$
where
$$W^{\triangleleft}_{L,\mathrm{in}}(X,Y) = 1 + X^6Y^7 + X^7Y^7 + X^{12}Y^8 + X^{13}Y^8 + X^{19}Y^{15} \ .$$

2. *If p ramifies completely in R_3 (i.e. if $(p) = \mathfrak{p}^3$ for some prime ideal $\mathfrak{p}$), then*

$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)\zeta_p(7s-8)\zeta_p(8s-14)(1+p^{7-5s}) \ .$$

3. *If p ramifies partially in R_3 (i.e. if $(p) = \mathfrak{p}^2\mathfrak{q}$ for prime ideals $\mathfrak{p} \neq \mathfrak{q}$),*

$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)^2\zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(8s-14)W^{\triangleleft}_{L,\mathrm{rp}}(p,p^{-s}),$$

where

$$\begin{aligned} W^{\triangleleft}_{L,\mathrm{rp}}(X,Y) = 1 &- X^6Y^5 + X^7Y^5 - X^7Y^7 - X^{13}Y^8 + X^{13}Y^{10} \\ &- X^{14}Y^{10} + X^{20}Y^{15} \ . \end{aligned}$$

4. *If p splits completely in R_3:*

$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)^3\zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(8s-14)W^{\triangleleft}_{L,\mathrm{sc}}(p,p^{-s}),$$

where $W^{\triangleleft}_{L,\mathrm{sc}} = W^{\triangleleft}_{\mathcal{H}^3}(X,Y)$ given above on p. 35.

5. *If p splits partially in R_3 (i.e. $(p) = \mathfrak{p}\mathfrak{q}$ for prime ideals $\mathfrak{p} \neq \mathfrak{q}$):*

$$\begin{aligned} \zeta^{\triangleleft}_{L,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)&\zeta_p(3s-6)\zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(6s-12)\zeta_p(8s-14) \\ &\times W^{\triangleleft}_{L,\mathrm{sp}}(p,p^{-s}) \ , \end{aligned}$$

where

$$\begin{aligned} W^{\triangleleft}_{L,\mathrm{sp}}(X,Y) = 1 &+ X^6Y^5 - X^6Y^7 - X^{12}Y^8 - X^{14}Y^{12} - X^{20}Y^{13} \\ &+ X^{20}Y^{15} + X^{26}Y^{20} \ . \end{aligned}$$

For all primes that do not ramify, this zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{L,p}(s)\big|_{p\to p^{-1}} = -p^{36-15s}\zeta^{\triangleleft}_{L,p}(s) \ .$$

The corresponding global function has abscissa of convergence $\alpha^{\triangleleft}_L = 6$.

Remark 2.10. 1. Cases 3 and 5 can only occur if the field K is not a normal extension of $\mathbb{Q}$.

2. As with the case with a quadratic number field, the p-local normal zeta function does satisfy a functional equation even when p ramifies. If f_p is the ramification degree of p in K, then

$$\zeta^{\triangleleft}_{L,p}(s)\big|_{p\to p^{-1}} = -p^{36-(13+2f_p)s}\zeta^{\triangleleft}_{L,p}(s)$$

for all primes p.

It is possible to write the global zeta function of L in terms of Riemann zeta functions, the zeta function of the number field and Euler products of these two variable polynomials. However, the end result is not as neat as (2.9):

Proposition 2.11. *If $(K:\mathbb{Q})=3$, R is the ring of integers of K and $L=U_3(R)$ then*

$$\zeta^{\triangleleft}_L(s)=\zeta_{\mathbb{Z}^6}(s)\zeta(5s-7)\zeta(7s-8)\zeta(8s-14)\zeta_K(3s-6)\prod_p W^{\triangleleft}_{L,p}(p,p^{-s}) \ ,$$

where

$$W^{\triangleleft}_{L,p}(X,Y)=\begin{cases} W^{\triangleleft}_{L,\mathrm{in}}(X,Y)(1-X^7Y^5) & \text{if } p \text{ is inert in } R,\\ 1-X^{14}Y^{10} & \text{if } p \text{ ramifies completely in } R,\\ W^{\triangleleft}_{L,\mathrm{rp}}(X,Y) & \text{if } p \text{ ramifies partially in } R,\\ W^{\triangleleft}_{L,\mathrm{sc}}(X,Y) & \text{if } p \text{ splits completely in } R,\\ W^{\triangleleft}_{L,\mathrm{sp}}(X,Y) & \text{if } p \text{ splits partially in } R.\end{cases}$$

2.6 Grenham's Lie Rings

The next examples are calculations made by Grenham in his D.Phil. thesis [28] of zeta functions of Lie rings $\mathcal{G}_n$ with the following presentation:

$$\mathcal{G}_n=\langle z,x_1,\dots,x_{n-1},y_1,\dots,y_{n-1}:[z,x_i]=y_i\ (1\le i\le n-1)\rangle \ .$$

These Lie rings are class-2 nilpotent. $\mathcal{G}_2\cong\mathcal{H}$, the Heisenberg Lie ring again. Grenham calculated $\zeta^{\triangleleft}_{\mathcal{G}_n,p}(s)$ and $\zeta^{\leq}_{\mathcal{G}_n,p}(s)$ for $n\le 5$. They all have the form of products of local Riemann zeta functions together with one of the palindromic polynomials.

Theorem 2.12 ([32, 28]).

$$\zeta^{\triangleleft}_{\mathcal{G}_3,p}(s)=\zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)^2\zeta_p(3s-4)\zeta_p(5s-6)\zeta_p(6s-6)^{-1} \ ,$$
$$\zeta^{\leq}_{\mathcal{G}_3,p}(s)=\zeta_{\mathbb{Z}^3,p}(s)\zeta_p(2s-4)\zeta_p(2s-5)\zeta_p(3s-6)W^{\leq}_{\mathcal{G}_3}(p,p^{-s}) \ ,$$

where

$$W^{\leq}_{\mathcal{G}_3}(X,Y)=1+X^3Y^2+X^4Y^2-X^4Y^3-X^5Y^3-X^8Y^5 \ .$$

These zeta functions satisfy the functional equations

$$\zeta^{\triangleleft}_{\mathcal{G}_3,p}(s)\Big|_{p\to p^{-1}}=-p^{10-8s}\zeta^{\triangleleft}_{\mathcal{G}_3,p}(s) \ ,$$
$$\zeta^{\leq}_{\mathcal{G}_3,p}(s)\Big|_{p\to p^{-1}}=-p^{10-5s}\zeta^{\leq}_{\mathcal{G}_3,p}(s) \ .$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{G}_3}=\alpha^{\leq}_{\mathcal{G}_3}=3$, with $\zeta^{\leq}_{\mathcal{G}_3}(s)$ having a double pole at $s=3$.

Theorem 2.13 ([28]).

$$\zeta^{\triangleleft}_{\mathcal{G}_4,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-6)\zeta_p(5s-10)\zeta_p(7s-12)W^{\triangleleft}_{\mathcal{G}_4}(p,p^{-s}) ,$$

where

$$W^{\triangleleft}_{\mathcal{G}_4}(X,Y) = 1 + X^4Y^3 + X^5Y^3 + X^8Y^5 + X^9Y^5 + X^{13}Y^8 ,$$

and

$$\begin{aligned}\zeta^{\leq}_{\mathcal{G}_4,p}(s) = {} & \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-5)\zeta_p(2s-6)\zeta_p(2s-7)\zeta_p(3s-10)\zeta_p(4s-12)\\ & \times W^{\leq}_{\mathcal{G}_4}(p,p^{-s})\end{aligned}$$

where $W^{\leq}_{\mathcal{G}_4}(X,Y)$ *is*

$$\begin{aligned}&1 + X^4Y^2 + X^5Y^2 + X^6Y^2 - X^5Y^3 - X^6Y^3 - X^7Y^3 + X^8Y^3 + X^9Y^3\\ &- X^9Y^4 - X^{10}Y^4 - X^{11}Y^4 - X^{14}Y^6 - X^{15}Y^6 - X^{16}Y^6 + X^{16}Y^7 + X^{17}Y^7\\ &- X^{18}Y^7 - X^{19}Y^7 - X^{20}Y^7 + X^{19}Y^8 + X^{20}Y^8 + X^{21}Y^8 + X^{25}Y^{10} .\end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned}\zeta^{\triangleleft}_{\mathcal{G}_4,p}(s)\Big|_{p\to p^{-1}} &= -p^{21-11s}\zeta^{\triangleleft}_{\mathcal{G}_4,p}(s) ,\\ \zeta^{\leq}_{\mathcal{G}_4,p}(s)\Big|_{p\to p^{-1}} &= -p^{21-7s}\zeta^{\leq}_{\mathcal{G}_4,p}(s) .\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{G}_4} = \alpha^{\leq}_{\mathcal{G}_4} = 4$, *with* $\zeta^{\leq}_{\mathcal{G}_4}(s)$ *having a double pole at* $s = 4$.

Theorem 2.14 ([28]).

$$\zeta^{\triangleleft}_{\mathcal{G}_5,p}(s) = \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(3s-8)\zeta_p(5s-14)\zeta_p(7s-18)\zeta_p(9s-20)W^{\triangleleft}_{\mathcal{G}_5}(p,p^{-s})$$

where $W^{\triangleleft}_{\mathcal{G}_5}(X,Y)$ *is*

$$\begin{aligned}&1 + X^5Y^3 + X^6Y^3 + X^7Y^3 + X^{10}Y^5 + X^{11}Y^5 + 2X^{12}Y^5 + X^{13}Y^5 + X^{15}Y^7\\ &+ X^{16}Y^7 + X^{17}Y^7 + X^{17}Y^8 + X^{18}Y^8 + X^{19}Y^8 + X^{21}Y^{10} + 2X^{22}Y^{10}\\ &+ X^{23}Y^{10} + X^{24}Y^{10} + X^{27}Y^{12} + X^{28}Y^{12} + X^{29}Y^{12} + X^{34}Y^{15} ,\end{aligned}$$

and

$$\begin{aligned}\zeta^{\leq}_{\mathcal{G}_5,p}(s) = {} & \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(2s-6)\zeta_p(2s-8)\zeta_p(2s-9)\zeta_p(3s-14)\zeta_p(4s-18)\\ & \times \zeta_p(5s-20)\zeta_p(s-2)^{-1}W^{\leq}_{\mathcal{G}_5}(p,p^{-s}) ,\end{aligned}$$

where $W^{\leq}_{\mathcal{G}_5}(X,Y)$ *is*

$$\begin{aligned}
&1 + X^2Y + X^4Y^2 + X^5Y^2 + X^6Y^2 + 2X^7Y^2 + X^8Y^2 + X^9Y^3 + 2X^{10}Y^3 \\
&+ X^{11}Y^3 + 2X^{12}Y^3 + X^{13}Y^3 + X^{12}Y^4 + 2X^{14}Y^4 + 2X^{15}Y^4 + X^{16}Y^4 \\
&+ X^{17}Y^4 + 2X^{17}Y^5 + X^{18}Y^5 + 2X^{19}Y^5 + X^{20}Y^5 - X^{18}Y^6 - X^{20}Y^6 \\
&+ X^{21}Y^6 + 2X^{22}Y^6 + 2X^{23}Y^6 + 2X^{24}Y^6 + X^{25}Y^6 - X^{22}Y^7 - 2X^{23}Y^7 \\
&- 2X^{24}Y^7 - 2X^{25}Y^7 - X^{26}Y^7 + X^{27}Y^7 + X^{29}Y^7 - X^{27}Y^8 - 2X^{28}Y^8 \\
&- X^{29}Y^8 - 2X^{30}Y^8 - X^{30}Y^9 - X^{31}Y^9 - 2X^{32}Y^9 - 2X^{33}Y^9 - X^{35}Y^9 \\
&- X^{34}Y^{10} - 2X^{35}Y^{10} - X^{36}Y^{10} - 2X^{37}Y^{10} - X^{38}Y^{10} - X^{39}Y^{11} \\
&- 2X^{40}Y^{11} - X^{41}Y^{11} - X^{42}Y^{11} - X^{43}Y^{11} - X^{45}Y^{12} - X^{47}Y^{13} .
\end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathcal{G}_5,p}(s)\Big|_{p\to p^{-1}} &= -p^{36-14s}\zeta^{\triangleleft}_{\mathcal{G}_5,p}(s) , \\
\zeta^{\leq}_{\mathcal{G}_5,p}(s)\Big|_{p\to p^{-1}} &= -p^{36-9s}\zeta^{\leq}_{\mathcal{G}_5,p}(s) .
\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{G}_5} = \alpha^{\leq}_{\mathcal{G}_5} = 5$*, with* $\zeta^{\leq}_{\mathcal{G}_5}(s)$ *having a triple pole at* $s = 5$*.*

In [61], Voll has given an explicit expression for $\zeta^{\triangleleft}_{\mathcal{G}_n,p}(s)$, and in a forthcoming paper, gives a similar expression for $\zeta^{\leq}_{\mathcal{G}_n,p}(s)$. In particular, he proves that

Theorem 2.15. *Let* $n > 1$*. Then for all primes* p*,* $\zeta^{\triangleleft}_{\mathcal{G}_n,p}(s)$ *and* $\zeta^{\leq}_{\mathcal{G}_n,p}(s)$ *satisfy the functional equations*

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathcal{G}_n,p}(s)\Big|_{p\to p^{-1}} &= -p^{\binom{2n-1}{2}-(3n-1)s}\zeta^{\triangleleft}_{\mathcal{G}_n,p}(s) , \\
\zeta^{\leq}_{\mathcal{G}_n,p}(s)\Big|_{p\to p^{-1}} &= -p^{\binom{2n-1}{2}-(2n-1)s}\zeta^{\leq}_{\mathcal{G}_n,p}(s) .
\end{aligned}$$

Grenham proved that the abscissa of convergence of $\zeta^{\triangleleft}_{\mathcal{G}_n}(s)$ is n. Voll gives in [61] an expression for the abscissa of convergence of $\zeta^{\leq}_{\mathcal{G}_n}(s)$, which agrees with an expression previously derived by Paajanen. In particular, $\alpha^{\leq}_{\mathcal{G}_6}(s) = 19/3$.

2.7 Free Class-2 Nilpotent Lie Rings

Let $F_{2,n}$ denote the free nilpotent Lie ring of class two on n generators. $F_{2,2}$ is the Heisenberg Lie ring once again.

2.7.1 Three Generators

Theorem 2.16 ([32, 57]). *Let the Lie ring* $F_{2,3}$ *have presentation*

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 : [x_1, x_2] = y_1, [x_1, x_3] = y_2, [x_2, x_3] = y_3 \rangle \ .$$

Then

$$\zeta^{\triangleleft}_{F_{2,3},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-5)\zeta_p(5s-8)\zeta_p(6s-9)W^{\triangleleft}_{F_{2,3}}(p, p^{-s}) \ ,$$

where

$$W^{\triangleleft}_{F_{2,3}}(X, Y) = 1 + X^3Y^3 + X^4Y^3 + X^6Y^5 + X^7Y^5 + X^{10}Y^8 \ ,$$

and

$$\begin{aligned} \zeta^{\leq}_{F_{2,3},p}(s) = {} & \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(2s-4)\zeta_p(2s-5)\zeta_p(2s-6)\zeta_p(3s-6)\zeta_p(3s-7) \\ & \times \zeta_p(3s-8)\zeta_p(4s-8)^{-1}W^{\leq}_{F_{2,3}}(p, p^{-s}) \ , \end{aligned}$$

where $W^{\leq}_{F_{2,3}}(X, Y)$ *is*

$$\begin{aligned} & 1 + X^3Y^2 + X^4Y^2 + X^5Y^2 - X^4Y^3 - X^5Y^3 - X^6Y^3 - X^7Y^4 - X^9Y^4 \\ & - X^{10}Y^5 - X^{11}Y^5 - X^{12}Y^5 + X^{11}Y^6 + X^{12}Y^6 + X^{13}Y^6 + X^{16}Y^8 \ . \end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned} \zeta^{\triangleleft}_{F_{2,3},p}(s)\Big|_{p\to p^{-1}} &= p^{15-9s}\zeta^{\triangleleft}_{F_{2,3},p}(s) \ , \\ \zeta^{\leq}_{F_{2,3},p}(s)\Big|_{p\to p^{-1}} &= p^{15-6s}\zeta^{\leq}_{F_{2,3},p}(s) \ . \end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{F_{2,3}} = 3$, $\alpha^{\leq}_{F_{2,3}} = 7/2$.

The zeta function counting all subrings is interesting since the abscissa of convergence is not an integer and is strictly greater than the rank of the abelianisation of G. This was the first such example calculated at nilpotency class 2.

2.7.2 n Generators

In [62], Voll gives an explicit formulae for the local ideal zeta functions of $F_{2,n}$ for all n. We shall not replicate Voll's explicit formulae for these functions, but we shall state some corollaries he deduces. Put $h(n) = \frac{1}{2}n(n+1)$, the rank of $F_{2,n}$.

Corollary 2.17. *The local zeta functions* $\zeta^{\triangleleft}_{F_{2,n},p}(s)$ *are uniform, i.e. are given by the same rational function in* p *and* p^{-s} *for all primes* p.

Corollary 2.18. *The local ideal zeta function of $F_{2,n}$ satisfies the local functional equation*

$$\zeta^{\triangleleft}_{F_{2,n},p}(s)\Big|_{p\to p^{-1}} = (-1)^{h(n)} p^{\binom{h(n)}{2}-(h(n)+n)s} \zeta^{\triangleleft}_{F_{2,n},p}(s)$$

for all primes p.

Corollary 2.19. *The abscissa of convergence of $\zeta^{\triangleleft}_{F_{2,n}}(s)$ is*

$$\alpha^{\triangleleft}_{F_{2,n}} = \max\left\{ n, \frac{\left(\binom{n}{2}-j\right)(n+j)+1}{h(n)-j} \;\middle|\; j \in \{1,\ldots,\binom{n}{2}-1\} \right\}$$

and $\zeta^{\triangleleft}_{F_{2,n}}(s)$ has a simple pole at $s = \alpha^{\triangleleft}_{F_{2,n}}$.

In particular, $F_{2,5}$ has abscissa of convergence $\alpha^{\triangleleft}_{F_{2,5}} = 51/10$. Indeed, this is the first Lie ring whose local ideal zeta function is known to have abscissa of convergence strictly greater than the rank of the abelianisation.

2.8 The 'Elliptic Curve Example'

Theorem 2.20 ([60]). *Let E denote the elliptic curve $y^2 = x^3 - x$. Define the nilpotent Lie ring L_E by the presentation*

$$L_E = \left\langle x_1,\ldots,x_6,y_1,y_2,y_3 : \begin{array}{c} [x_1,x_4]=y_3,[x_1,x_5]=y_1,[x_1,x_6]=y_2, \\ [x_2,x_4]=y_1,[x_2,x_5]=y_3, \\ [x_3,x_4]=y_2,[x_3,x_6]=y_1 \end{array} \right\rangle .$$

Then, for all but finitely many primes p, the local zeta function of L_E is given by

$$\begin{aligned}\zeta^{\triangleleft}_{L_E,p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(9s-18)\zeta_p(8s-14) \\ \times\left(P_1(p,p^{-s}) + |E(\mathbb{F}_p)|P_2(p,p^{-s})\right) ,\end{aligned}$$

where

$$\begin{aligned} |E(\mathbb{F}_p)| &= \left|\left\{ (x:y:z) \in \mathbb{P}^2(\mathbb{F}_p) : y^2z = x^3 - xz^2 \right\}\right| , \\ P_1(X,Y) &= (1 + X^6Y^7 + X^7Y^7 + X^{12}Y^8 + X^{13}Y^8 + X^{19}Y^{15})(1 - X^7Y^5) , \\ P_2(X,Y) &= X^6Y^5(1-Y^2)(1+X^{13}Y^8) . \end{aligned}$$

In [13] it was shown that this zeta function is not finitely uniform, thus answering in the negative a question posed by Grunewald, Segal and Smith in [32] that seemed 'plausible'. However, there was some doubt as to whether this zeta function would satisfy a functional equation similar to that satisfied by other local ideal zeta functions of Lie rings of class 2. The dependency on the number of points mod p on an elliptic curve did cast some doubt on this. However, it can easily be checked that

$$P_1(X^{-1}, Y^{-1}) = X^{-26}Y^{-20}P_1(X, Y) \; ,$$
$$P_2(X^{-1}, Y^{-1}) = X^{-25}Y^{-20}P_2(X, Y) \; .$$

Together with the functional equation of the Weil zeta function applied to $|E(\mathbb{F}_p)|$, this yields

Corollary 2.21 (Voll [60]). *For all but finitely many primes p,*

$$\zeta^{\triangleleft}_{L_E,p}(s)\big|_{p\to p^{-1}} = -p^{36-15s}\zeta^{\triangleleft}_{L_E,p}(s) \; .$$

2.9 Other Class Two Examples

We start with a number of Lie rings which appear in [32].

Theorem 2.22 ([32]). *Let $G(m, r)$ denote the direct product of $\mathbb{Z}^r$ with the central product of m copies of the Heisenberg Lie ring $\mathcal{H}$. Then $G(m, r)$ has Hirsch length $2m + r + 1$.*

$$\zeta^{\triangleleft}_{G(m,r),p}(s) = \zeta_{\mathbb{Z}^{2m+r},p}(s)\zeta_p((2m + 1)s - (2m + r)) \; .$$

For $m \leq 2$,

$$\zeta^{\leq}_{G(1,r),p}(s) = \zeta_{\mathbb{Z}^{r+2},p}(s)\zeta_p(2s - (r + 2))\zeta_p(2s - (r + 3))\zeta_p(3s - (r + 3))^{-1} \; ,$$
$$\zeta^{\leq}_{G(2,r),p}(s) = \zeta_{\mathbb{Z}^{r+4},p}(s)\zeta_p(3s - (r + 4))\zeta_p(3s - (r + 6))\zeta_p(3s - (r + 7)) \times W^{\leq}_{G(2,r)}(p, p^{-s}) \; ,$$

where

$$W^{\leq}_{G(2,r)}(X, Y) = 1 + X^{r+5}Y^3 - X^{r+5}Y^4 - X^{r+6}Y^4 - X^{r+7}Y^4 - X^{r+8}Y^4 + X^{r+8}Y^5 + X^{2r+13}Y^8 \; .$$

These zeta functions satisfy the functional equations

$$\zeta^{\triangleleft}_{G(m,r),p}(s)\Big|_{p\to p^{-1}} = (-1)^{2m+r+1}p^{\binom{2m+r+1}{2}-(4m+r+1)s}\zeta^{\triangleleft}_{G(m,r),p}(s) \; ,$$
$$\zeta^{\leq}_{G(m,r),p}(s)\Big|_{p\to p^{-1}} = (-1)^{2m+r+1}p^{\binom{2m+r+1}{2}-(2m+r+1)s}\zeta^{\leq}_{G(m,r),p}(s) \; (m = 1, 2) \; .$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{G(m,r)} = 2m + r$ for all $m \in \mathbb{N}_{>0}$, $r \in \mathbb{N}$ and $\alpha^{\leq}_{G(m,r)} = 2m + r$ for $m \in \{1, 2\}$, $r \in \mathbb{N}$.

Theorem 2.23 ([32]). *For $r \in \mathbb{N}$,*

$$\begin{aligned}\zeta^{\triangleleft}_{\mathcal{G}_3\times\mathbb{Z}^r,p}(s) &= \zeta_{\mathbb{Z}_p^{r+3}}(s)\zeta_p(3s-(r+4))\zeta_p(5s-(2r+6))(1+p^{r+3-3s}) ,\\ \zeta^{\leq}_{\mathcal{G}_3\times\mathbb{Z}^r,p}(s) &= \zeta_{\mathbb{Z}_p^{r+3}}(s)\zeta_p(2s-(r+4))\zeta_p(2s-(r+5))\zeta_p(3s-(2r+6))\\ &\qquad\times W^{\leq}_{\mathcal{G}_3\times\mathbb{Z}^r}(p,p^{-s}) ,\end{aligned}$$

where

$$W^{\leq}_{\mathcal{G}_3\times\mathbb{Z}^r}(p,p^{-s}) = 1 + X^{r+3}Y^2 + X^{r+4}Y^2 - X^{r+4}Y^3 - X^{r+5}Y^3 - X^{2r+8}Y^5 .$$

These zeta functions satisfy the functional equations

$$\begin{aligned}\zeta^{\triangleleft}_{\mathcal{G}_3\times\mathbb{Z}^r,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{r+5}p^{\binom{r+5}{2}-(r+8)s}\zeta^{\triangleleft}_{\mathcal{G}_3\times\mathbb{Z}^r,p}(s) ,\\ \zeta^{\leq}_{\mathcal{G}_3\times\mathbb{Z}^r,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{r+5}p^{\binom{r+5}{2}-(r+5)s}\zeta^{\leq}_{\mathcal{G}_3\times\mathbb{Z}^r,p}(s) .\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{G}_3\times\mathbb{Z}^r} = \alpha^{\leq}_{\mathcal{G}_3\times\mathbb{Z}^r} = r+3$.

The calculations of the ideal zeta functions were made by Grunewald, Segal and Smith in [32]. Note that they use the more cumbersome notation $F_{2,3}/\langle z\rangle$ in place of $\mathcal{G}_3$.

Theorem 2.24 ([32, 64]). *Let*

$$\mathfrak{g}_{6,4} = \langle x_1,x_2,x_3,x_4,y_1,y_2 : [x_1,x_2]=y_1,[x_1,x_3]=y_2,[x_2,x_4]=y_2\rangle .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,4},p}(s) &= \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(6s-9)\zeta_p(8s-9)^{-1} ,\\ \zeta^{\leq}_{\mathfrak{g}_{6,4},p}(s) &= \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-7)\zeta_p(3s-8)\zeta_p(4s-9)\\ &\qquad\times\zeta_p(4s-11)\zeta_p(5s-12)W^{\leq}_{\mathfrak{g}_{6,4}}(p,p^{-s}) ,\end{aligned}$$

where $W^{\leq}_{\mathfrak{g}_{6,4}}(X,Y)$ is given in Appendix A on p. 180. These zeta functions satisfy the functional equations

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,4},p}(s)\Big|_{p\to p^{-1}} &= p^{15-10s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,4},p}(s) ,\\ \zeta^{\leq}_{\mathfrak{g}_{6,4},p}(s)\Big|_{p\to p^{-1}} &= p^{15-6s}\zeta^{\leq}_{\mathfrak{g}_{6,4},p}(s) .\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,4}} = \alpha^{\leq}_{\mathfrak{g}_{6,4}} = 4$.

In [32], this Lie ring is given the more cumbersome name $F_{2,3}/\langle z\rangle \cdot \mathbb{Z}$. For brevity we have changed the name. The new name is borrowed from the classification of nilpotent Lie algebras of dimension 6 mentioned in Sect. 2.14 below.

Let T_n denote the maximal class-two quotient of the Lie ring of unitriangular $n \times n$ matrices. T_n has presentation

$$\langle x_1, \ldots, x_n, y_1, \ldots, y_{n-1} : [x_i, x_{i+1}] = y_i \text{ for } 1 \leq i \leq n-1\rangle \ .$$

T_2 is the Heisenberg Lie ring once again, and $T_3 \cong \mathcal{G}_3$, whose zeta functions are given in Sect. 2.6.

Theorem 2.25 ([57, 64]).

$$\zeta^{\triangleleft}_{T_4,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-5)^2\zeta_p(5s-6)\zeta_p(5s-8)\zeta_p(6s-10)\zeta_p(7s-12) \times W^{\triangleleft}_{T_4}(p,p^{-s}) \ ,$$

where $W^{\triangleleft}_{T_4}(X,Y)$ *is*

$$1 + X^4Y^3 - X^5Y^5 + X^8Y^5 - X^8Y^6 - X^9Y^6 - X^{10}Y^8 - X^{12}Y^8 - X^{13}Y^9 + X^{13}Y^{10} - 2X^{14}Y^{10} + X^{14}Y^{11} + X^{15}Y^{11} - X^{16}Y^{11} - X^{17}Y^{11} + 2X^{17}Y^{12} - X^{18}Y^{12} + X^{18}Y^{13} + X^{19}Y^{14} + X^{21}Y^{14} + X^{22}Y^{16} + X^{23}Y^{16} - X^{23}Y^{17} + X^{26}Y^{17} - X^{27}Y^{19} - X^{31}Y^{22} \ ,$$

and

$$\zeta^{\leq}_{T_4,p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-5)^2\zeta_p(2s-6)^2\zeta_p(3s-6)\zeta_p(3s-8)^2\zeta_p(3s-9) \times \zeta_p(4s-12)\zeta_p(5s-14)W^{\leq}_{T_4}(p,p^{-s}) \ ,$$

where the polynomial $W^{\leq}_{T_4}(X,Y)$ *is given in Appendix A on p. 180. These zeta functions satisfy the functional equations*

$$\zeta^{\triangleleft}_{T_4,p}(s)\Big|_{p\to p^{-1}} = -p^{21-11s}\zeta^{\triangleleft}_{T_4,p}(s) \ ,$$
$$\zeta^{\leq}_{T_4,p}(s)\Big|_{p\to p^{-1}} = -p^{21-7s}\zeta^{\leq}_{T_4,p}(s) \ .$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{T_4} = \alpha^{\leq}_{T_4} = 4$.

2.10 The Maximal Class Lie Ring M_3 and Variants

The most well-understood zeta functions of Lie rings are those for Lie rings of nilpotency class 2. However, as we move to higher nilpotency classes, there is much less in the way of theory to help us. In particular, as we mentioned in Chap. 1, the Mal'cev correspondence can be avoided for nilpotency class 2. There is no such shortcut in higher nilpotency classes.

Taylor [57] was the first to calculate the zeta functions of a class-3-nilpotent Lie ring, and since then the second author has greatly enlarged the stock of examples at class 3.

In some sense, the 'simplest' Lie rings of nilpotency class n are the Lie rings M_n, with presentation

$$M_n = \langle z, x_1, x_2, \ldots, x_n : [z, x_i] = x_{i+1} \text{ for } i = 1, \ldots, n-1 \rangle \ .$$

In particular, $\mathcal{H} = M_2$. We now consider M_3 and some variations.

Theorem 2.26. *For $r \in \mathbb{Z}$,*

$$\zeta^{\triangleleft}_{M_3\times\mathbb{Z}^r,p}(s) = \frac{\zeta_{\mathbb{Z}^{r+2},p}(s)\zeta_p(3s-(r+2))\zeta_p(4s-(r+2))\zeta_p(5s-(r+3))}{\zeta_p(5s-(r+2))} \ ,$$

and

$$\begin{aligned}\zeta^{\leq}_{M_3\times\mathbb{Z}^r,p}(s) = {} & \zeta_{\mathbb{Z}^{r+2},p}(s)\zeta_p(2s-(r+3))\zeta_p(3s-(r+5))\zeta_p(3s-(2r+4)) \\ & \times \zeta_p(4s-(2r+6))W^{\leq}_{M_3\times\mathbb{Z}^r}(p,p^{-s}) \ ,\end{aligned}$$

where

$$\begin{aligned}W^{\leq}_{M_3\times\mathbb{Z}^r}(p,p^{-s}) = {} & 1 + X^{r+2}Y^2 + X^{r+3}Y^2 - X^{r+3}Y^3 - X^{r+5}Y^4 + X^{2r+6}Y^4 \\ & - 2X^{2r+6}Y^5 - 2X^{2r+7}Y^5 + X^{2r+7}Y^6 - X^{3r+8}Y^6 \\ & - X^{3r+10}Y^7 + X^{3r+10}Y^8 + X^{3r+11}Y^8 + X^{4r+13}Y^{10} \ .\end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned}\zeta^{\triangleleft}_{M_3\times\mathbb{Z}^r,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{r+4}p^{\binom{r+4}{2}-(r+9)s}\zeta^{\triangleleft}_{M_3\times\mathbb{Z}^r,p}(s) \ , \\ \zeta^{\leq}_{M_3\times\mathbb{Z}^r,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{r+4}p^{\binom{r+4}{2}-(r+4)s}\zeta^{\leq}_{M_3\times\mathbb{Z}^r,p}(s) \ .\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{M_3\times\mathbb{Z}^r} = \alpha^{\leq}_{M_3\times\mathbb{Z}^r} = r+2$, with $\zeta^{\leq}_{M_3}(s)$ having a quadruple pole at $s=2$.

The zeta functions counting ideals or all subrings in M_3 were first calculated by Taylor in [57]. The second author generalised the results to $M_3 \times \mathbb{Z}^r$ for $r \in \mathbb{N}$.

Theorem 2.27 ([64]).

$$\begin{aligned}\zeta^{\triangleleft}_{\mathcal{H}\times M_3,p}(s) = {} & \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)^2\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(6s-5)\zeta_p(7s-6) \\ & \times \zeta_p(9s-10)W^{\triangleleft}_{\mathcal{H}\times M_3}(p,p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathcal{H}\times M_3}(X,Y)$ *is*

$$1 - 2X^4Y^5 + X^5Y^5 - X^4Y^6 + X^4Y^7 - 2X^5Y^7 + X^8Y^9 - 2X^9Y^9 + 3X^9Y^{11} - 2X^{10}Y^{11} + X^9Y^{12} + X^{10}Y^{13} + X^{13}Y^{14} + X^{14}Y^{15} - 2X^{13}Y^{16} + 3X^{14}Y^{16} - 2X^{14}Y^{18} + X^{15}Y^{18} - 2X^{18}Y^{20} + X^{19}Y^{20} - X^{19}Y^{21} + X^{18}Y^{22} - 2X^{19}Y^{22} + X^{23}Y^{27} .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathcal{H}\times M_3,p}(s)\Big|_{p\to p^{-1}} = -p^{21-14s}\zeta^{\triangleleft}_{\mathcal{H}\times M_3,p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}\times M_3} = 4$.

Theorem 2.28.

$$\begin{aligned}\zeta^{\triangleleft}_{\mathcal{H}^2\times M_3,p}(s) = {} & \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)^3\zeta_p(4s-6)\zeta_p(5s-7)\zeta_p(6s-7)\zeta_p(7s-8)\\ & \times \zeta_p(8s-8)\zeta_p(8s-14)\zeta_p(9s-9)\zeta_p(9s-14)\zeta_p(10s-15)\\ & \times \zeta_p(11s-16)\zeta_p(12s-21)W^{\triangleleft}_{\mathcal{H}^2\times M_3}(p,p^{-s})\end{aligned}$$

for some polynomial $W^{\triangleleft}_{\mathcal{H}^2\times M_3}(X,Y)$ *of degrees 113 in* X *and 85 in* Y. *This zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{\mathcal{H}^2\times M_3,p}(s)\Big|_{p\to p^{-1}} = p^{45-19s}\zeta^{\triangleleft}_{\mathcal{H}^2\times M_3,p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}^2\times M_3}=6$.

Theorem 2.29.

$$\begin{aligned}\zeta^{\triangleleft}_{M_3\times M_3,p}(s) = {} & \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-2)\zeta_p(3s-4)^2\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(6s-5)\\ & \times \zeta_p(7s-5)\zeta_p(7s-6)\zeta_p(8s-6)\zeta_p(9s-7)\zeta_p(9s-10)\\ & \times \zeta_p(10s-10)\zeta_p(11s-11)\zeta_p(12s-12)\zeta_p(13s-15)\\ & \times W^{\triangleleft}_{M_3\times M_3}(p,p^{-s})\end{aligned}$$

for some polynomial $W^{\triangleleft}_{M_3\times M_3}(X,Y)$ *of degrees 84 in* X *and 95 in* Y. *This zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{M_3\times M_3,p}(s)\Big|_{p\to p^{-1}} = p^{28-18s}\zeta^{\triangleleft}_{M_3\times M_3,p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{M_3\times M_3}=4$.

Theorem 2.30. *Let the Lie ring $M_3 \times_{\mathbb{Z}} M_3$ have presentation*

$$\langle z_1, z_2, w_1, w_2, x_1, x_2, y : [z_1, w_1] = x_1, [z_2, w_2] = x_2, [z_1, x_1] = y, [z_2, x_2] = y \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3, p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6) \\ \times \zeta_p(12s-10) W^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3}(X, Y)$ is

$$\begin{aligned}&1 - X^4Y^5 - 2X^4Y^8 + X^5Y^8 + X^4Y^9 - 2X^5Y^9 + X^8Y^{12} - 2X^9Y^{12} \\ &+ 3X^9Y^{13} - 2X^{10}Y^{13} + X^{10}Y^{14} + X^9Y^{17} + X^{14}Y^{17} + X^{13}Y^{20} - 2X^{13}Y^{21} \\ &+ 3X^{14}Y^{21} - 2X^{14}Y^{22} + X^{15}Y^{22} - 2X^{18}Y^{25} + X^{19}Y^{25} + X^{18}Y^{26} \\ &- 2X^{19}Y^{26} - X^{19}Y^{29} + X^{23}Y^{34} \ .\end{aligned}$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3, p}(s)\big|_{p \to p^{-1}} = -p^{21-17s}\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3, p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3} = 4$.

2.11 Lie Rings with Large Abelian Ideals

As we saw in Sect. 2.6, Voll has calculated $\zeta^{\triangleleft}_{\mathcal{G}_n,p}(s)$ and $\zeta^{\leq}_{\mathcal{G}_n,p}(s)$ for all $n \geq 2$. The Lie rings $\mathcal{G}_n$ have an abelian ideal of corank 1 (and thus of infinite index), and it is likely that this large ideal makes it easier to get a grasp on the structure of the lattices of ideals/subrings. Indeed the Lie rings M_n have this property too. In this section we consider some further Lie rings of nilpotency class 3 with this property.

Theorem 2.31 ([64]). *Let the Lie ring $L_{(3,3)}$ have presentation*

$$\langle z, w_1, w_2, x_1, x_2, y_1, y_2 : [z, w_1] = x_1, [z, w_2] = x_2, [z, x_1] = y_1, [z, x_2] = y_2 \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{L_{(3,3)},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-4)\zeta_p(4s-5)\zeta_p(5s-6)\zeta_p(6s-7)\zeta_p(7s-6) \\ \times \zeta_p(8s-10)\zeta_p(9s-12)\zeta_p(11s-12)\zeta_p(4s-4)^{-1} \\ \times W^{\triangleleft}_{L_{(3,3)}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{L_{(3,3)}}(X,Y)$ *is*

$$
\begin{aligned}
&1 + X^3Y^3 + 2X^4Y^4 - X^4Y^5 + X^6Y^5 + X^6Y^6 - X^6Y^7 + X^9Y^7 - X^6Y^8 \\
&+ 2X^8Y^8 - X^8Y^9 - X^{10}Y^9 - X^9Y^{10} + X^{12}Y^{10} - X^{10}Y^{11} - X^{12}Y^{11} \\
&- X^{13}Y^{12} - X^{12}Y^{13} - X^{14}Y^{13} - 2X^{16}Y^{13} - 2X^{15}Y^{14} - X^{14}Y^{15} - X^{16}Y^{15} \\
&- X^{18}Y^{15} + 2X^{16}Y^{16} - X^{18}Y^{16} - X^{19}Y^{16} - X^{18}Y^{17} - 2X^{20}Y^{17} + X^{18}Y^{18} \\
&+ X^{20}Y^{18} - X^{21}Y^{18} + X^{19}Y^{19} - X^{20}Y^{19} - X^{22}Y^{19} + 2X^{20}Y^{20} + X^{22}Y^{20} \\
&+ X^{21}Y^{21} + X^{22}Y^{21} - 2X^{24}Y^{21} + X^{22}Y^{22} + X^{24}Y^{22} + X^{26}Y^{22} + 2X^{25}Y^{23} \\
&+ 2X^{24}Y^{24} + X^{26}Y^{24} + X^{28}Y^{24} + X^{27}Y^{25} + X^{28}Y^{26} + X^{30}Y^{26} - X^{28}Y^{27} \\
&+ X^{31}Y^{27} + X^{30}Y^{28} + X^{32}Y^{28} - 2X^{32}Y^{29} + X^{34}Y^{29} - X^{31}Y^{30} + X^{34}Y^{30} \\
&- X^{34}Y^{31} - X^{34}Y^{32} + X^{36}Y^{32} - 2X^{36}Y^{33} - X^{37}Y^{34} - X^{40}Y^{37} \, .
\end{aligned}
$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{L_{(3,3)},p}(s)\Big|_{p\to p^{-1}} = -p^{21-15s}\zeta^{\triangleleft}_{L_{(3,3)},p}(s) \, .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{L_{(3,3)}} = 3$.

The second author also considered what happens when you delete generator y_2 from the presentation above:

Theorem 2.32 ([64]). *Let* $L_{(3,2)}$ *be given by the presentation*

$$\langle z, w_1, w_2, x_1, x_2, y : [z, w_1] = x_1, [z, w_2] = x_2, [z, x_1] = y \rangle \, .$$

Then

$$
\begin{aligned}
\zeta^{\triangleleft}_{L_{(3,2)},p}(s) = {}& \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6) \\
&\times \zeta_p(9s-11)W^{\triangleleft}_{L_{(3,2)}}(p,p^{-s}) \, ,
\end{aligned}
$$

where $W^{\triangleleft}_{L_{(3,2)}}(X,Y)$ *is*

$$
\begin{aligned}
&1 + X^3Y^3 - X^4Y^5 - X^6Y^7 - X^7Y^7 + X^8Y^7 - X^8Y^8 - X^9Y^9 - X^{10}Y^9 \\
&+ X^{10}Y^{10} - X^{11}Y^{10} + X^{10}Y^{11} - X^{11}Y^{11} + X^{11}Y^{12} - X^{14}Y^{12} + X^{13}Y^{13} \\
&- X^{14}Y^{13} + X^{14}Y^{14} + X^{15}Y^{14} + X^{17}Y^{16} + X^{18}Y^{17} + X^{20}Y^{18} - X^{21}Y^{21} \\
&- X^{24}Y^{23} \, ,
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta^{\leq}_{L_{(3,2)},p}(s) = {}& \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(2s-4)\zeta_p(2s-5)^2\zeta_p(3s-7)\zeta_p(3s-8)\zeta_p(4s-10) \\
&\times \zeta_p(5s-12)W^{\leq}_{L_{(3,2)}}(p,p^{-s}) \, ,
\end{aligned}
$$

where $W^{\leq}_{L_{(3,2)}}(X,Y)$ *is*

$$\begin{aligned}
&1 + X^3Y^2 + X^4Y^2 - X^4Y^3 - X^5Y^3 + X^6Y^3 + X^7Y^3 - 2X^7Y^4 - 2X^8Y^4 \\
&+ X^9Y^5 - 2X^{10}Y^5 - 3X^{11}Y^5 + X^{11}Y^6 + X^{12}Y^6 - 2X^{13}Y^6 - 3X^{14}Y^6 \\
&+ X^{13}Y^7 + X^{14}Y^7 + 3X^{15}Y^7 - 2X^{16}Y^7 - X^{17}Y^7 + X^{16}Y^8 + X^{17}Y^8 \\
&+ 2X^{18}Y^8 + 2X^{18}Y^9 + 2X^{21}Y^9 + 2X^{21}Y^{10} + X^{22}Y^{10} + X^{23}Y^{10} - X^{22}Y^{11} \\
&- 2X^{23}Y^{11} + 3X^{24}Y^{11} + X^{25}Y^{11} + X^{26}Y^{11} - 3X^{25}Y^{12} - 2X^{26}Y^{12} \\
&+ X^{27}Y^{12} + X^{28}Y^{12} - 3X^{28}Y^{13} - 2X^{29}Y^{13} + X^{30}Y^{13} - 2X^{31}Y^{14} \\
&- 2X^{32}Y^{14} + X^{32}Y^{15} + X^{33}Y^{15} - X^{34}Y^{15} - X^{35}Y^{15} + X^{35}Y^{16} + X^{36}Y^{16} \\
&+ X^{39}Y^{18} .
\end{aligned}$$

The local zeta function counting all subrings satisfies the functional equation

$$\zeta^{\leq}_{L_{(3,2)},p}(s)\Big|_{p\to p^{-1}} = p^{15-6s}\zeta^{\leq}_{L_{(3,2)},p}(s) .$$

However, the local ideal zeta function satisfies no such functional equation. The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{L_{(3,2)}} = \alpha^{\leq}_{L_{(3,2)}} = 3$*, with* $\zeta^{\leq}_{L_{(3,2)}}(s)$ *having a quadruple pole at* $s = 3$.

The zeta function counting ideals was the first calculated which satisfied no functional equation of the form (2.7).

A couple of Lie rings similar to $L_{(3,2)}$ were also considered. Their ideal zeta functions also satisfy no functional equation of the form seen numerous times before.

Theorem 2.33.

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathcal{H}\times L_{(3,2)},p}(s) = {}& \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(3s-5)\zeta_p(3s-6)\zeta_p(4s-6)\zeta_p(5s-7)\zeta_p(5s-10) \\
&\times \zeta_p(6s-7)\zeta_p(6s-10)\zeta_p(7s-8)\zeta_p(7s-12)\zeta_p(8s-12) \\
&\times \zeta_p(9s-14)\zeta_p(9s-17)\zeta_p(11s-19)\zeta_p(13s-20) \\
&\times \zeta_p(13s-23)W^{\triangleleft}_{\mathcal{H}\times L_{(3,2)}}(p,p^{-s})
\end{aligned}$$

for some polynomial $W^{\triangleleft}_{\mathcal{H}\times L_{(3,2)}}(X,Y)$ *of degrees 150 in* X *and 97 in* Y*. This local zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence* $\alpha^{\triangleleft}_{\mathcal{H}\times L_{(3,2)}} = 5$.

Theorem 2.34. *Let the Lie ring* $L_{(3,2,2)}$ *have presentation*

$$\left\langle z, w_1, w_2, w_3, x_1, x_2, x_3, y : \begin{array}{c} [z,w_1] = x_1, [z,w_2] = x_2, \\ [z,w_3] = x_3, [z,x_1] = y \end{array} \right\rangle .$$

Then

$$\begin{aligned}
\zeta^{\triangleleft}_{L_{(3,2,2)},p}(s) = {}& \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-3)\zeta_p(3s-6)\zeta_p(5s-7)\zeta_p(5s-10)\zeta_p(6s-10) \\
&\times \zeta_p(7s-12)\zeta_p(8s-12)\zeta_p(9s-17)\zeta_p(13s-23) \\
&\times W^{\triangleleft}_{L_{(3,2,2)}}(p,p^{-s}) ,
\end{aligned}$$

where $W^{\triangleleft}_{L_{(3,2,2)}}(X,Y)$ *is given in Appendix A on p. 181. This local zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence* $\alpha^{\triangleleft}_{L_{(3,2,2)}} = 4$.

2.12 $F_{3,2}$

On p. 40 we considered the zeta functions of the free class-2 nilpotent Lie rings. The second author has added the zeta functions of the class-3, 2-generator nilpotent Lie ring.

Theorem 2.35 ([64]). *Let the Lie ring* $F_{3,2}$ *have presentation*

$$\langle x_1, x_2, y_1, z_1, z_2 : [x_1, x_2] = y_1, [x_1, y_1] = z_1, [x_2, y_1] = z_2 \rangle \ .$$

Then

$$\zeta^{\triangleleft}_{F_{3,2},p}(s) = \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(3s-2)\zeta_p(4s-3)\zeta_p(5s-4)^2\zeta_p(7s-6)W^{\triangleleft}_{F_{3,2}}(p,p^{-s}) \ ,$$

where $W^{\triangleleft}_{F_{3,2}}(X,Y)$ *is*

$$1 + X^2Y^4 - X^2Y^5 - X^4Y^7 - X^6Y^9 - X^8Y^{11} + X^8Y^{12} + X^{10}Y^{16} \ ,$$

and

$$\begin{aligned}\zeta^{\leq}_{F_{3,2},p}(s) = {} & \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-8)\zeta_p(5s-8) \\ & \times \zeta_p(5s-9)W^{\leq}_{F_{3,2}}(p,p^{-s}) \ ,\end{aligned}$$

where $W^{\leq}_{F_{3,2}}(X,Y)$ *is*

$$\begin{aligned}& 1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 + 2X^5Y^3 - 2X^5Y^4 + 2X^7Y^4 - 2X^7Y^5 \\ & - 2X^8Y^5 - X^9Y^5 - X^{10}Y^6 - X^{11}Y^6 - X^{10}Y^7 - X^{13}Y^7 - 2X^{12}Y^8 \\ & - X^{13}Y^8 - X^{14}Y^8 - X^{15}Y^8 + X^{13}Y^9 - X^{16}Y^9 + X^{14}Y^{10} + X^{15}Y^{10} \\ & + X^{16}Y^{10} + 2X^{17}Y^{10} + X^{16}Y^{11} + X^{19}Y^{11} + X^{18}Y^{12} + X^{19}Y^{12} + X^{20}Y^{13} \\ & + 2X^{21}Y^{13} + 2X^{22}Y^{13} - 2X^{22}Y^{14} + 2X^{24}Y^{14} - 2X^{24}Y^{15} - X^{25}Y^{15} \\ & + X^{26}Y^{15} - X^{26}Y^{16} - X^{27}Y^{16} - X^{29}Y^{18} \ .\end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned}\zeta^{\triangleleft}_{F_{3,2},p}(s)\Big|_{p\to p^{-1}} &= -p^{10-10s}\zeta^{\triangleleft}_{F_{3,2},p}(s) \ , \\ \zeta^{\leq}_{F_{3,2},p}(s)\Big|_{p\to p^{-1}} &= -p^{10-5s}\zeta^{\leq}_{F_{3,2},p}(s) \ .\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{F_{3,2}} = 2$, $\alpha^{\leq}_{F_{3,2}} = 5/2$.

Theorem 2.36 ([64]).

$$\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(7s-8) \times W^{\triangleleft}_{F_{3,2}\times\mathbb{Z}}(p,p^{-s}) ,$$

where $W^{\triangleleft}_{F_{3,2}\times\mathbb{Z}}(X,Y)$ *is*

$$1+X^3Y^4-X^3Y^5-X^6Y^7-X^8Y^9-X^{11}Y^{11}+X^{11}Y^{12}+X^{14}Y^{16} .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z},p}(s)\Big|_{p\to p^{-1}} = p^{15-11s}\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{F_{3,2}\times\mathbb{Z}}=3$.

2.13 The Maximal Class Lie Rings M_4 and Fil_4

We saw above that M_3 is in some sense the simplest Lie ring of nilpotency class 3. The Lie ring M_4 can be defined in a similar way, and in some sense it is the simplest of nilpotency class 4. The M_n family of Lie rings are *filiform*, in that the nilpotency class is maximal given the rank.

Theorem 2.37 ([57]). *Let the Lie ring* M_4 *have presentation*

$$\langle z,x_1,x_2,x_3,x_4 : [z,x_1]=x_2, [z,x_2]=x_3, [z,x_3]=x_4\rangle .$$

Then

$$\zeta^{\triangleleft}_{M_4,p}(s) = \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6) \times \zeta_p(11s-6)\zeta_p(12s-7)\zeta_p(6s-3)^{-1}W^{\triangleleft}_{M_4}(p,p^{-s}) ,$$

where $W^{\triangleleft}_{M_4}(X,Y)$ *is*

$$\begin{aligned}&1+X^2Y^4-X^2Y^5+X^3Y^5-X^2Y^6+2X^3Y^6-X^3Y^7-X^5Y^9+X^6Y^{10}\\&-2X^5Y^{11}-X^7Y^{13}-X^8Y^{13}+X^7Y^{14}-X^8Y^{14}-X^8Y^{15}-X^9Y^{15}\\&+X^9Y^{16}-X^9Y^{17}-X^{10}Y^{17}+2X^9Y^{18}-X^{10}Y^{18}+X^{10}Y^{19}-2X^{11}Y^{19}\\&+X^{10}Y^{20}+X^{11}Y^{20}-X^{11}Y^{21}+X^{11}Y^{22}+X^{12}Y^{22}+X^{12}Y^{23}-X^{13}Y^{23}\\&+X^{12}Y^{24}+X^{13}Y^{24}+2X^{15}Y^{26}-X^{14}Y^{27}+X^{15}Y^{28}+X^{17}Y^{30}-2X^{17}Y^{31}\\&+X^{18}Y^{31}-X^{17}Y^{32}+X^{18}Y^{32}-X^{18}Y^{33}-X^{20}Y^{37}\end{aligned}$$

and

$$\zeta^{\leq}_{M_4,p}(s) = \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-7)\zeta_p(4s-8) \times \zeta_p(7s-12)W^{\leq}_{M_4}(p,p^{-s}) ,$$

where $W^{\leq}_{M_4}(X,Y)$ *is*

$$1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 + 2X^5Y^3 - 2X^5Y^4 + X^7Y^4 - 2X^7Y^5 - X^8Y^5 + X^9Y^5 - 2X^9Y^6 - 2X^{10}Y^6 - X^{11}Y^6 + X^{10}Y^7 - 2X^{12}Y^7 - X^{13}Y^7 + X^{13}Y^8 - X^{14}Y^8 - X^{16}Y^9 + X^{15}Y^{10} + X^{17}Y^{11} - X^{18}Y^{11} + X^{18}Y^{12} + 2X^{19}Y^{12} - X^{21}Y^{12} + X^{20}Y^{13} + 2X^{21}Y^{13} + 2X^{22}Y^{13} - X^{22}Y^{14} + X^{23}Y^{14} + 2X^{24}Y^{14} - X^{24}Y^{15} + 2X^{26}Y^{15} - 2X^{26}Y^{16} - X^{27}Y^{16} + X^{28}Y^{16} - X^{28}Y^{17} - X^{29}Y^{17} - X^{31}Y^{19} .$$

These zeta functions satisfy the functional equations

$$\zeta^{\lhd}_{M_4,p}(s)\Big|_{p\to p^{-1}} = -p^{10-14s}\zeta^{\lhd}_{M_4,p}(s) ,$$
$$\zeta^{\leq}_{M_4,p}(s)\Big|_{p\to p^{-1}} = -p^{10-5s}\zeta^{\leq}_{M_4,p}(s) .$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\lhd}_{M_4} = 2$, $\alpha^{\leq}_{M_4} = 5/2$.

Theorem 2.38 ([64]).

$$\zeta^{\lhd}_{M_4\times\mathbb{Z},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(5s-3)\zeta_p(7s-5)\zeta_p(8s-7)\zeta_p(9s-8) \times \zeta_p(11s-8)\zeta_p(12s-9)\zeta_p(6s-4)^{-1}W^{\lhd}_{M_4\times\mathbb{Z}}(p,p^{-s}) ,$$

where $W^{\lhd}_{M_4\times\mathbb{Z}}(X,Y)$ *is*

$$1 + X^3Y^4 - X^3Y^5 + X^4Y^5 - X^3Y^6 + 2X^4Y^6 - X^4Y^7 - X^7Y^9 + X^8Y^{10} - 2X^7Y^{11} - X^9Y^{13} - X^{11}Y^{13} + X^{10}Y^{14} - X^{11}Y^{14} - X^{11}Y^{15} - X^{12}Y^{15} + X^{12}Y^{16} - X^{12}Y^{17} - X^{13}Y^{17} + 2X^{12}Y^{18} - X^{13}Y^{18} + X^{14}Y^{19} - 2X^{15}Y^{19} + X^{14}Y^{20} + X^{15}Y^{20} - X^{15}Y^{21} + X^{15}Y^{22} + X^{16}Y^{22} + X^{16}Y^{23} - X^{17}Y^{23} + X^{16}Y^{24} + X^{18}Y^{24} + 2X^{20}Y^{26} - X^{19}Y^{27} + X^{20}Y^{28} + X^{23}Y^{30} - 2X^{23}Y^{31} + X^{24}Y^{31} - X^{23}Y^{32} + X^{24}Y^{32} - X^{24}Y^{33} - X^{27}Y^{37} .$$

This zeta function satisfies the functional equation

$$\zeta^{\lhd}_{M_4\times\mathbb{Z},p}(s)\Big|_{p\to p^{-1}} = p^{15-15s}\zeta^{\lhd}_{M_4\times\mathbb{Z},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\lhd}_{M_4\times\mathbb{Z}} = 3$.

M_4 is not the only filiform Lie ring of nilpotency class 4, up to isomorphism:

Theorem 2.39 ([64]). *Let the Lie ring* Fil_4 *have presentation*

$$\langle z, x_1, x_2, x_3, x_4 : [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4, [x_1, x_2] = x_4\rangle .$$

Then

$$\zeta^{\triangleleft}_{\mathrm{Fil}_4,p}(s) = \zeta_{\mathbb{Z}^2,p}(s)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6) \times \zeta_p(10s-6)\zeta_p(12s-7)W^{\triangleleft}_{\mathrm{Fil}_4}(p,p^{-s}) \,,$$

where $W^{\triangleleft}_{\mathrm{Fil}_4}(X,Y)$ *is*

$$\begin{aligned}
&1 + X^2Y^4 - X^2Y^5 + X^3Y^5 - X^2Y^6 + X^3Y^6 - X^3Y^7 - X^5Y^9 - X^5Y^{10}\\
&- X^6Y^{11} - X^6Y^{12} + X^6Y^{13} - X^7Y^{13} - X^8Y^{13} - X^8Y^{14} + X^7Y^{15}\\
&+ X^8Y^{15} - 2X^9Y^{15} + X^8Y^{17} + X^9Y^{17} - X^{10}Y^{17} + X^9Y^{19} + X^{10}Y^{19}\\
&+ X^{11}Y^{20} + 2X^{11}Y^{21} - X^{11}Y^{22} + 2X^{12}Y^{22} + 2X^{13}Y^{23} - X^{13}Y^{24}\\
&+ X^{14}Y^{24} - X^{13}Y^{25} + X^{14}Y^{25} + X^{15}Y^{25} - 2X^{14}Y^{27} + 2X^{15}Y^{27}\\
&- 2X^{15}Y^{28} + X^{16}Y^{28} - X^{15}Y^{29} - X^{16}Y^{29} + X^{17}Y^{29} - 2X^{17}Y^{30} + X^{18}Y^{30}\\
&- X^{18}Y^{31} - X^{18}Y^{32} - X^{18}Y^{33} - X^{20}Y^{35} + X^{20}Y^{36} - X^{21}Y^{36} + X^{20}Y^{37}\\
&- X^{21}Y^{37} + X^{21}Y^{38} + X^{23}Y^{42} \,.
\end{aligned}$$

This local zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathrm{Fil}_4} = 2$.

Despite repeated efforts, we have been unable to calculate $\zeta^{\leq}_{\mathrm{Fil}_4,p}(s)$. M_4 is the only Lie ring of nilpotency class 4 whose zeta function counting all subrings we have calculated.

Theorem 2.40 ([64]).

$$\zeta^{\triangleleft}_{\mathrm{Fil}_4\times\mathbb{Z},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(5s-3)\zeta_p(7s-5)\zeta_p(8s-7)\zeta_p(9s-8) \times \zeta_p(10s-8)\zeta_p(12s-9)W^{\triangleleft}_{\mathrm{Fil}_4\times\mathbb{Z}}(p,p^{-s}) \,,$$

where $W^{\triangleleft}_{\mathrm{Fil}_4\times\mathbb{Z}}(X,Y)$ *is*

$$\begin{aligned}
&1 + X^3Y^4 - X^3Y^5 + X^4Y^5 - X^3Y^6 + X^4Y^6 - X^4Y^7 - X^7Y^9 - X^7Y^{10}\\
&- X^8Y^{11} - X^8Y^{12} + X^8Y^{13} - X^9Y^{13} - X^{11}Y^{13} - X^{11}Y^{14} + X^{10}Y^{15}\\
&+ X^{11}Y^{15} - 2X^{12}Y^{15} + X^{11}Y^{17} + X^{12}Y^{17} - X^{13}Y^{17} + X^{12}Y^{19} + X^{14}Y^{19}\\
&+ X^{15}Y^{20} + 2X^{15}Y^{21} - X^{15}Y^{22} + 2X^{16}Y^{22} + X^{17}Y^{23} + X^{18}Y^{23} - X^{18}Y^{24}\\
&+ X^{19}Y^{24} - X^{18}Y^{25} + X^{19}Y^{25} + X^{20}Y^{25} - 2X^{19}Y^{27} + 2X^{20}Y^{27}\\
&- 2X^{20}Y^{28} + X^{21}Y^{28} - X^{20}Y^{29} - X^{22}Y^{29} + X^{23}Y^{29} - 2X^{23}Y^{30} + X^{24}Y^{30}\\
&- X^{24}Y^{31} - X^{24}Y^{32} - X^{24}Y^{33} - X^{27}Y^{35} + X^{27}Y^{36} - X^{28}Y^{36} + X^{27}Y^{37}\\
&- X^{28}Y^{37} + X^{28}Y^{38} + X^{31}Y^{42} \,.
\end{aligned}$$

This zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathrm{Fil}_4\times\mathbb{Z}} = 3$.

2.14 Nilpotent Lie Algebras of Dimension ≤ 6

A complete classification of the nilpotent Lie algebras over $\mathbb{R}$ of dimension ≤ 6 is given in [44].[2] We cannot hope to classify nilpotent Lie rings additively isomorphic to $\mathbb{Z}^d$ for some $d \leq 6$, but we can at least use a classification over $\mathbb{R}$ to produce Lie rings over $\mathbb{Z}$ which are guaranteed be non-isomorphic. For each Lie algebra, Magnin gives an $\mathbb{R}$-basis and a list of nonzero Lie brackets of the basis elements. The structure constants of each nilpotent Lie algebra L listed in [44] are (fortunately) all in $\mathbb{Z}$. Hence we can form Lie rings over $\mathbb{Z}$ (or $\mathbb{Z}_p$) by taking the $\mathbb{Z}$-span (or $\mathbb{Z}_p$-span) of the basis given.[3]

This approach has led to many new calculations of ideal zeta functions of Lie rings of rank 6, and some others arising from a Lie ring of rank 5:

Theorem 2.41 ([64]). *Let the Lie ring $\mathfrak{g}_{5,3}$ have presentation*

$$\langle x_1, x_2, x_3, x_4, x_5 : [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle \ .$$

Then

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathfrak{g}_{5,3}\times\mathbb{Z}^r,p}(s) &= \zeta_{\mathbb{Z}^{r+3},p}(s)\zeta_p(3s-(r+3))\zeta_p(5s-(r+4)) \ , \\
\zeta^{\leq}_{\mathfrak{g}_{5,3},p}(s) &= \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(2s-4)\zeta_p(3s-4)\zeta_p(3s-6)\zeta_p(6s-11)\zeta_p(6s-12) \\
&\quad \times W^{\leq}_{\mathfrak{g}_{5,3}}(p, p^{-s}) \ ,
\end{aligned}$$

where $W^{\leq}_{\mathfrak{g}_{5,3}}(X,Y)$ is

$$\begin{aligned}
&1 + X^3Y^2 - X^4Y^3 + X^5Y^3 - X^5Y^4 + X^7Y^4 + X^8Y^4 - 2X^7Y^5 - 2X^8Y^5 \\
&- X^9Y^5 + X^8Y^6 + X^9Y^6 + X^{10}Y^6 - X^{10}Y^7 - 2X^{11}Y^7 - 2X^{12}Y^7 + X^{11}Y^8 \\
&+ X^{12}Y^8 - X^{14}Y^8 - X^{15}Y^8 + X^{15}Y^{10} + X^{16}Y^{10} - X^{18}Y^{10} - X^{19}Y^{10} \\
&+ 2X^{18}Y^{11} + 2X^{19}Y^{11} + X^{20}Y^{11} - X^{20}Y^{12} - X^{21}Y^{12} - X^{22}Y^{12} + X^{21}Y^{13} \\
&+ 2X^{22}Y^{13} + 2X^{23}Y^{13} - X^{22}Y^{14} - X^{23}Y^{14} + X^{25}Y^{14} - X^{25}Y^{15} + X^{26}Y^{15} \\
&- X^{27}Y^{16} - X^{30}Y^{18} \ .
\end{aligned}$$

These zeta functions satisfy the functional equations

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathfrak{g}_{5,3}\times\mathbb{Z}^r,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{r+5}p^{\binom{r+5}{2}-(r+11)s}\zeta^{\triangleleft}_{\mathfrak{g}_{5,3}\times\mathbb{Z}^r,p}(s) \ , \\
\zeta^{\leq}_{\mathfrak{g}_{5,3},p}(s)\Big|_{p\to p^{-1}} &= -p^{10-5s}\zeta^{\leq}_{\mathfrak{g}_{5,3},p}(s) \ .
\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{5,3}} = \alpha^{\leq}_{\mathfrak{g}_{5,3}} = 3$.

[2] The classification was first given in [46], but we refer to [44] as this article is likely to be more accessible.

[3] We have permuted some of the bases of the Lie algebras from [44]; the bases we give are those that make the calculations of the zeta functions easiest.

Theorem 2.42.

$$\zeta^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{5,3},p}(s) = \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(3s-5)^2\zeta_p(5s-6)^2\zeta_p(7s-7)\zeta_p(5s-5)^{-1} \times \zeta_p(7s-6)^{-1} \; .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{5,3},p}(s)\Big|_{p\to p^{-1}} = p^{28-16s}\zeta^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{5,3},p}(s) \; .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{5,3}} = 5$*.*

Theorem 2.43.

$$\zeta^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3},p}(s) = \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)\zeta_p(3s-7)\zeta_p(5s-7)\zeta_p(5s-8)\zeta_p(5s-12) \times \zeta_p(7s-9)\zeta_p(7s-14)\zeta_p(9s-15)\zeta_p(11s-16) \times W^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3}}(p,p^{-s}) \; ,$$

where $W^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3}}(X,Y)$ *is given in Appendix A on p. 182. This zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3},p}(s)\Big|_{p\to p^{-1}} = p^{45-19s}\zeta^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3},p}(s) \; .$$

The corresponding global function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3}} = 6$*.*

We write $\mathfrak{g}_{6,n}$ for a Lie ring whose presentation is taken from that of the nth Lie algebra in the list in [44]. We have already seen several examples of rank 6, $\mathfrak{g}_{6,1} = L_{(3,2)}$, $\mathfrak{g}_{6,3} = F_{2,3}$, $\mathfrak{g}_{6,4} = F_{2,3}/\langle z\rangle \cdot \mathbb{Z}$ and $\mathfrak{g}_{6,5} = U_3(R_2)$ where R_2 is the ring of integers of a quadratic number field. $\mathfrak{g}_{6,2} = M_5$, whose local zeta functions we have been unable to calculate.

Theorem 2.44 ([64]). *Let the Lie ring* $\mathfrak{g}_{6,6}$ *have presentation*

$$\langle x_1,\ldots,x_6 : [x_1,x_2]=x_4, [x_1,x_3]=x_5, [x_1,x_4]=x_6, [x_2,x_3]=x_6\rangle \; .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,6},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6)\zeta_p(7s-8) \times \zeta_p(9s-11)W^{\triangleleft}_{\mathfrak{g}_{6,6}}(p,p^{-s}) \; ,$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,6}}(X,Y)$ *is*

$$1 + X^3Y^3 - X^6Y^7 - X^8Y^8 - X^9Y^9 - 2X^{11}Y^{10} - X^{14}Y^{12} + X^{14}Y^{14} - X^{15}Y^{14} + X^{15}Y^{15} + X^{17}Y^{16} + X^{17}Y^{17} + X^{19}Y^{17} + X^{20}Y^{19} + X^{21}Y^{19} - X^{21}Y^{20} + X^{22}Y^{20} - X^{25}Y^{24} - X^{28}Y^{26} \; .$$

This local zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,6}} = 3$*.*

Theorem 2.45 ([64]). *Let the Lie ring $\mathfrak{g}_{6,7}$ have presentation*

$$\langle x_1, \ldots, x_6 : [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_6 \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,7},p}(s) = {} & \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6) \\ & \times \zeta_p(7s-7)W^{\triangleleft}_{\mathfrak{g}_{6,7}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,7}}(X, Y)$ is

$$\begin{aligned}&1 + X^3Y^3 - X^3Y^5 - 2X^6Y^7 - X^7Y^8 - X^9Y^9 - X^{10}Y^{10} + X^9Y^{11} - X^{10}Y^{11} \\ &+ 2X^{10}Y^{12} + X^{12}Y^{14} + X^{13}Y^{14} + X^{13}Y^{15} + X^{16}Y^{16} - X^{16}Y^{19} - X^{19}Y^{21} \ .\end{aligned}$$

This local zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,7}} = 3$.

Theorem 2.46 ([64]). *Let the Lie ring $\mathfrak{g}_{6,8}$ have presentation*

$$\langle x_1, \ldots, x_6 : [x_1, x_2] = x_3 + x_4, [x_1, x_3] = x_5, [x_2, x_4] = x_6 \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s) = {} & \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(6s-6)\zeta_p(7s-7) \\ & \times \zeta_p(8s-8)(1 + p^{1-s})W^{\triangleleft}_{\mathfrak{g}_{6,8}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,8}}(X, Y)$ is

$$\begin{aligned}&1 - XY + X^2Y^2 - X^3Y^3 + X^3Y^4 + X^4Y^4 - 2X^3Y^5 - X^5Y^5 + 2X^4Y^6 \\ &+ X^6Y^6 - 2X^5Y^7 - 2X^6Y^7 + 3X^6Y^8 - 4X^7Y^9 + 4X^8Y^{10} - 4X^9Y^{11} \\ &- X^{10}Y^{11} + X^9Y^{12} + 4X^{10}Y^{12} - 4X^{11}Y^{13} + 4X^{12}Y^{14} - 3X^{13}Y^{15} \\ &+ 2X^{13}Y^{16} + 2X^{14}Y^{16} - X^{13}Y^{17} - 2X^{15}Y^{17} + X^{14}Y^{18} + 2X^{16}Y^{18} \\ &- X^{15}Y^{19} - X^{16}Y^{19} + X^{16}Y^{20} - X^{17}Y^{21} + X^{18}Y^{22} - X^{19}Y^{23} \ .\end{aligned}$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s)\Big|_{p \to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,8}} = 3$.

Theorem 2.47 ([64]). *Let the Lie ring $\mathfrak{g}_{6,9}$ have presentation*

$$\langle x_1, \ldots, x_6 : [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = x_6 \rangle \ .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,9},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(5s-5)\zeta_p(6s-6)\zeta_p(8s-7)\zeta_p(8s-8)\zeta_p(14s-15) \\ \times W^{\triangleleft}_{\mathfrak{g}_{6,9}}(p,p^{-s}) \ ,$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,9}}(X,Y)$ *is*

$$1 + X^3Y^3 + X^3Y^4 - X^3Y^5 + X^4Y^5 + X^6Y^6 + X^7Y^7 - X^6Y^8 - X^7Y^9 \\ + X^9Y^9 + X^{10}Y^{10} - X^9Y^{11} - X^{10}Y^{11} + X^{11}Y^{11} - X^{10}Y^{12} - X^{11}Y^{12} \\ + X^{12}Y^{12} - X^{11}Y^{13} + X^{13}Y^{13} - X^{12}Y^{14} - X^{13}Y^{14} - X^{13}Y^{15} + X^{13}Y^{16} \\ - X^{14}Y^{16} - X^{15}Y^{16} + X^{16}Y^{16} - X^{16}Y^{17} - X^{16}Y^{18} - X^{17}Y^{18} + X^{16}Y^{19} \\ - X^{18}Y^{19} + X^{17}Y^{20} - X^{18}Y^{20} - X^{19}Y^{20} + X^{18}Y^{21} - X^{19}Y^{21} - X^{20}Y^{21} \\ + X^{19}Y^{22} + X^{20}Y^{23} - X^{22}Y^{23} - X^{23}Y^{24} + X^{22}Y^{25} + X^{23}Y^{26} + X^{25}Y^{27} \\ - X^{26}Y^{27} + X^{26}Y^{28} + X^{26}Y^{29} + X^{29}Y^{32} \ .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,9},p}(s)\Big|_{p\to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,9},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,9}} = 3$.

Theorem 2.48 ([64]). *Let* $\gamma \in \mathbb{Z} \setminus \{0,1\}$ *be a squarefree integer. Let the Lie ring* $\mathfrak{g}_{6,10}(\gamma)$ *have presentation*

$$\left\langle x_1,\ldots,x_6 : \begin{matrix} [x_1,x_2]=x_4,[x_1,x_4]=x_6,[x_1,x_3]=x_5, \\ [x_2,x_3]=x_6,[x_2,x_4]=\alpha x_5+\beta x_6 \end{matrix} \right\rangle \ ,$$

where

$$\alpha x_5 + \beta x_6 = \begin{cases} \gamma x_5 & \text{if } \gamma \equiv 2,3 \pmod 4, \\ \frac{1}{4}(\gamma-1)x_5 + x_6 & \text{if } \gamma \equiv 1 \pmod 4. \end{cases}$$

Then, if p *is inert in* $\mathbb{Q}(\sqrt{\gamma})$,

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(5s-4)\zeta_p(5s-5)\zeta_p(6s-6) \\ \times \zeta_p(8s-8)\zeta_p(8s-6)^{-1}\zeta_p(10s-8)^{-1} \ .$$

If p *splits in* $\mathbb{Q}(\sqrt{\gamma})$ *and either*

- $\gamma \equiv 1 \pmod 4$ *and* $p \nmid \frac{1}{4}(\gamma-1)$, *or*
- $\gamma \not\equiv 1 \pmod 4$,

then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(6s-6)\zeta_p(7s-7) \\ \times \zeta_p(8s-8)(1+p^{1-s})W^{\triangleleft}_{\mathfrak{g}_{6,8}}(p,p^{-s}) \ ,$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,8}}(X,Y)$ *is given above on p. 57. For all but finitely many primes, the local zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s)\Big|_{p\to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s)\ .$$

Theorem 2.49 ([64]). *Let the Lie ring* $\mathfrak{g}_{6,12}$ *have presentation*

$$\langle x_1,\dots,x_6 : [x_1,x_3]=x_5, [x_1,x_5]=x_6, [x_2,x_4]=x_6\rangle\ .$$

Then

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathfrak{g}_{6,12},p}(s) &= \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(6s-4)\zeta_p(7s-5)\zeta_p(7s-4)^{-1}\ ,\\
\zeta^{\leq}_{\mathfrak{g}_{6,12},p}(s) &= \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-6)\zeta_p(4s-8)\zeta_p(4s-9)\\
&\quad\times \zeta_p(5s-12)\zeta_p(6s-12)\zeta_p(6s-13)\zeta_p(7s-16)\zeta_p(s-2)^{-1}\\
&\quad\times W^{\leq}_{\mathfrak{g}_{6,12}}(p,p^{-s})\ ,
\end{aligned}$$

where $W^{\leq}_{\mathfrak{g}_{6,12}}(X,Y)$ *is given in Appendix A on p. 183. These zeta functions satisfy the functional equations*

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathfrak{g}_{6,12},p}(s)\Big|_{p\to p^{-1}} &= p^{15-13s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,12},p}(s)\ ,\\
\zeta^{\leq}_{\mathfrak{g}_{6,12},p}(s)\Big|_{p\to p^{-1}} &= p^{15-6s}\zeta^{\leq}_{\mathfrak{g}_{6,12},p}(s)\ .
\end{aligned}$$

The corresponding global zeta functions have abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,12}} = \alpha^{\leq}_{\mathfrak{g}_{6,12}} = 4$.

It can easily be seen that $\mathfrak{g}_{6,12}$ is the direct product with central amalgamation of $\mathcal{H}$ with M_3.

Theorem 2.50.

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{6,12},p}(s) &= \zeta_{\mathbb{Z}^6,p}(s)\zeta_p(3s-6)^2\zeta_p(5s-7)\zeta_p(6s-6)\zeta_p(7s-7)\zeta_p(8s-7)\\
&\quad\times \zeta_p(9s-8)\zeta_p(11s-14)W^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{6,12}}(p,p^{-s})\ ,
\end{aligned}$$

where $W^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{6,12}}(X,Y)$ *is*

$$\begin{aligned}
&1 - X^6Y^5 - X^6Y^7 - X^6Y^8 + X^6Y^9 - 2X^7Y^9 + X^{12}Y^{11} - 2X^{13}Y^{11}\\
&+ 2X^{13}Y^{12} - X^{14}Y^{12} + 2X^{13}Y^{13} - X^{14}Y^{13} + X^{14}Y^{14} + 2X^{13}Y^{15}\\
&- X^{14}Y^{15} + X^{14}Y^{16} + X^{20}Y^{16} + X^{14}Y^{17} + X^{20}Y^{18} + X^{20}Y^{19} - 2X^{19}Y^{20}\\
&+ 2X^{21}Y^{20} - X^{20}Y^{21} - X^{20}Y^{22} - X^{26}Y^{23} - X^{20}Y^{24} - X^{26}Y^{24} + X^{26}Y^{25}\\
&- 2X^{27}Y^{25} - X^{26}Y^{26} + X^{26}Y^{27} - 2X^{27}Y^{27} + X^{26}Y^{28} - 2X^{27}Y^{28}\\
&+ 2X^{27}Y^{29} - X^{28}Y^{29} + 2X^{33}Y^{31} - X^{34}Y^{31} + X^{34}Y^{32} + X^{34}Y^{33} + X^{34}Y^{35}\\
&- X^{40}Y^{40}\ .
\end{aligned}$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{6,12},p}(s)\Big|_{p\to p^{-1}} = -p^{36-18s}\zeta^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{6,12},p}(s)\ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathcal{H}\times\mathfrak{g}_{6,12}}=6$.

Theorem 2.51 ([64]). *Let the Lie ring* $\mathfrak{g}_{6,13}$ *have presentation*

$$\langle x_1,\ldots,x_6 : [x_1,x_2]=x_5,[x_1,x_3]=x_4,[x_1,x_4]=x_6,[x_2,x_5]=x_6\rangle\ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,13},p}(s) = {}&\zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-4)\zeta_p(5s-6)\zeta_p(6s-4)\zeta_p(7s-5)\zeta_p(9s-8)\\ &\times W^{\triangleleft}_{\mathfrak{g}_{6,13}}(p,p^{-s})\ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,13}}(X,Y)$ *is*

$$1+X^3Y^3-X^4Y^7-X^7Y^9-X^8Y^{10}-X^{11}Y^{12}+X^{12}Y^{16}+X^{15}Y^{19}\ .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,13},p}(s)\Big|_{p\to p^{-1}} = p^{15-14s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,13},p}(s)\ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,13}}=3$.

Theorem 2.52 ([64]). *Let* $\gamma\in\mathbb{Z}$ *be a nonzero integer, and let* $\mathfrak{g}_{6,14}(\gamma)$ *have presentation*

$$\langle x_1,\ldots,x_6 : [x_1,x_3]=x_4,[x_1,x_4]=x_6,[x_2,x_3]=x_5,[x_2,x_5]=\gamma x_6\rangle\ .$$

Then, for all primes p not dividing γ,

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,14(\gamma)},p}(s) = {}&\zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(3s-4)\zeta_p(5s-6)\zeta_p(6s-3)\zeta_p(7s-5)\\ &\times\zeta_p(6s-6)^{-1}\zeta_p(7s-3)^{-1}\ .\end{aligned}$$

If $p\nmid\gamma$, *the local zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma),p}(s)\Big|_{p\to p^{-1}} = p^{15-14s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma),p}(s)\ .$$

For $\gamma=\pm1$, *the corresponding global zeta function has abscissa of convergence* $\alpha^{\triangleleft}_{\mathfrak{g}_{6,14}(\pm1)}=3$.

The following proposition has a routine proof which we do not repeat.

Proposition 2.53. *For* $\gamma_1,\gamma_2\neq0$, *let* $\mathfrak{g}_{6,14}(\gamma_1)$ *and* $\mathfrak{g}_{6,14}(\gamma_2)$ *be defined over any integral domain or field R. Then* $\mathfrak{g}_{6,14}(\gamma_1)\cong\mathfrak{g}_{6,14}(\gamma_2)$ *iff* $\gamma_1=u^2\gamma_2$ *for some* $u\in R^*$.

It can also be shown that the local zeta functions depend only on the power of p dividing γ. We therefore have the following

Corollary 2.54. *Let $\gamma \in \mathbb{Z}$ be a nonzero integer. Then $\mathfrak{g}_{6,14}(\gamma) \not\cong \mathfrak{g}_{6,14}(-\gamma)$ but $\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma)}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(-\gamma)}(s)$.*

The classification of six-dimensional Lie algebras has also given rise to some new calculations in nilpotency class 4. In particular, the second author found the following:

Theorem 2.55 ([64]). *Define the two Lie rings $\mathfrak{g}_{6,15}$ and $\mathfrak{g}_{6,17}$ by the presentations*

$$\mathfrak{g}_{6,15} = \left\langle x_1, x_2, x_3, x_4, x_5, x_6 : \begin{array}{c} [x_1, x_2] = x_3 + x_4, [x_1, x_4] = x_5, \\ [x_1, x_5] = x_6, [x_2, x_3] = x_6 \end{array} \right\rangle ,$$

$$\mathfrak{g}_{6,17} = \left\langle x_1, x_2, x_3, x_4, x_5, x_6 : \begin{array}{c} [x_1, x_2] = x_4, [x_1, x_4] = x_5, \\ [x_1, x_5] = x_6, [x_2, x_3] = x_6 \end{array} \right\rangle .$$

Then

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{6,15},p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{6,17},p}(s) &= \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(6s-4)\zeta_p(7s-5) \\ &\quad \times \zeta_p(9s-8)W^{\triangleleft}_{\mathfrak{g}_{6,15}}(p, p^{-s}) , \end{aligned} \tag{2.10}$$

where $W^{\triangleleft}_{\mathfrak{g}_{6,15}}(X, Y)$ is

$$1 - X^3Y^5 + X^4Y^5 - X^4Y^7 - X^7Y^9 + X^7Y^{11} - X^8Y^{11} + X^{11}Y^{16} .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,15},p}(s)\Big|_{p \to p^{-1}} = p^{15-16s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,15},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,15}} = 3$.

It follows from the classification [44] that $\mathfrak{g}_{6,15} \not\cong \mathfrak{g}_{6,17}$, but an appeal to a classification is not an enlightening proof. To be sure, we verify

Proposition 2.56. *$\mathfrak{g}_{6,15}$ and $\mathfrak{g}_{6,17}$ are not isomorphic.*

Proof. The rank of the centraliser of the derived subring is invariant under isomorphism. Firstly, $\mathfrak{g}'_{6,15} = \langle y_3 + y_4, y_5, y_6 \rangle$, which has centraliser $\langle y_3, y_4, y_5, y_6 \rangle$. Secondly, $\mathfrak{g}'_{6,17} = \langle x_4, x_5, x_6 \rangle$, which is centralised by $\langle x_2, x_3, x_4, x_5, x_6 \rangle$. Thus $\mathfrak{g}_{6,15} \not\cong \mathfrak{g}_{6,17}$. □

The only other calculation at nilpotency class 4 this classification leads to is the following:

Theorem 2.57 ([64]). *Let the Lie ring* $\mathfrak{g}_{6,16}$ *have presentation*

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6 : \begin{array}{c} [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, \\ [x_2, x_3] = x_5, [x_2, x_4] = x_6 \end{array} \right\rangle .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,16},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-3)\zeta_p(5s-4)\zeta_p(6s-3)\zeta_p(7s-5)\zeta_p(7s-3)^{-1} .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,16},p}(s)\Big|_{p\to p^{-1}} = p^{15-17s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,16},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{6,16}} = 3$.

2.15 Nilpotent Lie Algebras of Dimension 7

The Lie algebras of dimension 7 over algebraically closed fields and $\mathbb{R}$ were first classified successfully by Gong [26]. Once again, the structure constants of each Lie algebra are all rational integers. This includes the six one-parameter families, providing we restrict the parameter to $\mathbb{Z}$. Hence we can also use this classification to obtain presentations of $\mathbb{Z}$-Lie rings of rank 7.

We write $\mathfrak{g}_{\text{name}}$ for the $\mathbb{Z}$-Lie ring corresponding to the Lie algebra with the label (name) in [26]. For example, $\mathfrak{g}_{1357\text{F}}$ corresponds to (1357F) in [26]. The digits are the dimensions of the terms in the upper-central series, and the suffix letter (when shown) distinguishes non-isomorphic Lie algebras with the same upper-central series dimensions. We have encountered some of these Lie rings before, in particular $\mathfrak{g}_{17} \cong G(3,0)$, $\mathfrak{g}_{37\text{A}} \cong \mathcal{G}_4$, $\mathfrak{g}_{37\text{B}} \cong T_4$, $\mathfrak{g}_{137\text{A}} \cong M_3 \times_{\mathbb{Z}} M_3$ and $\mathfrak{g}_{247\text{A}} \cong L_{(3,3)}$. Furthermore, some of them arise as direct products with central amalgamation: $\mathfrak{g}_{157}$, $\mathfrak{g}_{257\text{K}}$, $\mathfrak{g}_{1457\text{A}}$ and $\mathfrak{g}_{1457\text{B}}$ are the direct products with central amalgamation of $\mathcal{H}$ with $\mathfrak{g}_{5,3}$, $F_{3,2}$, M_4 and Fil_4 respectively.

We saw above that $\mathfrak{g}_{6,15}$ and $\mathfrak{g}_{6,17}$ are non-isomorphic yet their ideal zeta functions are equal. Amongst those calculations in rank 7 we have so far completed, there are no less than seven pairs of normally isospectral Lie rings. We do not provide proof that the Lie rings are non-isomorphic, instead referring the curious reader to [26].

Theorem 2.58. *Let the Lie ring* $\mathfrak{g}_{27\text{A}}$ *have presentation*

$$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 : [x_1, x_2] = x_6, [x_1, x_4] = x_7, [x_3, x_5] = x_7 \rangle .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{27\text{A}},p}(s) = \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(3s-5)\zeta_p(5s-6)\zeta_p(7s-10)\zeta_p(8s-10)^{-1} .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{27\text{A}},p}(s)\Big|_{p\to p^{-1}} = -p^{21-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{27\text{A}},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{27\text{A}}} = 5$.

Theorem 2.59. *Let the Lie ring* $\mathfrak{g}_{27\mathrm{B}}$ *have presentation*

$$\langle x_1, \ldots, x_7 : [x_1, x_2] = x_6, [x_1, x_5] = x_7, [x_2, x_3] = x_7, [x_3, x_4] = x_6 \rangle \ .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{27\mathrm{B}},p}(s) = \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(7s-10)\zeta_p(10s-10)^{-1} \ .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{27\mathrm{B}},p}(s)\Big|_{p\to p^{-1}} = -p^{21-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{27\mathrm{B}},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{27\mathrm{B}}} = 5$.

Theorem 2.60. *Let the Lie ring* $\mathfrak{g}_{37\mathrm{C}}$ *have presentation*

$$\langle x_1, \ldots, x_7 : [x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_2, x_4] = x_7, [x_3, x_4] = x_5 \rangle \ .$$

Then $\zeta^{\triangleleft}_{\mathfrak{g}_{37\mathrm{C}},p}(s) = \zeta^{\triangleleft}_{T_4,p}(s)$ *(p. 45).*

Theorem 2.61. *Let the Lie ring* $\mathfrak{g}_{37\mathrm{D}}$ *have presentation*

$$\langle x_1, \ldots, x_7 : [x_1, x_2] = x_5, [x_1, x_3] = x_7, [x_2, x_4] = x_7, [x_3, x_4] = x_6 \rangle \ .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{37\mathrm{D}},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-5)\zeta_p(5s-6)\zeta_p(6s-10)\zeta_p(7s-12)W^{\triangleleft}_{\mathfrak{g}_{37\mathrm{D}}}(p,p^{-s}) \ ,$$

where $W^{\triangleleft}_{\mathfrak{g}_{37\mathrm{D}}}(X,Y)$ *is*

$$1 + X^4Y^3 + X^8Y^6 + X^9Y^6 - X^9Y^8 - X^{10}Y^8 - X^{14}Y^{11} - X^{18}Y^{14} \ .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{37\mathrm{D}},p}(s)\Big|_{p\to p^{-1}} = -p^{21-11s}\zeta^{\triangleleft}_{\mathfrak{g}_{37\mathrm{D}},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{37\mathrm{D}}} = 4$.

Theorem 2.62. *Let the Lie ring* $\mathfrak{g}_{137\mathrm{B}}$ *have presentation*

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 : \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_5] = x_7, [x_2, x_4] = x_7, \\ [x_3, x_4] = x_6, [x_3, x_6] = x_7 \end{array} \right\rangle \ .$$

Then $\zeta^{\triangleleft}_{\mathfrak{g}_{137\mathrm{B}},p}(s) = \zeta^{\triangleleft}_{M_3\times_{\mathbb{Z}}M_3,p}(s)$ *(p. 48).*

Theorem 2.63. *Let the Lie rings* $\mathfrak{g}_{137\mathrm{C}}$ *and* $\mathfrak{g}_{137\mathrm{D}}$ *have presentations*

$$\mathfrak{g}_{137\mathrm{C}} = \left\langle x_1, \ldots, x_7 : \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_4] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_3] = x_6, [x_3, x_5] = -x_7 \end{array} \right\rangle \ ,$$

$$\mathfrak{g}_{137\mathrm{D}} = \left\langle x_1, \ldots, x_7 : \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_4] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_3] = x_6, [x_2, x_4] = x_7, [x_3, x_5] = -x_7 \end{array} \right\rangle \ .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{137C},p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{137D},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(6s-9)\zeta_p(7s-4) \times \zeta_p(9s-6)\zeta_p(11s-10)\zeta_p(12s-10) \times \zeta_p(16s-11)W^{\triangleleft}_{\mathfrak{g}_{137C}}(p,p^{-s}) ,$$

where $W^{\triangleleft}_{\mathfrak{g}_{137C}}(X,Y)$ *is*

$$\begin{aligned}
&1 - X^4Y^8 + X^5Y^8 - X^9Y^8 - X^5Y^9 - X^9Y^{11} - X^{10}Y^{12} + X^9Y^{13} - X^{10}Y^{13} \\
&+ X^{13}Y^{15} - X^{14}Y^{15} - X^{10}Y^{16} + X^{14}Y^{16} - X^{15}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17} \\
&+ X^{15}Y^{17} + X^{14}Y^{19} - X^{15}Y^{19} + X^{19}Y^{19} + X^{15}Y^{20} + X^{19}Y^{20} + X^{14}Y^{21} \\
&+ X^{15}Y^{21} - X^{16}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22} + X^{18}Y^{23} + X^{19}Y^{23} - X^{20}Y^{23} \\
&- X^{18}Y^{24} - X^{19}Y^{24} + 3X^{20}Y^{24} + X^{15}Y^{25} - X^{23}Y^{26} + X^{24}Y^{26} + X^{19}Y^{27} \\
&- X^{19}Y^{28} + X^{20}Y^{28} + X^{21}Y^{28} - X^{23}Y^{28} - X^{24}Y^{28} + X^{25}Y^{28} - X^{25}Y^{29} \\
&- X^{20}Y^{30} + X^{21}Y^{30} - X^{29}Y^{31} - 3X^{24}Y^{32} + X^{25}Y^{32} + X^{26}Y^{32} + X^{24}Y^{33} \\
&- X^{25}Y^{33} - X^{26}Y^{33} - X^{28}Y^{34} + X^{29}Y^{34} + X^{28}Y^{35} - X^{29}Y^{35} - X^{30}Y^{35} \\
&- X^{25}Y^{36} - X^{29}Y^{36} - X^{25}Y^{37} + X^{29}Y^{37} - X^{30}Y^{37} - X^{29}Y^{39} + X^{33}Y^{39} \\
&- X^{34}Y^{39} + X^{29}Y^{40} - X^{30}Y^{40} + X^{34}Y^{40} + X^{30}Y^{41} - X^{31}Y^{41} + X^{34}Y^{43} \\
&- X^{35}Y^{43} + X^{34}Y^{44} + X^{35}Y^{45} + X^{39}Y^{47} + X^{35}Y^{48} - X^{39}Y^{48} + X^{40}Y^{48} \\
&- X^{44}Y^{56} .
\end{aligned}$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{137C},p}(s)\big|_{p\to p^{-1}} = -p^{21-17s}\zeta^{\triangleleft}_{\mathfrak{g}_{137C},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{137C}} = 4$.

Theorem 2.64. *Let the Lie rings* $\mathfrak{g}_{147A}$ *and* $\mathfrak{g}_{147B}$ *have presentations*

$$\mathfrak{g}_{147A} = \left\langle x_1,\ldots,x_7 : \begin{array}{c}[x_1,x_2]=x_4, [x_1,x_3]=x_5, [x_1,x_6]=x_7, \\ [x_2,x_5]=x_7, [x_3,x_4]=x_7\end{array} \right\rangle ,$$
$$\mathfrak{g}_{147B} = \left\langle x_1,\ldots,x_7 : \begin{array}{c}[x_1,x_2]=x_4, [x_1,x_3]=x_5, [x_1,x_4]=x_7, \\ [x_2,x_6]=x_7, [x_3,x_5]=x_7\end{array} \right\rangle .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{147A},p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{147B},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(3s-5)\zeta_p(5s-8)\zeta_p(7s-6) \times \zeta_p(6s-8)^{-1} .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{147A},p}(s)\big|_{p\to p^{-1}} = -p^{21-16s}\zeta^{\triangleleft}_{\mathfrak{g}_{147A},p}(s) .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{147A}} = 4$.

Theorem 2.65. *Let* $\mathfrak{g}_{157}$ *have presentation*

$$\langle x_1, \ldots, x_7 : [x_1, x_2] = x_3, [x_1, x_3] = x_7, [x_2, x_4] = x_7, [x_5, x_6] = x_7 \rangle \ .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{157},p}(s) = \zeta_{\mathbb{Z}^5,p}(s)\zeta_p(3s-5)\zeta_p(7s-6) \ .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{157\mathrm{A}},p}(s)\big|_{p \to p^{-1}} = -p^{21-15s}\zeta^{\triangleleft}_{\mathfrak{g}_{157\mathrm{A}},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{157}} = 5$.

Theorem 2.66. *Let the Lie ring* $\mathfrak{g}_{247\mathrm{B}}$ *have presentation*

$$\langle x_1, \ldots, x_7 : [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_3, x_5] = x_7 \rangle \ .$$

Then

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{247\mathrm{B}},p}(s) = \zeta_{\mathbb{Z}^3,p}(s)&\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-5) \\ &\times \zeta_p(6s-6)\zeta_p(7s-6)\zeta_p(7s-7)\zeta_p(8s-7)\zeta_p(8s-8) \\ &\times \zeta_p(9s-10)\zeta_p(9s-11)\zeta_p(10s-9)\zeta_p(10s-11)\zeta_p(11s-10) \\ &\times \zeta_p(11s-12)\zeta_p(12s-12)\zeta_p(13s-13)\zeta_p(s-1)^{-2} \\ &\times \zeta_p(2s-2)^{-1} W^{\triangleleft}_{\mathfrak{g}_{247\mathrm{B}}}(p, p^{-s}) \end{aligned}$$

for some polynomial $W^{\triangleleft}_{\mathfrak{g}_{247\mathrm{B}}}(X, Y)$ *of degrees 123 in* X *and 128 in* Y*. This zeta function satisfies the functional equation*

$$\zeta^{\triangleleft}_{\mathfrak{g}_{247\mathrm{B}},p}(s)\big|_{p \to p^{-1}} = -p^{21-15s}\zeta^{\triangleleft}_{\mathfrak{g}_{247\mathrm{B}},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{247\mathrm{B}}} = 3$.

Theorem 2.67. *Let the Lie rings* $\mathfrak{g}_{257\mathrm{A}}$ *and* $\mathfrak{g}_{257\mathrm{C}}$ *have presentations*

$$\begin{aligned} \mathfrak{g}_{257\mathrm{A}} &= \langle x_1, \ldots, x_7 : [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_5] = x_7, [x_2, x_4] = x_6 \rangle \ , \\ \mathfrak{g}_{257\mathrm{C}} &= \langle x_1, \ldots, x_7 : [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_2, x_4] = x_6, [x_2, x_5] = x_7 \rangle \ . \end{aligned}$$

Then

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{257\mathrm{A}},p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{257\mathrm{C}},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)&\zeta_p(3s-5)\zeta_p(5s-6)\zeta_p(5s-8)\zeta_p(7s-9) \\ &\times W^{\triangleleft}_{\mathfrak{g}_{257\mathrm{A}}}(p, p^{-s}) \ , \end{aligned}$$

where

$$W^{\triangleleft}_{\mathfrak{g}_{257\mathrm{A}}}(X, Y) = 1 + X^4Y^3 - X^9Y^8 - X^{13}Y^{10} \ .$$

This zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{257\mathrm{A}}} = 4$.

Theorem 2.68. *Let the Lie ring* $\mathfrak{g}_{257\mathrm{B}}$ *have presentation*

$$\langle x_1, \dots, x_7 : [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_5] = x_7 \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{257\mathrm{B}},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-6)\zeta_p(6s-9)\zeta_p(7s-9)\\ \times \zeta_p(8s-10)\zeta_p(12s-15)W^{\triangleleft}_{\mathfrak{g}_{257\mathrm{B}}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{257\mathrm{B}}}(X, Y)$ *is*

$$\begin{aligned}&1 - X^4Y^5 + X^5Y^5 - 2X^9Y^8 - X^9Y^9 - X^{13}Y^{10} + X^{13}Y^{11} - X^{14}Y^{11}\\ &+ 2X^{13}Y^{12} - 2X^{14}Y^{12} + X^{14}Y^{13} - X^{15}Y^{13} + 2X^{18}Y^{15} - X^{19}Y^{15}\\ &+ X^{18}Y^{16} + 2X^{19}Y^{17} - X^{20}Y^{17} + X^{23}Y^{18} - X^{22}Y^{19} + X^{23}Y^{19} - X^{23}Y^{20}\\ &+ 2X^{24}Y^{20} + X^{24}Y^{21} + X^{28}Y^{22} - X^{27}Y^{23} - X^{28}Y^{23} + X^{29}Y^{23} - 2X^{28}Y^{24}\\ &+ X^{29}Y^{24} - X^{33}Y^{27} - X^{33}Y^{28} - X^{33}Y^{29} - X^{38}Y^{30} + X^{37}Y^{32} + X^{42}Y^{35} \ .\end{aligned}$$

This zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{257\mathrm{B}}} = 4$.

Theorem 2.69. *Let* $\mathfrak{g}_{257\mathrm{K}}$ *have presentation*

$$\langle x_1, \dots, x_7 : [x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_2, x_5] = x_7, [x_3, x_4] = x_7 \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{257\mathrm{K}},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(6s-5)\zeta_p(7s-6)\\ \times \zeta_p(7s-8)\zeta_p(9s-10)W^{\triangleleft}_{\mathfrak{g}_{257\mathrm{K}}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{257\mathrm{K}}}(X, Y)$ *is*

$$\begin{aligned}&1 - X^4Y^5 - X^5Y^7 - X^8Y^9 - X^8Y^{10} + X^8Y^{11} - X^{10}Y^{11} + X^9Y^{12}\\ &+ X^{12}Y^{13} - X^{13}Y^{13} + X^{13}Y^{14} + 2X^{13}Y^{15} - X^{14}Y^{15} - X^{13}Y^{16} + 2X^{14}Y^{16}\\ &+ X^{14}Y^{17} - X^{14}Y^{18} + X^{15}Y^{18} + X^{18}Y^{19} - X^{17}Y^{20} + X^{19}Y^{20} - X^{19}Y^{21}\\ &- X^{19}Y^{22} - X^{22}Y^{24} - X^{23}Y^{26} + X^{27}Y^{31} \ .\end{aligned}$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{257\mathrm{K}},p}(s)\big|_{p\to p^{-1}} = -p^{21-14s}\zeta^{\triangleleft}_{\mathfrak{g}_{257\mathrm{K}},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{257\mathrm{K}}} = 4$.

Theorem 2.70. *Let the Lie ring* $\mathfrak{g}_{1357\mathrm{A}}$ *have presentation*

$$\left\langle x_1, \dots, x_7 : \begin{array}{c}[x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_7,\\ [x_2, x_3] = x_5, [x_2, x_6] = x_7, [x_3, x_4] = -x_7\end{array} \right\rangle \ .$$

Then

$$\zeta^{\triangleleft}_{\mathfrak{g}_{1357A},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(7s-6)\ .$$

This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{1357A},p}(s)\big|_{p\to p^{-1}} = -p^{21-19s}\zeta^{\triangleleft}_{\mathfrak{g}_{1357A},p}(s)\ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{1357A}} = 4$.

Theorem 2.71. *Let the Lie rings* $\mathfrak{g}_{1357B}$ *and* $\mathfrak{g}_{1357C}$ *have presentations*

$$\mathfrak{g}_{1357B} = \left\langle x_1,\dots,x_7 : \begin{array}{c}[x_1,x_2]=x_4,[x_1,x_4]=x_5,[x_1,x_5]=x_7\\ [x_2,x_3]=x_5,[x_3,x_4]=-x_7,[x_3,x_6]=x_7\end{array}\right\rangle\ ,$$

$$\mathfrak{g}_{1357C} = \left\langle x_1,\dots,x_7 : \begin{array}{c}[x_1,x_2]=x_4,[x_1,x_4]=x_5,[x_1,x_5]=x_7,\\ [x_2,x_3]=x_5,[x_2,x_4]=x_7,\\ [x_3,x_4]=-x_7,[x_3,x_6]=x_7\end{array}\right\rangle\ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{1357B},p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{1357C},p}(s) &= \zeta_{\mathbb{Z}^4,p}(s)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(9s-6)\\ &\times \zeta_p(11s-10)\zeta_p(16s-11)W^{\triangleleft}_{\mathfrak{g}_{1357B}}(p,p^{-s})\ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{1357B}}(X,Y)$ *is*

$$\begin{aligned}&1 - X^4Y^8 + X^5Y^8 - X^5Y^9 - X^9Y^{11} + X^9Y^{12} - X^{10}Y^{12} - X^{10}Y^{16}\\ &+ X^{10}Y^{17} - X^{11}Y^{17} + X^{14}Y^{19} - X^{15}Y^{19} + X^{15}Y^{20} + X^{15}Y^{25} + X^{19}Y^{27}\\ &- X^{19}Y^{28} + X^{21}Y^{28} - X^{25}Y^{36}\ .\end{aligned}$$

This zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{1357B}} = 4$.

Theorem 2.72. *Let the Lie rings* $\mathfrak{g}_{1357G}$ *and* $\mathfrak{g}_{1357H}$ *have presentations*

$$\mathfrak{g}_{1357G} = \left\langle x_1,\dots,x_7 : \begin{array}{c}[x_1,x_2]=x_3,[x_1,x_4]=x_6,[x_1,x_6]=x_7\\ [x_2,x_3]=x_5,[x_2,x_5]=x_7\end{array}\right\rangle\ ,$$

$$\mathfrak{g}_{1357H} = \left\langle x_1,\dots,x_7 : \begin{array}{c}[x_1,x_2]=x_3,[x_1,x_4]=x_6,[x_1,x_6]=x_7,\\ [x_2,x_3]=x_5,[x_2,x_5]=x_7,[x_2,x_6]=x_7,\\ [x_3,x_4]=-x_7\end{array}\right\rangle\ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{1357G},p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{1357H},p}(s) &= \zeta_{\mathbb{Z}^3,p}(s)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6)\\ &\times \zeta_p(6s-6)\zeta_p(7s-4)\zeta_p(7s-7)\zeta_p(8s-5)\\ &\times \zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-8)\zeta_p(12s-10)\\ &\times \zeta_p(12s-11)W^{\triangleleft}_{\mathfrak{g}_{1357G}}(p,p^{-s})\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{1357G}}(X,Y)$ *is given in Appendix A* *on p. 184**. This zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence* $\alpha^{\triangleleft}_{\mathfrak{g}_{1357G}} = 3$.

Theorem 2.73. *Let $\mathfrak{g}_{1457\mathrm{A}}$ have the presentation*

$$\langle x_1, \ldots, x_7 : [x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_1, x_6] = x_7, [x_3, x_4] = x_7 \rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{A}},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)&\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(9s-6)\\ &\times \zeta_p(10s-9)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(15s-10)\\ &\times \zeta_p(16s-11)W^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{A}}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{A}}}(X, Y)$ is given in Appendix A on p. 186. This zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{A}},p}(s)\big|_{p\to p^{-1}} = -p^{21-18s}\zeta^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{A}},p}(s) \ .$$

The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{A}}} = 4$.

Theorem 2.74. *Let $\mathfrak{g}_{1457\mathrm{B}}$ have presentation*

$$\left\langle x_1, \ldots, x_7 : \begin{array}{c}[x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_1, x_6] = x_7,\\ [x_2, x_5] = x_7, [x_3, x_4] = x_7\end{array} \right\rangle \ .$$

Then

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{B}},p}(s) = \zeta_{\mathbb{Z}^4,p}(s)&\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(9s-6)\\ &\times \zeta_p(10s-9)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(16s-11)\\ &\times W^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{B}}}(p, p^{-s}) \ ,\end{aligned}$$

where $W^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{B}}}(X, Y)$ is given in Appendix A on p. 187. This zeta function satisfies no functional equation. The corresponding global zeta function has abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{g}_{1457\mathrm{B}}} = 4$.

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