

Table of Contents

| | |
|--|------|
| Prerequisites | XIII |
| Notation | XIX |
| 1. Introduction and Historical Survey | 1 |
| 1.1 Liouville, Hermite, Lindemann, Gel'fond, Baker | 1 |
| 1.2 Lower Bounds for $ a_1^{b_1} \cdots a_m^{b_m} - 1 $ | 6 |
| 1.3 The Six Exponentials Theorem and the Four Exponentials Conjecture | 13 |
| 1.4 Algebraic Independence of Logarithms | 15 |
| 1.5 Diophantine Approximation on Linear Algebraic Groups | 19 |
| Exercises | 21 |
| <hr/> | |
| Part I. Transcendence | |
| <hr/> | |
| 2. Transcendence Proofs in One Variable | 29 |
| 2.1 Introduction to Transcendence Proofs | 29 |
| 2.2 Auxiliary Lemmas | 33 |
| 2.3 Schneider's Method with Alternants – Real Case | 37 |
| 2.4 Gel'fond's Method with Interpolation Determinants – Real Case .. | 43 |
| 2.5 Gel'fond-Schneider's Theorem in the Complex Case | 49 |
| 2.6 Hermite-Lindemann's Theorem in the Complex Case | 55 |
| Exercises | 59 |
| 3. Heights of Algebraic Numbers | 65 |
| 3.1 Absolute Values on a Number Field | 66 |
| 3.2 The Absolute Logarithmic Height (Weil) | 75 |
| 3.3 Mahler's Measure | 78 |
| 3.4 Usual Height and Size | 80 |
| 3.5 Liouville's Inequalities | 82 |
| 3.6 Lower Bound for the Height | 86 |
| Open Problems | 105 |
| Exercises | 106 |

| | |
|--|------------|
| Appendix – Inequalities Between Different Heights of a Polynomial – From a Manuscript by Alain Durand | 113 |
| 4. The Criterion of Schneider-Lang | 115 |
| 4.1 Algebraic Values of Entire Functions Satisfying Differential Equations | 115 |
| 4.2 First Proof of Baker’s Theorem | 118 |
| 4.3 Schwarz’ Lemma for Cartesian Products | 122 |
| 4.4 Exponential Polynomials | 130 |
| 4.5 Construction of an Auxiliary Function | 131 |
| 4.6 Direct Proof of Corollary 4.2 | 136 |
| Exercises | 141 |
| <hr/> | |
| Part II. Linear Independence of Logarithms and Measures | |
| <hr/> | |
| 5. Zero Estimate, by Damien Roy | 147 |
| 5.1 The Main Result | 147 |
| 5.2 Some Algebraic Geometry | 150 |
| 5.3 The Group G and its Algebraic Subgroups | 156 |
| 5.4 Proof of the Main Result | 164 |
| Exercises | 166 |
| 6. Linear Independence of Logarithms of Algebraic Numbers | 169 |
| 6.1 Applying the Zero Estimate | 170 |
| 6.2 Upper Bounds for Alternants in Several Variables | 175 |
| 6.3 A Second Proof of Baker’s Homogeneous Theorem | 181 |
| Exercises | 184 |
| 7. Homogeneous Measures of Linear Independence | 187 |
| 7.1 Statement of the Measure | 187 |
| 7.2 Lower Bound for a Zero Multiplicity | 192 |
| 7.3 Upper Bound for the Arithmetic Determinant | 195 |
| 7.4 Construction of a Nonzero Determinant | 199 |
| 7.5 The Transcendence Argument — General Case | 203 |
| 7.6 Proof of Theorem 7.1 — General Case | 208 |
| 7.7 The Rational Case: Fel’dman’s Polynomials | 214 |
| 7.8 Linear Dependence Relations between Logarithms | 222 |
| Open Problems | 227 |
| Exercises | 227 |

Part III. Multiplicities in Higher Dimension

| | |
|---|-----|
| 8. Multiplicity Estimates, by Damien Roy | 231 |
| 8.1 The Main Result | 231 |
| 8.2 Some Commutative Algebra | 234 |
| 8.3 The Group G and its Invariant Derivations | 238 |
| 8.4 Proof of the Main Result | 245 |
| Exercises | 247 |
| 9. Refined Measures | 251 |
| 9.1 Second Proof of Baker's Nonhomogeneous Theorem | 252 |
| 9.2 Proof of Theorem 9.1 | 262 |
| 9.3 Value of $C(m)$ | 286 |
| 9.4 Corollaries | 302 |
| Exercises | 314 |
| 10. On Baker's Method | 317 |
| 10.1 Linear Independence of Logarithms of Algebraic Numbers | 317 |
| 10.2 Baker's Method with Interpolation Determinants | 329 |
| 10.3 Baker's Method with Auxiliary Function | 356 |
| 10.4 The State of the Art | 360 |
| Exercises | 371 |

Part IV. The Linear Subgroup Theorem

| | |
|---|-----|
| 11. Points Whose Coordinates are Logarithms of Algebraic Numbers .. | 375 |
| 11.1 Introduction | 375 |
| 11.2 One Parameter Subgroups | 379 |
| 11.3 Six Variants of the Main Result | 381 |
| 11.4 Linear Independence of Logarithms | 387 |
| 11.5 Complex Toruses | 394 |
| 11.6 Linear Combinations of Logarithms with Algebraic Coefficients .. | 398 |
| 11.7 Proof of the Linear Subgroup Theorem | 404 |
| Exercises | 411 |
| 12. Lower Bounds for the Rank of Matrices | 417 |
| 12.1 Entries are Linear Polynomials | 418 |
| 12.2 Entries are Logarithms of Algebraic Numbers | 432 |
| 12.3 Entries are Linear Combinations of Logarithms | 435 |
| 12.4 Assuming the Conjecture on Algebraic Independence of Logarithms | 437 |
| 12.5 Quadratic Relations | 438 |

| | |
|--|------------|
| Exercises | 441 |
| <hr/> | |
| Part V. Simultaneous Approximation of Values of the Exponential Function in Several Variables | |
| <hr/> | |
| 13. A Quantitative Version of the Linear Subgroup Theorem | 445 |
| 13.1 The Main Result | 447 |
| 13.2 Analytic Estimates | 450 |
| 13.3 Exponential Polynomials | 459 |
| 13.4 Proof of Theorem 13.1 | 464 |
| 13.5 Directions for Use | 471 |
| 13.6 Introducing Feld'man's Polynomials | 476 |
| 13.7 Duality: the Fourier-Borel Transform | 480 |
| Exercises | 490 |
| 14. Applications to Diophantine Approximation | 495 |
| 14.1 A Quantitative Refinement to Gel'fond-Schneider's Theorem | 496 |
| 14.2 A Quantitative Refinement to Hermite-Lindemann's Theorem | 510 |
| 14.3 Simultaneous Approximation in Higher Dimension | 521 |
| 14.4 Measures of Linear Independence of Logarithms (Again) | 536 |
| Open Problems | 547 |
| Exercises | 549 |
| 15. Algebraic Independence | 555 |
| 15.1 Criteria: Irrationality, Transcendence, Algebraic Independence ... | 555 |
| 15.2 From Simultaneous Approximation to Algebraic Independence ... | 569 |
| 15.3 Algebraic Independence Results: Small Transcendence Degree ... | 587 |
| 15.4 Large Transcendence Degree: Conjecture on Simultaneous Approximation | 594 |
| 15.5 Further Results and Conjectures | 598 |
| Exercises | 606 |
| References | 615 |
| Index | 629 |



<http://www.springer.com/978-3-540-66785-8>

Diophantine Approximation on Linear Algebraic Groups
Transcendence Properties of the Exponential Function
in Several Variables

Waldschmidt, M.

2000, XXIII, 633 p., Hardcover

ISBN: 978-3-540-66785-8