Table of Contents

Volume A

1 Introduction ................................................................. 1
   1.1 Introduction ....................................................... 1
   1.2 Matchings ......................................................... 2
   1.3 But what about nonbipartite graphs? ............................. 4
   1.4 Hamiltonian circuits and the traveling salesman problem ... 5
   1.5 Historical and further notes ...................................... 6
       1.5a Historical sketch on polyhedral combinatorics .......... 6
       1.5b Further notes ................................................ 8

2 General preliminaries .................................................. 9
   2.1 Sets ...................................................................... 9
   2.2 Orders ................................................................... 11
   2.3 Numbers .................................................................. 11
   2.4 Vectors, matrices, and functions ............................... 11
   2.5 Maxima, minima, and infinity .................................... 14
   2.6 Fekete’s lemma ..................................................... 14

3 Preliminaries on graphs .................................................... 16
   3.1 Undirected graphs .................................................. 16
   3.2 Directed graphs ...................................................... 28
   3.3 Hypergraphs ........................................................ 36
       3.3a Background references on graph theory .................. 37

4 Preliminaries on algorithms and complexity ....................... 38
   4.1 Introduction ......................................................... 38
   4.2 The random access machine ..................................... 39
   4.3 Polynomial-time solvability ..................................... 39
   4.4 P ................................................................. 40
   4.5 NP ................................................................. 40
   4.6 co-NP and good characterizations ............................... 42
   4.7 Optimization problems ......................................... 42
   4.8 NP-complete problems ............................................ 43
   4.9 The satisfiability problem ........................................ 44
Table of Contents

4.10 NP-completeness of the satisfiability problem .......................... 44
4.11 NP-completeness of some other problems ............................ 46
4.12 Strongly polynomial-time ......................................... 47
4.13 Lists and pointers ............................................. 48
4.14 Further notes ................................................... 49
   4.14a Background literature on algorithms and complexity . 49
   4.14b Efficiency and complexity historically ............... 49

5 Preliminaries on polyhedra and linear and integer programming .................................................. 59
   5.1 Convexity and halfspaces ..................................... 59
   5.2 Cones ......................................................... 60
   5.3 Polyhedra and polytopes .................................... 60
   5.4 Farkas' lemma ................................................ 61
   5.5 Linear programming ......................................... 61
   5.6 Faces, facets, and vertices .................................. 63
   5.7 Polarity ....................................................... 65
   5.8 Blocking polyhedra ......................................... 65
   5.9 Antiblocking polyhedra ...................................... 67
   5.10 Methods for linear programming ............................ 67
   5.11 The ellipsoid method ....................................... 68
   5.12 Polyhedra and NP and co-NP ................................ 71
   5.13 Primal-dual methods ........................................ 72
   5.14 Integer linear programming ................................ 73
   5.15 Integer polyhedra ........................................... 74
   5.16 Totally unimodular matrices ............................... 75
   5.17 Total dual integrality ...................................... 76
   5.18 Hilbert bases and minimal TDI systems .................. 81
   5.19 The integer rounding and decomposition properties ..... 82
   5.20 Box-total dual integrality ................................ 83
   5.21 The integer hull and cutting planes ....................... 83
   5.21a Background literature ................................... 84

Part I: Paths and Flows ............................................. 85

6 Shortest paths: unit lengths ..................................... 87
   6.1 Shortest paths with unit lengths ............................ 87
   6.2 Shortest paths with unit lengths algorithmically:
       breadth-first search ...................................... 88
   6.3 Depth-first search .......................................... 89
   6.4 Finding an Eulerian orientation ............................. 91
   6.5 Further results and notes ................................... 91
       6.5a All-pairs shortest paths in undirected graphs ..... 91
       6.5b Complexity survey .................................... 93
Table of Contents

6.5c Ear-decomposition of strongly connected digraphs . . . 93
6.5d Transitive closure ........................................ 94
6.5e Further notes ................................................. 94

7 Shortest paths: nonnegative lengths .......................... 96
  7.1 Shortest paths with nonnegative lengths .................. 96
  7.2 Dijkstra's method ......................................... 97
  7.3 Speeding up Dijkstra's algorithm with k-heaps ............ 98
  7.4 Speeding up Dijkstra's algorithm with Fibonacci heaps ... 99
  7.5 Further results and notes ................................ 101
    7.5a Weakly polynomial-time algorithms .................... 101
    7.5b Complexity survey for shortest paths with
         nonnegative lengths ................................... 103
    7.5c Further notes ......................................... 105

8 Shortest paths: arbitrary lengths ............................ 107
  8.1 Shortest paths with arbitrary lengths but no negative
      circuits .................................................. 107
  8.2 Potentials ................................................... 107
  8.3 The Bellman-Ford method .................................. 109
  8.4 All-pairs shortest paths .................................. 110
  8.5 Finding a minimum-mean length directed circuit .......... 111
  8.6 Further results and notes ................................ 112
    8.6a Complexity survey for shortest path without
         negative-length circuits ............................. 112
    8.6b NP-completeness of the shortest path problem ........ 114
    8.6c Nonpolyminimality of Ford's method .................. 115
    8.6d Shortest and longest paths in acyclic graphs ......... 116
    8.6e Bottleneck shortest path ............................... 117
    8.6f Further notes ......................................... 118
    8.6g Historical notes on shortest paths .................... 119

9 Disjoint paths .................................................. 131
  9.1 Menger's theorem ......................................... 131
    9.1a Other proofs of Menger's theorem ..................... 133
  9.2 Path packing algorithmically ................................ 134
  9.3 Speeding up by blocking path packings .................... 135
  9.4 A sometimes better bound ................................ 136
  9.5 Complexity of the vertex-disjoint case .................. 137
  9.6 Further results and notes ................................ 138
    9.6a Complexity survey for the disjoint s–t paths
         problem ................................................. 138
    9.6b Partially disjoint paths ................................ 140
    9.6c Exchange properties of disjoint paths ................ 140
    9.6d Further notes ......................................... 141
# Table of Contents

9.6e  Historical notes on Menger’s theorem ............... 142

## 10 Maximum flow ........................................... 148
10.1 Flows: concepts ......................................... 148
10.2 The max-flow min-cut theorem ............................ 150
10.3 Paths and flows ........................................... 151
10.4 Finding a maximum flow ................................. 151
10.4a Nontermination for irrational capacities .............. 152
10.5 A strongly polynomial bound on the number of iterations 153
10.6 Dinits’ $O(n^2m)$ algorithm ............................ 154
10.6a Karzanov’s $O(n^3)$ algorithm ......................... 155
10.7 Goldberg’s push-relabel method ......................... 156
10.8 Further results and notes ............................ 159
10.8a A weakly polynomial bound ......................... 159
10.8b Complexity survey for the maximum flow problem ... 160
10.8c An exchange property ................................. 162
10.8d Further notes ........................................... 162
10.8e Historical notes on maximum flow .................... 164

## 11 Circulations and transshipments ......................... 170
11.1 A useful fact on arc functions .......................... 170
11.2 Circulations .............................................. 171
11.3 Flows with upper and lower bounds .................... 172
11.4 $b$-transshipments .................................... 173
11.5 Upper and lower bounds on excess $f$ .................. 174
11.6 Finding circulations and transshipments algorithmically 175
11.6a Further notes ........................................... 176

## 12 Minimum-cost flows and circulations .................... 177
12.1 Minimum-cost flows and circulations ................... 177
12.2 Minimum-cost circulations and the residual graph $D_f$ 178
12.3 Strongly polynomial-time algorithm ..................... 179
12.4 Related problems ....................................... 182
12.4a A dual approach ..................................... 183
12.4b A strongly polynomial-time algorithm using capacity-scaling 186
12.5 Further results and notes ............................. 190
12.5a Complexity survey for minimum-cost circulation ..... 190
12.5b Min-max relations for minimum-cost flows and circulations 191
12.5c Dynamic flows ...................................... 192
12.5d Further notes ....................................... 195
## Table of Contents

13 Path and flow polyhedra and total unimodularity ....... 198
   13.1 Path polyhedra ........................................... 198
       13.1a Vertices, adjacency, and facets ................. 202
       13.1b The s - t connector polytope ................. 203
   13.2 Total unimodularity .................................. 204
       13.2a Consequences for flows ......................... 205
       13.2b Consequences for circulations ................. 207
       13.2c Consequences for transshipments ............... 207
       13.2d Unions of disjoint paths and cuts ............ 210
   13.3 Network matrices ...................................... 213
   13.4 Cross-free and laminar families .................... 214

14 Partially ordered sets and path coverings .............. 217
   14.1 Partially ordered sets .............................. 217
   14.2 Dilworth's decomposition theorem .................. 218
   14.3 Path coverings ......................................... 219
   14.4 The weighted case ...................................... 220
   14.5 The chain and antichain polytopes .................. 221
       14.5a Path coverings algorithmically ............... 222
   14.6 Unions of directed cuts and antichains ............ 224
       14.6a Common saturating collections of chains ...... 226
   14.7 Unions of directed paths and chains ................ 227
       14.7a Common saturating collections of antichains .. 229
       14.7b Conjugacy of partitions ......................... 230
   14.8 Further results and notes ............................ 232
       14.8a The Gallai-Milgram theorem ..................... 232
       14.8b Partially ordered sets and distributive lattices 233
       14.8c Maximal chains .................................... 235
       14.8d Further notes ...................................... 236

15 Connectivity and Gomory-Hu trees ........................ 237
   15.1 Vertex-, edge-, and arc-connectivity ............... 237
   15.2 Vertex-connectivity algorithmically ............... 239
       15.2a Complexity survey for vertex-connectivity .... 241
       15.2b Finding the 2-connected components ........... 242
   15.3 Arc- and edge-connectivity algorithmically ....... 243
       15.3a Complexity survey for arc- and edge-connectivity 246
       15.3b Finding the 2-edge-connected components ....... 247
   15.4 Gomory-Hu trees ........................................ 248
       15.4a Minimum-requirement spanning tree .............. 251
   15.5 Further results and notes ............................ 252
       15.5a Ear-decomposition of undirected graphs ........ 252
       15.5b Further notes ...................................... 253
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>Cardinality bipartite matching and vertex cover</td>
<td>259</td>
</tr>
<tr>
<td>16.1</td>
<td>M-augmenting paths</td>
<td>259</td>
</tr>
<tr>
<td>16.2</td>
<td>Frobenius' and König's theorems</td>
<td>260</td>
</tr>
<tr>
<td>16.2a</td>
<td>Frobenius' proof of his theorem</td>
<td>262</td>
</tr>
<tr>
<td>16.2b</td>
<td>Linear-algebraic proof of Frobenius' theorem</td>
<td>262</td>
</tr>
<tr>
<td>16.2c</td>
<td>Rizzi's proof of König's matching theorem</td>
<td>263</td>
</tr>
<tr>
<td>16.3</td>
<td>Maximum-size bipartite matching algorithm</td>
<td>263</td>
</tr>
<tr>
<td>16.4</td>
<td>An O(n^{1/2}m) algorithm</td>
<td>264</td>
</tr>
<tr>
<td>16.5</td>
<td>Finding a minimum-size vertex cover</td>
<td>265</td>
</tr>
<tr>
<td>16.6</td>
<td>Matchings covering given vertices</td>
<td>265</td>
</tr>
<tr>
<td>16.7</td>
<td>Further results and notes</td>
<td>267</td>
</tr>
<tr>
<td>16.7a</td>
<td>Complexity survey for cardinality bipartite matching</td>
<td>267</td>
</tr>
<tr>
<td>16.7b</td>
<td>Finding perfect matchings in regular bipartite graphs</td>
<td>267</td>
</tr>
<tr>
<td>16.7c</td>
<td>The equivalence of Menger's theorem and König's theorem</td>
<td>275</td>
</tr>
<tr>
<td>16.7d</td>
<td>Equivalent formulations in terms of matrices</td>
<td>276</td>
</tr>
<tr>
<td>16.7e</td>
<td>Equivalent formulations in terms of partitions</td>
<td>276</td>
</tr>
<tr>
<td>16.7f</td>
<td>On the complexity of bipartite matching and vertex cover</td>
<td>277</td>
</tr>
<tr>
<td>16.7g</td>
<td>Further notes</td>
<td>277</td>
</tr>
<tr>
<td>16.7h</td>
<td>Historical notes on bipartite matching</td>
<td>278</td>
</tr>
<tr>
<td>17</td>
<td>Weighted bipartite matching and the assignment problem</td>
<td>285</td>
</tr>
<tr>
<td>17.1</td>
<td>Weighted bipartite matching</td>
<td>285</td>
</tr>
<tr>
<td>17.2</td>
<td>The Hungarian method</td>
<td>286</td>
</tr>
<tr>
<td>17.3</td>
<td>Perfect matching and assignment problems</td>
<td>288</td>
</tr>
<tr>
<td>17.4</td>
<td>Finding a minimum-size w-vertex cover</td>
<td>289</td>
</tr>
<tr>
<td>17.5</td>
<td>Further results and notes</td>
<td>290</td>
</tr>
<tr>
<td>17.5a</td>
<td>Complexity survey for maximum-weight bipartite matching</td>
<td>290</td>
</tr>
<tr>
<td>17.5b</td>
<td>Further notes</td>
<td>290</td>
</tr>
<tr>
<td>17.5c</td>
<td>Historical notes on weighted bipartite matching and optimum assignment</td>
<td>292</td>
</tr>
<tr>
<td>18</td>
<td>Linear programming methods and the bipartite matching polytope</td>
<td>301</td>
</tr>
<tr>
<td>18.1</td>
<td>The matching and the perfect matching polytope</td>
<td>301</td>
</tr>
<tr>
<td>18.2</td>
<td>Totally unimodular matrices from bipartite graphs</td>
<td>303</td>
</tr>
<tr>
<td>18.3</td>
<td>Consequences of total unimodularity</td>
<td>304</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>18.4</td>
<td>The vertex cover polytope</td>
<td>305</td>
</tr>
<tr>
<td>18.5</td>
<td>Further results and notes</td>
<td>305</td>
</tr>
<tr>
<td>18.5a</td>
<td>Derivation of König’s matching theorem from the matching polytope</td>
<td>305</td>
</tr>
<tr>
<td>18.5b</td>
<td>Dual, primal-dual, primal?</td>
<td>305</td>
</tr>
<tr>
<td>18.5c</td>
<td>Adjacency and diameter of the matching polytope</td>
<td>307</td>
</tr>
<tr>
<td>18.5d</td>
<td>The perfect matching space of a bipartite graph</td>
<td>308</td>
</tr>
<tr>
<td>18.5e</td>
<td>Up and down hull of the perfect matching polytope</td>
<td>309</td>
</tr>
<tr>
<td>18.5f</td>
<td>Matchings of given size</td>
<td>310</td>
</tr>
<tr>
<td>18.5g</td>
<td>Stable matchings</td>
<td>311</td>
</tr>
<tr>
<td>18.5h</td>
<td>Further notes</td>
<td>314</td>
</tr>
<tr>
<td>19</td>
<td>Bipartite edge cover and stable set</td>
<td>315</td>
</tr>
<tr>
<td>19.1</td>
<td>Matchings, edge covers, and Gallai’s theorem</td>
<td>315</td>
</tr>
<tr>
<td>19.2</td>
<td>The König-Rado edge cover theorem</td>
<td>317</td>
</tr>
<tr>
<td>19.3</td>
<td>Finding a minimum-weight edge cover</td>
<td>317</td>
</tr>
<tr>
<td>19.4</td>
<td>Bipartite edge covers and total unimodularity</td>
<td>318</td>
</tr>
<tr>
<td>19.5</td>
<td>The edge cover and stable set polytope</td>
<td>318</td>
</tr>
<tr>
<td>19.5a</td>
<td>Some historical notes on bipartite edge covers</td>
<td>319</td>
</tr>
<tr>
<td>20</td>
<td>Bipartite edge-colouring</td>
<td>321</td>
</tr>
<tr>
<td>20.1</td>
<td>Edge-colourings of bipartite graphs</td>
<td>321</td>
</tr>
<tr>
<td>20.1a</td>
<td>Edge-colouring regular bipartite graphs</td>
<td>322</td>
</tr>
<tr>
<td>20.2</td>
<td>The capacitated case</td>
<td>322</td>
</tr>
<tr>
<td>20.3</td>
<td>Edge-colouring polyhedrally</td>
<td>323</td>
</tr>
<tr>
<td>20.4</td>
<td>Packing edge covers</td>
<td>324</td>
</tr>
<tr>
<td>20.5</td>
<td>Balanced colours</td>
<td>325</td>
</tr>
<tr>
<td>20.6</td>
<td>Packing perfect matchings</td>
<td>326</td>
</tr>
<tr>
<td>20.6a</td>
<td>Polyhedral interpretation</td>
<td>327</td>
</tr>
<tr>
<td>20.6b</td>
<td>Extensions</td>
<td>328</td>
</tr>
<tr>
<td>20.7</td>
<td>Covering by perfect matchings</td>
<td>329</td>
</tr>
<tr>
<td>20.7a</td>
<td>Polyhedral interpretation</td>
<td>330</td>
</tr>
<tr>
<td>20.8</td>
<td>The perfect matching lattice of a bipartite graph</td>
<td>331</td>
</tr>
<tr>
<td>20.9</td>
<td>Further results and notes</td>
<td>333</td>
</tr>
<tr>
<td>20.9a</td>
<td>Some further edge-colouring algorithms</td>
<td>333</td>
</tr>
<tr>
<td>20.9b</td>
<td>Complexity survey for bipartite edge-colouring</td>
<td>334</td>
</tr>
<tr>
<td>20.9c</td>
<td>List-edge-colouring</td>
<td>335</td>
</tr>
<tr>
<td>20.9d</td>
<td>Further notes</td>
<td>336</td>
</tr>
<tr>
<td>21</td>
<td>Bipartite $b$-matchings and transportation</td>
<td>337</td>
</tr>
<tr>
<td>21.1</td>
<td>$b$-matchings and $w$-vertex covers</td>
<td>337</td>
</tr>
<tr>
<td>21.2</td>
<td>The $b$-matching polytope and the $w$-vertex cover polyhedron</td>
<td>338</td>
</tr>
<tr>
<td>21.3</td>
<td>Simple $b$-matchings and $b$-factors</td>
<td>339</td>
</tr>
<tr>
<td>21.4</td>
<td>Capacitated $b$-matchings</td>
<td>341</td>
</tr>
</tbody>
</table>
## Table of Contents

21.5 Bipartite $b$-matching and $w$-vertex cover algorithmically ... 342
21.6 Transportation .......................................................... 343
   21.6a Reduction of transshipment to transportation .... 345
   21.6b The transportation polytope ............................... 346
21.7 $b$-edge covers and $w$-stable sets ................................. 347
21.8 The $b$-edge cover and the $w$-stable set polyhedron .... 348
21.9 Simple $b$-edge covers .............................................. 349
21.10 Capacitated $b$-edge covers ....................................... 350
21.11 Relations between $b$-matchings and $b$-edge covers .... 351
21.12 Upper and lower bounds ........................................... 353
21.13 Further results and notes ......................................... 355
   21.13a Complexity survey on weighted bipartite $b$-matching and transportation .................. 355
   21.13b The matchable set polytope ................................. 359
   21.13c Existence of matrices ....................................... 359
   21.13d Further notes .................................................. 361
   21.13e Historical notes on the transportation and transshipment problems ................. 362

22 Transversals .............................................................. 378
   22.1 Transversals .............................................................. 378
   22.1a Alternative proofs of Hall's marriage theorem .......... 379
   22.2 Partial transversals .............................................. 380
   22.3 Weighted transversals ........................................... 382
   22.4 Min-max relations for weighted transversals .............. 382
   22.5 The transversal polytope ....................................... 383
   22.6 Packing and covering of transversals ......................... 385
   22.7 Further results and notes ....................................... 387
   22.7a The capacitated case .......................................... 387
   22.7b A theorem of Rado ............................................. 389
   22.7c Further notes .................................................. 389
   22.7d Historical notes on transversals ............................ 390

23 Common transversals ...................................................... 393
   23.1 Common transversals .............................................. 393
   23.2 Weighted common transversals ................................ 395
   23.3 Weighted common partial transversals ....................... 397
   23.4 The common partial transversal polytope ..................... 399
   23.5 The common transversal polytope ............................. 401
   23.6 Packing and covering of common transversals ............... 402
   23.7 Further results and notes ....................................... 407
   23.7a Capacitated common transversals ........................... 407
   23.7b Exchange properties ............................................ 407
   23.7c Common transversals of three families ..................... 408
   23.7d Further notes .................................................. 409
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Part III: Nonbipartite Matching and Covering</strong></td>
<td>411</td>
</tr>
<tr>
<td>24 Cardinality nonbipartite matching</td>
<td></td>
</tr>
<tr>
<td>24.1 Tutte's 1-factor theorem and the Tutte-Berge formula</td>
<td>413</td>
</tr>
<tr>
<td>24.1a Tutte's proof of his 1-factor theorem</td>
<td>415</td>
</tr>
<tr>
<td>24.1b Petersen's theorem</td>
<td>415</td>
</tr>
<tr>
<td>24.2 Cardinality matching algorithm</td>
<td>415</td>
</tr>
<tr>
<td>24.2a An $O(n^3)$ algorithm</td>
<td>418</td>
</tr>
<tr>
<td>24.3 Matchings covering given vertices</td>
<td>421</td>
</tr>
<tr>
<td>24.4 Further results and notes</td>
<td>422</td>
</tr>
<tr>
<td>24.4a Complexity survey for cardinality nonbipartite matching</td>
<td>422</td>
</tr>
<tr>
<td>24.4b The Edmonds-Gallai decomposition of a graph</td>
<td>423</td>
</tr>
<tr>
<td>24.4c Strengthening of Tutte's 1-factor theorem</td>
<td>425</td>
</tr>
<tr>
<td>24.4d Ear-decomposition of factor-critical graphs</td>
<td>425</td>
</tr>
<tr>
<td>24.4e Ear-decomposition of matching-covered graphs</td>
<td>426</td>
</tr>
<tr>
<td>24.4f Barriers in matching-covered graphs</td>
<td>427</td>
</tr>
<tr>
<td>24.4g Two-processor scheduling</td>
<td>428</td>
</tr>
<tr>
<td>24.4h The Tutte matrix and an algebraic matching algorithm</td>
<td>429</td>
</tr>
<tr>
<td>24.4i Further notes</td>
<td>430</td>
</tr>
<tr>
<td>24.4j Historical notes on nonbipartite matching</td>
<td>431</td>
</tr>
<tr>
<td>25 The matching polytope</td>
<td></td>
</tr>
<tr>
<td>25.1 The perfect matching polytope</td>
<td>438</td>
</tr>
<tr>
<td>25.2 The matching polytope</td>
<td>439</td>
</tr>
<tr>
<td>25.3 Total dual integrality; the Cunningham-Marsh formula</td>
<td>440</td>
</tr>
<tr>
<td>25.3a Direct proof of the Cunningham-Marsh formula</td>
<td>442</td>
</tr>
<tr>
<td>25.4 On the total dual integrality of the perfect matching constraints</td>
<td></td>
</tr>
<tr>
<td>25.5 Further results and notes</td>
<td>444</td>
</tr>
<tr>
<td>25.5a Adjacency and diameter of the matching polytope</td>
<td>444</td>
</tr>
<tr>
<td>25.5b Facets of the matching polytope</td>
<td>446</td>
</tr>
<tr>
<td>25.5c Polynomial-time solvability with the ellipsoid method</td>
<td>448</td>
</tr>
<tr>
<td>25.5d The matchable set polytope</td>
<td>450</td>
</tr>
<tr>
<td>25.5e Further notes</td>
<td>452</td>
</tr>
<tr>
<td>26 Weighted nonbipartite matching algorithmically</td>
<td></td>
</tr>
<tr>
<td>26.1 Introduction and preliminaries</td>
<td>453</td>
</tr>
<tr>
<td>26.2 Weighted matching algorithm</td>
<td>454</td>
</tr>
<tr>
<td>26.2a An $O(n^3)$ algorithm</td>
<td>456</td>
</tr>
<tr>
<td>26.3 Further results and notes</td>
<td>458</td>
</tr>
</tbody>
</table>
XVIII Table of Contents

26.3a Complexity survey for weighted nonbipartite matching .......................... 458
26.3b Derivation of the matching polytope characterization from the algorithm .......... 459
26.3c Further notes ........................................ 459

27 Nonbipartite edge cover .................................. 461
27.1 Minimum-size edge cover ................................ 461
27.2 The edge cover polytope and total dual integrality ............................... 462
27.3 Further notes on edge covers .................................. 464
27.3a Further notes ........................................ 464
27.3b Historical notes on edge covers .................................. 464

28 Edge-colouring ............................................. 465
28.1 Vizing's theorem for simple graphs ................................ 465
28.2 Vizing's theorem for general graphs ................................ 467
28.3 NP-completeness of edge-colouring ........................................ 468
28.4 Nowhere-zero flows and edge-colouring ........................................ 470
28.5 Fractional edge-colouring ........................................ 474
28.6 Conjectures ........................................ 475
28.7 Edge-colouring polyhedrally ........................................ 477
28.8 Packing edge covers ........................................ 478
28.9 Further results and notes ........................................ 480
28.9a Shannon's theorem ........................................ 480
28.9b Further notes ........................................ 480
28.9c Historical notes on edge-colouring ........................................ 482

29 T-joins, undirected shortest paths, and the Chinese postman ....................... 485
29.1 T-joins ........................................ 485
29.2 The shortest path problem for undirected graphs ........................................ 487
29.3 The Chinese postman problem ........................................ 487
29.4 T-joins and T-cuts ........................................ 488
29.5 The up hull of the T-join polytope ........................................ 490
29.6 The T-join polytope ........................................ 491
29.7 Sums of circuits ........................................ 493
29.8 Integer sums of circuits ........................................ 494
29.9 The T-cut polytope ........................................ 498
29.10 Finding a minimum-capacity T-cut ........................................ 499
29.11 Further results and notes ........................................ 500
29.11a Minimum-mean length circuit ........................................ 500
29.11b Packing T-cuts ........................................ 501
29.11c Packing T-joins ........................................ 507
29.11d Maximum joins ........................................ 510
29.11e Odd paths ........................................ 515
29.1lf Further notes ................................................... 517
29.1lg On the history of the Chinese postman problem ........ 519

30 2-matchings, 2-covers, and 2-factors .......................... 520
30.1 2-matchings and 2-vertex covers ............................. 520
30.2 Fractional matchings and vertex covers ...................... 521
30.3 The fractional matching polytope ............................ 522
30.4 The 2-matching polytope ..................................... 522
30.5 The weighted 2-matching problem ........................... 523
   30.5a Maximum-size 2-matchings and maximum-size matchings ............................. 524
30.6 Simple 2-matchings and 2-factors ........................... 526
30.7 The simple 2-matching polytope and the 2-factor polytope ... 528
30.8 Total dual integrality ........................................... 531
30.9 2-edge covers and 2-stable sets ............................. 531
30.10 Fractional edge covers and stable sets ..................... 532
30.11 The fractional edge cover polyhedron ....................... 533
30.12 The 2-edge cover polyhedron ............................... 533
30.13 Total dual integrality of the 2-edge cover constraints .... 534
30.14 Simple 2-edge covers .......................................... 535
30.15 Graphs with \( \nu(G) = \tau(G) \) and \( \alpha(G) = \rho(G) \) ...................... 536
30.16 Excluding triangles ............................................ 539
   30.16a Excluding higher polygons ............................. 544
30.16b Packing edges and factor-critical subgraphs ............. 544
   30.16c 2-factors without short circuits .................... 545

31 \( b \)-matchings ................................................. 546
31.1 \( b \)-matchings ............................................... 546
31.2 The \( b \)-matching polytope .................................. 547
31.3 Total dual integrality ........................................... 550
31.4 The weighted \( b \)-matching problem ......................... 554
31.5 If \( b \) is even ................................................. 556
31.6 If \( b \) is constant ............................................... 558
31.7 Further results and notes ..................................... 559
   31.7a Complexity survey for the \( b \)-matching problem ...... 559
   31.7b Facets and minimal systems for the \( b \)-matching polytope ................................................. 559
   31.7c Regularizable graphs ....................................... 560
   31.7d Further notes ............................................... 561

32 Capacitated \( b \)-matchings ....................................... 562
32.1 Capacitated \( b \)-matchings ................................... 562
32.2 The capacitated \( b \)-matching polytope ..................... 564
32.3 Total dual integrality ........................................... 566
32.4 The weighted capacitated \( b \)-matching problem .......... 567
33 Simple $b$-matchings and $b$-factors ........................................ 569
33.1 Simple $b$-matchings and $b$-factors ........................................ 569
33.2 The simple $b$-matching polytope and the $b$-factor polytope . 570
33.3 Total dual integrality ....................................................... 570
33.4 The weighted simple $b$-matching and $b$-factor problem ....... 571
33.5 If $b$ is constant ......................................................... 572
33.6 Further results and notes .................................................. 573
  33.6a Complexity results .................................................... 573
  33.6b Degree-sequences ..................................................... 573
  33.6c Further notes ......................................................... 574

34 $b$-edge covers ................................................................. 575
34.1 $b$-edge covers ............................................................. 575
34.2 The $b$-edge cover polyhedron ........................................... 576
34.3 Total dual integrality ...................................................... 576
34.4 The weighted $b$-edge cover problem ................................ 577
34.5 If $b$ is even ............................................................... 578
34.6 If $b$ is constant ........................................................... 578
34.7 Capacitated $b$-edge covers .............................................. 579
34.8 Simple $b$-edge covers .................................................... 581
  34.8a Simple $b$-edge covers and $b$-matchings ....................... 582
  34.8b Capacitated $b$-edge covers and $b$-matchings ............... 583

35 Upper and lower bounds ..................................................... 584
35.1 Upper and lower bounds ................................................ 584
35.2 Convex hull ............................................................... 586
35.3 Total dual integrality .................................................... 589
35.4 Further results and notes ............................................... 591
  35.4a Further results on subgraphs with prescribed degrees .... 591
  35.4b Odd walks ............................................................. 593

36 Bidirected graphs ............................................................ 594
36.1 Bidirected graphs ........................................................ 594
36.2 Convex hull ............................................................... 597
36.3 Total dual integrality .................................................... 598
36.4 Including parity conditions ........................................... 600
36.5 Convex hull ............................................................... 604
  36.5a Convex hull of vertex-disjoint circuits ....................... 605
36.6 Total dual integrality .................................................... 605
36.7 Further results and notes ............................................... 607
  36.7a The Chvátal rank ...................................................... 607
  36.7b Further notes ........................................................ 608
Table of Contents

37 The dimension of the perfect matching polytope .......... 609
  37.1 The dimension of the perfect matching polytope .......... 609
  37.2 The perfect matching space .......................... 611
  37.3 The brick decomposition .............................. 612
  37.4 The brick decomposition of a bipartite graph .......... 613
  37.5 Braces .............................................. 614
  37.6 Bricks .............................................. 614
  37.7 Matching-covered graphs without nontrivial tight cuts ... 617

38 The perfect matching lattice ..................................... 619
  38.1 The perfect matching lattice .......................... 619
  38.2 The perfect matching lattice of the Petersen graph .... 620
  38.3 A further fact on the Petersen graph .................. 621
  38.4 Various useful observations .......................... 622
  38.5 Simple barriers ..................................... 624
  38.6 The perfect matching lattice of a brick ................ 630
  38.7 Synthesis and further consequences of the previous results 643
  38.8 What further might (not) be true .................... 644
  38.9 Further results and notes ............................ 646
    38.9a The perfect 2-matching space and lattice ........... 646
    38.9b Further notes .................................... 647
Volume B

### Part IV: Matroids and Submodular Functions

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>39 Matroids</td>
<td>651</td>
</tr>
<tr>
<td>39.1 Matroids</td>
<td>651</td>
</tr>
<tr>
<td>39.2 The dual matroid</td>
<td>652</td>
</tr>
<tr>
<td>39.3 Deletion, contraction, and truncation</td>
<td>653</td>
</tr>
<tr>
<td>39.4 Examples of matroids</td>
<td>654</td>
</tr>
<tr>
<td>39.4a Relations between transversal matroids and gammoids</td>
<td>659</td>
</tr>
<tr>
<td>39.5 Characterizing matroids by bases</td>
<td>662</td>
</tr>
<tr>
<td>39.6 Characterizing matroids by circuits</td>
<td>662</td>
</tr>
<tr>
<td>39.6a A characterization of Lehman</td>
<td>663</td>
</tr>
<tr>
<td>39.7 Characterizing matroids by rank functions</td>
<td>664</td>
</tr>
<tr>
<td>39.8 The span function and flats</td>
<td>666</td>
</tr>
<tr>
<td>39.8a Characterizing matroids by span functions</td>
<td>666</td>
</tr>
<tr>
<td>39.8b Characterizing matroids by flats</td>
<td>667</td>
</tr>
<tr>
<td>39.8c Characterizing matroids in terms of lattices</td>
<td>668</td>
</tr>
<tr>
<td>39.9 Further exchange properties</td>
<td>669</td>
</tr>
<tr>
<td>39.9a Further properties of bases</td>
<td>671</td>
</tr>
<tr>
<td>39.10 Further results and notes</td>
<td>671</td>
</tr>
<tr>
<td>39.10a Further notes</td>
<td>671</td>
</tr>
<tr>
<td>39.10b Historical notes on matroids</td>
<td>672</td>
</tr>
<tr>
<td><strong>40 The greedy algorithm and the independent set polytope</strong></td>
<td>688</td>
</tr>
<tr>
<td>40.1 The greedy algorithm</td>
<td>688</td>
</tr>
<tr>
<td>40.2 The independent set polytope</td>
<td>690</td>
</tr>
<tr>
<td>40.3 The most violated inequality</td>
<td>693</td>
</tr>
<tr>
<td>40.3a Facets and adjacency on the independent set polytope</td>
<td>698</td>
</tr>
<tr>
<td>40.3b Further notes</td>
<td>699</td>
</tr>
<tr>
<td><strong>41 Matroid intersection</strong></td>
<td>700</td>
</tr>
<tr>
<td>41.1 Matroid intersection theorem</td>
<td>700</td>
</tr>
<tr>
<td>41.1a Applications of the matroid intersection theorem</td>
<td>702</td>
</tr>
<tr>
<td>41.1b Woodall's proof of the matroid intersection theorem</td>
<td>704</td>
</tr>
<tr>
<td>41.2 Cardinality matroid intersection algorithm</td>
<td>705</td>
</tr>
<tr>
<td>41.3 Weighted matroid intersection algorithm</td>
<td>707</td>
</tr>
<tr>
<td>41.3a Speeding up the weighted matroid intersection algorithm</td>
<td>710</td>
</tr>
<tr>
<td>41.4 Intersection of the independent set polytopes</td>
<td>712</td>
</tr>
<tr>
<td>41.4a Facets of the common independent set polytope</td>
<td>717</td>
</tr>
<tr>
<td>41.4b Up and down hull of the common base polytope</td>
<td>719</td>
</tr>
</tbody>
</table>
41.5 Further results and notes ........................................ 720
41.5a Menger’s theorem for matroids ................................ 720
41.5b Exchange properties ........................................... 721
41.5c Jump systems ................................................... 722
41.5d Further notes ................................................... 723

42 Matroid union ....................................................... 725
42.1 Matroid union theorem ......................................... 725
  42.1a Applications of the matroid union theorem ............ 727
  42.1b Horn’s proof ................................................ 729
42.2 Polyhedral applications ....................................... 730
42.3 Matroid union algorithm ....................................... 731
42.4 The capacitated case: fractional packing and covering of
dbases ............................................................. 732
42.5 The capacitated case: integer packing and covering of bases 734
42.6 Further results and notes ....................................... 736
  42.6a Induction of matroids ....................................... 736
  42.6b List-colouring ............................................... 737
  42.6c Strongly base orderable matroids ....................... 738
  42.6d Blocking and antithetical polyhedra .................... 741
  42.6e Further notes .............................................. 743
  42.6f Historical notes on matroid union ...................... 743

43 Matroid matching .................................................. 745
43.1 Infinite matroids ............................................... 745
43.2 Matroid matchings ............................................. 746
43.3 Circuits ....................................................... 747
43.4 A special class of matroids .................................. 747
43.5 A min-max formula for maximum-size matroid matching .. 751
43.6 Applications of the matroid matching theorem .......... 753
43.7 A Gallai theorem for matroid matching and covering ..... 756
43.8 Linear matroid matching algorithm ....................... 757
43.9 Matroid matching is not polynomial-time solvable in
genral ............................................................. 762
43.10 Further results and notes .................................... 763
  43.10a Optimal path-matching .................................. 763
  43.10b Further notes .............................................. 764

44 Submodular functions and polymatroids ...................... 766
44.1 Submodular functions and polymatroids ................... 766
  44.1a Examples .................................................. 768
44.2 Optimization over polymatroids by the greedy method .. 771
44.3 Total dual integrality ....................................... 773
44.4 $f$ is determined by $EP_f$ .................................. 773
44.5 Supermodular functions and contrapollmatroids .......... 774
XXIV Table of Contents

44.6 Further results and notes ........................................ 775
44.6a Submodular functions and matroids .......................... 775
44.6b Reducing integer polymatroids to matroids ................. 776
44.6c The structure of polymatroids ............................... 776
44.6d Characterization of polymatroids ............................ 779
44.6e Operations on submodular functions and polymatroids ........................................ 781
44.6f Duals of polymatroids ........................................ 782
44.6g Induction of polymatroids ..................................... 782
44.6h Lovász’s generalization of König’s matching theorem ... 783
44.6i Further notes .................................................. 784

45 Submodular function minimization ............................. 786
45.1 Submodular function minimization ............................ 786
45.2 Orders and base vectors ........................................ 787
45.3 A subroutine ................................................... 787
45.4 Minimizing a submodular function ............................ 789
45.5 Running time of the algorithm ................................. 790
45.6 Minimizing a symmetric submodular function ............... 792
45.7 Minimizing a submodular function over the odd sets ....... 793

46 Polymatroid intersection ........................................ 795
46.1 Box-total dual integrality of polymatroid intersection ...... 795
46.2 Consequences ................................................... 796
46.3 Contrapolymatroid intersection ............................... 797
46.4 Intersecting a polymatroid and a contrapolymatroid ...... 798
46.5 Frank’s discrete sandwich theorem ........................... 799
46.6 Integer decomposition .......................................... 800
46.7 Further results and notes ...................................... 801
46.7a Up and down hull of the common base vectors ............. 801
46.7b Further notes ................................................ 804

47 Polymatroid intersection algorithmically ...................... 805
47.1 A maximum-size common vector in two polymatroids ....... 805
47.2 Maximizing a coordinate of a common base vector .......... 807
47.3 Weighted polymatroid intersection in polynomial time .... 809
47.4 Weighted polymatroid intersection in strongly polynomial time ........................................ 811
47.5 Contrapolymatroids ............................................. 818
47.6 Intersecting a polymatroid and a contrapolymatroid ...... 818
47.6a Further notes ................................................ 819
48 Dilworth truncation ............................................. 820
  48.1 If \( f(0) < 0 \) ........................................ 820
  48.2 Dilworth truncation .................................... 821
    48.2a Applications and interpretations .................. 823
  48.3 Intersection .......................................... 825

49 Submodularity more generally ............................. 826
  49.1 Submodular functions on a lattice family ............ 826
  49.2 Intersection .......................................... 828
  49.3 Complexity ........................................... 829
  49.4 Submodular functions on an intersecting family .... 832
  49.5 Intersection .......................................... 833
  49.6 From an intersecting family to a lattice family ...... 834
  49.7 Complexity .......................................... 835
  49.8 Intersecting a polymatroid and a contrapolymatroid 837
  49.9 Submodular functions on a crossing family .......... 838
  49.10 Complexity .......................................... 840
    49.10a Nonemptiness of the base polyhedron .............. 841
  49.11 Further results and notes ............................ 842
    49.11a Minimizing a submodular function over a
    subcollection of a lattice family ...................... 842
    49.11b Generalized polymatroids ........................ 845
    49.11c Supermodular colourings .......................... 849
    49.11d Further notes .................................... 851

Part V: Trees, Branchings, and Connectors .................. 853

50 Shortest spanning trees .................................. 855
  50.1 Shortest spanning trees ................................ 855
  50.2 Implementing Prim’s method ............................ 857
  50.3 Implementing Kruskal’s method ......................... 858
    50.3a Parallel forest-merging ............................ 859
    50.3b A dual greedy algorithm ............................ 859
  50.4 The longest forest and the forest polytope .......... 860
  50.5 The shortest connector and the connector polytope ... 862
  50.6 Further results and notes ............................. 864
    50.6a Complexity survey for shortest spanning tree .... 864
    50.6b Characterization of shortest spanning trees ....... 865
    50.6c The maximum reliability problem ................. 866
    50.6d Exchange properties of forests .................... 867
    50.6e Uniqueness of shortest spanning tree .............. 868
    50.6f Forest covers ...................................... 869
    50.6g Further notes ..................................... 870
    50.6h Historical notes on shortest spanning trees .... 871
51 Packing and covering of trees ........................................... 877
  51.1 Unions of forests ............................................. 877
  51.2 Disjoint spanning trees ...................................... 877
  51.3 Covering by forests .......................................... 878
  51.4 Complexity .................................................... 879
  51.5 Further results and notes .................................... 889
    51.5a Complexity survey for tree packing and covering .... 889
    51.5b Further notes .............................................. 892

52 Longest branchings and shortest arborescences .................. 893
  52.1 Finding a shortest r-arborescence .......................... 893
    52.1a r-arborescences as common bases of two matroids .... 895
  52.2 Related problems ............................................. 895
  52.3 A min-max relation for shortest r-arborescences ........ 896
  52.4 The r-arborescence polytope ................................ 897
    52.4a Uncrossing cuts .......................................... 899
  52.5 A min-max relation for longest branchings ............... 900
  52.6 The branching polytope ...................................... 901
  52.7 The arborescence polytope ................................... 901
  52.8 Further results and notes ................................... 902
    52.8a Complexity survey for shortest r-arborescence ...... 902
    52.8b Concise LP formulation for shortest r-arborescence .. 902
    52.8c Further notes ............................................. 903

53 Packing and covering of branchings and arborescences .......... 904
  53.1 Disjoint branchings ......................................... 904
  53.2 Disjoint r-arborescences ................................... 905
  53.3 The capacitated case ........................................ 907
  53.4 Disjoint arborescences ...................................... 908
  53.5 Covering by branchings ...................................... 908
  53.6 An exchange property of branchings ....................... 909
  53.7 Covering by r-arborescences ................................ 911
  53.8 Minimum-length unions of k r-arborescences ............. 913
  53.9 The complexity of finding disjoint arborescences .......... 918
  53.10 Further results and notes .................................. 921
    53.10a Complexity survey for disjoint arborescences ....... 921
    53.10b Arborescences with roots in given subsets .......... 923
    53.10c Disclaimers ............................................. 925
    53.10d Further notes ........................................... 926

54 Biconnectors and biflareings .................................... 928
  54.1 Shortest $R - S$ biconnectors ................................ 928
  54.2 Longest $R - S$ biflareings ................................ 930
  54.3 Disjoint $R - S$ biconnectors ................................ 931
  54.4 Covering by $R - S$ biflareings ................................ 934
Table of Contents

54.5 Minimum-size bibranchings ........................................ 934
54.6 Shortest bibranchings ........................................ 935
  54.6a Longest bifurcations ....................................... 937
54.7 Disjoint bibranchings .......................................... 940
  54.7a Proof using supermodular colourings .................... 943
  54.7b Covering by bifurcations ................................ 943
  54.7c Disjoint $R - S$ biconnectors and $R - S$ bibranchings .. 944
  54.7d Covering by $R - S$ biforests and by $R - S$
  bifurcations .................................................. 944

55 Minimum directed cut covers and packing directed cuts ....... 946
  55.1 Minimum directed cut covers and packing directed cuts .... 946
  55.2 The Lucchesi-Younger theorem ................................ 947
  55.3 Directed cut $k$-covers ...................................... 949
  55.4 Feedback arc sets ........................................... 951
  55.5 Complexity ................................................ 953
    55.5a Finding a dual solution ................................ 954
  55.6 Further results and notes .................................... 956
    55.6a Complexity survey for minimum-size directed cut
    cover .......................................................... 956
    55.6b Feedback arc sets in linklessly embeddable digraphs .. 956
    55.6c Feedback vertex sets ...................................... 958
    55.6d The bipartite case ...................................... 959
    55.6e Further notes ............................................ 960

56 Minimum directed cuts and packing directed cut covers ...... 962
  56.1 Minimum directed cuts and packing directed cut covers .... 962
  56.2 Source-sink connected digraphs .............................. 964
  56.3 Other cases where Woodall’s conjecture is true .......... 967
    56.3a Further notes ............................................ 968

57 Strong connectors ................................................ 969
  57.1 Making a directed graph strongly connected ................ 969
  57.2 Shortest strong connectors .................................. 970
  57.3 Polyhedrally ................................................. 973
  57.4 Disjoint strong connectors .................................. 973
  57.5 Complexity ................................................ 975
    57.5a Crossing families ........................................ 976

58 The traveling salesman problem ................................ 981
  58.1 The traveling salesman problem .............................. 981
  58.2 NP-completeness of the TSP ................................ 982
  58.3 Branch-and-bound techniques ................................ 982
  58.4 The symmetric traveling salesman polytope ................ 983
  58.5 The subtour elimination constraints ........................ 984
### Table of Contents

58.6 1-trees and Lagrangean relaxation ................................. 985
58.7 The 2-factor constraints ........................................... 986
58.8 The clique tree inequalities ....................................... 987
58.8a Christofides’ heuristic for the TSP ......................... 989
58.8b Further notes on the symmetric traveling salesman problem ........................................... 990
58.9 The asymmetric traveling salesman problem ................. 992
58.10 Directed 1-trees ..................................................... 993
58.10a An integer programming formulation ..................... 993
58.10b Further notes on the asymmetric traveling salesman problem ........................................... 994
58.11 Further notes on the traveling salesman problem ......... 995
58.11a Further notes ....................................................... 995
58.11b Historical notes on the traveling salesman problem .... 996

59 Matching forests ...................................................... 1005
59.1 Introduction ......................................................... 1005
59.2 The maximum size of a matching forest .................... 1006
59.3 Perfect matching forests ......................................... 1007
59.4 An exchange property of matching forests ............... 1008
59.5 The matching forest polytope .................................... 1011
59.6 Further results and notes ......................................... 1015
59.6a Matching forests in partitionable mixed graphs ....... 1015
59.6b Further notes ....................................................... 1017

60 Submodular functions on directed graphs .................... 1018
60.1 The Edmonds-Giles theorem ..................................... 1018
60.1a Applications ...................................................... 1020
60.1b Generalized polymatroids and the Edmonds-Giles theorem ........................................... 1020
60.2 A variant ............................................................. 1021
60.2a Applications ...................................................... 1023
60.3 Further results and notes ......................................... 1025
60.3a Lattice polyhedra .................................................. 1025
60.3b Polymatroidal network flows .................................. 1028
60.3c A general model .................................................. 1029
60.3d Packing cuts and Győri’s theorem ............................ 1030
60.3e Further notes ....................................................... 1034

61 Graph orientation .................................................... 1035
61.1 Orientations with bounds on in- and outdegrees .......... 1035
61.2 2-edge-connectivity and strongly connected orientations ... 1037
61.2a Strongly connected orientations with bounds on degrees ........................................... 1038
61.3 Nash-Williams’ orientation theorem ......................... 1040
Table of Contents

64.9a Graphs with polynomial-time stable set algorithm . 1099
64.9b Colourings and orientations . 1101
64.9c Algebraic methods . 1102
64.9d Approximation algorithms . 1103
64.9e Further notes . 1104

65 Perfect graphs: general theory . 1106
65.1 Introduction to perfect graphs . 1106
65.2 The perfect graph theorem . 1108
65.3 Replication . 1109
65.4 Perfect graphs and polyhedra . 1110
65.4a Lovász's proof of the replication lemma . 1111
65.5 Decomposition of Berge graphs . 1112
65.5a 0- and 1-joins . 1112
65.5b The 2-join . 1113
65.6 Pre-proof work on the strong perfect graph conjecture . 1115
65.6a Partitionable graphs . 1116
65.6b More characterizations of perfect graphs . 1118
65.6c The stable set polytope of minimally imperfect graphs . 1118
65.6d Graph classes . 1120
65.6e The \( P_3 \)-structure of a graph and a semi-strong perfect graph theorem . 1122
65.6f Further notes on the strong perfect graph conjecture . 1123
65.7 Further results and notes . 1125
65.7a Perz and Rolewicz's proof of the perfect graph theorem . 1125
65.7b Kernel solvability . 1126
65.7c The amalgam . 1130
65.7d Diperfect graphs . 1131
65.7e Further notes . 1133

66 Classes of perfect graphs . 1135
66.1 Bipartite graphs and their line graphs . 1135
66.2 Comparability graphs . 1137
66.3 Chordal graphs . 1138
66.3a Chordal graphs as intersection graphs of subtrees of a tree . 1142
66.4 Meyniel graphs . 1143
66.5 Further results and notes . 1145
66.5a Strongly perfect graphs . 1145
66.5b Perfectly orderable graphs . 1146
66.5c Unimodular graphs . 1147
66.5d Further classes of perfect graphs . 1148
67 Perfect graphs: polynomial-time solvability ........................................ 1152
   67.1 Optimum clique and colouring in perfect graphs algorithmically ............... 1152
   67.2 Weighted clique and colouring algorithmically ..................................... 1155
   67.3 Strong polynomial-time solvability .................................................. 1159
   67.4 Further results and notes ..................................................................... 1159
      67.4a Further on \( \theta(G) \) ................................................................. 1159
      67.4b The Shannon capacity \( \Theta(G) \) .................................................... 1167
      67.4c Clique cover numbers of products of graphs ....................................... 1172
      67.4d A sharper upper bound \( \theta'(G) \) on \( \alpha(G) \) ............................ 1173
      67.4e An operator strengthening convex bodies .......................................... 1173
      67.4f Further notes ................................................................................. 1175
      67.4g Historical notes on perfect graphs .................................................. 1176

68 T-perfect graphs ...................................................................................... 1186
   68.1 T-perfect graphs .................................................................................. 1186
   68.2 Strongly t-perfect graphs ..................................................................... 1187
   68.3 Strong t-perfection of odd-\( K_4 \)-free graphs ...................................... 1188
   68.4 On characterizing t-perfection ............................................................. 1194
   68.5 A combinatorial min-max relation ....................................................... 1196
   68.6 Further results and notes ................................................................. 1200
      68.6a The \( w \)-stable set polyhedron ........................................................ 1200
      68.6b Bidirected graphs ........................................................................... 1201
      68.6c Characterizing odd-\( K_4 \)-free graphs by mixing stable sets and vertex covers ............................................................... 1203
      68.6d Orientations of discrepancy 1 ........................................................... 1204
      68.6e Colourings and odd \( K_4 \)-subdivisions ............................................. 1206
      68.6f Homomorphisms ........................................................................... 1207
      68.6g Further notes ................................................................................. 1207

69 Claw-free graphs ...................................................................................... 1208
   69.1 Introduction ......................................................................................... 1208
   69.2 Maximum-size stable set in a claw-free graph ....................................... 1208
   69.3 Maximum-weight stable set in a claw-free graph .................................... 1213
   69.4 Further results and notes ................................................................. 1216
      69.4a On the stable set polytope of a claw-free graph ............................... 1216
      69.4b Further notes ................................................................................. 1217
Volume C

Part VII: Multiflows and Disjoint Paths

70 Multiflows and disjoint paths ................................. 1221
  70.1 Directed multflow problems ............................. 1221
  70.2 Undirected multflow problems .......................... 1222
  70.3 Disjoint paths problems ............................... 1223
  70.4 Reductions ............................................. 1223
  70.5 Complexity of the disjoint paths problem .......... 1224
  70.6 Complexity of the fractional multflow problem .... 1225
  70.7 The cut condition for directed graphs .............. 1227
  70.8 The cut condition for undirected graphs .......... 1228
  70.9 Relations between fractional, half-integer, and integer solutions ........................................ 1230
  70.10 The Euler condition .................................... 1233
  70.11 Survey of cases where a good characterization has been found ........................................ 1234
  70.12 Relation between the cut condition and fractional cut packing ........................................ 1236
    70.12a Sufficiency of the cut condition sometimes implies an integer multflow ........................ 1238
    70.12b The cut condition and integer multiflows in directed graphs ..................................... 1241
  70.13 Further results and notes ............................. 1242
    70.13a Fixing the number of commodities in undirected graphs ........................................ 1242
    70.13b Fixing the number of commodities in directed graphs ........................................ 1243
    70.13c Disjoint paths in acyclic digraphs ............... 1244
    70.13d A column generation technique for multiflows ... 1245
    70.13e Approximate max-flow min-cut theorems for multiflows ........................................ 1247
    70.13f Further notes ....................................... 1248
    70.13g Historical notes on multicommodity flows ...... 1249

71 Two commodities .............................................. 1251
  71.1 The Rothvlcht-Whinston theorem and Hu's 2-commodity flow theorem .................................... 1251
    71.1a Nash-Williams' proof of the Rothvlcht-Whinston theorem ........................................ 1254
  71.2 Consequences ............................................ 1255
  71.3 2-commodity cut packing ................................ 1257
  71.4 Further results and notes ............................. 1261
Table of Contents

71.4a Two disjoint paths in undirected graphs ........ 1261
71.4b A directed 2-commodity flow theorem ........ 1262
71.4c Kleitman, Martin-Löf, Rothschild, and Whinston's theorem ........ 1263
71.4d Further notes .................. 1265

72 Three or more commodities .................. 1266
72.1 Demand graphs for which the cut condition is sufficient ........ 1266
72.2 Three commodities .................. 1271
72.2a The $K_{3,3}$-metric condition ........ 1273
72.2b Six terminals .................. 1275
72.3 Cut packing .................. 1276

73 T-paths .................. 1279
73.1 Disjoint T-paths .................. 1279
73.1a Disjoint T-paths with the matroid matching algorithm ........ 1283
73.1b Polynomial-time findability of edge-disjoint T-paths ........ 1285
73.1c A feasibility characterization for integer $K_3$-flows ........ 1286
73.2 Fractional packing of T-paths .................. 1287
73.2a Direct proof of Corollary 73.2d .................. 1288
73.3 Further results and notes .................. 1289
73.3a Further notes on Mader's theorem .................. 1289
73.3b A generalization of fractionally packing T-paths ........ 1290
73.3c Lockable collections .................. 1291
73.3d Mader matroids .................. 1292
73.3e Minimum-cost maximum-value multiflows .................. 1294
73.3f Further notes .................. 1295

74 Planar graphs .................. 1296
74.1 All nets spanned by one face: the Okamura-Seymour theorem .................. 1296
74.1a Complexity survey .................. 1299
74.1b Graphs on the projective plane .................. 1299
74.1c If only inner vertices satisfy the Euler condition .................. 1302
74.1d Distances and cut packing .................. 1304
74.1e Linear algebra and distance realizability .................. 1305
74.1f Directed planar graphs with all terminals on the outer boundary .................. 1307
74.2 $G + H$ planar .................. 1307
74.2a Distances and cut packing .................. 1308
74.2b Deleting the Euler condition if $G + H$ is planar .................. 1309
74.3 Okamura's theorem .................. 1311
74.3a Distances and cut packing .................. 1313
XXXIV  Table of Contents

74.3b  The Klein bottle ........................................ 1314
74.3c  Commodities spanned by three or more faces .... 1316
74.4  Further results and notes .............................. 1318
74.4a  Another theorem of Okamura ......................... 1318
74.4b  Some other planar cases where the cut condition is sufficient ....... 1320
74.4c  Vertex-disjoint paths in planar graphs ............ 1320
74.4d  Grid graphs ........................................... 1323
74.4e  Further notes ......................................... 1325

75  Cuts, odd circuits, and multilows ....................... 1326
75.1  Weakly and strongly bipartite graphs ............... 1326
75.1a  NP-completeness of maximum cut .................. 1328
75.1b  Planar graphs ......................................... 1328
75.2  Signed graphs .......................................... 1329
75.3  Weakly, evenly, and strongly bipartite signed graphs .... 1330
75.4  Characterizing strongly bipartite signed graphs ... 1331
75.5  Characterizing weakly and evenly bipartite signed graphs .... 1334
75.6  Applications to multilows .............................. 1341
75.7  The cut cone and the cut polytope .................... 1342
75.8  The maximum cut problem and semidefinite programming .... 1345
75.9  Further results and notes ............................. 1348
    75.9a  Cuts and stable sets ............................... 1348
    75.9b  Further notes ...................................... 1350

76  Homotopy and graphs on surfaces ......................... 1352
76.1  Graphs, curves, and their intersections: terminology and notation .... 1352
76.2  Making curves minimally crossing by Reidemeister moves .... 1353
76.3  Decomposing the edges of an Eulerian graph on a surface .... 1354
76.4  A corollary on lengths of closed curves .............. 1356
76.5  A homotopic circulation theorem ........................ 1357
76.6  Homotopic paths in planar graphs with holes .......... 1361
76.7  Vertex-disjoint paths and circuits of prescribed homotopies .... 1367
    76.7a  Vertex-disjoint circuits of prescribed homotopies .... 1367
    76.7b  Vertex-disjoint homotopic paths in planar graphs with holes .... 1368
    76.7c  Disjoint trees ..................................... 1371
# Table of Contents

## Part VIII: Hypergraphs

### 77 Packing and blocking in hypergraphs: elementary notions
- 77.1 Elementary hypergraph terminology and notation  
- 77.2 Deletion, restriction, and contraction  
- 77.3 Duplication and parallelization  
- 77.4 Clutters  
- 77.5 Packing and blocking  
- 77.6 The blocker  
- 77.7 Fractional matchings and vertex covers  
- 77.8 k-matchings and k-vertex covers  
- 77.9 Further results and notes  
  - 77.9a Bottleneck extrema  
  - 77.9b The ratio of and \( r^* \)  
  - 77.9c Further notes

### 78 Ideal hypergraphs
- 78.1 Ideal hypergraphs  
- 78.2 Characterizations of ideal hypergraphs  
- 78.3 Minimally nonideal hypergraphs  
- 78.4 Properties of minimally nonideal hypergraphs: Lehman's theorem  
  - 78.4a Application of Lehman's theorem: Guenin's theorem  
  - 78.4b Ideality is in co-NP  
- 78.5 Further results and notes  
  - 78.5a Composition of clutters  
  - 78.5b Further notes

### 79 Mengerian hypergraphs
- 79.1 Mengerian hypergraphs  
- 79.2 Minimally non-Mengerian hypergraphs  
- 79.3 Further results and notes  
  - 79.3a Packing hypergraphs  
  - 79.3b Restrictions instead of parallelizations  
  - 79.3c Equivalences for k-matchings and k-vertex covers  
  - 79.3d A general technique  
  - 79.3e Further notes
XXXVI  Table of Contents

80  Binary hypergraphs ............................................. 1406
  80.1  Binary hypergraphs ......................................... 1406
  80.2  Binary hypergraphs and binary matroids .................... 1406
  80.3  The blocker of a binary hypergraph ......................... 1407
    80.3a  Further characterizations of binary clutters .......... 1408
  80.4  On characterizing binary ideal hypergraphs ............... 1408
  80.5  Seymour's characterization of binary Mengerian
        hypergraphs ............................................. 1409
    80.5a  Applications of Seymour's theorem ................... 1413
  80.6  Mengerian matroids .......................................... 1415
  80.6a  Oriented matroids ......................................... 1415
  80.7  Further results and notes ................................... 1416
    80.7a  $\tau_2(H) = 2\tau(H)$ for binary hypergraphs $H$ .... 1416
    80.7b  Application $T$-joins and $T$-cuts ..................... 1417
    80.7c  Box-integrality of $k \cdot P_H$ ......................... 1418

81  Matroids and multiflows ........................................ 1419
  81.1  Multiflows in matroids ..................................... 1419
  81.2  Integer $k$-flowing ........................................ 1420
  81.3  1-flowing and 1-cycling ................................... 1421
  81.4  2-flowing and 2-cycling ................................... 1421
  81.5  3-flowing and 3-cycling ................................... 1422
  81.6  4-flowing, $4$-cycling, $\infty$-flowing, and $\infty$-cycling.. 1423
  81.7  The circuit cone and cycle polytope of a matroid .......... 1424
  81.8  The circuit space and circuit lattice of a matroid .......... 1425
  81.9  Nonnegative integer sums of circuits ..................... 1425
  81.10  Nowhere-zero flows and circuit double covers in matroids . 1426

82  Covering and antiblocking in hypergraphs ....................... 1428
  82.1  Elementary concepts ......................................... 1428
  82.2  Fractional edge covers and stable sets .................... 1429
  82.3  $k$-edge covers and $k$-stable sets ........................ 1429
  82.4  The antiblocker and conformality .......................... 1430
    82.4a  Gilmore's characterization of conformality .......... 1431
  82.5  Perfect hypergraphs ........................................ 1431
  82.6  Further notes ................................................ 1434
    82.6a  Some equivalences for the $k$-parameters ............. 1434
    82.6b  Further notes .......................................... 1437

83  Balanced and unimodular hypergraphs .......................... 1439
  83.1  Balanced hypergraphs ....................................... 1439
  83.2  Characterizations of balanced hypergraphs .................. 1440
    83.2a  Totally balanced matrices ............................... 1444
    83.2b  Examples of balanced hypergraphs ....................... 1447
    83.2c  Balanced 0, $\pm1$ matrices ............................. 1447
Table of Contents  XXXVII

83.3  Unimodular hypergraphs .......................... 1448
83.3a  Further notes .................................... 1450

Survey of Problems, Questions, and Conjectures ............. 1453

References .............................................. 1463

Name Index ............................................. 1767

Subject Index ........................................... 1807

Greek graph and hypergraph functions ....................... 1880
Combinatorial Optimization
Polyhedra and Efficiency
Schrijver, A.
2003, CIV, 1879 p. In 3 volumes, not available separately., Hardcover
ISBN: 978-3-540-44389-6