Nonlinear Subdivision Schemes: Applications to Image Processing

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Extended Abstract

In classical subdivision schemes, some initial discrete data set v^0 is refined iteratively, following a prescribed linear rule which is summarized by

$$v^j = \mathcal{S}v^{j-1} = \cdots = \mathcal{S}^j v^0,$$

where v^j are the numerical data at resolution 2^{-j} and $\mathcal{S}$ the subdivision operator. One is usually interested in the convergence properties of this process to some limit function $f = \mathcal{S}^\infty v^0$. In the simplest setting the data v^j belongs to the uniform grid $2^{-j}\mathbb{Z}$ and convergence means that $\sup_k |f(2^{-j}k) - v_k^j|$ goes to zero as j tends to $+\infty$. The analysis of convergence can be performed by various methods, including Fourier analysis by Laurent polynomials [4], when the scheme is uniform.

An important motivation for the study of subdivision algorithms is their relation to multiresolution analysis and wavelets bases (see e.g. [2] or [6]). In particular, the contribution of a single wavelet coefficient in the representation of a discrete signal is precisely obtained by applying a subdivision scheme from the scale of the coefficient up to the signal discretization scale. Therefore, understanding the convergence of subdivision algorithms is fundamental in the context of applications of wavelets to data compression or signal denoising, in which certain coefficients are quantized or discarded.

We consider the situation where this rule is *nonlinear* in the sense that the refinement operator depends itself on the data to be refined. We are therefore facing a rule of the type

$$v^j = \mathcal{S}(v^{j-1})v^{j-1} = \cdots = \mathcal{S}^j v^0,$$

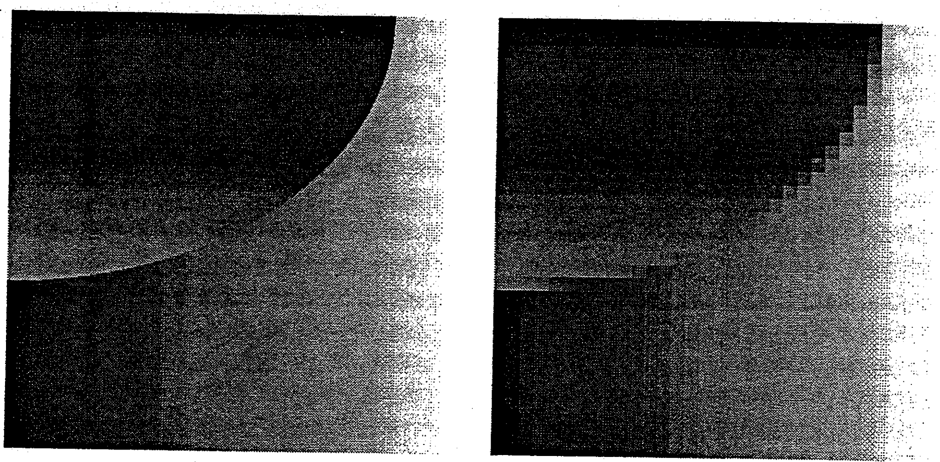
We are especially interested in a class of data dependent operators which has been introduced by Ami Harten in the context of the numerical simulation of conservation laws, and applied in [7] to the derivation of generalized multiscale representations. The nonlinear refinement rule involves a data dependent stencil selection which aims at making the refinement process more accurate in the presence of isolated singularities such as shocks. Consider the simple $1D$ case where v_k^j is interpreted as the cell-averages of a function v on the interval $I_{j,k} = 2^{-j}[k, k+1]$, i.e. $a_k^j(v) = 2^j \int_{I_{j,k}} v(t)dt$. Note that the function v is not unique. We are interested in a refinement rule such that if

$v_k^j = a_k^j(v)$ for a function v in some class of interest, then v_k^{j+1} approximates as best as possible the averages $a_k^{j+1}(v)$ at the next finer scale. This allows us to make a comparison between several types of refinement rule:

- Linear refinement by polynomial reconstruction: for some fixed M, consider for each k the unique polynomial p_k of degree $2M$ which agrees with the cell averages data on the stencil $\{k-M,\cdots,k+M\}$, in the sense that $a_{k+l}^j(p_k) = v_{k+l}^j$ for $l=-M,\cdots,M$. One then defines the refined average on the half intervals by $v_{2k}^{j+1} = a_{2k}^{j+1}(p_k)$ and $v_{2k+1}^{j+1} = a_{2k+1}^{j+1}(p_k)$. This linear refinement process is associated to a multiresolution decomposition into biorthogonal wavelets and its convergence is well established. The polynomial p_k provides accuracy $\mathcal{O}(2^{-(2M+1)j})$ on $I_{j,k}$ if the function v is sufficiently *smooth* - at least C^{2M+1} - on the stencil interval $2^{-j}[k-M,k+M+1]$. Therefore the presence of an isolated jump discontinuity of v at some point $x \in I_{j,k}$ deteriorates the accuracy down to $\mathcal{O}(1)$ on the $2M+1$ intervals $I_{j,k-M},\cdots,I_{j,k+M}$.
- Nonlinear refinement by essentially non oscillatory (ENO) stencil selection: in order to improve the situation in the neighbourhood of a jump point $x \in I_{j,k}$, the idea is to systematically replace p_k by a polynomial $\tilde{p}_k = p_l$ selected within $\{p_{k-M},\cdots,p_{k+M}\}$ as the *least oscillatory*. This is typically done by minimizing a cost function such as $C(l) = \sum_{n=l-M+1,l+M} |v_n^j - v_{n-1}^j|$, which measures the oscillation within the stencil of p_l. With such a process, we expect to restore the accuracy on all the intervals which do not contain x, since the selection mechanism will tend to avoid the interval $I_{j,k}$.
- Nonlinear refinement by using stencil selection and subcell resolution (ENO-SR): from the above remark, one can *detect* the intervals which might contain jump singularities by the fact that the selection mechanism tends to avoid them. In order to improve further the accuracy in such intervals $I_{j,k}$, one can replace $\tilde{p}_k$ by a piecewise polynomial function $q_k(t)$ which agrees with $\tilde{p}_{k-1}(t)$ and $\tilde{p}_{k+1}(t)$, respectively for $t < x$ and $t > x$. Since however the position of x is unknown, we use the following strategy to estimate it: we set $q_k(t) = \tilde{p}_{k-1}(t)\chi_{t<y} + \tilde{p}_{k+1}(t)\chi_{t>y}$ and set y in such a way that the average of q_k agrees with the numerical data i.e. $a_k^j(q_k) = v_k^j$.

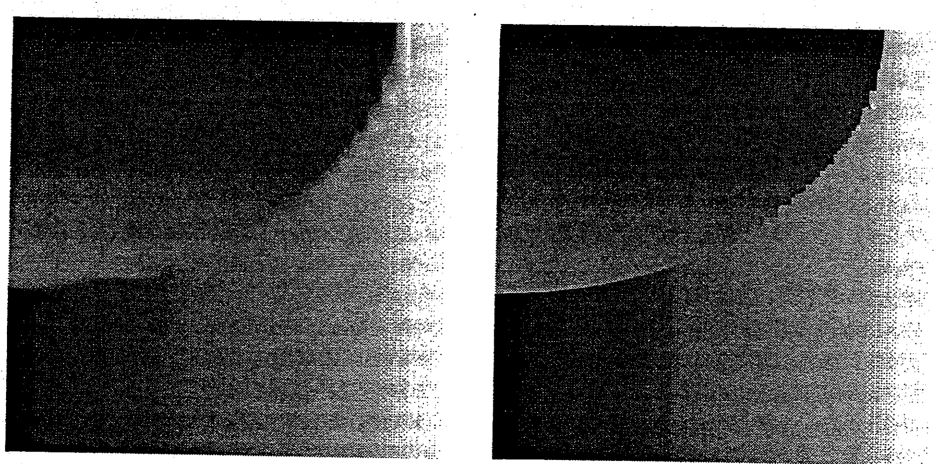
More details on these methods can be found in [1], [3] and [8]. The analysis of the subdivision process based on ENO and ENO-SR data dependent refinement rules is more complicated than for standard linear rules. In particular, Fourier analysis cannot be used. Nevertheless some basic principle remains, in particular the possibility of deriving a (data dependent) subdivision scheme for the finite differences, and analyzing convergence and smoothness of the limit functions through the contractivity properties of this auxiliary scheme, as detailed in [3].

It is no surprise that the above refinement rules have recently been applied to image compression. In this context, it is hoped that a better adapted



(a) (b)

Fig. 1. Piecewise smooth image 512 × 512 pixels (**a**) and coarse scale averages (**b**).



(a) (b)

Fig. 2. Reconstruction by linear subdivision (**a**) and by tensor product ENO-SR subdivision (**b**).

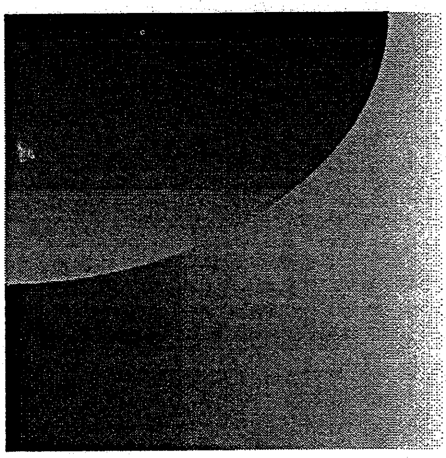


Fig. 3. Reconstruction by EA subdivision.

treatment of the singularities corresponding to edges improves the sparsity of the multiscale representations of images, and in turn the rate/distortion performance of compression algorithms based on such representations. He we use a particularly simple piecewise smooth image in order to illustrate the benefit of using nonlinear rules for the treatment of edge singularities.

We display in Figure 1 (a) the original 512×512 image. We start from its averages on 16×16 blocks, represented on Figure 1 (b), and apply to this coarse resolution image various subdivision schemes in order to refine it. Figure 2 (a) displays the result of the linear subdivision scheme based on a tensor product generalization of the quadratic polynomial reconstruction, i.e. the case $M = 2$. As expected, accuracy is lost near the edge singularities. Figure 2 (b) shows the results of the ENO-SR strategy applied in a tensor product fashion as introduced in [1], which leads to a good treatment only for vertical and horizontal edges. In order to obtain a good treatment of edges in any direction, it is necessary to give up tensor products and introduce an intrinsically bidimensional approach. As in the 1D case, the strategy is divided in two steps: based on the stencil selection one first detects the pixels $I_{j,k} \times I_{j,l}$ which might contain an edge, then the parameters (p, q, r) of the line edge $\{px + qy = r\}$ are estimated from the numerical data, as explained in [5] or [8]. We refer to this approach as *edge-adapted* (EA) refinement. As illustrated by Figure 3, it yields much better results for the approximation of piecewise smooth functions. Its application to real image compression is the object of active current research, see [8].

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