Contents

Preface .......................................................... V

Chapter 1. Affine Connections .................................... 1
§1. Connection on a Manifold ...................................... 1
§2. Covariant Differentiation and Parallel Translation Along a Curve ... 3
§3. Geodesics .................................................. 4
§4. Exponential Mapping and Normal Neighborhoods ............... 7
§5. Whitehead Theorem ......................................... 9
§7. Existence of Leray Coverings .................................. 13

Chapter 2. Covariant Differentiation. Curvature ................. 14
§1. Covariant Differentiation ..................................... 14
§2. The Case of Tensors of Type $(r, 1)$ .......................... 16
§3. Torsion Tensor and Symmetric Connections .................... 18
§5. Commutativity of Second Covariant Derivatives ............... 21
§6. Curvature Tensor of an Affine Connection .................... 22
§7. Space with Absolute Parallelism ............................. 24
§8. Bianci Identities ............................................ 24
§9. Trace of the Curvature Tensor ................................ 27
§10. Ricci Tensor ................................................ 27

Chapter 3. Affine Mappings. Submanifolds ..................... 29
§1. Affine Mappings ............................................ 29
§2. Affinities .................................................. 32
§3. Affine Coverings ............................................ 33
§4. Restriction of a Connection to a Submanifold .................. 35
§5. Induced Connection on a Normalized Submanifold ............. 37
§6. Gauss Formula and the Second Fundamental Form of a Normalized Submanifold ........................................... 38
§7. Totally Geodesic and Auto-Parallel Submanifolds ............. 40
§8. Normal Connection and the Weingarten Formula .............. 42
§9. Van der Waerden–Bortolotti Connection ...................... 42

Chapter 4. Structural Equations. Local Symmetries ............ 44
§1. Torsion and Curvature Forms ................................ 44
§2. Cartan Structural Equations in Polar Coordinates ............. 47
§3. Existence of Affine Local Mappings .......................... 50
§4. Locally Symmetric Affine Connection Spaces ................. 51
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.</td>
<td>Symmetric Spaces</td>
<td>55</td>
</tr>
<tr>
<td>§1.</td>
<td>Globally Symmetric Spaces</td>
<td>55</td>
</tr>
<tr>
<td>§2.</td>
<td>Germs of Smooth Mappings</td>
<td>55</td>
</tr>
<tr>
<td>§3.</td>
<td>Extensions of Affine Mappings</td>
<td>56</td>
</tr>
<tr>
<td>§4.</td>
<td>Uniqueness Theorem</td>
<td>58</td>
</tr>
<tr>
<td>§5.</td>
<td>Reduction of Locally Symmetric Spaces to Globally Symmetric Spaces</td>
<td>59</td>
</tr>
<tr>
<td>§6.</td>
<td>Properties of Symmetries in Globally Symmetric Spaces</td>
<td>60</td>
</tr>
<tr>
<td>§7.</td>
<td>Symmetric Spaces</td>
<td>61</td>
</tr>
<tr>
<td>§8.</td>
<td>Examples of Symmetric Spaces</td>
<td>62</td>
</tr>
<tr>
<td>§9.</td>
<td>Coincidence of Classes of Symmetric and Globally Symmetric Spaces</td>
<td>63</td>
</tr>
<tr>
<td>6.</td>
<td>Connections on Lie Groups</td>
<td>67</td>
</tr>
<tr>
<td>§1.</td>
<td>Invariant Construction of the Canonical Connection</td>
<td>67</td>
</tr>
<tr>
<td>§2.</td>
<td>Morphisms of Symmetric Spaces as Affine Mappings</td>
<td>69</td>
</tr>
<tr>
<td>§3.</td>
<td>Left-Invariant Connections on a Lie Group</td>
<td>70</td>
</tr>
<tr>
<td>§4.</td>
<td>Cartan Connections</td>
<td>71</td>
</tr>
<tr>
<td>§5.</td>
<td>Left Cartan Connection</td>
<td>73</td>
</tr>
<tr>
<td>§6.</td>
<td>Right-Invariant Vector Fields</td>
<td>74</td>
</tr>
<tr>
<td>§7.</td>
<td>Right Cartan Connection</td>
<td>76</td>
</tr>
<tr>
<td>7.</td>
<td>Lie Functor</td>
<td>77</td>
</tr>
<tr>
<td>§1.</td>
<td>Categories</td>
<td>77</td>
</tr>
<tr>
<td>§2.</td>
<td>Functors</td>
<td>78</td>
</tr>
<tr>
<td>§3.</td>
<td>Lie Functor</td>
<td>79</td>
</tr>
<tr>
<td>§4.</td>
<td>Kernel and Image of a Lie Group Homomorphism</td>
<td>80</td>
</tr>
<tr>
<td>§5.</td>
<td>Campbell–Hausdorff Theorem</td>
<td>82</td>
</tr>
<tr>
<td>§6.</td>
<td>Dynkin Polynomials</td>
<td>83</td>
</tr>
<tr>
<td>§7.</td>
<td>Local Lie Groups</td>
<td>84</td>
</tr>
<tr>
<td>§8.</td>
<td>Bijectivity of the Lie Functor</td>
<td>85</td>
</tr>
<tr>
<td>8.</td>
<td>Affine Fields and Related Topics</td>
<td>87</td>
</tr>
<tr>
<td>§1.</td>
<td>Affine Fields</td>
<td>87</td>
</tr>
<tr>
<td>§2.</td>
<td>Dimension of the Lie Algebra of Affine Fields</td>
<td>89</td>
</tr>
<tr>
<td>§3.</td>
<td>Completeness of Affine Fields</td>
<td>91</td>
</tr>
<tr>
<td>§4.</td>
<td>Mappings of Left and Right Translation on a Symmetric Space</td>
<td>94</td>
</tr>
<tr>
<td>§5.</td>
<td>Derivations on Manifolds with Multiplication</td>
<td>95</td>
</tr>
<tr>
<td>§6.</td>
<td>Lie Algebra of Derivations</td>
<td>96</td>
</tr>
<tr>
<td>§ 7. Involutive Automorphism of the Derivation Algebra of a Symmetric Space</td>
<td>97</td>
<td></td>
</tr>
<tr>
<td>§ 8. Symmetric Algebras and Lie Ternaries</td>
<td>98</td>
<td></td>
</tr>
<tr>
<td>§ 9. Lie Ternary of a Symmetric Space</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Chapter 9. Cartan Theorem ........................................ 101

| § 1. Functor $s$ | 101 |
| § 2. Comparison of the Functor $s$ with the Lie Functor $l$ | 103 |
| § 3. Properties of the Functor $s$ | 104 |
| § 4. Computation of the Lie Ternary of the Space $(\mathcal{G}/\mathcal{H})_\sigma$ | 105 |
| § 5. Fundamental Group of the Quotient Space | 107 |
| § 6. Symmetric Space with a Given Lie Ternary | 109 |
| § 7. Coverings | 109 |
| § 8. Cartan Theorem | 110 |
| § 9. Identification of Homogeneous Spaces with Quotient Spaces | 111 |
| § 10. Translations of a Symmetric Space | 112 |
| § 11. Proof of the Cartan Theorem | 112 |

Chapter 10. Palais and Kobayashi Theorems .................... 114

| § 1. Infinite-Dimensional Manifolds and Lie Groups | 114 |
| § 2. Vector Fields Induced by a Lie Group Action | 115 |
| § 3. Palais Theorem | 117 |
| § 4. Kobayashi Theorem | 124 |
| § 5. Affine Automorphism Group | 125 |
| § 6. Automorphism Group of a Symmetric Space | 125 |
| § 7. Translation Group of a Symmetric Space | 126 |

Chapter 11. Lagrangians in Riemannian Spaces ................. 127

| § 1. Riemannian and Pseudo-Riemannian Spaces | 127 |
| § 2. Riemannian Connections | 129 |
| § 3. Geodesics in a Riemannian Space | 133 |
| § 4. Simplest Problem of the Calculus of Variations | 134 |
| § 5. Euler–Lagrange Equations | 135 |
| § 6. Minimum Curves and Extremals | 137 |
| § 7. Regular Lagrangians | 139 |
| § 8. Extremals of the Energy Lagrangian | 139 |

Chapter 12. Metric Properties of Geodesics .................... 141

| § 1. Length of a Curve in a Riemannian Space | 141 |
| § 2. Natural Parameter | 142 |
| § 3. Riemannian Distance and Shortest Arcs | 142 |
| § 4. Extremals of the Length Lagrangian | 143 |
| § 5. Riemannian Coordinates | 144 |
Contents

§6. Gauss Lemma ................................................ 145
§7. Geodesics are Locally Shortest Arcs ...................... 148
§8. Smoothness of Shortest Arcs .............................. 149
§9. Local Existence of Shortest Arcs ......................... 150
§10. Intrinsic Metric ............................................. 151
§11. Hopf–Rinow Theorem ........................................ 153

Chapter 13. Harmonic Functionals and Related Topics ....... 159

§1. Riemannian Volume Element ................................. 159
§2. Discriminant Tensor ......................................... 159
§3. Foss–Weyl Formula .......................................... 160
§4. Case $n = 2$ .................................................... 162
§5. Laplace Operator on a Riemannian Space ................ 164
§6. The Green Formulas .......................................... 165
§7. Existence of Harmonic Functions with a Nonzero Differential .................................................... 166
§8. Conjugate Harmonic Functions ............................. 170
§9. Isothermal Coordinates ...................................... 172
§10. Semi-Cartesian Coordinates ............................... 173
§11. Cartesian Coordinates ....................................... 175

Chapter 14. Minimal Surfaces ..................................... 176

§1. Conformal Coordinates ....................................... 176
§2. Conformal Structures ....................................... 177
§3. Minimal Surfaces ............................................. 178
§4. Explanation of Their Name ................................... 181
§5. Plateau Problem ............................................... 181
§6. Free Relativistic Strings ..................................... 182
§7. Simplest Problem of the Calculus of Variations for Functions of Two Variables .................................................... 184
§8. Extremals of the Area Functional .......................... 186
§9. Case $n = 3$ .................................................... 188
§10. Representation of Minimal Surfaces Via Holomorphic Functions .................................................... 189
§11. Weierstrass Formulas ........................................ 190
§12. Adjoined Minimal Surfaces ............................... 191

Chapter 15. Curvature in Riemannian Space ..................... 193

§1. Riemannian Curvature Tensor ............................... 193
§2. Symmetries of the Riemannian Tensor .................. 193
§3. Riemannian Tensor as a Functional ........................ 198
§4. Walker Identity and Its Consequences ................... 199
§5. Recurrent Spaces ............................................. 200
§6. Virtual Curvature Tensors ................................... 201
§7. Reconstruction of the Bianci Tensor from Its Values on Bivectors .................................................... 202
Contents XI

§8. Sectional Curvatures ........................................ 204
§9. Formula for the Sectional Curvature ......................... 205

Chapter 16. Gaussian Curvature ................................. 207
§1. Bianchi Tensors as Operators ................................. 207
§2. Splitting of Trace-Free Tensors ............................... 208
§3. Gaussian Curvature and the Scalar Curvature ............... 209
§4. Curvature Tensor for $n = 2$ ................................. 210
§5. Geometric Interpretation of the Sectional Curvature ....... 210
§6. Total Curvature of a Domain on a Surface ................. 212
§7. Rotation of a Vector Field on a Curve ..................... 214
§8. Rotation of the Field of Tangent Vectors .................. 215
§9. Gauss–Bonnet Formula ...................................... 218
§10. Triangulated Surfaces ...................................... 220
§11. Gauss–Bonnet Theorem .................................... 221

Chapter 17. Some Special Tensors ............................... 223
§1. Characteristic Numbers ....................................... 223
§2. Euler Characteristic Number .................................. 223
§3. Hodge Operator ............................................... 225
§4. Euler Number of a 4$m$-Dimensional Manifold ............... 226
§5. Euler Characteristic of a Manifold of an Arbitrary Dimension ... 228
§6. Signature Theorem ........................................... 229
§7. Ricci Tensor of a Riemannian Space ......................... 230
§8. Ricci Tensor of a Bianchi Tensor ............................. 231
§9. Einstein and Weyl Tensors ................................... 232
§10. Case $n = 3$ ................................................... 234
§11. Einstein Spaces ............................................. 234
§12. Thomas Criterion ............................................ 236

Chapter 18. Surfaces with Conformal Structure ................. 238
§1. Conformal Transformations of a Metric ....................... 238
§2. Conformal Curvature Tensor ................................ 240
§3. Conformal Equivalences ...................................... 241
§4. Conformally Flat Spaces ..................................... 242
§5. Conformally Equivalent Surfaces ............................. 243
§6. Classification of Surfaces with a Conformal Structure ..... 243
  6.1. Surfaces of Parabolic Type ................................ 244
  6.2. Surfaces of Elliptic Type .................................. 245
  6.3. Surfaces of Hyperbolic Type ............................... 246
Chapter 19. Mappings and Submanifolds I ................................................. 248
§1. Locally Isometric Mapping of Riemannian Spaces .................................... 248
§2. Metric Coverings ......................................................................................... 249
§3. Theorem on Expanding Mappings ................................................................ 250
§4. Isometric Mappings of Riemannian Spaces ................................................. 251
§5. Isometry Group of a Riemannian Space ...................................................... 252
§6. Elliptic Geometry ......................................................................................... 252
§7. Proof of Proposition 18.1 .............................................................................. 253
§8. Dimension of the Isometry Group ................................................................. 253
§9. Killing Fields .................................................................................................. 254
§10. Riemannian Connection on a Submanifold of a Riemannian Space .......... 255
§11. Gauss and Weingarten Formulas for Submanifolds of Riemannian Spaces ........................................................ 257
§12. Normal of the Mean Curvature .................................................................... 258
§13. Gauss, Peterson–Codazzi, and Ricci Relations ............................................ 259
§14. Case of a Flat Ambient Space ....................................................................... 260

Chapter 20. Submanifolds II .............................................................................. 262
§1. Locally Symmetric Submanifolds ................................................................. 262
§2. Compact Submanifolds .................................................................................. 267
§3. Chern–Kuiper Theorem ............................................................................... 268
§4. First and Second Quadratic Forms of a Hypersurface .................................. 269
§5. Hypersurfaces Whose Points are All Umbilical ........................................... 271
§6. Principal Curvatures of a Hypersurface ....................................................... 272
§7. Scalar Curvature of a Hypersurface ............................................................. 273
§8. Hypersurfaces That are Einstein Spaces ....................................................... 274
§9. Rigidity of the Sphere .................................................................................... 275

Chapter 21. Fundamental Forms of a Hypersurface ........................................... 276
§1. Sufficient Condition for Rigidity of Hypersurfaces ....................................... 276
§2. Hypersurfaces with a Given Second Fundamental Form ............................ 277
§3. Hypersurfaces with Given First and Second Fundamental Forms ............. 278
§4. Proof of the Uniqueness ............................................................................... 280
§5. Proof of the Existence .................................................................................. 281
§6. Proof of a Local Variant of the Existence and Uniqueness Theorem ............ 282

Chapter 22. Spaces of Constant Curvature ....................................................... 288
§1. Spaces of Constant Curvature ....................................................................... 288
§2. Model Spaces of Constant Curvature .......................................................... 290
§3. Model Spaces as Hypersurfaces .................................................................... 292
§4. Isometries of Model Spaces .......................................................................... 294
§5. Fixed Points of Isometries ............................................................................ 296
Chapter 27. Jacobi Theory ........................................... 344

§1. Conjugate Points .................................................. 344
§2. Second Variation of Length ..................................... 345
§3. Formula for the Second Variation .............................. 346
§4. Reduction of the Problem ........................................ 348
§5. Minimal Fields and Jacobi Fields ............................... 349
§6. Jacobi Variation .................................................... 351
§7. Jacobi Fields and Conjugate Points ............................ 353
§8. Properties of Jacobi Fields ....................................... 353
§9. Minimality of Normal Jacobi Fields ............................ 355
§10. Proof of the Jacobi Theorem .................................. 358

Chapter 28. Some Additional Theorems I ......................... 360

§1. Cut Points .......................................................... 360
§2. Lemma on Continuity ............................................. 361
§3. Cut Loci and Maximal Normal Neighborhoods ............... 362
§4. Proof of Lemma 28.1 .............................................. 364
§5. Spaces of Strictly Positive Ricci Curvature ................... 367
§6. Mayers Theorem ................................................... 368
§7. Spaces of Strictly Positive Sectional Curvature ............... 369
§8. Spaces of Nonpositive Sectional Curvature .................... 370

Chapter 29. Some Additional Theorems II ......................... 371

§1. Cartan–Hadamard Theorem ...................................... 371
§2. Consequence of the Cartan–Hadamard Theorem ............... 374
§3. Cartan–Killing Theorem for $K = 0$ ............................ 375
§4. Bochner Theorem .................................................. 375
§5. Operators $A_X$ .................................................... 376
§6. Infinitesimal Variant of the Bochner Theorem ................. 378
§7. Isometry Group of a Compact Space ............................ 378

Addendum .................................................................... 381

Chapter 30. Smooth Manifolds ...................................... 381

§1. Introductory Remarks ............................................. 381
§2. Open Sets in the Space $\mathbb{R}^n$ and Their Diffeomorphisms .... 381
§3. Charts and Atlases .................................................. 383
§4. Maximal Atlases .................................................... 385
§5. Smooth Manifolds .................................................. 386
§6. Smooth Manifold Topology ..................................... 386
§7. Smooth Structures on a Topological Space .................... 390
§8. DIFF Category ..................................................... 391
§9. Transport of Smooth Structures ................................ 392
Chapter 31. Tangent Vectors ........................................... 394

§1. Vectors Tangent to a Smooth Manifold ......................... 394
§2. Oriented Manifolds .................................................. 396
§3. Differential of a Smooth Mapping ................................. 397
§4. Chain Rule .......................................................... 398
§5. Gradient of a Smooth Function ..................................... 399
§6. Étale Mapping Theorem .............................................. 400
§7. Theorem on a Local Coordinate Change ......................... 400
§8. Locally Flat Mappings ............................................... 401
§9. Immersions and Submersions ....................................... 402

Chapter 32. Submanifolds of a Smooth Manifold .................. 404

§1. Submanifolds of a Smooth Manifold .............................. 404
§2. Subspace Tangent to a Submanifold ............................... 405
§3. Local Representation of a Submanifold ......................... 405
§4. Uniqueness of a Submanifold Structure ......................... 407
§5. Case of Embedded Submanifolds .................................. 407
§6. Tangent Space of a Direct Product ............................... 408
§7. Theorem on the Inverse Image of a Regular Value ............. 409
§8. Solution of Sets of Equations ..................................... 410
§9. Embedding Theorem ............................................... 411

Chapter 33. Vector and Tensor Fields. Differential Forms ......... 413

§1. Tensor Fields ....................................................... 413
§2. Vector Fields and Derivations .................................... 416
§3. Lie Algebra of Vector Fields ...................................... 419
§4. Integral Curves of Vector Fields ................................ 421
§5. Vector Fields and Flows .......................................... 422
§6. Transport of Vector Fields via Diffeomorphisms ............... 423
§7. Lie Derivative of a Tensor Field ................................ 425
§8. Linear Differential Forms .......................................... 426
§9. Differential Forms of an Arbitrary Degree ...................... 428
§10. Differential Forms as Functionals on Vector Fields ......... 429
§11. Inner Product of Vector Fields and Differential Forms ...... 430
§12. Transport of a Differential Form via a Smooth Mapping .... 431
§13. Exterior Differential ............................................. 433

Chapter 34. Vector Bundles .............................................. 436

§1. Bundles and Their Morphisms .................................... 436
§2. Vector Bundles ..................................................... 438
§3. Sections of Vector Bundles ....................................... 439
§4. Morphisms of Vector Bundles .................................... 440
§5. Trivial Vector Bundles ............................................ 442
Geometry VI
Riemannian Geometry
Postnikov, M.M.
2001, XVIII, 504 p., Hardcover
ISBN: 978-3-540-41108-6