Preface

Orthogonal Polynomials and Special Functions are a very old branch of mathematics having a very rich history. Many famous mathematicians have contributed to the subject in the past: Euler's work on the gamma function, Gauss's and Riemann's work on the hypergeometric functions and the hypergeometric differential equation, Abel's and Jacobi's work on elliptic functions, and so on. Usually the special functions have been introduced to solve a specific problem, and many of them occurred in solving the differential equations describing a physical problem, e.g., the astronomer Bessel introduced the functions named after him in his work on Kepler's problem of three bodies moving under mutual gravitation.

So the subject of *Orthogonal Polynomials and Special Functions* is very classical, but there have been very interesting developments through the centuries, and there have been numerous applications to various branches of mathematics, such as combinatorics, representation theory, number theory, and applications to physics and astronomy, such as the aforementioned classical physical problems, but also integrable systems, optics, quantum chemistry, etcetera. So *Orthogonal Polynomials and Special Functions* is well-established, and very much driven by applications. At first, the advent of the computer has been thought to be fatal to the subject, but it turned out that it has been a great stimulus for *Orthogonal Polynomials and Special Functions*. First of all because more detailed computations can be done, which require special numerical algorithms, but mainly because it has led to automatic summation routines, notably the WZ-method (see Koepf's contribution). So *Orthogonal Polynomials and Special Functions* is a very lively branch of mathematics, and there are many research activities in this field.

Since more advanced courses on *Orthogonal Polynomials and Special Functions* are usually not included in the curriculum, we have felt the need for such courses for young researchers (graduate students and post-docs). A series of European summer schools in *Orthogonal Polynomials and Special Functions* was started with a summer school in Laredo, Spain (2000) and in Inzell, Germany (2001). This book contains the lecture notes for the lectures of the summer school in Belgium in 2002, which has taken place from August 12–16, 2002, at the Katholieke Universiteit Leuven, Belgium. In 2003 a summer school in *Orthogonal Polynomials and Special Functions* will be held in Coimbra, Portugal.

As is clear from the previous paragraphs, there are many different aspects of *Orthogonal Polynomials and Special Functions* and the Leuven summer school has focussed on the following subjects: computer algebra, representation theory and harmonic analysis, combinatorics, and asymptotics. The relation between *computer algebra* and special functions has been revolutionised by the introduction of very clever algorithms that allow to decide, e.g., for summability of hypergeometric series (Gosper's algorithm, Zeilberger's algorithm, WZ-method, etc.). This makes computer algebra a very important tool in research involving special functions but also a valuable source of research within computer science. The relation between *representation theory* of groups, *harmonic analysis* and special functions is approximately fifty years old, and hence relatively young. The interaction has turned out to be very fruitful on both sides, and it is still developing rapidly also because of its applications in physics, e.g., Racah-Wigner theory of angular momentum, integrable systems (Calogero-Moser-Sutherland), quantum groups and basic hypergeometric series, and Knizhnik-Zamolodchikov equations. One relation between *combinatorics* and special functions is via enumeration, and typical results are the famous Rogers-Ramanujan identities and other identities for partitions of integers. In this field there are many open problems that can be formulated in an elementary way. *Asymptotics*, and related error estimates, are very important in order to describe phenomena for large time or for a large number of degrees of freedom. The classical asymptotic expansions for special functions have recently greatly been improved by allowing exponentially small terms, leading to exponetial asymptotics and hyperasymptotics. Sometimes one obtains asymptotics from an integral representation, or from a differential equation. Another recent development is that boundary value problems can be used, and a Riemann-Hilbert approach combined with a steepest descent method then allows to find uniform asymptotics. There have been six series of lectures each of six hours. The six contributions in this book come from the lecturers at the 2002 Leuven summer school in *Orthogonal Polynomials and Special Functions.*

Wolfram Koepf discusses the interaction between computer algebra and special functions. The automatic summation algorithms of Gosper, Wilf-Zeilberger and Petkovšek are discussed. Also algorithms for definite and indefinite integration, obtaining generating functions, obtaining hypergeometric expressions, solving recurrence relations and differential, difference and q-difference equations are discussed. This subject is easiest understood during hands-on sessions, as has been the case during the Leuven summer school, and to make this possible for the reader of this book the Maple worksheets, including references to other lectures, are available, see Koepf's contribution for the web address. Quite a few of the identities appearing in the other lectures can be obtained using the software described in Koepfs's contribution.

Joris Van der Jeugt discusses the link between Clebsch-Gordan and Racah coefficients for 2 and 3-fold tensor products of simple Lie algebras ($\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$) and orthogonal polynomials of hypergeometric type, in particular the Hahn and Racah polynomials. He extends this to multivariable orthogonal polynomials by going to n-fold tensor products related to combinatorics on rooted trees. This is closely related to the Racah-Wigner algebra in the theory of angular momentum.

Margit Rösler discusses the Dunkl transform, which is a non-trivial multidimensional generalisation of the one-dimensional Fourier and Hankel transforms. The definition of the Dunkl kernel is in terms of finite reflection groups, and for particular choices of the parameters this transform has a group theoretical interpretation as spherical Fourier transform. Many familiar features of the Fourier transform, such as inversion formula, L^2-theory, generalised Hermite functions as eigenfunctions, positivity of the kernel, asymptotic behaviour of the kernel, have analogues for the Dunkl transform, and are discussed. She describes the analogues of the Laplacian, heat equation, heat semi-group, and the link to Calogero-Moser-Sutherland models (n-particle systems).

Dennis Stanton discusses the interaction between combinatorics, enumeration, additive number theory, and special functions. He uses the q-binomial theorem to derive Ramanujan's congruences for the partition function and other important identities such as the Jacobi triple product identity and the Rogers-Ramanujan identities. Unimodality of the q-binomial coefficient is proved using representation theory as described in the contribution of Joris Van der Jeugt and the Macdonald identity of type B_2 is proved using the Weyl group, which is described in Margit Rösler's contribution. A combinatorial interpretation of the three-term recurrence relation is given for some sets of orthogonal polynomials using Motzkin paths, which allows a combinatorial interpretation of moments, Hankel determinants, and continued fractions. Some open problems, like the Borwein conjecture which is related to the representation theory of the Virasoro algebra, are presented as well.

Arno Kuijlaars discusses asymptotics of orthogonal polynomials using the so-called Riemann-Hilbert method. This method characterises orthogonal polynomials and their Cauchy transforms in terms of matrix valued analytic functions having a jump over a system of contours, typically the real line or an interval. This is a very strong method that has arisen from recent work of Fokas, Its and Kitaev on isomonodromy problems in $2D$ quantum gravity. Deformation of contours combined with a steepest descent method for oscillatory Riemann-Hilbert problems, which was developed by Deift and Zhou for the analysis of the MKdV equation, gives a very powerful tool for obtaining asymptotics for orthogonal polynomials.

Adri Olde Daalhuis discusses asymptotics of functions defined by integrals or as solutions of differential equations. He shows how re-expansions of divergent asymptotic series can be used to obtain exponentially improved asymptotics, both locally and globally. He also discusses the notion of Stokes multipliers and Stokes lines, and he shows how the Stokes lines can be determined from the saddle point method, and how to compute the Stokes multipliers with great accuracy from asymptotic expansions. This method is worked out in detail in several examples.

The lecture notes are aimed at graduate students and post-docs, or anyone who wants to have an introduction to (and learn about) the subjects mentioned. Each of the contributions is self-contained, and contains up to date references to the literature so that anyone who wants to apply the results to his own advantage has a good starting point. The knowledge required for the lectures is (real and complex) analysis, some basic notions of algebra and discrete mathematics, and some elementary facts of orthogonal polynomials. A computer equipped with Maple software is useful for the lecture related to computer algebra. Exercises are supplied in each of the contributions, and some open problems are discussed in most of them. An extensive index of keywords at the end will be useful for locating the topics of interest. So having mastered the lecture notes gives a good level to read research papers in this field, and to start doing research as well. This has been one of the main scientific goals of the summer school, another main goal being to enhance the interaction between young researchers from various European countries.

The summer school in Orthogonal Polynomials and Special Functions in Leuven has been attended by 60 participants, most of whom are at the beginning of their research. The following institutions have supported the Leuven summer school financially or otherwise: Fonds voor Wetenschappelijk Onderzoek – Vlaanderen (Belgium), the Netherlands Organisation for Scientific Research (NWO), FWO Research Network WO.011.96N "Fundamentele Methoden en Technieken in de Wiskunde" (Belgium), Thomas Stieltjes Institute for Mathematics (the Netherlands), Stichting Computer Algebra Nederland (the Netherlands), Katholieke Universiteit Leuven (Belgium), SIAM Activity Group on Orthogonal Polynomials and Special Functions. The summer school is also part of the Socrates/Erasmus Intensive Programme *Orthogonal Polynomials and Special Functions* of the European Union (29242-IC-1-2001-PT-ERASMUS-IP-13). We thank all these organisations for their support. We thank the lecturers for giving excellent lectures and for preparing the contributions in this volume. We also thank Eric Opdam for his generous support.

Delft and Leuven, *Erik Koelink*
October 2002 *Walter Van Assche*

Springer

http://www.springer.com/978-3-540-40375-3

Orthogonal Polynomials and Special Functions
Leuven 2002
Koelink, E.; Van Assche, W. (Eds.)
2003, X, 250 p., Softcover
ISBN: 978-3-540-40375-3