1

Cobordism and oriented cohomology

In this chapter, we introduce the axiomatic framework of oriented cohomology theories, and state our main results.

1.1 Oriented cohomology theories

Fix a base scheme S. For $z \in Z \in \mathbf{Sm}_S$ we denote by $\dim_S(Z, z)$ the dimension over S of the connected component of Z containing z

Let $d \in \mathbb{Z}$ be an integer. A morphism $f : Y \to X$ in $\mathbf{Sm}_S$ has *relative dimension* d if, for each $y \in Y$, we have $\dim_S(Y, y) - \dim_S(X, f(y)) = d$. We shall also say in that case that f has relative codimension $-d$.

For a fixed base-scheme S, $\mathcal{V}$ will usually denote a full subcategory of $\mathbf{Sch}_S$ satisfying the following conditions

1. S and the empty scheme are in $\mathcal{V}$.
2. If $Y \to X$ is a smooth quasi-projective morphism in $\mathbf{Sch}_S$ with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
3. If X and Y are in $\mathcal{V}$, then so is the product $X \times_S Y$.
4. If X and Y are in $\mathcal{V}$, so is $X \coprod Y$.

(1.1)

In particular, $\mathcal{V}$ contains $\mathbf{Sm}_S$. We call such a subcategory of $\mathbf{Sch}_S$ *admissible.*

Definition 1.1.1. *Let $f : X \to Z$, $g : Y \to Z$ be morphisms in an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$. We say that f and g are* transverse in $\mathcal{V}$ *if*

1. $\mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ *for all* $q > 0$.
2. *The fiber product* $X \times_Z Y$ *is in* $\mathcal{V}$.

If $\mathcal{V} = \mathbf{Sm}_S$ we just say f and g are transverse; if $\mathcal{V} = \mathbf{Sch}_S$, we sometimes say instead that f and g are Tor-independent.

We let $\mathbf{R}^*$ denote the category of *commutative, graded rings with unit*. Observe that a commutative graded ring is not necessarily *graded commutative*. We say that a functor $A^* : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{R}^*$ is *additive* if $A^*(\emptyset) = 0$ and for any pair $(X, Y) \in \mathbf{Sm}_S^2$ the canonical ring map $A^*(X \amalg Y) \to A^*(X) \times A^*(Y)$ is an isomorphism.

The following notion is directly taken from Quillen's paper [30]:

Definition 1.1.2. *Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_S$. An* oriented cohomology theory *on $\mathcal{V}$ is given by*

(D1). An additive functor $A^* : \mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$.
(D2). For each projective morphism $f : Y \to X$ in $\mathcal{V}$ of relative codimension d, a homomorphism of graded $A^*(X)$-modules:

$$f_* : A^*(Y) \to A^{*+d}(X)$$

Observe that the ring homomorphism $f^* : A^*(X) \to A^*(Y)$ gives $A^*(Y)$ the structure of an $A^*(X)$-module.

These satisfy

(A1). One has $(\mathrm{Id}_X)_* = \mathrm{Id}_{A^*(X)}$ for any $X \in \mathcal{V}$. Moreover, given projective morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$, with f of relative codimension d and g of relative codimension e, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X).$$

(A2). Let $f : X \to Z$, $g : Y \to Z$ be transverse morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Suppose that f is projective of relative dimension d (thus so is f'). Then $g^* f_* = f'_* g'^*$.
(PB). Let $E \to X$ be a rank n vector bundle over some X in $\mathcal{V}$, $O(1) \to \mathbb{P}(E)$ the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\xi \in A^1(\mathbb{P}(E))$ by

$$\xi := s^*(s_*(1)).$$

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module, with basis

$$(1, \xi, \ldots, \xi^{n-1}).$$

(EH). Let $E \to X$ be a vector bundle over some X in $\mathcal{V}$, and let $p : V \to X$ be an E-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

A morphism of oriented cohomology theories on $\mathcal{V}$ is a natural transformation of functors $\mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$ which commutes with the maps f_*.

The morphisms of the form f^* are called *pull-backs* and the morphisms of the form f_* are called *push-forwards.* Axiom (PB) will be referred to as the *projective bundle formula* and axiom (EH) as the *extended homotopy property.*

We now specialize to $S = \operatorname{Spec} k$, $\mathcal{V} = \mathbf{Sm}_k$, k a field. Given an oriented cohomology theory A^*, one may use Grothendieck's method [11] to define Chern classes $c_i(E) \in A^i(X)$ of a vector bundle $E \to X$ of rank n over X as follows: Using the notations of the previous definition, axiom (PB) implies that there exists unique elements $c_i(E) \in A^i(X)$, $i \in \{0, \dots, n\}$, such that $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i \, c_i(E) \, \xi^{n-i} = 0.$$

One can check all the standard properties of Chern classes as in [11] using the axioms listed above (see §4.1.7 for details). Moreover, these Chern classes are characterized by the following properties:

1) For any line bundle L over $X \in \mathbf{Sm}_k$, $c_1(L)$ equals $s^* s_*(1) \in A^1(X)$, where $s : X \to L$ denotes the zero section.
2) For any morphism $Y \to X \in \mathbf{Sm}_k$, and any vector bundle E over X, one has for each $i \geq 0$
$$c_i(f^* E) = f^*(c_i(E)).$$
3) Whitney product formula: if
$$0 \to E' \to E \to E'' \to 0$$
is an exact sequence of vector bundles, then one has for each integer $n \geq 0$:
$$c_n(E) = \sum_{i=0}^{n} c_i(E') \, c_{n-i}(E'').$$

Sometime, to avoid confusion, we will write $c_i^A(E)$ for the Chern classes of E computed in the oriented cohomology theory A^*.

The fundamental insight of Quillen in [30], and the main difference with Grothendieck's axioms in [11], is that it is not true in general that one has the formula

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

for line bundles L and M over the same base. In other words the map

$$\begin{aligned} c_1 : \operatorname{Pic}(X) &\to A^1(X) \\ L &\mapsto c_1(L), \end{aligned}$$

is not assumed to be a group homomorphism, but is only a natural transformation of *pointed sets.* In fact, a classical remark due to Quillen [30, Proposition 2.7] describes the way c_1 is not additive as follows (see proposition 5.2.4 for a proof of this lemma):

Lemma 1.1.3. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Then for any line bundle L on $X \in \mathbf{Sm}_k$ the class $c_1(L)^n$ vanishes for n large enough*[1]*. Moreover, there is a unique power series*

$$F_A(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A^*(k)[[u,v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$, such that, for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L, M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

In addition, the pair $(A^*(k), F_A)$ is a commutative formal group law of rank one.

Recall from [16] that a commutative formal group law of rank one with coefficients in A is a pair (A, F) consisting of a commutative ring A and a formal power series

$$F(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A[[u,v]]$$

such that the following holds:

1. $F(u,0) = F(0,u) = u \in A[[u]]$.
2. $F(u,v) = F(v,u) \in A[[u,v]]$.
3. $F(u, F(v,w)) = F(F(u,v),w) \in A[[u,v,w]]$.

These properties of F_A reflect the fact that, for line bundles L, M, N on $X \in \mathbf{Sm}_k$, one has:

1′. $L \otimes O_X = O_X \otimes L = L \in \mathrm{Pic}(X)$.
2′. $L \otimes M = M \otimes L \in \mathrm{Pic}(X)$.
3′. $L \otimes (M \otimes N) = (L \otimes M) \otimes N \in \mathrm{Pic}(X)$.

Lazard pointed out in [16] that there exists a universal commutative formal group law of rank one $(\mathbb{L}, F_{\mathbb{L}})$ and proved that the ring $\mathbb{L}$ (now called the *Lazard ring*) is a polynomial ring with integer coefficients on a countable set of variables x_i, $i \geq 1$. The construction of $(\mathbb{L}, F_{\mathbb{L}})$ is rather easy. Set $\tilde{\mathbb{L}} := \mathbb{Z}[\{A_{i,j} |\ (i,j) \in \mathbb{N}^2\}]$, and $\tilde{F}(u,v) = \sum_{i,j} A_{i,j}\, u^i\, v^j \in \tilde{\mathbb{L}}[[u,v]]$. Then define $\mathbb{L}$ to be the quotient ring of $\tilde{\mathbb{L}}$ by the relations obtained by imposing the relations (1), (2) and (3) above to $\tilde{F}$, and let

$$F_{\mathbb{L}} = \sum_{i,j} a_{i,j}\, u^i\, v^j \in \mathbb{L}[[u,v]]$$

denote the image of F by the homomorphism $\tilde{\mathbb{L}} \to \mathbb{L}$. It is clear that the pair $(\mathbb{L}, F_{\mathbb{L}})$ is the universal commutative formal group law of rank one, which

[1] In fact we will prove later on that $n > \dim_k(X)$ suffices; this follows from theorem 2.3.13 and proposition 5.2.4.

means that to define a commutative formal group law of rank one (F, A) on A is equivalent to giving a ring homomorphism $\Phi_F : \mathbb{L} \to A$.

The Lazard ring can be graded by assigning the degree $i + j - 1$ to the coefficient $a_{i,j}$. We denote by $\mathbb{L}_*$ this commutative graded ring. We could as well have graded it by assigning the degree $1 - i - j$ to the coefficient $a_{i,j}$, in which case we denote by $\mathbb{L}^*$ the corresponding commutative graded ring. For instance $\mathbb{L}^0 = \mathbb{L}_0 = \mathbb{Z}$ and $\mathbb{L}^{-n} = \mathbb{L}_n = 0$ if $n < 0$.

One can then check that for any oriented cohomology theory A^* the homomorphism of rings induced by the formal group law given by lemma 1.1.3 is indeed a homomorphism of graded rings

$$\Phi_A : \mathbb{L}^* \to A^*(k)$$

Example 1.1.4. The Chow ring $X \mapsto \mathrm{CH}^*(X)$ is a basic example of an oriented cohomology theory on $\mathbf{Sm}_k$; this follows from [9]. In that case, the formal group law obtained on $\mathbb{Z} = \mathrm{CH}^*(k)$ by lemma 1.1.3 is the *additive* formal group law $F_a(u, v) = u + v$.

Example 1.1.5. Another fundamental example of oriented cohomology theory is given by the Grothendieck K^0 functor $X \mapsto K^0(X)$, where for X a smooth k-scheme, $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves on X. For $\mathcal{E}$ a locally free sheaf on X we denote by $[\mathcal{E}] \in K^0(X)$ its class. The tensor product of sheaves induces a unitary, commutative ring structure on $K^0(X)$. In fact we rather consider the graded ring $K^0(X)[\beta, \beta^{-1}] := K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$, where $\mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomial in a variable β of degree -1.

It is endowed with pull-backs for any morphism $f : Y \to X$ by the formula:

$$f^*([\mathcal{E}] \cdot \beta^n) := [f^*(\mathcal{E})] \cdot \beta^n$$

for $\mathcal{E}$ a locally free coherent sheaf on X and $n \in \mathbb{Z}$. We identify $K^0(X)$ with the Grothendieck group $G_0(X)$ of all coherent sheaves on X by taking a finite locally free resolution of a coherent sheaf (X is assumed to be regular). This allows one to define push-forwards for a projective morphism $f : Y \to X$ of pure codimension d by the formula

$$f_*([\mathcal{E}] \cdot \beta^n) := \Sigma_{i=0}^{\infty} (-1)^i [R^i f_*(\mathcal{E})] \cdot \beta^{n-d} \in K_0(X)[\beta, \beta^{-1}]$$

for $\mathcal{E}$ a locally free sheaf on Y and $n \in \mathbb{Z}$. One can easily check using standard results that this is an oriented cohomology theory.

Moreover, for a line bundle L over X with with projection $\pi : L \to X$, zero section $s : X \to L$ and sheaf of sections $\mathcal{L}$, one has

$$s^*(s_*(1_X)) = s^*([\mathcal{O}_{s(X)}]\beta^{-1}) = s^*(1 - [\pi^*(\mathcal{L})^\vee])\beta^{-1} = (1 - [\mathcal{L}^\vee])\beta^{-1}$$

so that $c_1^K(L) := (1 - [\mathcal{L}^\vee])\beta^{-1}$. We thus find that the associated power series F_K is the *multiplicative formal group law*

$$F_m(u,v) := u + v - \beta uv$$

as this follows easily from the relation

$$(1 - [(\mathcal{L} \otimes \mathcal{M})^\vee]) = (1 - [\mathcal{L}^\vee]) + (1 - [\mathcal{M}^\vee]) - (1 - [\mathcal{L}^\vee])(1 - [\mathcal{M}^\vee])$$

in $K^0(X)$, where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X.

1.2 Algebraic cobordism

Definition 1.2.1. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$ with associated formal group law F_A.*

1) We shall say that A^* is ordinary *if $F_A(u,v)$ is the additive formal group law.*
2) We shall say that A^* is multiplicative *if $F_A(u,v) = u + v - buv$ for some (uniquely determined) $b \in A^{-1}(k)$; we shall say moreover that A^* is* periodic *if b is a unit in $A^*(k)$.*

Our main results on oriented cohomology theories are the following three theorems. In each of these statements, A^* denoted a fixed oriented cohomology theory on $\mathbf{Sm}_k$:

Theorem 1.2.2. *Let k be a field of characteristic zero. If A^* is ordinary then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^{\mathrm{CH}} : \mathrm{CH}^* \to A^*.$$

Theorem 1.2.3. *Let k be a field. If A^* is multiplicative and periodic then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^K : K^0[\beta, \beta^{-1}] \to A^*.$$

Theorem 1.2.2 says that, in characteristic zero, the Chow ring functor is the universal ordinary oriented cohomology theory on $\mathbf{Sm}_k$. It seems reasonable to conjecture that this statement still holds over any field. Theorem 1.2.3 says that $K^0[\beta, \beta^{-1}]$ is the universal multiplicative and periodic oriented cohomology theory on $\mathbf{Sm}_k$.

Remark 1.2.4. The classical Grothendieck-Riemann-Roch theorem can be easily deduced from theorem 1.2.3, see remark 4.2.11.

Remark 1.2.5. Using theorem 1.2.3 and the fact that for any smooth k-scheme the Chern character induces an isomorphism

$$ch : K^0(X) \otimes \mathbb{Q} \cong \mathrm{CH}(X) \otimes \mathbb{Q}$$

(where CH denotes the ungraded Chow ring), it is possible to prove $\mathbb{Q}$-versions of theorem 1.2.2 and theorem 1.2.6 below over *any* field.

Well-known examples of ordinary cohomology theories are given by the "classical" ones: the even part of étale ℓ-adic cohomology theory (with $\ell \neq \mathrm{char}(k)$ a prime number), the even de Rham cohomology theory over a field of characteristic zero, the even part of Betti cohomology associated to a complex embedding of the base field. In some sense theorem 1.2.2 and its rational analog over any field explains, a priori, the existence of the cycle map in all these classical cohomology theories.

The following result introduces our main object of study:

Theorem 1.2.6. *Assume k has characteristic zero. Then there exists a universal oriented cohomology theory on $\mathbf{Sm}_k$, denoted by*

$$X \mapsto \Omega^*(X),$$

which we call algebraic cobordism*. Thus, given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism*

$$\vartheta : \Omega^* \to A^*$$

of oriented cohomology theories.

In addition, we have two main results describing properties of the universal theory Ω^* which do not obviously follow from universality. The first may be viewed as an algebraic version of Quillen's identification of $MU^*(pt)$ with $\mathbb{L}$:

Theorem 1.2.7. *For any field k of characteristic zero, the canonical homomorphism classifying F_Ω*

$$\Phi_\Omega : \mathbb{L}^* \to \Omega^*(k)$$

is an isomorphism.

The second reflects the strongly algebraic nature of Ω^*:

Theorem 1.2.8. *Let $i : Z \to X$ be a closed immersion between smooth varieties over k, d the codimension of Z in X and $j : U \to X$ the open immersion of the complement of Z. Then the sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(U) \to 0$$

is exact.

The construction of Ω^* is directly inspired by Quillen's description of complex cobordism [30]: For $f : Y \to X$ a projective morphism of codimension d from a smooth k-scheme Y to X denote by $[f : Y \to X]_A \in A^d(X)$ the element $f_*(1_Y)$. For each X, $\Omega^d(X)$ is generated as a group by the isomorphism classes of projective morphisms $Y \to X$ of codimension d with Y smooth. The morphism ϑ necessarily maps $f : Y \to X$ to $[f : Y \to X]_A$, which proves uniqueness of ϑ. Observe that $\Omega^n(X) = 0$ for $n > \dim(X)$. When $X = \mathrm{Spec}\, k$ we simply denote by $[Y] \in \Omega^{-d}(k)$ and $[Y]_A \in A^{-d}(k)$ the class of the projective smooth variety $Y \to \mathrm{Spec}\, k$ of dimension d.

Remark 1.2.9. One should note that the relations defining Ω^* are not just the obvious "algebraization" of the complex cobordism relations. Indeed, one can consider projective morphisms of the form $f : Y \to X \times \mathbb{P}^1$ with Y smooth and f transverse to the inclusion $X \times \{0,1\} \to X \times \mathbb{P}^1$. Letting $f_0 : Y_0 \to X$, $f_1 : Y_1 \to X$ be the pull-backs of f via $X \times 0 \to X \times \mathbb{P}^1$ and $X \times 1 \to X \times \mathbb{P}^1$, respectively, we do have the relation

$$[f_0 : Y_0 \to X] = [f_1 : Y_1 \to X]$$

in $\Omega^*(X)$. However, imposing only relations of this form on the free abelian group of isomorphism classes of projective morphisms $f : Y \to X$ (with Y irreducible and smooth over k) does not give $\Omega^*(X)$, even for $X = \operatorname{Spec} k$, and even for algebraically closed k. To see this, consider $\Omega^{-1}(k)$, i.e., the part of $\Omega^*(k)$ generated by the classes of smooth projective curves C over k. Clearly, the genus and the number of (geometrically) connected components is invariant under the "naive" cobordisms given by maps $Y \to \mathbb{P}^1$, but we know that $\mathbb{L}^{-1} \cong \mathbb{Z}$, generated by the class of $\mathbb{P}^1$. Thus, if one uses only the naive notion of algebraic cobordism, it would not be possible to make a curve of genus $g > 0$ equivalent to $(1-g)\mathbb{P}^1$, as it should be.

Example 1.2.10. In [30], Quillen defines a notion of *complex oriented cohomology theory* on the category of differentiable manifolds and pointed out that complex cobordism theory $X \mapsto MU^*(X)$ can be interpreted as the universal such theory. Our definition 1.1.2 is so inspired by Quillen's axioms that given a complex imbedding $\sigma : k \to \mathbb{C}$, it is clear that the functor $X \mapsto MU^{2*}(X_\sigma(\mathbb{C}))$ admits a canonical structure of oriented cohomology theory ($X_\sigma(\mathbb{C})$ denoting the differentiable manifold of complex points of $X \times_k \mathbb{C}$). From the universality of algebraic cobordism we get for any $X \in \mathbf{Sm}_k$ a canonical morphism of graded rings

$$\Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C})).$$

Given a complex embedding $\sigma : k \to \mathbb{C}$ the previous considerations define a ring homomorphism

$$\Phi^{top} : \Omega^* \to MU^{2*}.$$

In very much the same way, given an extension of fields $k \subset K$ and a k-scheme X denote by X_K the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. For any oriented cohomology theory A^* on $\mathbf{Sm}_K$, the functor

$$(\mathbf{Sm}_k)^{op} \to R^*, X \mapsto A^*(X_K)$$

is an oriented cohomology theory on $\mathbf{Sm}_k$. In particular, we get natural morphisms $\Omega^*(X) \to \Omega^*(X_K)$, giving in the case $X = \operatorname{Spec} k$ a canonical ring homomorphism

$$\Omega^*(k) \to \Omega^*(K).$$

Theorem 1.2.7 easily implies:

Corollary 1.2.11. *Let k be a field of characteristic zero.*

(1) Given a complex embedding $\sigma : k \to \mathbb{C}$ the canonical homomorphism

$$\Phi^{top} : \Omega^*(k) \to MU^{2*}(pt)$$

is an isomorphism.

(2) Given a field extension $k \subset F$, the canonical homomorphism

$$\Omega^*(k) \to \Omega^*(F)$$

is an isomorphism

Remark 1.2.12. Suppose $\mathrm{char}(k) = 0$. Let X be a smooth irreducible quasi-projective k-scheme, with field of functions K. One then has a canonical homomorphism of rings $\Omega^*(X) \to \Omega^*(K)$ defined as the composition of the canonical morphism $\Omega^*(X) \to \Omega^*(X_K)$ (extension of scalars) with the restriction $\Omega^*(X_K) \to \Omega^*(K)$ to the tautological K-point of X_K. It corresponds to "taking the generic fiber" in the sense that given a projective morphism $f : Y \to X$ of relative codimension d and generic fiber $Y_K \to \mathrm{Spec}\, K$, a smooth projective K-scheme, its image by the previous homomorphism is the class $[Y_K] \in \Omega^d(K)$.

The composition $\Omega^*(k) \to \Omega^*(X) \to \Omega^*(K)$ is an isomorphism by corollary 1.2.11(2). We denote by

$$\deg : \Omega^*(X) \to \Omega^*(k)$$

the composition of $\Omega^*(X) \to \Omega^*(K)$ and the inverse isomorphism $\Omega^*(K) \to \Omega^*(k)$. Now, for a morphism $f : Y \to X$ of relative codimension 0, we have the *degree* of f, denoted $\deg(f)$, which is zero if f is not dominant and equal to the degree of the field extension $k(X) \to k(Y)$ if f is dominant. We observe that $\Omega^0(k)$ is canonically isomorphic to $\mathbb{Z}$ and that through this identification, $\deg([f : Y \to X]) = \deg(f)$ in case f has relative codimension zero.

From theorem 1.2.8 and corollary 1.2.11 we get the following result, which is a very close analogue of the fundamental results in [30] concerning complex cobordism.

Corollary 1.2.13. *Let k be a field of characteristic zero and let X be in $\mathbf{Sm}_k$. Then $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the classes of non-negative degrees (Recall that $\mathbb{L}^*$ is concentrated in degrees ≤ 0).*

Indeed corollary 1.2.11, with $F = k(X)$, implies that a given element $\eta \in \Omega^*(X)$ is "constant" over some open subscheme $j : U \to X$ of X:

$$j^*\eta = \deg(\eta) \cdot 1_U.$$

By theorem 1.2.8, the difference $\eta - \deg(\eta) \cdot 1_X$ comes from Ω^* of some proper closed subscheme Z (after removing the singular locus of Z), and noetherian induction completes the proof. In fact, since each reduced closed subscheme Z of X has a smooth birational model $\tilde{Z} \to Z$, we get the following more precise version, which we call the *generalized degree formula*:

Theorem 1.2.14. *Let k be a field of characteristic zero. Let X be in $\mathbf{Sm}_k$. For each closed integral subscheme $Z \subset X$ let $\tilde{Z} \to Z$ be a projective birational morphism with $\tilde{Z}$ smooth quasi-projective over k and let $[\tilde{Z} \to X] \in \Omega^*(X)$ denote the class of the projective morphism $\tilde{Z} \to X$. Then $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the classes $[\tilde{Z} \to X]$.*

In particular, for any irreducible $X \in \mathbf{Sm}_k$, $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the unit $1 \in \Omega^0(X)$ and by the elements $[\tilde{Z} \to X]$ with $\dim(\tilde{Z}) < \dim(X)$, that is to say of degrees > 0 in $\Omega^*(X)$. More precisely, for $\eta \in \Omega^*(X)$, there are integral proper closed subschemes Z_i of X, and elements $\alpha_i \in \Omega^*(k)$, $i = 1, \ldots, r$, such that

$$\eta = \deg(\eta) \cdot [\mathrm{Id}_X] + \sum_{i=1}^{r} \alpha_i \cdot [\tilde{Z}_i \to X]. \tag{1.2}$$

Given a smooth projective irreducible k-scheme X of dimension $d > 0$, Rost introduces (see [23]) the ideal $M(X) \subset \mathbb{L}^* = \Omega^*(\mathrm{Spec}\, k)$ generated by classes $[Y] \in \mathbb{L}^*$ of smooth projective k-schemes Y of dimension $< d$ for which there exists a morphism $Y \to X$ over k. The following result establishes Rost's degree formula as conjectured in [23]. It is an obvious corollary to theorem 1.2.14 and remark 1.2.12.

Theorem 1.2.15. *Let k be a field of characteristic zero. For any morphism $f : Y \to X$ between smooth projective irreducible k-schemes the class $[Y] - \deg(f)[X]$ of $\mathbb{L}^*$ lies in the ideal $M(X)$. In other words, one has the following equality in the quotient ring $\mathbb{L}^*/M(X)$:*

$$[Y] = \deg(f) \cdot [X] \in \mathbb{L}^*/M(X).$$

We shall also deduce the following

Theorem 1.2.16. *Let k be a field of characteristic zero. Let X be a smooth projective k-variety.*

1. *The ideal $M(X)$ is a birational invariant of X.*
2. *The class of X modulo $M(X)$:*

$$[X] \in \mathbb{L}^*/M(X)$$

is a birational invariant of X as well.

For instance, let $d \geq 1$ be an integer and let N_d be the d-th Newton polynomial,

$$N_d(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d].$$

Recall that if σ_i is the ith elementary symmetric function in variables $t_1, t_2, \ldots$, then

$$N_d(\sigma_1, \ldots, \sigma_d) = \Sigma_i t_i^d.$$

If X is smooth projective of dimension d, we set

$$S_d(X) := -\deg N_d(c_1, \ldots, c_d)(T_X) \in \mathbb{Z},$$

T_X denoting the tangent bundle of X and $\deg : \mathrm{CH}^d(X) \to \mathbb{Z}$ the usual degree homomorphism. One checks that if X and Y are smooth projective k-schemes of dimension d and d', one has $S_{d+d'}(X \times Y) = 0$ if both $d > 0$ and $d' > 0$. We also know (see [3]) that if d is of the form $p^n - 1$ where p is a prime number and $n > 0$, then $s_d(X) := \frac{S_d(X)}{p}$ is always an integer. In that case, using theorem 1.2.14 and observing that if $\dim(Z) < d$ then $s_d([W] \cdot [Z]) \neq 0$ implies $\dim(Z) = 0$ one obtains the following result:

Corollary 1.2.17. *Let $f : Y \to X$ be a morphism between smooth projective varieties of dimensions $d > 0$. Assume that $d = p^n - 1$ where p is a prime number and $n > 0$. Then there exists a 0-cycle on X with integral coefficients whose degree is the integer*

$$s_d(Y) - \deg(f) \cdot s_d(X).$$

This formula was first proven by Rost[2], and then generalized further by Borghesi [4].

Consider now the graded ring homomorphisms

$$\Phi_a : \mathbb{L}^* \to \mathbb{Z}$$

and

$$\Phi_m : \mathbb{L}^* \to \mathbb{Z}[\beta, \beta^{-1}]$$

classifying respectively the additive and multiplicative formal group laws.

Theorem 1.2.6 obviously implies that, over a field of characteristic zero, the ordinary oriented cohomology theory

$$X \mapsto \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$$

obtained by extension of scalars from Ω^* via Φ_a is the universal ordinary oriented cohomology theory. In the same way theorem 1.2.6 implies that, over a field of characteristic zero, the multiplicative oriented cohomology theory

$$X \mapsto \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}]$$

obtained by extending the scalars from Ω^* via Φ_m is the universal multiplicative periodic oriented cohomology theory. Over a field of characteristic zero, we get from theorem 1.2.6 canonical morphisms of oriented cohomology theories

$$\Omega^* \to \mathrm{CH}^*$$

and

$$\Omega^* \to K^0[\beta, \beta^{-1}].$$

We immediately deduce from theorems 1.2.3 and 1.2.6 the following result:

[2] V. Voevodsky had considered weaker forms before.

Theorem 1.2.18. *Over a field of characteristic zero, the canonical morphism*

$$\Omega^* \to K^0[\beta, \beta^{-1}]$$

induces an isomorphism

$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \cong K^0[\beta, \beta^{-1}].$$

Theorem 1.2.18 is the analogue of a well-known theorem of Conner and Floyd [5]. Theorems 1.2.2 and 1.2.6 similarly imply the analogous relation between Ω^* and CH^*:

Theorem 1.2.19. *Let k be a field of characteristic zero. Then the canonical morphism*

$$\Omega^* \to \mathrm{CH}^*$$

induces an isomorphism

$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}^*.$$

In fact, we prove theorem 1.2.19 before theorem 1.2.2, using theorem 1.2.3, theorem 1.2.6, theorem 1.2.7 and some explicit computations of the class of a blow-up of a smooth variety along a smooth subvariety. We then deduce theorem 1.2.2 from theorems 1.2.6 and 1.2.19.

Remark 1.2.20. The hypothesis of characteristic zero in theorems 1.2.6, and the related theorem 1.2.18 is needed only to allow the use of resolution of singularities, and so these results are valid over any field admitting resolution of singularities in the sense of Appendix A. Theorem 1.2.7 uses resolution of singularities as well as the weak factorization theorem of [2] and [37]. Thus theorems 1.2.2 and 1.2.19 rely on both resolution of singularities and the weak factorization theorem.

Our definition of the homomorphism deg, on the other hand, relies at present on the generic smoothness of a morphism $Y \to X$ of smooth k-schemes, hence is restricted to characteristic zero, regardless of any assumptions on resolution of singularities. Thus, the explicit formula (1.2) in theorem 1.2.14 relies on characteristic zero for its very definition. Since the proof of theorem 1.2.19 relies on theorem 1.2.14, this result also requires characteristic zero in the same way.

Remark 1.2.21. Theorem 1.2.19, together with the natural transformation described in example 1.2.10, immediately implies a result of B. Totaro [34] constructing for any smooth $\mathbb{C}$-variety X, a map

$$\mathrm{CH}^*(X) \to MU^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$$

factoring the topological cycle class map $\mathrm{CH}^*(X) \to H^{2*}(X, \mathbb{Z})$ through the natural map $MU^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \to H^{2*}(X, \mathbb{Z})$.

Remark 1.2.22. Unoriented cobordism. Let $X \mapsto MO^*(X)$ denote unoriented cobordism theory and $MO^* := MO^*(pt)$ the unoriented cobordism of a point, as studied by Thom [33]. Given a real embedding $\sigma : k \to \mathbb{R}$, then for any smooth k-scheme X of dimension d denote by $X_\sigma(\mathbb{R})$ the differentiable manifold (of dimension d) of real points of X. Then clearly, the assignment

$$X \mapsto MO^*(X_\sigma(\mathbb{R}))$$

has a structure of oriented cohomology theory on $\mathbf{Sm}_k$ (one can use [30]; observe that the associated theory of Chern classes is this case is nothing but the theory of Stiefel-Whitney classes in $MO^*(X_\sigma(\mathbb{R}))$). Thus we get from the universality of Ω^* a natural transformation

$$\Omega^*(X) \to MO^*(X_\sigma(\mathbb{R})).$$

From theorem 1.2.7 we thus get for any real embedding $k \to \mathbb{R}$ a natural homomorphism:

$$\Psi_{k\to\mathbb{R}} : \mathbb{L}^* \cong \Omega^*(k) \to MO^*$$

which (using corollary 1.2.11) does not depend on k; to compute $\Psi_{k\to\mathbb{R}}$, we may thus assume $k = \mathbb{R}$. Concretely, $\Psi_\mathbb{R} : \mathbb{L}^* = \Omega^*(\operatorname{Spec}\mathbb{R}) \to MO^*$ is the map which sends the class $[X]$ of a smooth projective variety X over $\mathbb{R}$ to the unoriented class of the differentiable manifold $X(\mathbb{R})$ of real points.

From [30], the theory of Stiefel-Whitney classes in MO^* defines an isomorphism of rings

$$\mathbb{L}^*/[2] \to MO^*$$

where [2] denotes the ideal generated by the coefficients of the power series $[2](u) := F_\mathbb{L}(u, u)$. One easily checks that the induced epimorphism $\mathbb{L}^* \to MO^*$ is the homomorphism $\Psi_\mathbb{R}$ above.

From all this follows a geometric interpretation of the map $\Psi : \mathbb{L}^* \to \mathbb{L}^*/[2]$ using the identifications $\mathbb{L}^* = \Omega^*(\mathbb{R}) = MU^{2*}$ and $\mathbb{L}^*/[2] = MO^*$: let $x \in MU^{2n}$ be an element represented by a smooth projective variety X over $\mathbb{R}$. Then $\Psi(x)$ is equal to the unoriented cobordism class $[X(\mathbb{R})]$ (which thus only depends on x).

1.3 Relations with complex cobordism

At this point, let's give some heuristic explanation of the whole picture.

For X a finite CW-complex, one can define its singular cohomology groups with integral coefficients $H^*(X;\mathbb{Z})$, its complex K-theory $K^*(X)$, and its complex cobordism $MU^*(X)$ (see [3], for instance). These are complex oriented cohomology theories, they admit a theory of Chern classes and the analogue of lemma 1.1.3 implies the existence of a canonical ring homomorphism from $\mathbb{L}^*$ to the coefficient ring of the theory (which double the degrees with our conventions).

Quillen in [29] refined Milnor's [22] and Novikov's [26] computations that the complex cobordism MU^* of a point is a polynomial algebra with integral coefficient by showing that the map

$$\Phi^{top} : \mathbb{L}^* \to MU^{2*}$$

is an isomorphism (here we mean that Φ^{top} double the degrees and that the odd part of MU^* vanishes). Then in [30], Quillen produced a geometric proof of that fact emphasizing that MU^* is the universal complex oriented cohomology theory on the category of differentiable manifolds.

The theorem of Conner-Floyd [5] now asserts that for each CW-complex X the map

$$MU^*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K^*(X)$$

is an isomorphism (beware that in topology β has degree -2).

However, in general for a CW-complex X the homomorphism

$$MU^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \to H^*(X; \mathbb{Z})$$

is not an isomorphism (not even surjective), even when restricted to the even part. Thus contrary to theorem 1.2.18, theorem 1.2.19 has no obvious counterpart in topology.

To give a heuristic explanation of our results we should mention that for smooth varieties over a field singular cohomology is replaced by motivic cohomology $H^{*,*}(X; \mathbb{Z})$, complex K-theory by Quillen's algebraic K-theory $K^{*,*}(X)$ and complex cobordism by the theory $MGL^{*,*}$ represented by the algebraic Thom complex MGL (in the setting of $\mathbb{A}^1$-homotopy theory, see [36]). One should note that these theories take values in the category of bigraded rings, the first degree corresponding to the cohomological degree and the second to the weight. In this setting, one should still have the Conner-Floyd isomorphism[3]

$$MGL^{*,*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \cong K^{*,*}(X)[\beta, \beta^{-1}]$$

for any simplicial smooth k-variety X (beware here that β has bidegree $(-2, -1)$). However the map $MGL^{*,*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \to H^{*,*}(X)$ would almost never be an isomorphism. Instead one expects a spectral sequence[4] from motivic cohomology to $MGL^{*,*}(X)$; the filtration considered in §4.5.2 should by the way be the one induced by that spectral sequence. Then theorem 1.2.19 is explained by the degeneration of this spectral sequence in the area computing the bidegrees of the form $(2n, n)$.

In fact, the geometric approach taken in the present work only deal with bidegrees of the form $(2n, n)$. Indeed, one can check that for any oriented

[3] This has been proven over any field by the second author jointly with M. Hopkins, unpublished.

[4] This spectral sequence has been announced in characteristic zero by the second author jointly with M. Hopkins, in preparation.

bigraded cohomology theory $A^{*,*}$ in the setting of $\mathbb{A}^1$ homotopy theory, the associated functor $X \mapsto \oplus_n A^{2n,n}(X)$ has a structure of oriented cohomology theory on $\mathbf{Sm}_k$ in our sense. In particular the universal property of Ω^* yields a natural transformation

$$\Omega^*(X) \to MGL^{2*,*}(X)$$

which we conjecture to be an isomorphism.

We are hopeful that our geometric approach can be extended to describe the whole bigraded algebraic cobordism, and that our results are only the first part of a general description of the functor $MGL^{*,*}$.

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