3

Induced Representations of Linear Groups

From now on, we concentrate on the group $G = \mathrm{GL}_2(F)$ over a non-Archimedean local field F. The group G inherits a locally profinite topology from the base field F, as in 1.4. Our objective is the classification of the irreducible smooth representations of G, although we shall not achieve it until the end of Chapter IV.

In this chapter, the algebraic subgroups B, N, T of G (as in 5.1) play a pivotal rôle. In parallel with the representation theory of the finite group $\mathrm{GL}_2(\boldsymbol{k})$ worked out in §6, the irreducible smooth representations of G fall into two broad classes. First, there is a "principal series" of representations: these are the composition factors of representations obtained from characters of T by a process of inflation to B and then induction to G. Frobenius Reciprocity characterizes them, among the irreducible smooth representations of G, as those admitting a non-trivial quotient on which N acts trivially. The irreducible smooth representations not obtainable this way are called "cuspidal". The main result of this chapter (9.11) gives a complete classification of the principal series representations.

There is a further subgroup of G which plays a surprisingly important part in the classification process. This is the "mirabolic subgroup" M of matrices $(x_{ij}) \in B$ with $x_{22} = 1$. The group M has a very simple representation theory: besides an obvious family of characters, it has a unique irreducible smooth representation. Further, irreducible representations of G decompose very little when restricted to M. This is the basis of our detailed analysis of the principal series representations of G.

In this chapter, we make only rather general remarks about the irreducible cuspidal representations of G. We give a characterization of them more helpful than that of not being in the principal series, and a speculative method for constructing them. This prepares the ground for the analysis in Chapter IV.

Some of the arguments and results here, particularly in §11, apply to quite arbitrary locally profinite groups: we point these out as they arise. Most of the time, we work exclusively with $\mathrm{GL}_2(F)$ or its subgroups, and exploit this restriction as much as we can to simplify and abbreviate the treatment. We are rarely unwilling to substitute an explicit matrix calculation for a more general abstract argument.

7. Linear Groups over Local Fields

As noted in §1, the group $G = \mathrm{GL}_2(F)$ has many compact open subgroups, of which a small number are of particular importance. This is expressed first via various coset decompositions of G, beyond the universal Bruhat decomposition of 5.2. Using these decompositions, one can turn the general measure theory of §3 into an effective computational tool, necessary for handling the integrals arising within the representation theory of G.

This section thus amounts to a course of calculus on $\mathrm{GL}_2(F)$, which can be skimmed at first reading and referred back to at need. The only result to which we will return is the Duality Theorem at the end, but the general techniques developed here are used frequently.

7.1. For this section, we set $V = F \oplus F$, and think of it as the space of column vectors with G acting on the left. The standard subgroups B, N, T, Z are as in §5. These are all closed subgroups of G. The group isomorphisms $B/N \cong T$ and $B \cong T \ltimes N$ are homeomorphisms.

7.2. Reflecting the special nature of the base field F, the group G admits decompositions besides the Bruhat decomposition of 5.2. The first of these is:

(7.2.1) ***Iwasawa decomposition.*** *Let B be the standard Borel subgroup of G and set $K = \mathrm{GL}_2(\mathfrak{o})$; then $G = BK$.*

Proof. Take $g \in G$; if the $(2,1)$-entry of g is zero, then $g \in B$. Otherwise, post-multiplying by the permutation matrix $w \in K$ if necessary, we can assume $\upsilon_F(g_{21}) \geqslant \upsilon_F(g_{22})$. We can then post-multiply by a lower triangular matrix in K to achieve $g_{21} = 0$. $\square$

Consequently, the quotient space $B\backslash G$ is a continuous image of the compact group $K = \mathrm{GL}_2(\mathfrak{o})$, and so:

Corollary. *The quotient space $B\backslash G$ is compact.*

Continuing with the notation $K = \mathrm{GL}_2(\mathfrak{o})$, we also have:

(7.2.2) ***Cartan decomposition.*** *Let ϖ be a prime element of F. The matrices*

$$\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}, \quad a, b \in \mathbb{Z},\ a \leqslant b,$$

form a set of representatives for the coset space $K\backslash G/K$.

Proof. Permuting rows and columns, using the permutation matrix in K, we can arrange for the largest entry of g (in absolute value) to be in the $1,1$ place. Multiplying by elementary matrices from K, we can then arrange for g to be diagonal, and unit factors can be absorbed into K. This gives

$$G = \bigcup_{a \leqslant b} K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K.$$

We have to prove this union is disjoint. That is, we have to recover the integers a, b from the coset KgK, where

$$g = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}.$$

First, we have $a+b = \upsilon_F(\det h)$, for any $h \in KgK$. Next, the group index $(K : K \cap hKh^{-1})$ depends only on the coset KhK, and $(K : K \cap gKg^{-1}) = 1$ if $b = a$, or $(q+1)q^{b-a-1}$ if $b > a$. □

Corollary. *If K is a compact open subgroup of G, the set G/K is countable.*

Proof. As observed in 2.6, it is enough to show that G/K is countable for one choice of K: we take $K = \mathrm{GL}_2(\mathfrak{o})$. The space $K\backslash G/K$ is certainly countable, and each double coset KgK contains only finitely many cosets $g'K$. □

That is, G satisfies the countability hypothesis of 2.6.

Exercise. *Let K be a compact subgroup of G. Show that $gKg^{-1} \subset \mathrm{GL}_2(\mathfrak{o})$, for some $g \in G$. Deduce that, up to G-conjugacy, $\mathrm{GL}_2(\mathfrak{o})$ is the unique maximal compact subgroup of G.*

Hint. There are two steps. One first shows that there exists a K-stable $\mathfrak{o}$-lattice in V: consider the $\mathfrak{o}$-span of KL, for a randomly chosen $\mathfrak{o}$-lattice L. The second consists of showing that the only $\mathrm{GL}_2(\mathfrak{o})$-stable lattices in V are the obvious ones $\mathfrak{p}^j \oplus \mathfrak{p}^j$, $j \in \mathbb{Z}$.

7.3. The *standard Iwahori subgroup* of G is the compact open subgroup

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in U_F,\ b \in \mathfrak{o},\ c \in \mathfrak{p} \right\}.$$

Let $N' = N^w$ denote the group of lower triangular unipotent matrices in G.

(7.3.1) *Iwahori decomposition.* *We have $I = (I \cap N')(I \cap T)(I \cap N)$. More precisely, the product map*

$$I \cap N' \times I \cap T \times I \cap N \longrightarrow I$$

is bijective, and a homeomorphism, for any ordering of the factors on the left hand side.

Proof. The product map is certainly continuous. It is elementary to write down its inverse and observe that it is continuous. $\square$

Set $K = \mathrm{GL}_2(\mathfrak{o})$; under the canonical surjection $K \to \mathrm{GL}_2(\boldsymbol{k})$, the image of I is the standard Borel subgroup of $\mathrm{GL}_2(\boldsymbol{k})$. The Bruhat decomposition for $\mathrm{GL}_2(\boldsymbol{k})$ implies

$$K = I \cup IwI. \tag{7.3.2}$$

Combining (7.3.2) with the Iwasawa decomposition (7.2.1) for G, we obtain the more symmetric double-coset decomposition

$$G = BI \cup BwI = B(I \cap N') \cup Bw(I \cap N). \tag{7.3.3}$$

The cosets BwI, BI are both open in G.

Remark. Let $L = \mathfrak{o} \oplus \mathfrak{o}$, $L' = \mathfrak{o} \oplus \mathfrak{p}$. The Iwahori subgroup I is then the common G-stabilizer of the two lattices L, L'.

7.4. We now describe the Haar measures attached to the various locally profinite groups under discussion. We start with the basic example of the field F itself.

Lemma. *The vector space $C_c^\infty(F)$ is spanned by the characteristic functions of sets $a+\mathfrak{p}^m$, $a \in F$, $m \in \mathbb{Z}$.*

Proof. Surely the characteristic function of $a+\mathfrak{p}^m$ lies in $C_c^\infty(F)$. Conversely, let $\Phi \in C_c^\infty(F)$. Since Φ has compact support, there exists $n \in \mathbb{Z}$ such that $\operatorname{supp}\Phi \subset \mathfrak{p}^n$. Also, Φ is fixed under translation by a compact open subgroup of F, hence by $\mathfrak{p}^m$, for some $m \in \mathbb{Z}$. Thus Φ is a linear combination of characteristic functions of sets $a+\mathfrak{p}^m$, $a \in \mathfrak{p}^n/\mathfrak{p}^m$. $\square$

If Φ_0 denotes the characteristic function of $\mathfrak{o}$ and μ is a Haar measure on F, we have

$$\int_F \Phi_0(x)\, d\mu(x) = c_0,$$

for some $c_0 > 0$. If Φ_1 is the characteristic function of a coset $a+\mathfrak{p}^b$, $a \in F$, $b \in \mathbb{Z}$, then

$$\int_F \Phi_1(x)\, d\mu(x) = c_0 q^{-b}.$$

This identity suffices for integrating any function $\varPhi \in C_c^\infty(F)$.

Now take $\varPhi \in C_c^\infty(F)$ and $y \in F^\times$. Using the identity above, we find

$$\int_F \varPhi(xy)\, d\mu(x) = \|y\|^{-1} \int_F \varPhi(x)\, d\mu(x),$$

where, we recall, $\|y\| = q^{-\upsilon_F(y)}$. We accordingly define a measure $\mu^\times$ on $F^\times$ by $d\mu^\times(x) = d\mu(x)/\|x\|$, meaning the following. If $\varPhi \in C_c^\infty(F^\times)$, the function $x \mapsto \|x\|^{-1}\varPhi(x)$ (vanishing at 0) lies in $C_c^\infty(F)$, so we can put

$$\int_{F^\times} \varPhi(x)\, d\mu^\times(x) = \int_F \varPhi(x)\|x\|^{-1}\, d\mu(x), \quad \varPhi \in C_c^\infty(F^\times). \tag{7.4.1}$$

A simple manipulation shows that (7.4.1) defines a Haar integral on $F^\times$.

7.5. The matrix ring $A = \mathrm{M}_2(F)$ is (as additive group) a product of 4 copies of F and a Haar measure is obtained by taking a (tensor) product of 4 copies of a Haar measure on F.

Proposition. *Let μ be a Haar measure on A. For $\varPhi \in C_c^\infty(G)$, the function $x \mapsto \varPhi(x)\|\det x\|^{-2}$ (vanishing on $A \smallsetminus G$) lies in $C_c^\infty(A)$. The functional*

$$\varPhi \mapsto \int_A \varPhi(x)\|\det x\|^{-2}\, d\mu(x), \quad \varPhi \in C_c^\infty(G),$$

is a left and right Haar integral on G. In particular, G is unimodular.

Proof. Let $g \in G$ and consider the functionals

$$\varPhi \longmapsto \begin{cases} \displaystyle\int_A \varPhi(gx)\, d\mu(x), \\ \displaystyle\int_A \varPhi(xg)\, d\mu(x), \end{cases} \qquad \varPhi \in C_c^\infty(A).$$

Each is a Haar integral on A and differs from the initial one by a positive constant (depending on g). To evaluate this constant, we take $\varPhi$ to be the characteristic function of $\mathfrak{m} = \mathrm{M}_2(\mathfrak{o})$. In the first instance, the function $x \mapsto \varPhi(gx)$ is the characteristic function of the lattice $\mathfrak{m}' = g^{-1}\mathfrak{m}$. Thus

$$\int_A \varPhi(gx)\, d\mu(x) = \mu(\mathfrak{m}') = \mu(\mathfrak{m})\,(\mathfrak{m}' : \mathfrak{m} \cap \mathfrak{m}')/(\mathfrak{m} : \mathfrak{m} \cap \mathfrak{m}').$$

This quotient of indices depends only on the double coset KgK, $K = \mathrm{GL}_2(\mathfrak{o})$. Taking g in diagonal form (7.2.2), one gets

$$\int_A \varPhi(gx)\, d\mu(x) = \|\det g\|^{-2} \int_A \varPhi(x)\, d\mu(x).$$

The second instance is treated in same way to get

$$\int_A \Phi(xg)\,d\mu(x) = \|\det g\|^{-2} \int_A \Phi(x)\,d\mu(x).$$

The proposition then follows from simple manipulations. For example, if $\Phi \in C_c^\infty(G)$,

$$\begin{aligned}\int_A \Phi(xg)\|\det x\|^{-2}\,d\mu(x) &= \|\det g\|^{-2} \int_A \Phi(x)\|\det xg^{-1}\|^{-2}\,d\mu(x)\\ &= \int_A \Phi(x)\|\det x\|^{-2}\,d\mu(x),\end{aligned}$$

as required. □

7.6. We turn to the subgroups B, N, T of G. Since $N \cong F$ and $T \cong F^\times \times F^\times$, there is nothing more to say about them. We have $B = T \ltimes N$; we define a linear functional on the space $C_c^\infty(B) = C_c^\infty(T) \otimes C_c^\infty(N)$ by

$$\Phi \longmapsto \int_T \int_N \Phi(tn)\,d\mu_T(t)d\mu_N(n), \quad \Phi \in C_c^\infty(B),$$

where μ_T, μ_N are Haar measures on T, N respectively. One verifies immediately that this functional is left B-invariant, so it is a left Haar integral on B. We are so justified in denoting it

$$\Phi \longmapsto \int_B \Phi(b)\,d\mu_B(b).$$

The Haar measure μ_B may be thought of as the tensor product, $\mu_B = \mu_T \otimes \mu_N$, but the two factors do not commute. This reflects the fact that the group B is not unimodular. In the language of 3.3:

Proposition. *The module δ_B of the group B is given by*

$$\delta_B : tn \longmapsto \|t_2/t_1\|, \qquad n \in N,\ t = \left(\begin{smallmatrix} t_1 & 0\\ 0 & t_2\end{smallmatrix}\right) \in T. \tag{7.6.1}$$

Proof. Setting

$$c = sm, \quad m \in N,\ s = \left(\begin{smallmatrix} s_1 & 0\\ 0 & s_2\end{smallmatrix}\right) \in T,$$

we get

$$\int_B \Phi(bc)\,d\mu_B(b) = \int_T \int_N \Phi(ts\,s^{-1}ns\,m)\,d\mu_T(t)d\mu_N(n).$$

We use the obvious isomorphism $N \to F$ to identify μ_N with a certain Haar measure μ_F on F. For $\phi \in C_c^\infty(N)$, we then have

$$\begin{aligned}\int_N \phi(s^{-1}ns)\,d\mu_N(n) &= \int_F \phi\left(\begin{smallmatrix} 1 & s_1^{-1}xs_2\\ 0 & 1\end{smallmatrix}\right)\,d\mu_F(x)\\ &= \|s_1 s_2^{-1}\| \textstyle\int_N \phi(n)\,d\mu_N(n).\end{aligned}$$

By definition,

$$\int_B \Phi(bc)\,d\mu_B(b) = \delta_B(c)^{-1} \int_B \Phi(b)\,d\mu_B(b),$$

and the result follows. $\square$

In the notation of 3.4, we now have:

Corollary. *The space $C_c^\infty(B\backslash G, \delta_B^{-1})$ admits a positive semi-invariant measure $\dot\mu$, where δ_B is given by (7.6.1). If $K = \mathrm{GL}_2(\mathfrak{o})$, there is a Haar measure μ_K on K such that*

$$\int_{B\backslash G} f(g)\,d\dot\mu(g) = \int_K f(k)\,d\mu_K(k),$$

for $f \in C_c^\infty(B\backslash G, \delta_B^{-1})$.

Proof. The character δ_B is trivial on the compact group $K \cap B$. Restriction of functions is an isomorphism $C_c^\infty(B\backslash G, \delta_B^{-1}) \to C_c^\infty(K\cap B\backslash K, 1)$, where 1 denotes the trivial character of $K\cap B$. The semi-invariant measure $\dot\mu$ thus restricts to a semi-invariant measure on $C_c^\infty(K\cap B\backslash K, 1)$, but so does any Haar measure on K. $\square$

We observe that μ_K is effectively just the restriction of a Haar measure μ_G on G. Comparing with the proof of 3.4 Proposition, there is a left Haar measure μ_B on B such that

$$\int_G \phi(g)\,d\mu_G(g) = \int_K \int_B \phi(bk)\,d\mu_B(b)d\mu_G(k), \quad \phi \in C_c^\infty(G). \tag{7.6.2}$$

Exercises.

(1) *Let I be the standard Iwahori subgroup; let dn', dt, dn be Haar measures on the groups $I\cap N'$, $I\cap T$, $I\cap N$ respectively. Show that the functional*

$$f \longmapsto \iiint f(n'tn)\,dn'dt\,dn, \quad f \in C_c^\infty(I),$$

is a Haar integral on I.

(2) *Let $C = N'TN$. Show that C is open and dense in G, and that the product map $N'\times T\times N \to C$ is a homeomorphism.*

(3) *Let dg be a Haar measure on G. Show that there are Haar measures dn', dt, dn on N', T, N such that*

$$\int_G f(g)\,dg = \iiint f(n'tn)\delta_B(t)^{-1}\,dn'dt\,dn, \quad f \in C_c^\infty(G).$$

7.7. Let σ be a smooth representation of T, viewed as representation of B trivial on N. Corollary 7.2 implies that the canonical inclusion map

$$c\text{-}\mathrm{Ind}_B^G \sigma \longrightarrow \mathrm{Ind}_B^G \sigma$$

is an isomorphism. We can therefore apply the Duality Theorem of 3.5 to get:

Duality Theorem. *Let σ be a smooth representation of T, viewed as representation of B trivial on N, and fix a positive semi-invariant measure $\dot{\mu}$ on $C_c^\infty(B\backslash G, \delta_B^{-1})$. There is a canonical isomorphism*

$$\left(\mathrm{Ind}_B^G \sigma\right)^\vee \cong \mathrm{Ind}_B^G \delta_B^{-1} \otimes \check{\sigma},$$

depending only on the choice of $\dot{\mu}$.

8. Representations of the Mirabolic Group

Before starting on the representation theory of the group $G = \mathrm{GL}_2(F)$, we study the representations of a certain subgroup of G, the so-called *mirabolic subgroup*

$$M = \left\{ \left(\begin{smallmatrix} a & x \\ 0 & 1 \end{smallmatrix}\right) : a \in F^\times, x \in F \right\}.$$

Thus M is the semi-direct product of N by the group $S = T \cap M \cong F^\times$.

8.1. To start with, let (π, V) be a smooth representation of N and let ϑ be a character of N. We denote by $V(\vartheta)$ the linear subspace of V spanned by the vectors $\pi(n)v - \vartheta(n)v$, $n \in N$, $v \in V$. We set $V_\vartheta = V/V(\vartheta)$: this is the unique maximal N-quotient of V on which N acts via the character ϑ.

If ϑ_0 is the trivial character of N, we have $V(\vartheta_0) = V(N)$ (notation of 2.3) and we write $V_{\vartheta_0} = V_N$.

Lemma. *Let μ_N be a Haar measure on N and ϑ a character of N.*

(1) *Let (π, V) be a smooth representation of N and $v \in V$. The vector v lies in $V(\vartheta)$ if and only if there is a compact open subgroup N_0 of N such that*
$$\int_{N_0} \vartheta(n)^{-1} \pi(n) v \, d\mu_N(n) = 0. \tag{8.1.1}$$

(2) *The process $(\pi, V) \mapsto V_\vartheta$ is an exact functor from* $\mathrm{Rep}(N)$ *to the category of complex vector spaces.*

Proof. We assume first that ϑ is the trivial character of N.

The group $N \cong F$ is the union of an ascending sequence of compact open subgroups. So, if

$$v = \sum_{i=1}^{r} v_i - \pi(n_i)v_i \in V(N),$$

there is a compact open subgroup N_0 of N containing all n_i. The relation (8.1.1) then holds for this choice of N_0.

Conversely, let $v \in V$ and suppose (8.1.1) holds. There is an open normal subgroup N_1 of N_0 such that $v \in V^{N_1}$. The space V^{N_1} carries a representation of the finite group N_0/N_1. Therefore, in the obvious notation, $V^{N_1} = V^{N_1}(N_0/N_1) \oplus V^{N_0}$ (*cf.* 2.3) and the map

$$w \longmapsto \mu_N(N_0)^{-1} \int_{N_0} \pi(n)w \, d\mu_N(n), \quad w \in V^{N_1},$$

is the N_0-projection $V^{N_1} \to V^{N_0}$. This has kernel $V^{N_1}(N_0/N_1) \subset V(N)$ and we have proved (1) for the trivial character of N.

Now let ϑ be an arbitrary character of N, and consider the representation (π', V') of N, where $V' = V$ and $\pi'(n) = \vartheta(n)^{-1}\pi(n)$. We then have $V(\vartheta) = V'(N)$ and so (1) follows in general.

Part (2) is an immediate consequence of (1). $\square$

We mention some simple consequences of the lemma. If (π, V) is a smooth representation of N (or of M), then $V(N)$ is an N- (or M-) subspace of V. The exact sequence

$$0 \to V(N) \longrightarrow V \longrightarrow V_N \to 0$$

gives an exact sequence

$$0 \to V(N)_N \longrightarrow V_N \longrightarrow V_N \to 0$$

in which the map $V_N \to V_N$ is the identity. Therefore

$$V(N)_N = 0 \quad \text{and} \quad V(N)(N) = V(N). \tag{8.1.2}$$

Suppose that $\vartheta \neq 1$. As N acts trivially on $V/V(N)$, we have $(V/V(N))_\vartheta = 0$ and so the inclusion $V(N) \to V$ induces an isomorphism

$$V(N)_\vartheta \cong V_\vartheta. \tag{8.1.3}$$

Proposition. *Let (π, V) be a smooth representation of N, and let $v \in V$, $v \neq 0$. There exists a character ϑ of N such that $v \notin V(\vartheta)$.*

Proof. We write

$$N_j = \left(\begin{smallmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{smallmatrix}\right), \quad j \in \mathbb{Z}. \tag{8.1.4}$$

Take $v \in V$, $v \neq 0$. We choose $j_0 \in \mathbb{Z}$ such that N_{j_0} fixes v. For $j \leqslant j_0$, let V_j denote the N_j-space generated by v. This is the direct sum of isotypic components V_j^η, as η ranges over the characters of N_j trivial on N_{j_0}. For each $j \leqslant j_0$, there exists η_j such that $V_j^{\eta_j} \neq 0$; by definition, we have

$$\int_{N_j} \eta_j(n)^{-1}\pi(n)v\,dn \neq 0.$$

The N_{j-1}-space generated by $V_j^{\eta_j}$ is contained in V_{j-1}, so we may choose η_{j-1} such that $\eta_{j-1} \mid N_j = \eta_j$. It follows (compare the argument in 1.7) that there exists a character ϑ of N such that, for all $j \leqslant j_0$, we have

$$\int_{N_j} \vartheta(n)^{-1}\pi(n)v\,dn \neq 0.$$

Therefore $v \notin V(\vartheta)$, as required. $\square$

Corollary 1. *Let (π, V) be a smooth representation of N. If $V_\vartheta = 0$ for all characters ϑ of N, then $V = 0$.*

Now let (π, V) be a smooth representation of M. The space $V(N)$ is then an M-subspace of V and V_N carries a natural representation of $M/N = S$. On the other hand, $\pi(S)$ permutes the subspaces $V(\vartheta)$, $\vartheta \neq 1$, transitively. Explicitly, for $s \in S$, $\pi(s)V(\vartheta) = V(\vartheta')$, where $\vartheta'(n) = \vartheta(s^{-1}ns)$. We can therefore sharpen Corollary 1 for representations of M:

Corollary 2. *Let (π, V) be a smooth representation of M. Suppose that $V_N = 0$ and that $V_\vartheta = 0$ for some non-trivial character ϑ of N. Then $V = 0$.*

8.2. We now fix a non-trivial character ϑ of N, and consider the two M-spaces $\mathrm{Ind}_N^M \vartheta$, c-$\mathrm{Ind}_N^M \vartheta$. Observe that, if ϑ' is some other non-trivial character of N, then $\mathrm{Ind}_N^M \vartheta'$ is M-isomorphic to $\mathrm{Ind}_N^M \vartheta$ and similarly for the compactly induced representations.

Proposition. *Let ϑ be a non-trivial character of N and set $\mathcal{W} = \mathrm{Ind}_N^M \vartheta$, $\mathcal{W}^c = c$-$\mathrm{Ind}_N^M \vartheta$. Let $\alpha : \mathcal{W} \to \mathbb{C}$ denote the canonical map $f \mapsto f(1)$.*

(1) *We have $\mathcal{W}(N) = \mathcal{W}^c(N) = \mathcal{W}^c$ and $(\mathcal{W}/\mathcal{W}^c)(N) = 0$.*
(2) *The map α induces isomorphisms $\mathcal{W}_\vartheta \cong \mathbb{C}$ and $\mathcal{W}^c_\vartheta \cong \mathbb{C}$.*

Proof. Let $f \in \mathcal{W}$ and $n \in N$. For $a \in S$, we have $f(an) = \vartheta(ana^{-1})f(a)$. As there is an integer j such that N_j (as in (8.1.4)) fixes f, we see that $f(a) = 0$ if $\|\det a\|$ is sufficiently large. On the other hand, $f(an) = f(a)$ if $\|\det a\|$ is sufficiently small, so $nf - f$ vanishes at a if $\|\det a\|$ is sufficiently small. This implies that $nf - f \in \mathcal{W}^c$. Thus $\mathcal{W}(N) \subset \mathcal{W}^c$ and N acts trivially on $\mathcal{W}/\mathcal{W}^c$.

We next prove that $\mathcal{W}^c(N) = \mathcal{W}^c$. Let ψ be the character of F defined by

$$\psi(x) = \vartheta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Take $a \in F^\times$, $j \in \mathbb{Z}$, $j \geqslant 1$. Let $f_{a,j} \in \mathcal{W}^c$ be the function such that

$$f_{a,j} : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} au & 0 \\ 0 & 1 \end{pmatrix} \longmapsto \psi(x), \quad u \in U_F^j,$$

and which vanishes elsewhere. The various functions $f_{a,j}$ span $\mathcal{W}^c$ over $\mathbb{C}$. We have

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f_{a,j} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} = \psi(bx) f_{a,j} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}.$$

We deduce that $nf_{a,j}$ has the same support as $f_{a,j}$, $n \in N$. We can certainly find $x \in F$ such that the function $u \mapsto \psi(aux)$, $u \in U_F^j$, is constant, equal to c say, with $c \neq 1$. If $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, then $nf_{a,j} - f_{a,j} = (c-1)f_{a,j}$, whence $f_{a,j} \in \mathcal{W}^c(N)$ and so $\mathcal{W}^c(N) = \mathcal{W}^c$. This implies $\mathcal{W}(N) = \mathcal{W}^c$ also, and we have proved (1).

The map α induces a surjection $\mathcal{W}_\vartheta \to \mathbb{C}$. On the other hand, since N acts trivially on $\mathcal{W}/\mathcal{W}^c$, the inclusion $\mathcal{W}^c \hookrightarrow \mathcal{W}$ induces an isomorphism $\mathcal{W}^c_\vartheta \cong \mathcal{W}_\vartheta$. To prove (2), therefore, it is enough to show that any $f \in \mathcal{W}^c$ with $f(1) = 0$ belongs to $\mathcal{W}^c(\vartheta)$. A function f, vanishing at 1, is a finite linear combination of functions $f_{a,j}$ with $a \notin U_F^j$, so it is enough to treat such functions. However, as $a \notin U_F^j$, there exists $x \in F$ such that the function $u \mapsto \psi(aux) - \psi(x)$, $u \in U_F^j$, is a non-zero constant. Taking $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, the same calculation as before shows that $nf_{a,j} - \vartheta(n)f_{a,j}$ is a non-zero constant multiple of $f_{a,j}$, whence $f_{a,j} \in \mathcal{W}^c(\vartheta)$ and $\mathcal{W}^c(\vartheta) = \mathcal{W}^c$, as required. □

Corollary. *The representation c-$\mathrm{Ind}_N^M \vartheta$ is irreducible over M.*

Proof. Let V be an M-subspace of $\mathcal{W}^c$. As $\mathcal{W}^c_N = 0$, the spaces V_N, $(\mathcal{W}^c/V)_N$ are both zero. The sequence

$$0 \to V_\vartheta \longrightarrow \mathcal{W}^c_\vartheta \longrightarrow (\mathcal{W}^c/V)_\vartheta \to 0$$

is exact. As $\dim \mathcal{W}^c_\vartheta = 1$, we conclude that $\dim V_\vartheta$ is 0 or 1. In the first case, $V = 0$ by 8.1 Corollary 2. In the second, $(\mathcal{W}^c/V)_\vartheta = 0$ and so $\mathcal{W}^c = V$. □

We display some of the remarks made in the course of the proof of the proposition:

Gloss.

(1) *A function $f \in \mathcal{W}$ is determined by its restriction to $S \cong F^\times$. The restriction $f \mid F^\times$ is a smooth function on $F^\times$.*

(2) *A smooth function ϕ on $F^\times$ is of the form $\phi = f \mid F^\times$, for some $f \in \mathcal{W}$, if and only if there exists $c > 0$ such that $\phi(x) = 0$ for all x satisfying $\|x\| > c$.*

(3) *A function $f \in \mathcal{W}$ lies in $\mathcal{W}^c$ if and only if $f \mid F^\times \in C_c^\infty(F^\times)$.*

Remark. Part (3) implies that the representation $\mathrm{Ind}_N^M \vartheta$ is never irreducible, for any non-trivial character ϑ of N. The Duality Theorem 3.5 implies

$$(c\text{-}\mathrm{Ind}_N^M \vartheta)^\vee \cong \mathrm{Ind}_N^M \check{\vartheta}.$$

Thus $c\text{-}\mathrm{Ind}_N^M \vartheta$ provides an example of an irreducible smooth representation with *reducible* contragredient (*cf.* 2.10).

8.3. Again let ϑ be a non-trivial character of N. Let (π, V) be a smooth representation of M. Frobenius Reciprocity (2.4) gives a canonical isomorphism

$$\mathrm{Hom}_N(V, V_\vartheta) \cong \mathrm{Hom}_M(V, \mathrm{Ind}_N^M V_\vartheta).$$

Let $q : V \to V_\vartheta$ denote the quotient map and let $q_\star$ be the map $V \to \mathrm{Ind}_N^M V_\vartheta$ corresponding to q under this isomorphism. Explicitly, for $v \in V$, $q_\star(v)$ is the function $m \mapsto q(\pi(m)v)$.

Theorem. *Let (π, V) be a smooth representation of M. The M-homomorphism $q_\star : V \to \mathrm{Ind}_N^M V_\vartheta$ induces an isomorphism $V(N) \cong c\text{-}\mathrm{Ind}_N^M V_\vartheta$.*

Proof. The N-space V_ϑ is a direct sum of copies of ϑ. Therefore $\mathrm{Ind}_N^M V_\vartheta$ is a direct sum of copies of $\mathrm{Ind}_N^M \vartheta$. Proposition 8.2 so yields

$$(\mathrm{Ind}_N^M V_\vartheta)(N) = c\text{-}\mathrm{Ind}_N^M V_\vartheta = (c\text{-}\mathrm{Ind}_N^M V_\vartheta)(N). \tag{8.3.1}$$

The M-homomorphism $q_\star : V \to \mathrm{Ind}_N^M V_\vartheta$ surely maps $V(N)$ to $(\mathrm{Ind}_N^M V_\vartheta)(N) = c\text{-}\mathrm{Ind}_N^M V_\vartheta$.

Let $W = \mathrm{Ker}\, q_\star \cap V(N)$ and $C = c\text{-}\mathrm{Ind}_N^M V_\vartheta / q_\star(V(N))$. The natural map $W_N \to V(N)_N$ is injective, by 8.1 Lemma (2), so $W_N = 0$. Likewise, $(c\text{-}\mathrm{Ind}\, V_\vartheta)_N$ is zero, so $C_N = 0$.

The map $q_\star : V \to c\text{-}\mathrm{Ind}\, V_\vartheta$ induces a map

$$q_{\star,\vartheta} : V_\vartheta = V(N)_\vartheta \longrightarrow (c\text{-}\mathrm{Ind}\, V_\vartheta)_\vartheta.$$

By 8.2 Proposition (2), the canonical N-map $\mathrm{Ind}\, V_\vartheta \to V_\vartheta$ induces an isomorphism $\alpha_\vartheta : (c\text{-}\mathrm{Ind}\, V_\vartheta)_\vartheta \to V_\vartheta$. The composite map $\alpha_\vartheta \circ q_{\star,\vartheta} : V_\vartheta \to V_\vartheta$ is the identity. However, the kernel of this map is W_ϑ and its cokernel is isomorphic to C_ϑ. We have shown that the spaces W_N, W_ϑ, C_N, C_ϑ are all zero. By 8.1 Corollary 2, therefore, both W and C are trivial and $q_\star : V(N) \to c\text{-}\mathrm{Ind}\, V_\vartheta$ is an isomorphism, as desired. $\square$

Corollary. *Let* (π, V) *be an irreducible smooth representation of* M. *Either:*

(1) $\dim V = 1$ *and* π *is the inflation of a character of* $M/N \cong F^\times$, *or*
(2) $\dim V$ *is infinite and* $\pi \cong c\text{-}\mathrm{Ind}_N^M \vartheta$, *for any character* $\vartheta \neq 1$ *of* N.

In case (1), $\dim V_N = 1$ *and* $V_\vartheta = 0$ *for* $\vartheta \neq 1$. *In case* (2), $V_N = 0$ *and* $\dim V_\vartheta = 1$ *for all* $\vartheta \neq 1$.

Proof. If $V(N) = 0$, then N acts trivially on V. The group M/N is abelian, so Schur's Lemma 2.6 implies $\dim V = 1$ and we are in case (1).

If $V(N) \neq 0$, then $V(N) = V$ and $V_N = 0$. Therefore $V_\vartheta \neq 0$ for all characters $\vartheta \neq 1$ of N, and so $\dim V$ is infinite. The theorem implies that $V = V(N)$ is M-isomorphic to $c\text{-}\mathrm{Ind}_N^M V_\vartheta$. The N-space V_ϑ is a direct sum of copies of ϑ, so V is a direct sum of copies of $c\text{-}\mathrm{Ind}_N^M \vartheta$ and, since it is irreducible, $V \cong c\text{-}\mathrm{Ind}_N^M \vartheta$. $\square$

9. Jacquet Modules and Induced Representations

We start the process of classifying the irreducible smooth representations of the locally profinite group $G = \mathrm{GL}_2(F)$. In this section, we deal completely with those irreducible smooth representations (π, V) of G (the "principal series") for which $V_N \neq 0$.

9.1. Let (π, V) be a smooth representation of G. As in 8.1, $V(N)$ denotes the subspace of V spanned by the vectors $v - \pi(x)v$, for $v \in V$ and $x \in N$. The space $V_N = V/V(N)$ inherits a representation π_N of $B/N = T$, which is smooth. We call (π_N, V_N) the *Jacquet module* of (π, V) at N.

In particular, *the Jacquet functor*

$$\begin{aligned} \mathrm{Rep}(G) &\longrightarrow \mathrm{Rep}(T), \\ (\pi, V) &\longmapsto (\pi_N, V_N), \end{aligned} \tag{9.1.1}$$

is exact and additive.

Let (σ, W) be a smooth representation of T. We view σ as a smooth representation of B which is trivial on N, and form the smooth induced representation $\mathrm{Ind}_B^G \sigma$. (We sometimes abbreviate $\mathrm{Ind}_B^G \sigma = \mathrm{Ind}\,\sigma$, since B and G are the only groups involved for most of the time.)

If (π, V) is a smooth representation of G, Frobenius Reciprocity (2.4) gives an isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\,\sigma) \cong \mathrm{Hom}_B(\pi, \sigma).$$

However, σ is trivial on N so any B-homomorphism $\pi \to \sigma$ factors through the quotient map $\pi \to \pi_N$. We deduce

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\,\sigma) \cong \mathrm{Hom}_T(\pi_N, \sigma). \tag{9.1.2}$$

This has the following consequence:

Proposition. *Let (π, V) be an irreducible smooth representation of G. The following are equivalent:*

(1) *The Jacquet module V_N is non-zero.*
(2) *The representation π is isomorphic to a G-subspace of a representation $\operatorname{Ind}_B^G \chi$, for some character χ of T.*

Proof. Suppose (2) holds. From (9.1.2) we get

$$\operatorname{Hom}_T(\pi_N, \chi) \cong \operatorname{Hom}_G(\pi, \operatorname{Ind} \chi) \neq 0,$$

so surely $\pi_N \neq 0$.

To prove (1)⇒(2), we have to show that the Jacquet module V_N admits an irreducible T- (or B-) quotient.

Choose $v \in V$, $V \neq 0$. Since V is irreducible over G, any element of V is a finite linear combination of translates $\pi(g)v$ of v, for various $g \in G$. Write $K = \mathrm{GL}_2(\mathfrak{o})$. The vector v is fixed by a subgroup K' of K of finite index; let $\{v_1, v_2, \dots, v_r\}$ be the distinct elements of the form $\pi(k)v$, $k \in K$. In particular, $r \leqslant (K{:}K')$. Since $G = BK$, the elements $v_1, \dots, v_r$ generate V over B, and their images generate V_N over T.

Thus V_N is finitely generated as a representation of T. We choose a minimal generating set $\{u_1, \dots, u_t\}$, $t \geqslant 1$, say. A standard Zorn's Lemma argument shows that V_N has a T-subspace U, containing $u_1, \dots, u_{t-1}$, and maximal for the property $u_t \notin U$. Then U is a maximal T-subspace of V_N and V_N/U is an irreducible representation of T, hence a character (2.6 Corollary 2). □

An irreducible smooth representation (π, V) of G is called *cuspidal* if V_N is zero. In the literature, cuspidal representations are usually called *supercuspidal* or *absolutely cuspidal.* On the other hand, if $V_N \neq 0$, one says that π is in the *principal series.*

9.2. In the case of a finite field $\boldsymbol{k}$, we divided the irreducible representations according to whether or not they contained the trivial character of N. For $\mathrm{GL}_2(F)$ we use the existence of an N-trivial quotient, for the following reason:

Exercise 1. *Let (π, V) be an irreducible smooth representation of G with a non-trivial $\pi(N)$-fixed vector. Show that $\pi = \phi \circ \det$, for some character ϕ of $F^\times$.*

Hint. If $v \in V$ is fixed by N, it is fixed by the subgroup H of G generated by N and some open subgroup K. Show that, since H contains a lower triangular unipotent matrix, it also contains $\mathrm{SL}_2(F)$.

Exercise 2. *Let (π, V) be an irreducible smooth representation of G such that $\dim V$ is finite. Show that V has a non-zero $\pi(N)$-fixed vector. Deduce that $\dim V = 1$ and π is of the form $\phi \circ \det$, for some character ϕ of $F^\times$.*

These exercises help to explain the direction we take, although they play no part in the argument to follow. (We will, however, need Exercise 1 at a later stage.)

In this connection, we note:

Proposition. *Any character of G is of the form $\phi \circ \det$, for some character ϕ of $F^\times$.*

Proof. If χ is a character of G, its kernel contains the commutator subgroup of G. Since F is infinite, this commutator subgroup is $\mathrm{SL}_2(F)$, so $\chi = \phi \circ \det$, for some homomorphism $\phi : F^\times \to \mathbb{C}^\times$. The determinant map is surjective and open, so ϕ is a character. $\square$

9.3. An important fact concerns the structure of the Jacquet module $(\mathrm{Ind}_B^G \chi)_N$ of an induced representation. Here, it is no more difficult to give a very general result.

Let μ_N be a Haar measure on N and let $t \in T$. The measure $S \mapsto \mu_N(t^{-1}St)$ is the Haar measure $\delta_B(t)\mu_N$, for δ_B as in (7.6.1):

$$\int_N f(txt^{-1})\, d\mu_N(x) = \delta_B(t) \int_N f(x)\, d\mu_N(x), \quad f \in C_c^\infty(N).$$

As before, let w denote the permutation matrix

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If σ is a smooth representation of T, we can form the representation $\sigma^w : t \mapsto \sigma(wtw^{-1})$, and view it as a representation of B which is trivial on N.

As in 2.4, α_σ denotes the canonical B-map $\mathrm{Ind}\,\sigma \to \sigma$ given by $f \mapsto f(1)$. It induces a canonical T-map $(\mathrm{Ind}\,\sigma)_N \to \sigma$, which we continue to denote α_σ.

Restriction-Induction Lemma. *Let (σ, U) be a smooth representation of T and set $(\Sigma, X) = \mathrm{Ind}_B^G \sigma$. There is an exact sequence of representations of T:*

$$0 \to \sigma^w \otimes \delta_B^{-1} \longrightarrow \Sigma_N \xrightarrow{\alpha_\sigma} \sigma \to 0.$$

Proof. By definition, X is the space of G-smooth functions $f : G \to U$ such that $f(bg) = \sigma(b)f(g)$, $b \in B$, $g \in G$. The canonical map $\alpha_\sigma : X \to U$ amounts to restriction of functions to B. Set $V = \mathrm{Ker}\,\alpha_\sigma$. Thus V provides a smooth representation of B and there is an exact sequence

$$0 \to V_N \longrightarrow X_N \longrightarrow U \to 0.$$

We have to identify the T-representation V_N.

We recall that $G = B \cup BwN$. A function $f \in X$ thus lies in V if and only if $\operatorname{supp} f \subset BwN$. More precisely:

Lemma. *Let $f \in X$; then $f \in V$ if and only if there is a compact open subgroup N_0 of N (depending on f) such that* $\operatorname{supp} f \subset BwN_0$.

Proof. A function $f \in X$ lies in V if and only if $f(1) = 0$. Since f is G-smooth, such a function f vanishes on a set BN_0', where N_0' is some compact open subgroup of N'. The identity (for $x \neq 0$)

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \in Bw \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

implies that $\operatorname{supp} f \subset BwN_0$, for some compact open subgroup N_0 of N, as required. $\square$

Let $f \in V$; in view of the lemma, we can define a function $f_N : T \to U$ by

$$f_N(x) = \int_N f(xwn)\, dn = \sigma(x)\, f_N(1), \quad x \in T.$$

By 8.1 Lemma, the kernel of the map $f \mapsto f_N$ is $V(N)$, and so $f \mapsto f_N(1)$ gives a bijective map $V_N \to U$. Taking $t \in T$ and $f \in V$, we have

$$\begin{aligned} (tf)_N(x) &= \int_N f(xwnt)\, dn \\ &= \delta_B(t^{-1}) \int_N f(xt^w wn)\, dn = \delta_B^{-1}(t)\, (t^w f_N)(x). \end{aligned}$$

Thus $f \mapsto f_N(1)$ is a B-homomorphism $V \to \sigma^w \otimes \delta_B^{-1}$ inducing a T-isomorphism $V_N \cong \sigma^w \otimes \delta_B^{-1}$. $\square$

9.4. The irreducible representations of G exhibit a helpful finiteness property:

Proposition. *Let (π, V) be an irreducible smooth representation of G which is not cuspidal. The representation π is admissible.*

Proof. By definition, $V_N \neq 0$. By 9.1 Proposition, π is equivalent to a subrepresentation of $\operatorname{Ind}_B^G \chi$, for some character χ of T. It is enough, therefore, to prove that $\operatorname{Ind} \chi$ is admissible.

We fix a compact open subgroup K of G; shrinking it if necessary, we may assume that $K \subset K_0 = \mathrm{GL}_2(\mathfrak{o})$. The space X^K of K-fixed points in $\operatorname{Ind} \chi$ consists of the functions $f : G \to \mathbb{C}$ satisfying

$$f(bgk) = \chi(b) f(g), \quad b \in B,\ g \in G,\ k \in K. \tag{9.4.1}$$

We have $G = BK_0$, so the set $B \backslash G / K$ is finite, and each double coset BgK supports, at most, a one-dimensional space of functions satisfying (9.4.1) (*cf.* 3.5). Thus X^K is finite-dimensional, as required. $\square$

Remark. The irreducible cuspidal representations of G are likewise admissible, but the proof requires different techniques: see 10.2 Corollary below.

9.5. We introduce another notation. If (π, V) is a smooth representation of G and ϕ is a character of $F^\times$, we define a smooth representation $(\phi\pi, V)$ of G by setting

$$\phi\pi(g) = \phi(\det g)\,\pi(g), \quad g \in G. \tag{9.5.1}$$

One calls $\phi\pi$ the *twist* of π by ϕ.

Similarly for characters of T: if $\chi = \chi_1 \otimes \chi_2$ is a character of T and ϕ is a character of $F^\times$, then we put $\phi \cdot \chi = \phi\chi_1 \otimes \phi\chi_2$. If we inflate $\phi \cdot \chi$ to a representation of B trivial on N, we get $\phi \cdot \chi = (\phi \circ \det \mid B) \otimes \chi$. It follows immediately that

$$\mathrm{Ind}_B^G(\phi \cdot \chi) \cong \phi\,\mathrm{Ind}_B^G\,\chi. \tag{9.5.2}$$

This allows us to make convenient adjustments to the character χ without changing the essential structure of the induced representation.

9.6. We aim to give a precise account of the structure of representations of the form $\mathrm{Ind}_B^G\,\chi$. The main step is:

Irreducibility Criterion. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, and set $(\Sigma, X) = \mathrm{Ind}_B^G\,\chi$.*

(1) *The representation (Σ, X) is reducible if and only if $\chi_1\chi_2^{-1}$ is either the trivial character or the character $x \mapsto \|x\|^2$ of $F^\times$.*
(2) *Suppose that (Σ, X) is reducible. Then:*
   (a) *the G-composition length of X is 2;*
   (b) *one composition factor of X has dimension 1, the other is of infinite dimension;*
   (c) *X has a 1-dimensional G-subspace if and only if $\chi_1\chi_2^{-1} = 1$;*
   (d) *X has a 1-dimensional G-quotient if and only if $\chi_1\chi_2^{-1}(x) = \|x\|^2$, $x \in F^\times$.*

We will refine this to a classification of the irreducible principal series representations in 9.11 below. The proof of the theorem occupies paragraphs 9.7–9.9 to follow.

9.7. We use the notation of 9.6. Let

$$V = \{f \in X : f(1) = 0\}.$$

This is a B-subspace of X and we have an exact sequence

$$0 \to V \longrightarrow X \longrightarrow \mathbb{C} \to 0,$$

where the one-dimensional space $\mathbb{C} = X/V$ carries the character χ of T. By the Restriction-Induction Lemma (9.3), $V_N \cong \delta_B^{-1}\chi^w$.

Proposition. *Let W be the kernel $V(N)$ of the canonical map $V \to V_N$. The space W is irreducible over B.*

Proof. We shall actually prove that W is irreducible as a representation of the mirabolic group M of §8. We observe that, by (8.1.2), $W_N = 0$ and $W = W(N)$.

Lemma. *For $f \in V$, define a function $f_N \in C_c^\infty(N)$ by $f_N(n) = f(wn)$, $n \in N$. The map*

$$\begin{aligned} V &\longrightarrow C_c^\infty(N), \\ f &\longmapsto f_N, \end{aligned} \tag{9.7.1}$$

is an N-isomorphism.

Proof. As in 9.3 Lemma, the support of $f \in V$ is contained in $Bw\mathcal{N}$, for some compact open subgroup $\mathcal{N}$ of N. The assertions follow immediately (observing that the notation f_N here is not the same as that in 9.3). □

For $\phi \in C_c^\infty(N)$ and $a \in F^\times$, we define $a\phi \in C_c^\infty(N)$ by

$$a\phi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \chi_2(a)\, \phi \begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}.$$

This gives an action of $F^\times$ on $C_c^\infty(N)$ which we regard as an action of the group S of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We combine this with the natural action of N to give $C_c^\infty(N)$ the structure of a smooth representation of M. With this structure, the map (9.7.1) is a M-isomorphism.

Let ϑ be a non-trivial character of N. The map $f \mapsto \vartheta f$ is a linear automorphism of $V = C_c^\infty(N)$ carrying $V(N)$ to $V(\vartheta)$. Since $V/V(N)$ has dimension 1, we deduce that $\dim V_\vartheta = 1$ also. However, since N acts trivially on V_N, the inclusion $W \to V$ induces an isomorphism $W_\vartheta \cong V_\vartheta$, whence $W_\vartheta \cong \vartheta$. We apply 8.3 Theorem to get $W = W(N) \cong c\text{-}\mathrm{Ind}_N^M \vartheta$ which, by 8.2 Corollary, is irreducible. □

As a direct consequence of the Proposition, we have:

Corollary. *As a representation of B or of M, $\mathrm{Ind}_B^G \chi$ has composition length 3. Two of the composition factors have dimension one, and the third is of infinite dimension. In particular, the G-composition length of the representation $\mathrm{Ind}_B^G \chi$ is at most 3.*

9.8. We continue with the same notation and observe:

Proposition. *The following are equivalent:*

(1) $\chi_1 = \chi_2$;
(2) *X has a one-dimensional N-subspace.*

When these conditions hold,

(3) *X has a unique one-dimensional N-subspace X_0;*
(4) *X_0 is a G-subspace of X, and it is not contained in V.*

Proof. If (1) holds, we may as well take $\chi_1 = \chi_2 = 1$ (*cf.* (9.5.2)). The (non-zero) constant function then spans a one-dimensional G-subspace of X.

Conversely, let $f \in X$ span an N-stable subspace of dimension 1. Thus N acts on f (by right translation) as a character. The support of f is left-invariant under B, so $\operatorname{supp} f$ is either G or BwN. The second case is impossible: if $f(1) = 0$, then the support of f is confined to BwN_0, for some compact open subgroup N_0 of N (9.3 Lemma). Thus $\operatorname{supp} f = G$ and f vanishes nowhere. In particular $f(1) \neq 0$, and $f \notin V$. The canonical N-map $X \to \mathbb{C} = X/V$ identifies the N-space $\mathbb{C}f$ with the trivial N-space $\mathbb{C}$. It follows that N fixes f under right translation. Take $x \in F^\times$, and consider the identity

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

If $\|x\|$ is sufficiently large, then f is fixed under right translation by $\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$. So, as f is fixed by N, we have

$$f(w) = \chi_1(-1)\,\chi_1^{-1}\chi_2(x)\,f(1),$$

for all $x \in F^\times$ of sufficiently large absolute value. Thus $\chi_1 = \chi_2 = \phi$, say, and $f(g) = \phi(\det g)f(1)$. We have proved (1) $\Leftrightarrow$ (2), and that the one-dimensional N-subspace is uniquely determined. We have already shown that it is not contained in V. $\square$

9.9. We now finish the proof of the Irreducibility Criterion 9.6. Assume X is reducible. Its G-length is 2 or 3, and it has either a finite-dimensional G-subspace or a finite-dimensional G-quotient (9.7 Corollary).

Assume the first alternative. Thus X has a one-dimensional N-subspace, and we are in the situation of 9.8: X has a one-dimensional G-subspace L, and $\chi_1 = \chi_2 = \phi$, say. Moreover, G acts on L as the character $\phi \circ \det$ and $L \cap V = 0$ (notation of 9.7). The quotient $Y = X/L$ is therefore B-isomorphic to V. If X has G-length 3, then Y has G-length 2. However, V has B-length 2 and a unique B-quotient, which is of dimension 1. This gives a G-quotient of Y on which G must act as a character $\phi' \circ \det$ (9.2). This would force $\phi' \otimes \phi'$ to appear as a factor in the Jacquet module $Y_N \cong \phi \cdot \delta_B^{-1}$, which it cannot. Thus X has G-length 2 and we are in case (2)(c) of 9.6.

In the other alternative, X has a finite-dimensional G-quotient. The representation $\check{X}$ therefore has a finite-dimensional G-subspace, and we are back with the first alternative. By the Duality Theorem of 7.7, $\check{X} \cong \operatorname{Ind}_B^G \delta_B^{-1}\check{\chi}$, so we are in the case (2)(d) of 9.6.

Thus, in part (1) of the theorem, we have shown that X reducible implies χ has the stated form. The converse is given by 9.8 and the dual case. We have also proved statements (a)–(d). $\square$

9.10. To get a classification of the irreducible, non-cuspidal representations of G, we need to investigate the homomorphisms between induced representations:

Proposition. *Let χ, ξ be characters of T. The space* $\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi)$ *has dimension* 1 *if $\xi = \chi$ or $\chi^w \delta_B^{-1}$, zero otherwise.*

Proof. We use Frobenius Reciprocity (9.1.2):

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) \cong \mathrm{Hom}_T((\mathrm{Ind}\,\chi)_N, \xi).$$

The Jacquet module $(\mathrm{Ind}\,\chi)_N$ fits into the exact sequence

$$0 \to \chi^w \delta_B^{-1} \longrightarrow (\mathrm{Ind}\,\chi)_N \longrightarrow \chi \to 0.$$

If we assume that $\chi \neq \chi^w \delta_B^{-1}$, then this sequence splits and the result follows immediately. The equation $\chi = \chi^w \delta_B^{-1}$ amounts to $\chi_1(x) = \|x\|\chi_2(x)$, $x \in F^\times$. In this case, $\mathrm{Ind}\,\chi$ is irreducible and the result again follows. □

By way of some examples, we examine in more detail the case where $\mathrm{Ind}_B^G \chi$ is *reducible.* Thus there is a character ϕ of $F^\times$ such that $\chi = \phi \cdot 1_T$ or $\chi = \phi \cdot \delta_B^{-1}$. Twisting does not affect the situation materially, so we assume $\phi = 1$.

Consider first the case $\mathrm{Ind}_B^G 1_T$. The irreducible G-quotient of $\mathrm{Ind}_B^G 1_T$ is called the *Steinberg representation* of G, and is denoted St_G:

$$0 \to 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow \mathrm{St}_G \to 0. \tag{9.10.3}$$

Its dimension is infinite and $(\mathrm{St}_G)_N \cong \delta_B^{-1}$. Likewise, if ϕ is a character of $F^\times$, we have an exact sequence

$$0 \to \phi_G \longrightarrow \mathrm{Ind}_B^G \phi_T \longrightarrow \phi \cdot \mathrm{St}_G \to 0,$$

where $\phi_G = \phi \circ \det$ and $\phi_T = \phi \otimes \phi$. (Representations of the form $\phi \cdot \mathrm{St}_G$ are sometimes called *special.*)

Taking the smooth dual of (9.10.3), we get an exact sequence

$$0 \to \mathrm{St}_G^\vee \longrightarrow \mathrm{Ind}_B^G \delta_B^{-1} \longrightarrow 1_G \to 0. \tag{9.10.4}$$

The proposition implies

$$\mathrm{St}_G \cong \mathrm{St}_G^\vee. \tag{9.10.5}$$

Remark. The proposition also implies that the space $\mathrm{End}_G(\mathrm{Ind}\,1_T)$ has dimension 1, while $\mathrm{Ind}\,1_T$ is not irreducible. Thus the converse of Schur's Lemma fails in this context, as remarked in 2.6.

Observe also the imperfect parallelism between the proposition above and the corresponding result (6.3) for the finite field case.

9.11. We introduce a new notation. If σ is a smooth representation of T, we define

$$\boldsymbol{\iota}_B^G\,\sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2}\otimes\sigma). \tag{9.11.1}$$

This provides another exact functor $\mathrm{Rep}(T) \to \mathrm{Rep}(G)$, known as *normalized* or *unitary* smooth induction. It gives rise to more convenient combinatorics, for example:

$$\big(\boldsymbol{\iota}_B^G\,\sigma\big)^\vee \cong \boldsymbol{\iota}_B^G\,\check{\sigma}. \tag{9.11.2}$$

In this language, the Irreducibility Criterion (9.6) and 9.10 Proposition say:

Lemma.

(1) *Let $\chi = \chi_1\otimes\chi_2$ be a character of T. The representation $\boldsymbol{\iota}_B^G\,\chi$ is reducible if and only if $\chi_1\chi_2^{-1}$ is one of the characters $x \mapsto \|x\|^{\pm1}$ of $F^\times$ or, equivalently, $\chi = \phi\cdot\delta_B^{\pm1/2}$ for some character ϕ of $F^\times$.*
(2) *Let χ, ξ be characters of T. The space $\mathrm{Hom}_G(\boldsymbol{\iota}_B^G\,\chi, \boldsymbol{\iota}_B^G\,\xi)$ is not zero if and only if $\xi = \chi$ or $\xi = \chi^w$.*

Gathering up our earlier arguments and results, we get:

Classification Theorem. *The following is a complete list of the isomorphism classes of irreducible, non-cuspidal representations of G:*

(1) *the irreducible induced representations $\boldsymbol{\iota}_B^G\,\chi$, where $\chi \neq \phi\cdot\delta_B^{\pm1/2}$ for any character ϕ of $F^\times$;*
(2) *the one-dimensional representations $\phi\circ\det$, where ϕ ranges over the characters of $F^\times$;*
(3) *the special representations $\phi\cdot\mathrm{St}_G$, where ϕ ranges over the characters of $F^\times$.*

The classes in this list are all distinct except that, in (1), we have $\boldsymbol{\iota}_B^G\,\chi \cong \boldsymbol{\iota}_B^G\,\chi^w$.

Proof. The examples in 9.10 show that every irreducible, non-cuspidal representation of G appears in this list. The relations between the irreducibly induced ones are given by 9.10 Proposition. The same result also implies that no special representation $\phi\cdot\mathrm{St}_G$ can be equivalent to $\phi'\cdot\mathrm{St}_G$, $\phi'\neq\phi$, or any irreducibly induced representation. □

10. Cuspidal Representations and Coefficients

In 9.1, we defined an irreducible smooth representation (π, V) of $\mathrm{GL}_2(F)$ to be cuspidal if its Jacquet module (π_N, V_N) is trivial or, equivalently, if it is not isomorphic to a composition factor of an induced representation $\mathrm{Ind}_B^G\,\chi$, for any character χ of T. Such a negative and exclusive approach yields essentially no information about this particularly important class of representations. We now give an alternative definition, valid in a much wider context, and show it is equivalent to the original one. It provides the starting point for a method of constructing cuspidal representations.

From this new viewpoint, irreducible cuspidal representations have a striking algebraic property: they are projective objects in the appropriate subcategory of $\mathrm{Rep}(G)$. We have no direct need for this result, but we have included it as an appendix.

10.1. Let (π, V) be a smooth representation of $G = \mathrm{GL}_2(F)$; from vectors $v \in V$, $\check{v} \in \check{V}$, we get a smooth function on G by

$$\gamma_{\check{v}\otimes v} : g \longmapsto \langle \check{v}, \pi(g)v \rangle.$$

We let $\mathcal{C}(\pi)$ be the vector space spanned by the functions $\gamma_{\check{v}\otimes v}$, $\check{v}\otimes v \in \check{V}\otimes V$. The functions $f \in \mathcal{C}(\pi)$ are called the *(matrix) coefficients* of π.

The space $\check{V}\otimes V$ carries a smooth representation of the group $G\times G$, while $G \times G$ acts on the function space $\mathcal{C}(\pi)$ by translation: the first factor acts by left translation and the second by right translation. The map $\check{v}\otimes v \mapsto \gamma_{\check{v}\otimes v}$ is then a surjective $G \times G$-homomorphism $\check{V}\otimes V \to \mathcal{C}(\pi)$.

The space $\mathcal{C}(\pi)$ is primarily, but not exclusively, of interest in the case where π is irreducible. When π is irreducible, the centre Z of G acts on V via the central character ω_π of π and

$$\gamma(zg) = \omega_\pi(z)\gamma(g), \quad z \in Z, g \in G, \gamma \in \mathcal{C}(\pi).$$

The support of a coefficient is therefore invariant under translation by Z.

Definition. *Let (π, V) be an irreducible smooth representation of G; one says that π is* γ-cuspidal *if every $\gamma \in \mathcal{C}(\pi)$ is compactly supported modulo Z.*

The term "γ-cuspidal" is a convenient, but temporary, expedient.

Convention. To save adjectives, if a representation is described as cuspidal or γ-cuspidal, it is implicitly assumed to be smooth.

We first achieve some technical control:

Proposition.

(1) *If (π, V) is an irreducible γ-cuspidal representation of G, then π is admissible.*
(2) *Let (π, V) be an irreducible admissible representation of G, and suppose that some non-zero coefficient of π is compactly supported modulo Z; then π is γ-cuspidal.*

Proof. In part (1), we suppose for a contradiction that π is not admissible. We choose a compact open subgroup K such that V^K has infinite dimension. This dimension, we note, is countable (2.6). The dimension of $\check{V}^K \cong \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ is therefore uncountable.

We fix a non-zero $v \in V^K$ and consider the map $\Gamma_v : \check{V}^K \to \mathcal{C}(\pi)$ given by $\check{v} \mapsto \gamma_{\check{v}\otimes v}$. Since the translates gv, $g \in G$, span V, the map Γ_v is injective. Its image is a space of functions f on G, satisfying

$$f(zkgk') = \omega_\pi(z) f(g), \quad g \in G,\ z \in Z,\ k, k' \in K, \tag{10.1.1}$$

and supported on a finite union of cosets $ZKgK$. The dimension of $\Gamma_v(\check{V}^K)$ is therefore at most countable, while Γ_v is injective and $\dim \check{V}^K$ is uncountable. This gives the desired contradiction.

We turn to part (2). The smooth dual $(\check{\pi}, \check{V})$ is irreducible and admissible (2.10). We view the space $\check{V} \otimes V$ as a smooth representation of $G \times G$, and hence as a smooth module over $\mathcal{H}(G \times G) = \mathcal{H}(G) \otimes \mathcal{H}(G)$.

If K is a compact open subgroup of G, we have

$$(\check{V} \otimes V)^{K\times K} = (e_K \otimes e_K) * (\check{V} \otimes V) = \check{V}^K \otimes V^K.$$

If K is sufficiently small, the spaces V^K, $\check{V}^K$ are finite-dimensional simple modules over $\mathcal{H}(G,K)$. The Jacobson Density Theorem implies that $\check{V}^K \otimes V^K$ is a simple module over $\mathcal{H}(G,K) \otimes \mathcal{H}(G,K) \cong \mathcal{H}(G \times G, K \times K)$. This holds for all sufficiently small K, so $\check{V} \otimes V$ is an irreducible admissible $G \times G$-space (4.3 Corollary).

The surjective $G \times G$-homomorphism $\gamma : \check{V} \otimes V \to \mathcal{C}(\pi)$ is therefore an isomorphism and $\mathcal{C}(\pi)$ is irreducible over $G \times G$. If $\gamma \in \mathcal{C}(\pi)$ is non-zero, any $\gamma' \in \mathcal{C}(\pi)$ is a finite linear combination of functions $(g,h)\gamma$, $(g,h) \in G \times G$. If γ is compactly supported modulo Z, then so is γ'. $\square$

Remark 1. All of the preceding definitions and arguments apply in the general case, where G is a unimodular locally profinite group satisfying 2.6 Hypothesis. Indeed, 2.6 is only used at one point, in the proof of part (1) of the proposition. Even this can be avoided by noting that the dual of a vector space W has dimension strictly greater than $\dim W$ except when $\dim W$ is finite.

Remark 2. In the general context of Remark 1, part (2) of the proposition *fails* when the irreducible smooth representation (π, V) is not admissible. An example is given by the representation $c\text{-}\mathrm{Ind}_N^M \vartheta$ considered in 8.2: see especially 8.2 Remark.

10.2. The reason for introducing the notion of γ-cuspidality is explained by:

Theorem. *Let (π, V) be an irreducible smooth representation of G; then π is cuspidal if and only if it is γ-cuspidal.*

Proof. We first assume that π is cuspidal, and show it is γ-cuspidal. Let ϖ be a prime element of F, and put

$$t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

The set T^+ of powers t^n, $n \geqslant 0$, then provides a family of representatives for $ZK\backslash G/K$, $K = \mathrm{GL}_2(\mathfrak{o})$ (7.2.2).

Lemma. *Let $v \in V$, $\check{v} \in \check{V}$. There exists $m \geqslant 0$ such that $\gamma_{\check{v}\otimes v}(t^n) = 0$, for all $n \geqslant m$.*

Proof. We choose a compact open subgroup N_1 of N which fixes $\check{v}$. Since $V_N = 0$, we have $v \in V(N)$ and (8.1) there is a compact open subgroup N_2 of N such that

$$\int_{N_2} \pi(x)v\,dx = 0.$$

We then have

$$\int_{N_0} \pi(x)v\,dx = 0,$$

for any compact open subgroup N_0 of N containing N_2. However, there exists $m \geqslant 0$ such that $t^a N_2 t^{-a} \subset N_1$ for all $a \geqslant m$. For such a we have (for certain positive constants k_1, k_2)

$$\begin{aligned}
\langle \check{v}, \pi(t^a)v\rangle &= k_1 \int_{N_1} \langle \check{\pi}(x^{-1})\check{v}, \pi(t^a)v\rangle\,dx \\
&= k_1 \int_{N_1} \langle \check{\pi}(t^{-a})\check{v}, \pi(t^{-a}xt^a)v\rangle\,dx \\
&= k_2 \int_{t^{-a}N_1t^a} \langle \check{\pi}(t^{-a})\check{v}, \pi(x)v\rangle\,dx \\
&= 0,
\end{aligned}$$

since $t^{-a}N_1t^a \supset N_2$. $\square$

Continuing with the proof of the theorem, we fix a non-zero coefficient $f = \gamma_{\check{v}\otimes v}$ of π. We write $K = \mathrm{GL}_2(\mathfrak{o})$ and let K' be an open normal subgroup of K fixing both $\check{v}$ and v. We let $k_1, k_2, \ldots, k_r$ be a set of coset representatives for K/K'. Thus, if $g \in G$, there exists $n \geqslant 0$ such that

$$ZKgK = ZKt^nK = \bigcup_{i,j} ZK'k_i^{-1}t^nk_jK'.$$

It follows that

$$\operatorname{supp} f \subset \bigcup_{1\leqslant i,j\leqslant r} ZK'\,(\operatorname{supp} f_{ij} \cap T^+)\,K',$$

where f_{ij} denotes the coefficient function $x \mapsto f(k_i x k_j^{-1})$. This set is compact modulo Z, by the lemma. It follows that all coefficients $\gamma_{\check{v}\otimes v}$ of π are compactly supported modulo Z, and π is therefore γ-cuspidal.

Combining this argument with 9.4 Proposition and 10.1 Proposition, we have shown:

Corollary. *Every irreducible smooth representation of* $G = \mathrm{GL}_2(F)$ *is admissible.*

We now prove the converse statement in the theorem. Let (π, V) be an irreducible γ-cuspidal representation of G. In particular, (π, V) is admissible. By 2.10 Proposition, the dual $(\check{\pi}, \check{V})$ is irreducible and admissible. Let K_n denote the group $1+\mathfrak{p}^n \mathrm{M}_2(\mathfrak{o})$, $n \geqslant 1$. We take $v \in V$ and choose $n \geqslant 1$ so that v is fixed by $\pi(K_n)$. We take t as before.

For $\check{v} \in \check{V}^{K_n}$, the function $g \mapsto \langle \check{v}, \pi(g)v\rangle$ is compactly supported modulo Z; we deduce that $\langle \check{v}, \pi(t^a)v\rangle = 0$ for all $a \in \mathbb{Z}$ sufficiently large. Since $\check{V}^{K_n}$ is of finite dimension there is a constant c such that $\langle \check{v}, \pi(t^a)v\rangle = 0$ for all $\check{v} \in \check{V}^{K_n}$ and all $a \geqslant c$. This implies $\pi(e_{K_n})\pi(t^a)v = 0$ for $a \geqslant c$. We write, for $j \in \mathbb{Z}$,

$$N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}, \quad N'_j = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^j & 1 \end{pmatrix}, \quad T_n = K_n \cap T,$$

so that $K_n = N_n T_n N'_n$. Set $K_n^{(a)} = t^{-a} K_n t^a = N_{n-a} T_n N'_{n+a}$; we then have, for $a \geqslant c$,

$$\begin{aligned} 0 &= \pi(e_{K_n})\pi(t^a)v = \pi(t^a)\pi(e_{K_n^{(a)}})v \\ &= \pi(t^a) \sum_{x \in N_{n-a}/N_n} \pi(x)\pi(e_{K_n^{(a)} \cap K_n})v. \end{aligned}$$

However, v is fixed by $K_n^{(a)} \cap K_n$, so this equation reduces to

$$0 = k\,\pi(t^a) \int_{N_{n-a}} \pi(x)v\,dx,$$

for a constant $k > 0$ depending on the choice of a Haar measure dx on N. We deduce that $v \in V(N)$ (8.1 Lemma). This applies to all $v \in V$, so π is cuspidal, as required. $\square$

10a. Appendix: Projectivity Theorem

We give another property of γ-cuspidal representations. We will not use this result, but we have included it for its power and beauty. It also holds in a very general context.

10a.1. Let G be a locally profinite group with centre Z. Let χ be a character of Z and let (π, V) be a smooth representation of G. We recall that π *admits* χ *as central character* if $\pi(z)v = \chi(z)v$, for $z \in Z$, $v \in V$.

Projectivity Theorem. *Let G be a unimodular locally profinite group, satisfying* (2.6) *and with centre Z. Assume that any character $\chi : Z \to \mathbb{R}_+^\times$ extends to a character $G \to \mathbb{R}_+^\times$.*

Let (π, V) be an irreducible γ-cuspidal representation of G, and let (τ, U) be a smooth representation of G admitting ω_π as central character. Let $f : U \to V$ be a surjective G-homomorphism. There exists a G-homomorphism $\phi : V \to U$ such that $f \circ \phi = 1_V$.

10a.2. The group G/Z is locally profinite. One sees easily that it is unimodular. Indeed, let μ_G, μ_Z be Haar measures on G, Z respectively. By 3.4 Proposition, there is a unique right Haar measure $\dot\mu$ on G/Z such that

$$\int_G f(g)\, d\mu_G(g) = \int_{G/Z} \int_Z f(zg)\, d\mu_Z(z)\, d\dot\mu(g), \quad f \in C_c^\infty(G).$$

Symmetrically, $\dot\mu$ is also a left Haar measure on G/Z.

Schur's orthogonality relation. *Let $d\dot g$ be a Haar measure on G/Z, and let $v_1, v_2 \in V$, $\check v_1, \check v_2 \in \check V$. The function*

$$g \longmapsto \langle \check\pi(g)\check v_1, v_1\rangle \langle \check v_2, \pi(g)v_2\rangle, \quad g \in G,$$

is invariant under translation by Z and

$$\int_{G/Z} \langle \check\pi(g)\check v_1, v_1\rangle \langle \check v_2, \pi(g)v_2\rangle\, d\dot g = d(\pi)^{-1}\, \langle \check v_1, v_2\rangle\, \langle \check v_2, v_1\rangle,$$

for a constant $d(\pi) > 0$ depending only on π and the measure $d\dot g$.

Proof. Since π is γ-cuspidal, the integrand has compact support in G/Z and the integral converges. If we fix, say, $\check v_1$ and v_2, the integral determines a G-invariant pairing $\check V \times V \to \mathbb{C}$. Such a pairing is given by a G-homomorphism $\Theta : \check V \to \check V$ (*cf.* Exercise 2.10). Since V is admissible (10.1 Proposition) and irreducible, the same applies to $\check V$ (2.10) and Schur's Lemma (2.6) implies that any G-invariant pairing $\check V \times V \to \mathbb{C}$ is a scalar multiple of the standard one. Therefore, there is a constant $c_{\check v_1, v_2}$ such that

$$\int_{G/Z} \langle \check\pi(g)\check v_1, v_1\rangle \langle \check v_2, \pi(g)v_2\rangle\, d\dot g = c_{\check v_1, v_2}\, \langle \check v_2, v_1\rangle.$$

The function $(\check v_1, v_2) \mapsto c_{\check v_1, v_2}$ is again a G-invariant bilinear pairing $\check V \times V \to \mathbb{C}$, so

$$c_{\check v_1, v_2} = c_\pi\, \langle \check v_1, v_2\rangle,$$

for a constant c_π.

It remains only to prove that $c_\pi > 0$. The assumption on G allows us to replace π by a twist and assume that $|\omega_\pi| = 1$. The space V then admits a positive definite, G-invariant Hermitian form h, constructed as follows. One chooses a nonzero element $\check{v} \in \check{V}$ and sets

$$h(v_1, v_2) = \int_{G/Z} \overline{\langle \check{v}, \pi(g)v_1\rangle} \langle \check{v}, \pi(g)v_2\rangle \, d\dot{g}.$$

There is then a complex anti-linear G-isomorphism $\Theta : (\pi, V) \to (\check{\pi}, \check{V})$ such that $h(v_1, v_2) = \langle \Theta v_1, v_2\rangle$. Schur's Lemma again implies that h is the unique G-invariant, positive definite Hermitian form on V, up to a positive constant factor.

Going through the same argument, one sees that

$$\int_{G/Z} h(\pi(g)v_1, v_2)\, h(v_3, \pi(g)v_4)\, d\dot{g} = b_\pi\, h(v_1, v_4)\, h(v_3, v_2),$$

for a constant b_π. On taking $v_1 = v_2 = v_3 = v_4 \neq 0$, one sees that $b_\pi > 0$. On the other hand,

$$\begin{aligned} \int_{G/Z} h(\pi(g)v_1, v_2)\, h(v_3, \pi(g)v_4)\, d\dot{g} &= \int_{G/Z} \langle \check{\pi}(g)\Theta(v_1), v_2)\rangle \langle \Theta(v_3), \pi(g)v_4\rangle \, d\dot{g} \\ &= c_\pi \, \langle \Theta(v_1), v_4\rangle \langle \Theta(v_3), v_2\rangle \\ &= c_\pi \, h(v_1, v_4)\, h(v_3, v_2). \end{aligned}$$

Therefore $c_\pi = b_\pi > 0$, as required. □

Remark. Let (π, V) be an irreducible smooth representation of G such that $|\omega_\pi| = 1$. One says that π is *square-integrable modulo Z* if

$$\int_{G/Z} |\langle \check{v}, \pi(g)v\rangle|^2 \, d\dot{g} < \infty$$

for all $\check{v} \in \check{V}$, $v \in V$. The orthogonality relation then holds for π, with exactly the same proof. The positive constant $d(\pi)$ is called the *formal degree* of π, relative to the measure $d\dot{g}$. (For a full discussion of square-integrable representations of $\mathrm{GL}_2(F)$, see 17.4 *et seq.* below.)

10a.3. We now prove the Projectivity Theorem. First, we need to generalize the constructions of 4.1, 4.2. Let χ be a character of $F^\times$. Let $\mathcal{H}_\chi(G)$ be the space of locally constant functions $f : G \to \mathbb{C}$, which are compactly supported modulo Z, such that $f(zg) = \chi(z)^{-1} f(g)$, $z \in Z$, $g \in G$. Using a Haar measure on G/Z, we define convolution on $\mathcal{H}_\chi(G)$ as in (4.1). If (σ, W) is a smooth representation of G admitting χ as central character, we extend the action of G on W to one of $\mathcal{H}_\chi(G)$, just as in 4.2.

We let (π, V) be an irreducible γ-cuspidal representation of G, as in the theorem. We abbreviate $\omega = \omega_\pi$. We take a smooth representation (τ, U) of G, admitting ω as central character, and a G-surjection $f : U \to V$. If $u \in U$ satisfies $f(u) \neq 0$, the restriction of f to the G-space $\tau(\mathcal{H}_\omega(G))u$ generated by u is still surjective. Composing it with the obvious G-surjection $\mathcal{H}_\omega(G) \to \tau(\mathcal{H}_\omega(G))u$, we get a G-surjection

$$\begin{aligned} \varPi : \mathcal{H}_\omega(G) &\longrightarrow V, \\ \phi \longmapsto f(\tau(\phi)u) &= \pi(\phi)v_0, \end{aligned}$$

where $v_0 = f(u)$. It is enough to show that $\varPi$ splits over G.

We choose a vector $\check{v}_0 \in \check{V}$ such that $d(\pi)^{-1}\langle \check{v}_0, v_0\rangle = 1$. The function $\phi_v : g \mapsto \langle \check{\pi}(g)\check{v}_0, v\rangle$ lies in $\mathcal{H}_\omega(G)$ and the map

$$\begin{aligned} \varPhi : V &\longrightarrow \mathcal{H}_\omega(G), \\ v &\longmapsto \phi_v, \end{aligned}$$

is a G-homomorphism. The composite map $\varPi \circ \varPhi$ is given by

$$w \longmapsto \pi(\phi_w)v_0 = \int_{G/Z} \pi(g)\phi_w(g)v_0\, d\dot{g}, \quad w \in V.$$

For $\check{w} \in \check{V}$, this gives

$$\langle \check{w}, \varPi\varPhi(w)\rangle = \int_{G/Z} \langle \check{w}, \check{\pi}(g)v_0\rangle\langle \check{\pi}(g)\check{v}_0, w\rangle\, d\dot{g} = \langle \check{w}, w\rangle,$$

whence $\varPi\varPhi(w) = w$, as required. □

11. Intertwining, Compact Induction and Cuspidal Representations

We describe a method for constructing irreducible cuspidal representations of $G = \mathrm{GL}_2(F)$, using compact induction from open subgroups. At this stage, it is a purely formal matter: it is not clear that the necessary hypotheses are satisfied sufficiently often to give useful results. Such issues are the subject of the next chapter. Here, we have to be content with one interesting example.

11.1. We start with general considerations so, for the time being, G is a unimodular locally profinite group with the countability property 2.6. Throughout, Z denotes the centre of G. If K is a compact open subgroup of G, we write $\widehat{K}$ for the set of isomorphism classes of irreducible smooth representations of K.

Definition 1. *For $i = 1, 2$, let K_i be a compact open subgroup of G and let $\rho_i \in \widehat{K}_i$. Let $g \in G$. The element g* intertwines *ρ_1* with *ρ_2 if*

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0,$$

where ρ_1^g denotes the representation $x \mapsto \rho_1(gxg^{-1})$ of the group $K_1^g = g^{-1}K_1g$.

As a property of g, this depends only on the double coset K_1gK_2.

The definition applies equally if the K_i are just closed subgroups of G; we will often need it in the case where the K_i are open and compact modulo the centre of G.

Definition 2. *Let K be a compact open subgroup of G, and let (π, V) be a smooth representation of G. We say that π* contains *ρ, or ρ* occurs in *π, if* $\mathrm{Hom}_K(\rho, \pi) \neq 0$.

Again, we can use the same definition in more general contexts, for example, if K is open and compact modulo the centre of G and π admits a central character (see 2.7). We also use it when G is compact and K is a closed subgroup of G.

Remaining with the compact open case for the time being, the significance of the concept of intertwining is first indicated by the following.

Proposition 1. *For $i = 1, 2$, let K_i be a compact open subgroup of G and let $\rho_i \in \widehat{K}_i$. Let (π, V) be an irreducible smooth representation of G which contains both ρ_1 and ρ_2. There then exists $g \in G$ which intertwines ρ_1 with ρ_2.*

Proof. For each i, we have the decomposition of V into K_i-isotypic components (2.3 Proposition). The hypothesis is equivalent to $V^{\rho_i} \neq 0$, $i = 1, 2$.

Let e_2 denote the K_2-projection $V \to V^{\rho_2}$. Since π is irreducible and $V^{\rho_1} \neq 0$, the spaces $\pi(g^{-1})V^{\rho_1} = V^{\rho_1^g}$, $g \in G$, span V. We can therefore choose $g \in G$ such that $e_2 \circ \pi(g^{-1})$ induces a non-zero map $V^{\rho_1} \to V^{\rho_2}$: this is the required element g. $\square$

Take (K_i, ρ_i) as in the Proposition. The representations ρ_1^g, ρ_2 of $K_1^g \cap K_2$ are semisimple, so the spaces

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2), \qquad \mathrm{Hom}_{K_1^g \cap K_2}(\rho_2, \rho_1^g) \cong \mathrm{Hom}_{K_1 \cap K_2^{g^{-1}}}(\rho_2^{g^{-1}}, \rho_1)$$

have the same dimension. Therefore g intertwines ρ_1 with ρ_2 if and only if g^{-1} intertwines ρ_2 with ρ_1.

Language.

(1) We say that the ρ_i *intertwine in* G if there exists $g \in G$ which intertwines ρ_1 with ρ_2. The relation of G-intertwining between pairs (K_i, ρ_i) is therefore symmetric and reflexive; it is not transitive.
(2) If we have a single pair (K, ρ), we say that g intertwines ρ if it intertwines ρ with itself.

Remark. We will often wish to use this approach when K is not compact, but only an open subgroup of G which is *compact modulo* Z. One cannot, in general, decompose a smooth representation (π, V) of G into a direct sum of K-isotypic components. Such a decomposition does exist (2.7) if (π, V) admits a central character ω_π, in particular, if π is irreducible. With this caveat, we can treat open, compact modulo centre subgroups of G in the same way as compact open subgroups.

We will later (in Chapter VI) need another intertwining criterion. (We use the notation of 4.4 here.)

Proposition 2. *Let* K *be a compact open subgroup of* G, *let* $g \in G$, *and* $\rho \in \widehat{K}$. *The following are equivalent:*

(1) *there exists* $f \in e_\rho * \mathcal{H}(G) * e_\rho$ *such that* $f \mid KgK \neq 0$;
(2) g *intertwines* ρ.

Proof. Consider the space $C^\infty(KgK)$ of G-smooth functions on the coset KgK. This carries a smooth representation of $K \times K$ by

$$(k_1, k_2)f : x \longmapsto f(k_1^{-1} x k_2).$$

Let H denote the group of pairs $(k, g^{-1}kg) \in K \times K$, $k \in K \cap gKg^{-1}$. The map $f \mapsto f(g)$ is then an H-homomorphism $C^\infty(KgK) \to \mathbb{C}$ (with H acting trivially). By Frobenius Reciprocity (2.4), this induces a $K \times K$-homomorphism

$$C^\infty(KgK) \longrightarrow \mathrm{Ind}_H^{K\times K}(1_H). \tag{11.1.1}$$

We show this is an isomorphism.

The space $V = \mathrm{Ind}_H^{K\times K}(1_H)$ consists, by definition, of smooth functions $\phi : K \times K \to \mathbb{C}$ such that $\phi(hk_1, g^{-1}hgk_2) = \phi(k_1, k_2)$, $k_i \in K$, $h \in K \cap gKg^{-1}$. Given such a function ϕ, we can define $f_\phi \in C^\infty(KgK)$ by setting $f_\phi(k_1 g k_2) = \phi(k_1^{-1}, k_2)$; the map $\phi \mapsto f_\phi$ is the inverse of the map (11.1.1).

In these terms, condition (1) amounts to $e_\rho * C^\infty(KgK) * e_\rho \cong V^{\rho\otimes\check{\rho}} \neq 0$. Equivalently,

$$\mathrm{Hom}_{K\times K}(\rho \otimes \check{\rho}, V) \cong \mathrm{Hom}_H(\rho \otimes \check{\rho}, 1_H) \neq 0.$$

The last relation is equivalent to the representation $k \mapsto \rho(k) \otimes \check{\rho}(g^{-1}kg)$ of $K \cap gKg^{-1}$ having a fixed vector, that is, $\mathrm{Hom}_{K\cap gKg^{-1}}(\rho, \rho^{g^{-1}}) \neq 0$, as required. $\square$

11.2. Let K be an open subgroup of G, containing and compact modulo the centre Z of G. Let (ρ, W) be an irreducible smooth representation of K. Let $\mathcal{H}(G,\rho)$ be the space of functions $f : G \to \mathrm{End}_{\mathbb{C}}(W)$ which are compactly supported modulo Z and satisfy

$$f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2), \quad k_i \in K,\ g \in G.$$

Observe that the support of any $f \in \mathcal{H}(G,\rho)$ is a finite union of double cosets KgK.

Let $\dot\mu$ be a Haar measure on G/Z. For $\phi_1, \phi_2 \in \mathcal{H}(G,\rho)$, we set

$$\phi_1 * \phi_2(g) = \int_{G/Z} \phi_1(x)\phi_2(x^{-1}g)\, d\dot\mu(x), \quad g \in G.$$

The function $\phi_1 * \phi_2$ lies in $\mathcal{H}(G,\rho)$ and, under this operation of convolution, the space $\mathcal{H}(G,\rho)$ is an associative $\mathbb{C}$-algebra with 1.

Remark. The algebra $\mathcal{H}(G,\rho)$ is called the *ρ-spherical Hecke algebra* of G, or the *intertwining algebra* of ρ in G. It is closely related to the algebra $e_\rho * \mathcal{H}(G) * e_\rho$: there is a canonical algebra isomorphism $e_\rho * \mathcal{H}(G) * e_\rho \cong \mathcal{H}(G,\rho) \otimes \mathrm{End}_{\mathbb{C}}(W)$. (In the literature, the algebra we have defined is sometimes denoted $\mathcal{H}(G,\check\rho)$.)

Lemma. *Let $g \in G$; there exists $\phi \in \mathcal{H}(G,\rho)$ with support KgK if and only if g intertwines ρ.*

Proof. Let $f \in \mathrm{End}_{\mathbb{C}}(W)$; for a fixed $g \in G$, the assignment $kgk' \mapsto \rho(k) f \rho(k')$, $k, k' \in K$, gives an element of $\mathcal{H}(G,\rho)$ if and only if, for $k \in K^g \cap K$, we have $f \circ \rho(k) = \rho^g(k) \circ f$. That is, if and only if $f \in \mathrm{Hom}_{K^g \cap K}(\rho, \rho^g)$. The representations ρ, ρ^g of $K \cap K^g$ are semisimple, so the spaces $\mathrm{Hom}_{K^g\cap K}(\rho,\rho^g)$, $\mathrm{Hom}_{K^g\cap K}(\rho^g,\rho)$ have the same dimension. The Lemma now follows. □

We have actually shown that the space of functions $f \in \mathcal{H}(G,\rho)$ supported on KgK is canonically isomorphic to $\mathrm{Hom}_{K^g\cap K}(\rho,\rho^g)$.

11.3. With (K,ρ) as in 11.2, we consider the compactly induced representation $c\text{-}\mathrm{Ind}_K^G\, \rho$, as in 2.5. The space X underlying this representation consists of the functions $f : G \to W$, which are compactly supported modulo Z, and satisfy $f(kg) = \rho(k) f(g)$, $k \in K$, $g \in G$. The group G acts by right translation. (All functions $f \in X$ are G-smooth for this action, since K is open: see 2.5 Exercise 2.)

For $\phi \in \mathcal{H}(G,\rho)$ and $f \in c\text{-}\mathrm{Ind}\, \rho$, we define

$$\phi * f(g) = \int_{G/Z} \phi(x) f(x^{-1}g)\, d\dot\mu(x), \quad g \in G.$$

Clearly, $\phi * f \in X$, and this action gives a homomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}(G,\rho) \longrightarrow \mathrm{End}_G(c\text{-}\mathrm{Ind}\, \rho). \tag{11.3.1}$$

Proposition. *The map* (11.3.1) *is an isomorphism of* $\mathbb{C}$*-algebras.*

Proof. We use the relation $\mathrm{End}_G(c\text{-Ind}\,\rho) \cong \mathrm{Hom}_K(\rho, c\text{-Ind}\,\rho)$ of (2.5.2). Let $\phi^0 : w \to \phi^0_w$ be the canonical map $W \to c\text{-Ind}\,\rho$, corresponding to the identity endomorphism of $c\text{-Ind}\,\rho$: the function ϕ^0_w has support K and $\phi^0_w(k) = \rho(k)w$. The isomorphism $\mathrm{End}_G(c\text{-Ind}\,\rho) \to \mathrm{Hom}_K(\rho, c\text{-Ind}\,\rho)$ is composition with ϕ^0. Composing (11.3.1) with ϕ^0, we get a map $\mathcal{H}(G,\rho) \to \mathrm{Hom}_K(W, c\text{-Ind}\,\rho)$. We write down its inverse. Let

$$\begin{aligned} \phi : W &\longrightarrow c\text{-Ind}\,\rho, \\ w &\longmapsto \phi_w, \end{aligned}$$

be a K-homomorphism. We define a function $\varPhi : G \to \mathrm{End}_{\mathbb{C}}(W)$ by

$$\varPhi(g) : w \longmapsto \phi_w(g).$$

For $k \in K$, we have $\varPhi(kg) : w \mapsto \phi_w(kg) = \rho(k)\phi_w(g)$, so $\varPhi(kg) = \rho(k)\varPhi(g)$. Also, $\varPhi(gk) : w \mapsto \phi_w(gk) = \phi_{\rho(k)w}(g)$, since ϕ is a K-map. Therefore $\varPhi \in \mathcal{H}(G,\rho)$ and $\phi \mapsto \dot{\mu}(K/Z)^{-1}\varPhi$ is the required inverse map. □

11.4. The central result of this section is:

Theorem. *Let K be an open subgroup of $G = \mathrm{GL}_2(F)$, containing and compact modulo Z. Let (ρ, W) be an irreducible smooth representation of K and suppose that an element $g \in G$ intertwines ρ if and only if $g \in K$. The compactly induced representation $c\text{-Ind}_K^G\,\rho$ is then irreducible and cuspidal.*

Proof. We write $(\varSigma, X) = c\text{-Ind}_K^G\,\rho$. We first show that the representation $\varSigma$ has a non-zero coefficient which is compactly supported modulo Z. To see this, we use the canonical K-embedding $\phi^0 : W \to X$ of the preceding proof, which identifies W with the space of functions in X that are supported in K (2.5 Lemma).

The groups K, G are unimodular, so the Duality Theorem of 3.5 implies that $\check{X} \cong \mathrm{Ind}_K^G\,\check{\rho}$. The induced representation $\mathrm{Ind}_K^G\,\check{\rho}$ contains $c\text{-Ind}_K^G\,\check{\rho}$ as G-subspace. The canonical K-embedding $\check{W} \to c\text{-Ind}_K^G\,\check{\rho}$ identifies $\check{W}$ with the space of functions in $\check{X}$ with support contained in K. We take non-zero functions $\check{w} \in \check{W} \subset \check{X}$ and $w \in W \subset X$: the coefficient $\gamma_{\check{w}\otimes w}$ is then non-zero and supported in K.

Consequently, we need only prove that X is irreducible: it is then admissible (10.2 Corollary) and we can apply 10.1 Proposition (2) to show it is γ-cuspidal, hence cuspidal.

The centre Z of G acts on X via the character ω_ρ, where $\rho(z)w = \omega_\rho(z)w$, $z \in Z$, $w \in W$. Therefore X is the direct sum of its K-isotypic components (2.7). Any K-map $W \to X$ has image contained in X^ρ, so:

$$\mathrm{Hom}_K(W, X^\rho) = \mathrm{Hom}_K(W, X) \cong \mathrm{End}_G(X) \cong \mathcal{H}(G, \rho).$$

However, the intertwining condition implies that $\dim \mathcal{H}(G, \rho) = 1$. The space $\mathrm{Hom}_K(W, X^\rho)$ therefore has dimension 1, and we conclude that $W = X^\rho$.

Let Y be a non-zero G-subspace of X. Therefore

$$0 \neq \mathrm{Hom}_G(Y, X) \subset \mathrm{Hom}_G(Y, \mathrm{Ind}_K^G \rho) \cong \mathrm{Hom}_K(Y, \rho).$$

Since Y is semisimple over K (2.7), we have $Y^\rho \neq 0$. Thus $Y \supset W = X^\rho$, since W is irreducible over K. As W generates X over G, we conclude that $Y = X$. Thus X is irreducible, as required. □

Remark 1. The theorem holds (with the conclusion that c-$\mathrm{Ind}\,\rho$ is γ-cuspidal), with the same proof, in considerable generality. It is valid for a unimodular locally profinite group G, satisfying 2.6 Hypothesis, and such that any irreducible smooth representation of G is admissible.

Remark 2. The converse of the theorem also holds. If ρ is intertwined by some $g \in G \smallsetminus K$, then $\mathcal{H}(G, \rho) \cong \mathrm{End}_G(c\text{-}\mathrm{Ind}\,\rho)$ has dimension > 1. Thus c-$\mathrm{Ind}\,\rho$ has a non-scalar endomorphism and cannot be irreducible.

Remark 3. In the situation of the theorem, the smooth dual $(c\text{-}\mathrm{Ind}\,\rho)^\vee$ is irreducible. It is, however, isomorphic to $\mathrm{Ind}\,\check\rho$. We deduce that $\mathrm{Ind}\,\check\rho = c\text{-}\mathrm{Ind}\,\check\rho \cong (c\text{-}\mathrm{Ind}\,\rho)^\vee$. Since these representations are all admissible, we can dualize again to get c-$\mathrm{Ind}\,\rho = \mathrm{Ind}\,\rho$.

11.5. We give an example illustrative of the above procedures. Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathfrak{o})$ and $K_1 = 1{+}\mathfrak{p}\mathrm{M}_2(\mathfrak{o})$. Thus K_1 is an open normal subgroup of K and $K/K_1 \cong \mathrm{GL}_2(\boldsymbol{k})$. We also let I_1 denote the group of matrices

$$I_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}.$$

Thus I_1 is the inverse image in K of the standard group $N(\boldsymbol{k})$ of upper triangular unipotent matrices in $\mathrm{GL}_2(\boldsymbol{k})$.

Theorem. *Let (π, V) be an irreducible smooth representation of G, and suppose that π contains the trivial character of K_1. Exactly one of the following holds:*

(1) *π contains a representation λ of K, inflated from an irreducible cuspidal representation $\tilde\lambda$ of $\mathrm{GL}_2(\boldsymbol{k})$;*
(2) *π contains the trivial character of I_1.*

In the first case, π is cuspidal, and there exists a representation Λ of ZK such that $\Lambda \mid K \cong \lambda$ and

$$\pi \cong c\text{-}\mathrm{Ind}_{ZK}^G \Lambda.$$

Proof. The group K stabilizes the space V^{K_1}, which is therefore a direct sum of irreducible representations of K which are trivial on K_1, that is, they are inflated from $\mathrm{GL}_2(\boldsymbol{k})$. Let λ be one of these, inflated from $\tilde{\lambda}$. Either $\tilde{\lambda}$ is cuspidal (in the sense of §6), or it is not. In the latter case, it contains the trivial character of $N(\boldsymbol{k})$, whence λ contains the trivial character of I_1.

We have to show that the two cases cannot occur together. To do this, we interpolate a useful general lemma.

Lemma. *For $i = 1, 2$, let $\tilde{\rho}_i$ be an irreducible representation of $\mathrm{GL}_2(\boldsymbol{k})$, and let ρ_i denote the inflation of $\tilde{\rho}_i$ to a representation of K. Suppose that $\tilde{\rho}_1$ is cuspidal.*

(1) *The representations ρ_i intertwine in G if and only if $\tilde{\rho}_1 \cong \tilde{\rho}_2$.*
(2) *An element $g \in G$ intertwines ρ_1 if and only if $g \in ZK$.*

Proof. Let $g \in G$ intertwine ρ_2 with ρ_1. It is only the coset $KgZK$ which intervenes, so we can take g of the form

$$g = \begin{pmatrix} \varpi^a & 0 \\ 0 & 1 \end{pmatrix},$$

for some $a \geqslant 0$. If $a = 0$, we have $g = 1$ and there is nothing to do. We therefore assume $a \geqslant 1$. The group $K_1^g \cap K$ contains the group

$$N_0 = \begin{pmatrix} 1 & \mathfrak{o} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathfrak{p} \\ 0 & 1 \end{pmatrix}^g$$

on which ρ_2^g is trivial. Since $\tilde{\rho}_1$ is cuspidal, ρ_1 does not contain the trivial character of N_0, so g cannot intertwine the ρ_i. All assertions now follow. □

It follows from 11.1 Proposition 1 that, in the theorem, the two cases cannot occur together. We now assume that $\tilde{\lambda}$ is cuspidal. Surely π contains some representation Λ of ZK extending λ. Thus we have a non-trivial ZK-homomorphism $\Lambda \to \pi$, giving a non-trivial G-homomorphism $c\text{-}\mathrm{Ind}_{ZK}^G \Lambda \to \pi$. However, by part (2) of the lemma and 11.4 Theorem, the representation $c\text{-}\mathrm{Ind}\, \Lambda$ is irreducible, so $\pi \cong c\text{-}\mathrm{Ind}\, \Lambda$, as desired. □

Remark. We will eventually see (14.5) that the theorem has a kind of converse. If (π, V) is an irreducible representation of G containing the trivial character of K_1, then it is cuspidal if and only if it satisfies condition (1) of the theorem.

Further reading.

Although we have focused exclusively on $G = \mathrm{GL}_2(F)$, many elements reflect the much more general discussions in the papers [5,6] of Bernstein and Zelevinsky. These apply in the context of connected reductive algebraic groups over

F and centre on general versions of the Restriction-Induction Lemma (9.3) and the homomorphism theorem in the form 9.11 Lemma 2. That programme culminates in a classification of the non-cuspidal representations of $\mathrm{GL}_n(F)$, [90]. Rodier's report [71] is a helpful introduction. The eternal pre-print [25] is also written in these terms, from a slightly different point of view. Only in very few cases, however, does one have a good command of the non-cuspidal representations of groups besides $\mathrm{GL}_n(F)$.

The initial analysis of cuspidal representations in this chapter is quite general in tone, and holds very widely. Even 11.5 and its converse have close analogues for completely general reductive groups [67], [64].

http://www.springer.com/978-3-540-31486-8

The Local Langlands Conjecture for GL(2)
Bushnell, C.J.; Henniart, G.
2006, XII, 340 p., Hardcover
ISBN: 978-3-540-31486-8