Preface

This volume discusses a construction situated at the intersection of two different mathematical fields: Abstract harmonic analysis, understood as the theory of group representations and their decomposition into irreducibles on the one hand, and wavelet (and related) transforms on the other. In a sense the volume reexamines one of the roots of wavelet analysis: The paper [60] by Grossmann, Morlet and Paul may be considered as one of the initial sources of wavelet theory, yet it deals with a unitary representation of the affine group, citing results on discrete series representations of nonunimodular groups due to Duflo and Moore. It was also observed in [60] that the discrete series setting provided a unified approach to wavelet as well as other related transforms, such as the windowed Fourier transform.

We consider generalizations of these transforms, based on a representation-theoretic construction. The construction of continuous and discrete wavelet transforms, and their many relatives which have been studied in the past twenty years, involves the following steps: Pick a suitable basic element (the *wavelet*) in a Hilbert space, and construct a system of vectors from it by the action of certain prescribed operators on the basic element, with the aim of expanding arbitrary elements of the Hilbert space in this system. The associated *wavelet transform* is the map which assigns each element of the Hilbert space its expansion coefficients, i.e. the family of scalar products with all elements of the system. A *wavelet inversion formula* allows the reconstruction of an element from its expansion coefficients.

Continuous wavelet transforms, as studied in the current volume, are obtained through the action of a group via a unitary representation. Wavelet inversion is achieved by integration against the left Haar measure of the group. The key questions that are treated –and solved to a large extent– by means of abstract harmonic analysis are: Which representations can be used? Which vectors can serve as wavelets?

The representation-theoretic formulation focusses on one aspect of wavelet theory, the inversion formula, with the aim of developing general criteria and providing a more complete understanding. Many other aspects that have made wavelets such a popular tool, such as discretization with fast algorithms and the many ensuing connections and applications to signal and image processing, or, on the more theoretical side, the use of wavelets for the characterization of large classes of function spaces such as Besov spaces, are lost when we move on to the more general context which is considered here. One of the reasons for this is that these aspects often depend on a specific *realization* of a representation, whereas abstract harmonic analysis does not differentiate between unitarily equivalent representations.

In view of these shortcomings there is a certain need to justify the use of techniques such as direct integrals, entailing a fair amount of technical detail, for the solution of problems which in concrete settings are often amenable to more direct approaches. Several reasons could be given: First of all, the inversion formula is a crucial aspect of wavelet and Gabor analysis. Analogous formulae have been – and are being – constructed for a wide variety of settings, some with, some without a group-theoretic background. The techniques developed in the current volume provide a systematic, unified and powerful approach which for type I groups yields a complete description of the possible choices of representations and vectors. As the discussion in Chapter 5 shows, many of the existing criteria for wavelets in higher dimensions, but also for Gabor systems, are covered by the approach.

Secondly, Plancherel theory provides an attractive theoretical context which allows the unified treatment of related problems. In this respect, my prime example is the discretization and sampling of continuous transforms. The analogy to real Fourier analysis suggests to look for nonabelian versions of Shannon's sampling theorem, and the discussion of the Heisenberg group in Chapter 6 shows that this intuition can be made to work at least in special cases. The proofs for the results of Chapter 6 rely on a combination of direct integral theory and the theory of Weyl-Heisenberg frames. Thus the connection between wavelet transforms and the Plancherel formula can serve as a source of new problems, techniques and results in representation theory.

The third reason is that the connection between the initial problem of characterizing wavelet transforms on one side and the Plancherel formula on the other is beneficial also for the development and understanding of Plancherel theory. Despite the close connection, the answers to the above key questions require more than the straightforward application of known results. It was necessary to prove new results in Plancherel theory, most notably a precise description of the scope of the pointwise inversion formula. In the nonunimodular case, the Plancherel formula is obscured by the *formal dimension operators*, a family of unbounded operators needed to make the formula work. As we will see, these operators are intimately related to *admissibility conditions* characterizing the possible wavelets, and the fact that the operators are unbounded has rather surprising consequences for the existence of such vectors. Hence, the drawback of having to deal with unbounded operators, incurring the necessity to check domains, turns into an asset.

Finally the study of admissibility conditions and wavelet-type inversion formulae offers an excellent opportunity for getting acquainted with the Plancherel formula for locally compact groups. My own experience may serve as an illustration to this remark. The main part of the current is concerned with the question how Plancherel theory can be employed to derive admissibility criteria. This way of putting it suggests a fixed hierarchy: First comes the general theory, and the concrete problem is solved by applying it. However, for me a full understanding of the Plancherel formula on the one hand, and of its relations to admissibility criteria on the other, developed concurrently rather than consecutively. The exposition tries to reproduce this to some extent. Thus the volume can be read as a problem-driven – and reasonably self-contained– introduction to the Plancherel formula.

As the volume connects two different fields, it is intended to be open to researchers from both of them. The emphasis is clearly on representation theory. The role of group theory in constructing the continuous wavelet transform or the windowed Fourier transform is a standard issue found in many introductory texts on wavelets or time-frequency analysis, and the text is intended to be accessible to anyone with an interest in these aspects. Naturally more sophisticated techniques are required as the text progresses, but these are explained and motivated in the light of the initial problems, which are existence and characterization of admissible vectors. Also, a number of well-known examples, such as the windowed Fourier transform or wavelet transforms constructed from semidirect products, keep reappearing to provide illustration to the general results. Specifically the Heisenberg group will occur in various roles.

A further group of potential readers are mathematical physicists with an interest in generalized coherent states and their construction via group representations. In a sense the current volume may be regarded as a complement to the book by Ali, Antoine and Gazeau [1]: Both texts consider generalizations to the discrete series case. [1] replaces the square-integrability requirement by a weaker condition, but mostly stays within the realm of irreducible representations, whereas the current volume investigates the irreducibility condition. Note however that we do not comment on the relevance of the results presented here to mathematical physics, simply for lack of competence.

In any case it is only assumed that the reader knows the basics of locally compact groups and their representation theory. The exposition is largely self-contained, though for known results usually only references are given. The somewhat introductory Chapter 2 can be understood using only basic notions from group theory, with the addition of a few results from functional and Fourier analysis which are also explained in the text. The more sophisticated tools, such as direct integrals, the Plancherel formula or the Mackey machine, are introduced in the text, though mostly by citation and somewhat concisely. In order to accomodate readers of varying backgrounds, I have marked some of the sections and subsections according to their relation to the core material of the text. The core material is the study of admissibility conditions, discretization and sampling of the transforms. Sections and subsections with the superscript * contain predominantly technical results and arguments which are indispensable for a rigorous proof, but not necessarily for an understanding and assessment of results belonging to the core material. Sections and subsections marked with a superscript ** contain results which may be considered diversions, and usually require more facts from representation theory than we can present in the current volume. The marks are intended to provide some orientation and should not be taken too literally; it goes without saying that distinctions of this kind are subjective.

Acknowledgements. The current volume was developed from the papers [52, 53, 4], and I am first and foremost indebted to my coauthors, which are in chronological order: Matthias Mayer, Twareque Ali and Anna Krasowska. The results in Section 2.7 were developed with Keith Taylor.

Volkmar Liebscher, Markus Neuhauser and Olaf Wittich read parts of the manuscript and made many useful suggestions and corrections. Needless to say, I blame all remaining mistakes, typos etc. on them.

In addition, I owe numerous ideas, references, hints etc. to Jean-Pierre Antoine, Larry Baggett, Hans Feichtinger, Karlheinz Gröchenig, Rolf Wim Henrichs, Rupert Lasser, Michael Lindner, Wally Madych, Arlan Ramsay, Günter Schlichting, Bruno Torrésani, Guido Weiss, Edward Wilson, Gerhard Winkler and Piotr Wojdyłło.

I would also like to acknowledge the support of the Institute of Biomathematics and Biometry at GSF National Research Center for Environment and Health, Neuherberg, where these lecture notes were written, as well as additional funding by the EU Research and Training Network *Harmonic Analysis and Statistics in Signal and Image Processing* (HASSIP).

Finally, I would like to thank Marina Reizakis at Springer, as well as the editors of the Lecture Notes series, for their patience and cooperation. Thanks are also due to the referees for their constructive criticism.

Neuherberg, December 5, 2004 *Hartmut Führ* http://www.springer.com/978-3-540-24259-8

Abstract Harmonic Analysis of Continuous Wavelet Transforms
Führ, H.
2005, X, 193 p., Softcover
ISBN: 978-3-540-24259-8