4

Vector Bundles

This chapter provides indispensable tools in the study of complex manifolds: connections, curvature, and Chern classes. In contrast to previous sections, we will not focus on the holomorphic tangent bundle of a complex manifold but allow arbitrary holomorphic vector bundles. However, we will not be in the position to undertake an indepth analysis of certain fundamental questions. E.g. the question whether there exist non-trivial bundles on a given complex manifold (or holomorphic structures on a given complex bundle) will not be addressed. This is partially due to the limitations of the book, but also to the state of art. The situation is fairly well understood only for curves and projective surfaces.

In Sections 5.1 to 5.3 the reader can find a number of central results in complex algebraic geometry. Except for the Hirzebruch–Riemann–Roch theorem, complete proofs, in particular of Kodaira's vanishing and embedding theorems, are provided. These three results are of fundamental importance in the global theory of complex manifolds. Roughly, in conjunction they allow to determine the size of linear systems on a manifold X and, if X is projective, how it can be embedded into a projective space.

In the appendices we discuss the interplay between the complex geometry of holomorphic vector bundles and related structures: Appendix 4.A tries to clarify the relation between Riemannian and Kähler geometry. In particular, we will show that for Kähler manifolds the Levi-Civita connection coincides with the Chern connection. The concept of holonomy, well-known in classical Riemannian geometry, allows to view certain features in complex geometry from a different angle. Appendix 4.B outlines fundamental results about Kähler–Einstein and Hermite–Einstein metrics. Before, the hermitian structure on a holomorphic vector bundle was used as an additional datum in order to apply Hodge theory, etc. One might wonder, whether natural hermitian structures, satisfying certain compatibility conditions, can be found. This leads to the concept of Hermite–Einstein metrics, which exist on certain privileged holomorphic bundles. If the holomorphic bundle happens to be the tangent bundle, this is related to Kähler–Einstein metrics.

4.1 Hermitian Vector Bundles and Serre Duality

In Chapter 3 we studied complex manifolds together with a compatible Riemannian metric, so called hermitian manifolds or, more restrictive, Kähler manifolds. The Riemannian metric gives rise to an hermitian metric on the (holomorphic) tangent bundle. More generally, one could and should be interested in hermitian metrics on arbitrary holomorphic and complex vector bundles. This twisted version will be discussed now. Many of the arguments will be familiar to the reader. Repeating Hodge theory on compact hermitian manifolds, this time for vector bundles, might help to get used to this important technique.

Let E be a complex vector bundle over a real manifold M.

Definition 4.1.1 An *hermitian structure* h on $E \to M$ is an hermitian scalar product h_x on each fibre $E(x)$ which depends differentiably on x. The pair (E,h) is called an *hermitian vector bundle.*

The latter condition can be made more precise in terms of local trivializations. Let $\psi : E|_U \cong U \times \mathbb{C}^r$ be a trivialization over some open subset U. Then, for any $x \in U$ the form $h_x(\psi_x^{-1}(\ ), \psi_x^{-1}(\ ))$ defines an hermitian scalar product on $\mathbb{C}^r$. In other words, h_x is given by a positive-definite hermitian matrix $(h_{ij}(x))$ (which depends on ψ) and we require the map $(h_{ij}) : U \to \mathrm{Gl}(r,\mathbb{C})$ to be differentiable.

Examples 4.1.2 i) Let L be a (holomorphic) line bundle and let $s_1, \ldots, s_k$ be global (holomorphic) sections generating L everywhere, i.e. at every point at least one of them is non-trivial. Then one defines an hermitian structure on L by

$$h(t) = \frac{|\psi(t)|^2}{\sum |\psi(s_i)|^2},$$

where t is a point in the fibre $L(x)$ and ψ is a local trivialization of L around the point x. The definition does not depend on the chosen trivialization, as two of them only differ by a scalar factor. Observe that h is not holomorphic, i.e. even if a trivialization of L over an open subset is chosen holomorphic, the induced map $h : U \to \mathbb{C}^*$ is usually not holomorphic. By abuse of language, one sometimes says that h is given by $(\sum |s_i|^2)^{-1}$.

The standard example is $L = \mathcal{O}(1)$ over the projective space $\mathbb{P}^n$ and the standard globally generating sections $z_0, \ldots, z_n \in H^0(\mathbb{P}^n, \mathcal{O}(1))$.

ii) If (X,g) is an hermitian manifold then the tangent, the cotangent, and all form bundles $\bigwedge^{p,q} X$ have natural hermitian structures.

iii) If E and F are endowed with hermitian structures, then the associated bundles $E \oplus F$, $E \otimes F$, $\mathrm{Hom}(E,F)$, etc., inherit natural hermitian structures.

iv) If (E,h) is an hermitian vector bundle and $F \subset E$ is a subbundle, then the restriction of h to F endows F with an hermitian structure. One can define the orthogonal complement, $F^\perp \subset E$ of F with respect to h. It is easy to see, that the pointwise condition indeed yields a complex vector bundle. Moreover, the bundle E can be decomposed as $E = F \oplus F^\perp$ and $F^\perp$ is canonically (as a complex vector bundle) isomorphic to the quotient E/F. In particular, h also induces an hermitian structure on the quotient E/F.

v) If (E, h) is an hermitian vector bundle over an hermitian manifold (X, g), then the twisted form bundles $\bigwedge^{p,q} X \otimes E$ have natural hermitian structures.

Example 4.1.3 Let us consider the projective space $\mathbb{P}^n$ and the Euler sequence twisted by $\mathcal{O}(-1)$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n}(-1) \longrightarrow 0\,.$$

The constant standard hermitian structure on $\mathcal{O}^{\oplus n+1}$ induces canonical hermitian structures h_1 on $\mathcal{O}(-1)$ and h_2 on $\mathcal{T}_{\mathbb{P}^n}(-1)$ (see the previous example).

The hermitian structure h_1 on $\mathcal{O}(-1)$ is nothing but the dual of the canonical hermitian structure on $\mathcal{O}(1)$ determined by the choice of the basis $z_0, \ldots, z_n \in H^0(\mathbb{P}^n, \mathcal{O}(-1))$ as in i) of Examples 4.1.2. This is straightforward to verify and we leave the general version of this assertion as Exercise 4.1.1.

We next wish to identify h_2 as the tensor product of the Fubini–Study metric on $\mathcal{T}_{\mathbb{P}^n}$ and h_1 (up to the constant factor 2π).

The verification is done on the open subset $U_0 = \{(z_0 : \ldots : z_n) \mid z_0 \neq 0\}$ with coordinates $w_i = \frac{z_i}{z_0}$, $i = 1, \ldots, n$. Hence, with respect to the bases $\frac{\partial}{\partial w_i}$ the Fubini–Study metric on $\mathcal{T}_{\mathbb{P}^n}|_{U_i}$ is given (up to the factor 2π) by the matrix

$$H := (1 + \sum |w_i|^2)^{-2} \cdot \Big((1 + \sum |w_i|^2)\delta_{ij} - \bar{w}_i w_j\Big)_{ij}$$

(see i), Examples 3.1.9). The induced hermitian structure on the dual $\Omega_{\mathbb{P}^n}|_{U_0}$ with respect to the dual basis $dw_1, \ldots, dw_n$ corresponds thus to the matrix $\bar{H}^{-1} = (1 + \sum |w_i|^2) \cdot (\delta_{ij} + \bar{w}_i w_j)_{ij}$

On the other hand, the inclusion $\Omega_{\mathbb{P}^n} \subset \mathcal{O}(-1)^{\oplus n+1}$ given by the Euler sequence is on U_0 explicitly given by $dw_i \mapsto e_i - w_i \cdot e_0$ (see the proof of Proposition 2.4.4). Since h_1^* on $\mathcal{O}(-1)|_{U_0}$ is the scalar function $(1 + \sum |w_i|^2)$, one finds that the hermitian structure on $\Omega_{\mathbb{P}^n}|_{U_0}$ induced by this inclusion is

$$(1 + \sum |w_i|^2) \cdot \Big((e_i - w_i \cdot e_0, e_j - w_j \cdot e_0)\Big)_{ij} = \bar{H}^{-1}.$$

Note that choosing another basis of $H^0(\mathbb{P}^n, \mathcal{O}(1))$, which in general results in a different hermitian structure on $\mathcal{O}(1)$, amounts to choosing a different, though still constant, hermitian structure on the trivial bundle $\mathcal{O}^{\oplus n+1}$ on the middle term of the Euler sequence. So, more invariantly, one could work with the Euler sequence on $\mathbb{P}(V)$, where the middle term is $V \otimes \mathcal{O}$, and an hermitian structure on V.

An hermitian structure h on a vector bundle E defines a $\mathbb{C}$-antilinear isomorphism (of real bundles) $E \cong E^*$. Here, E^* is the dual complex bundle of E. Generalizing Exercise 3.1.1 we observe the following

Proposition 4.1.4 *Every complex vector bundle admits an hermitian metric.*

Proof. Choose an open covering $X = \bigcup U_i$ trivializing a given vector bundle E. Then one might glue the constant hermitian structures on the trivial vector bundles $U_i \times \mathbb{C}^r$ over U_i by means of a partition of unity.

Here, we use that any positive linear combination of positive definite hermitian products on $\mathbb{C}^n$ is again positive definite and hermitian. □

Let $f : M \to N$ be a differentiable map and let E be a vector bundle on N endowed with an hermitian structure h. Then the pull-back vector bundle f^*E gets a natural hermitian structure f^*h by $(f^*h)_x = h_{f(x)}$ on $(f^*E)(x) = E(f(x))$.

Example 4.1.5 Let X be a complex manifold and let $s_0, \ldots, s_k$ be globally generating holomorphic sections of a holomorphic line bundle L (see Remark 2.3.27, ii)). By Proposition 2.3.26 there exists an induced morphism $\varphi : X \to \mathbb{P}^k$, $x \mapsto (s_0(x) : \ldots : s_k(x))$ with $\varphi^*\mathcal{O}(1) = L$ and $\varphi^*(z_i) = s_i$. The natural hermitian structures h on $\mathcal{O}(1)$ and h' on L induced by $z_0, \ldots, z_k$ and $s_0, \ldots, s_k$, respectively, (see Example 4.1.2, i)) are compatible under φ, i.e. $\varphi^*h = h'$.

Let (X, g) be an hermitian manifold and let (E, h) be an hermitian vector bundle on X. Then the induced hermitian structures on $\bigwedge^{p,q} X \otimes E$ will be denoted $(\ ,\ )$.

Definition 4.1.6 Let E be a complex vector bundle over an hermitian manifold (X, g) of complex dimension n. An hermitian structure h on E is interpreted as a $\mathbb{C}$-antilinear isomorphism $h : E \cong E^*$. Then

$$\bar{*}_E : \bigwedge^{p,q} X \otimes E \longrightarrow \bigwedge^{n-p,n-q} X \otimes E^*$$

is defined by $\bar{*}_E(\varphi \otimes s) = \bar{*}(\varphi) \otimes h(s) = \overline{*(\varphi)} \otimes h(s) = *(\bar{\varphi}) \otimes h(s)$. (Recall that $*$ is $\mathbb{C}$-linear on $\bigwedge^{p,q} X$.)

Clearly, $\bar{*}_E$ is a $\mathbb{C}$-antilinear isomorphism that depends on g and h. Note that with this definition we have

$$(\alpha, \beta) * 1 = \alpha \wedge \bar{*}_E(\beta)$$

for α, β sections of $\bigwedge^{p,q} X \otimes E$, where "$\wedge$" is the exterior product in the form part and the evaluation map $E \otimes E^* \to \mathbb{C}$ in the bundle part. It is not difficult to verify that, as for the usual Hodge $*$-operator, one has $\bar{*}_{E^*} \circ \bar{*}_E = (-1)^{p+q}$ on $\bigwedge^{p,q} X \otimes E$.

The aim of this section is to generalize Poincaré duality for compact manifolds (or rather Serre duality (cf. Remark 3.2.7, ii) and Exercise 3.2.2) to a duality for the cohomology groups of holomorphic vector bundles. In order to do this we need to discuss Hodge theory in analogy to the discussion in Section 3.2 for hermitian vector bundles. Let us begin with the definition of the adjoint operator of $\bar\partial_E$.

Definition 4.1.7 Let (E,h) be a holomorpic vector bundle together with an hermitian structure h on an hermitian manifold (X,g). The operator $\bar\partial_E^* : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q-1}(E)$ is defined as

$$\bar\partial_E^* := -\bar{*}_{E^*} \circ \bar\partial_{E^*} \circ \bar{*}_E.$$

Remark 4.1.8 For $E = \mathcal{O}_X$ with a constant hermitian structure one recovers the adjoint operator $\bar\partial^* = -*\circ\partial\circ *$. Indeed, $-\bar{*}(\bar\partial(\bar{*}\varphi)) = -\bar{*}(\bar\partial(\overline{*\varphi})) = -\bar{*}(\overline{\partial * \varphi}) = -*(\partial * \varphi)$.

Definition 4.1.9 Let E be a holomorphic vector bundle endowed with an hermitian structure h on an hermitian manifold (X,g), then the *Laplace operator* on $\mathcal{A}^{p,q}(E)$ is defined by

$$\Delta_E := \bar\partial_E^*\bar\partial_E + \bar\partial_E\bar\partial_E^*.$$

Definition 4.1.10 Let (E,h) be an hermitian holomorphic vector bundle over an hermitian manifold (X,g). A section α of $\bigwedge^{p,q} X \otimes E$ is called *harmonic* if $\Delta_E(\alpha) = 0$. The space of all harmonic forms is denoted $\mathcal{H}^{p,q}(X,E)$, where we omit g and h in the notation.

Observe that $\bar{*}_E$ induces a $\mathbb{C}$-antilinear isomorphism

$$\bar{*}_E : \mathcal{H}^{p,q}(X,E) \cong \mathcal{H}^{n-p,n-q}(X,E^*).$$

Definition 4.1.11 Let (E,h) be an hermitian vector bundle on a compact hermitian manifold (X,g). Then a natural hermitian scalar product on $\mathcal{A}^{p,q}(X,E)$ is defined by

$$(\alpha,\beta) := \int_X (\alpha,\beta) * 1,$$

where $(\ ,\ )$ is the hermitian product on $\bigwedge^{p,q} X \otimes E$ depending on h and g (cf. Example 4.1.2).

Lemma 4.1.12 *Let (E,h) be an hermitian holomorphic vector bundle on a compact hermitian manifold (X,g). Then, with respect to $(\ ,\ )$, the operator $\bar\partial_E^*$ on $\mathcal{A}^{p,q}(X,E)$ is adjoint to $\bar\partial_E$ and Δ_E is self-adjoint.*

Proof. By definition, the second assertion follows from the first one which in turn is proved by the following purely formal calculation:

For $\alpha \in \mathcal{A}^{p,q}(X,E)$ and $\beta \in \mathcal{A}^{p,q+1}(X,E)$ one has

$$\begin{aligned}(\alpha, \bar\partial_E^* \beta) &= -(\alpha, \bar{*}_{E^*} \circ \bar\partial_{E^*} \circ \bar{*}_E \beta)\\ &= -\int_X \alpha \wedge \bar{*}_E \bar{*}_{E^*} \bar\partial_{E^*} \bar{*} \beta\\ &= (-1)^{n-p+n-q-1} \int_X \alpha \wedge \bar\partial_{E^*} \bar{*}_E \beta\\ &= \int_X \bar\partial_E(\alpha) \wedge \bar{*}_E \beta = (\alpha, \beta).\end{aligned}$$

Here we use the Leibniz rule $\bar\partial(\alpha \wedge \bar{*}_E \beta) = \bar\partial_E(\alpha) \wedge \bar{*}_E \beta + (-1)^{p+q} \alpha \wedge \bar\partial_{E^*} \bar{*}_E \beta$ and Stokes' theorem $\int_X \bar\partial(\alpha \wedge \bar{*}_E \beta) = \int_X d(\alpha \wedge \bar{*}_E \beta) = 0$. □

Using the lemma the reader may check that a form $\alpha \in \mathcal{A}^{p,q}(X,E)$ over a compact manifold X is harmonic if and only if α is $\bar\partial_E$- and $\bar\partial_E^*$-closed (cf. Lemma 3.2.5).

Theorem 4.1.13 (Hodge decomposition) *Let E be a holomorphic vector bundle together with an hermitian structure h on a compact hermitian manifold (X, g). Then*

$$\mathcal{A}^{p,q}(X,E) = \bar\partial_E \mathcal{A}^{p,q-1}(X,E) \oplus \mathcal{H}^{p,q}(X,E) \oplus \bar\partial_E^* \mathcal{A}^{p,q+1}(X,E) \tag{4.1}$$

and $\mathcal{H}^{p,q}(X,E)$ is finite-dimensional. □

The case of the trivial vector bundle $E \cong \mathcal{O}_X$ with a constant hermitian structure corresponds to Hodge decomposition of compact hermitian manifolds (cf. Theorem 3.2.8). As in this case, we obtain

Corollary 4.1.14 *The natural projection $\mathcal{H}^{p,q}(X,E) \to H^{p,q}(X,E)$ is bijective. In particular, $H^{p,q}(X,E) \cong H^q(X, E \otimes \Omega_X^p)$ is finite-dimensional.*

Proof. Indeed, as any harmonic section of $\bigwedge^{p,q} X \otimes E$ is $\bar\partial_E$-closed, the projection is well-defined. Moreover, the space of $\bar\partial_E$-closed forms in $\mathcal{A}^{p,q}(X,E)$ is $\bar\partial_E \mathcal{A}^{p,q-1}(X,E) \oplus \mathcal{H}^{p,q}(X,E)$, as $(\bar\partial_E \bar\partial_E^* \alpha, \alpha) = \|\bar\partial_E^* \alpha\|^2 \neq 0$ for $\bar\partial_E^* \alpha \neq 0$.

Thus, the projection is surjective and its kernel is the space of forms, which are $\bar\partial_E$-exact and harmonic. But since the decomposition (4.1) in theorem 4.1.13 is direct, this space is trivial. □

Let E be a holomorphic vector bundle over a compact manifold X of dimension n and consider the natural pairing

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \longrightarrow \mathbb{C}, \quad (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta,$$

where as before $\alpha \wedge \beta$ is the exterior product in the form part and the evaluation map in the bundle part. The pairing is well-defined, i.e. does not depend on the $\bar\partial$-closed representatives $\alpha \in \mathcal{A}^{p,q}(E)$ and $\beta \in \mathcal{A}^{n-p,n-q}(E)$.

Proposition 4.1.15 (Serre duality) *Let X be a compact complex manifold. For any holomorphic vector bundle E on X the natural pairing*

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \longrightarrow \mathbb{C}$$

is non-degenerate.

Proof. Fix hermitian structures h and g on E and X, respectively. Then consider the pairing $\mathcal{H}^{p,q}(X,E) \times \mathcal{H}^{n-p,n-q}(X,E^*) \to \mathbb{C}$. In order to show that this pairing is non-degenerate, we have to show that for any $0 \neq \alpha \in \mathcal{H}^{p,q}(X,E)$ there exists an element $\beta \in \mathcal{H}^{n-p,n-q}(X,E^*)$ with $\int_X \alpha \wedge \beta \neq 0$. Now choose $\beta := \bar{*}_E\alpha$, then $\int \alpha \wedge \beta = \int \alpha \wedge \bar{*}_E\alpha = \int (\alpha,\alpha) * 1 = \|\alpha\|^2 \neq 0$. □

Serre duality (together with the Hirzebruch–Riemann–Roch theorem 5.1.1 and Kodaira vanishing theorem 5.2.2) is one of the most useful tools to control the cohomology of holomorphic vector bundles.

Let us mention a few special cases and reformulations.

Corollary 4.1.16 *For any holomorphic vector bundle E over a compact complex manifold X there exist natural $\mathbb{C}$-linear isomorphisms* (Serre duality)*:*

$$\begin{aligned} H^{p,q}(X,E) &\cong H^{n-p,n-q}(X,E^*)^* \\ H^q(X,\Omega^p \otimes E) &\cong H^{n-q}(X,\Omega^{n-p} \otimes E^*)^* \\ H^q(X,E) &\cong H^{n-q}(X,K_X \otimes E^*)^* \end{aligned}$$

□

For the trivial bundle this yields $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$ (cf. Exercise 3.2.2). Moreover, if X is Kähler these isomorphisms are compatible with the bidegree decomposition $H^k(X,\mathbb{C}) = \bigoplus H^{p,q}(X)$ and Poincaré duality (cf. Exercise 3.2.3).

Remark 4.1.17 The isomorphism $\bar{*}_E : \mathcal{H}^{p,q}(X,E) \cong \mathcal{H}^{n-p,n-q}(X,E^*)$ induces an isomorphism $H^{p,q}(X,E) \cong H^{n-p,n-q}(X,E^*)$. But this isomorphism is only $\mathbb{C}$-antilinear and depends on the chosen hermitian structures g and h. Thus, Serre duality $H^{p,q}(X,E) \cong H^{n-p,n-q}(X,E^*)^*$ is better behaved in both respects.

Exercises

4.1.1 Let L be a holomorphic line bundle which is globally generated by sections $s_1,\ldots,s_k \in H^0(X,L)$. Then L admits a canonical hermitian structure h defined in Example 4.1.2. The dual bundle L^* obtains a natural hermitian structure h' via the inclusion $L^* \subset \mathcal{O}^{\oplus k}$. Describe h' and and show that $h' = h^*$.

4.1.2 Let L be a holomorphic line bundle of degree $d > 2g(C) - 2$ on a compact curve C. Show that $H^1(C, L) = 0$. Here, for our purpose we define the *genus* $g(C)$ of C by the formula $\deg(K_X) = 2g(C) - 2$.

In other words, $H^1(C, K_C \otimes L) = 0$ for any holomorphic line bundle L with $\deg(L) > 0$. In this form, it will later be generalized to the Kodaira vanishing theorem for arbitrary compact Kähler manifolds.

4.1.3 Show, e.g. by writing down an explicit basis, that

$$h^n(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} 0 & k > -n-1 \\ \binom{-k-1}{-n-1-k} & k \leq -n-1 \end{cases}$$

4.1.4 Let E be an hermitian holomorphic vector bundle on a compact Kähler manifold X. Show that any section $s \in H^0(X, \Omega^p \otimes E)$ is harmonic.

4.1.5 Compare this section with the discussion in Sections 3.2 and 3.3. In particular, check whether the Lefschetz operator L is defined on $H^{p,q}(X, E)$ and whether it defines isomorphisms $H^{p,k-p}(X, E) \to H^{n+p-k,n-p}(X, E)$ (cf. Remark 3.2.7, iii)).

Comments: Serre duality does in fact hold, in an appropriately modified form, for arbitrary coherent sheaves. Moreover, it is a special case of the so called Grothendieck–Verdier duality which is a duality statement for the direct image of coherent sheaves under proper morphisms. An algebraic proof, i.e. without using any metrics, can be given in case the manifold is projective

4.2 Connections

Let M be a real manifold and let $\pi : E \to M$ be a complex vector bundle on M. As before, we denote by $\mathcal{A}^i(E)$ the sheaf of i-forms with values in E. In particular, $\mathcal{A}^0(E)$ is just the sheaf of sections of E. Sections of E cannot be differentiated canonically, i.e. the exterior differential is in general not defined (see the discussion in Section 2.6). A substitute for the exterior differential is provided by a connection on E, which is not canonical, but always available.

This section introduces the reader to the fundamental notion of a connection and studies various compatibility conditions with additional data, like holomorphic or hermitian structures. At the end of this section, a short discussion of a purely holomorphic and more rigid analogue is introduced.

We will focus on complex vector bundles, but for almost everything a real version exists. We leave it to the reader to work out the precise formulation in each case.

Definition 4.2.1 A *connection* on a vector bundle E is a $\mathbb{C}$-linear sheaf homomorphism $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisfies the *Leibniz rule*

$$\nabla(f \cdot s) = d(f) \otimes s + f \cdot \nabla(s) \tag{4.2}$$

for any local function f on M and any local section s of E.

Definition 4.2.2 A section s of a vector bundle E is called *parallel* (or *flat* or *constant*) with respect to a connection ∇ on E if $\nabla(s) = 0$.

Proposition 4.2.3 *If ∇ and ∇' are two connections on a vector bundle E, then $\nabla - \nabla'$ is $\mathcal{A}^0_M$-linear and can, therefore, be considered as an element in $\mathcal{A}^1(M, \mathrm{End}(E))$. If ∇ is a connection on E and $a \in \mathcal{A}^1(M, \mathrm{End}(E))$, then $\nabla + a$ is again a connection on E.*

Proof. We have to show that $(\nabla - \nabla')(f \cdot s) = f \cdot (\nabla - \nabla')(s)$, which is an immediate consequence of the Leibniz rule (4.2).

An element $a \in \mathcal{A}^1(M, \mathrm{End}(E))$ acts on $\mathcal{A}^0(E)$ by multiplication in the form part and evaluation $\mathrm{End}(E) \times E \to E$ on the bundle component. In order to prove the second assertion, one checks $(\nabla + a)(f \cdot s) = \nabla(f \cdot s) + a(f \cdot s) = d(f) \otimes s + f \cdot \nabla(s) + f a(s) = d(f) \otimes s + f \cdot (\nabla + a)(s)$. Thus, $\nabla + a$ satisfies the Leibniz rule and is, therefore, a connection. □

As a consequence of this proposition and Exercise 4.2.1 one obtains

Corollary 4.2.4 *The set of all connections on a vector bundle E is in a natural way an affine space over the (infinite-dimensional) complex vector space $\mathcal{A}^1(M, \mathrm{End}(E))$.* □

Remark 4.2.5 Often, local calculations are performed by using the following statement: Any connection ∇ on a vector bundle E can locally be written as $d + A$, where A is a matrix valued one-form.

Indeed, if E is the trivial vector bundle $E = M \times \mathbb{C}^r$, then $\mathcal{A}^k(E) = \bigoplus_{i=1}^r \mathcal{A}_M^k$ and one defines the trivial connection $d : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ on E by applying the usual exterior differential to each component. Any other connection ∇ is then of the form $\nabla = d + A$, where $A \in \mathcal{A}^1(M, \mathrm{End}(E))$. For the trivial vector bundle, the latter is just a matrix valued one-form.

Let E be an arbitrary vector bundle on M endowed with a connection ∇. With respect to a trivialization $\psi : E|_U \cong U \times \mathbb{C}^r$ we may write $\nabla = d + A$ or, more precisely, $\nabla = \psi^{-1} \circ (d + A) \circ \psi$. If the trivialization is changed by $\phi : U \to \mathrm{Gl}(r, \mathbb{C})$, i.e. one considers $\psi'_x = \phi(x) \circ \psi_x$, then $\nabla = \psi'^{-1} \circ (d + A') \circ \psi'$ with $A' = \phi^{-1} d(\phi) + \phi^{-1} A \phi$.

For a given point $x_0 \in M$ one can always choose the local trivialization such that $A(x_0) = 0$. Indeed, a given trivialization can be changed by a local $\phi : U \to \mathrm{Gl}(r, \mathbb{C})$, whose Taylor expansion is of the form

$$\phi(x) = \mathrm{Id} - \sum x_i A_i(0) + \text{higher order terms}.$$

Here, $x_1, \ldots, x_n$ are local coordinates with x_0 as the origin and the connection matrix A is written as $A = \sum A_i dx_i$.

Given connections induce new connections on associated vector bundles. Here is a list of the most important examples of this principle:

Examples 4.2.6 i) Let E_1 and E_2 be vector bundles on M endowed with connections ∇_1 and ∇_2, respectively. If s_1, s_2 are local section of E_1 and E_2, respectively, we set

$$\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2).$$

This defines a natural connection on the direct sum $E_1 \oplus E_2$.

ii) In order to define a connection on the tensor product $E_1 \otimes E_2$ one defines

$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2).$$

iii) A natural connection on $\mathrm{Hom}(E_1, E_2)$ can be defined by:

$$\nabla(f)(s_1) = \nabla_2(f(s_1)) - f(\nabla_1(s_1)).$$

Here, f is a local homomorphism $E_1 \to E_2$. Then $f(s_1)$ is a local section of E_2 and ∇_2 can be applied. In the second term the homomorphism f is applied to the one-form $\nabla_1(s_1)$ with values in E_1 according to $f(\alpha \otimes t) = \alpha \otimes f(t)$, for $\alpha \in \mathcal{A}^1$ and $t \in \mathcal{A}^0(E)$.

iv) If we endow the trivial bundle with the natural connection given by the exterior differential, then the last construction yields as a special case a connection ∇^* on the dual E^* of any bundle E equipped with a connection ∇. Explicitly, one has

$$\nabla^*(f)(s) = d(f(s)) - f(\nabla(s)).$$

Clearly, changing any of the given connections by a one-form as in Proposition 4.2.3 induces new connections on the associated bundles. E.g. if $\nabla' = \nabla + a$, then $\nabla'^* = \nabla^* - a^*$. The other cases are left to the reader (cf. Exercise 4.2.2).

v) Let $f : M \to N$ be a differentiable map and let ∇ be a connection on a vector bundle E over N. Let ∇ over an open subset $U_i \subset N$ be of the form $d+A_i$ (after trivializing $E|_{U_i}$). Then the pull-back connection $f^*\nabla$ on the pull-back vector bundle f^*E over M is locally defined by $f^*\nabla|_{f^{-1}(U_i)} = d + f^*A_i$. It is straightforward to see that the locally given connections glue to a global one on f^*E.

Next we shall describe how conversely a connection ∇ on the direct sum $E = E_1 \oplus E_2$ induces connections ∇_1 and ∇_2 on E_1 and E_2, respectively. Denote by p_1 and p_2 the two projections $E_1 \oplus E_2 \to E_i$. Clearly, any section s_i of E_i can also be regarded as a section of E and thus ∇ can be applied. Then we set $\nabla_i(s_i) := p_i(\nabla(s_i))$. The verification of the Leibniz rule for ∇_i is straightforward. Thus we obtain

Lemma 4.2.7 *The connection ∇ on $E = E_1 \oplus E_2$ induces natural connections ∇_1, ∇_2 on E_1 and E_2, respectively.* □

The difference between the direct sum $\nabla_1 \oplus \nabla_2$ of the two induced connections and the connection ∇ on E we started with is measured by the second fundamental form. Let E_1 be a subbundle of a vector bundle E and assume that a connection on the latter is given.

Definition 4.2.8 The *second fundamental form* of $E_1 \subset E$ with respect to the connection ∇ on E is the section $b \in \mathcal{A}^1(M, \mathrm{Hom}(E_1, E/E_1))$ defined for any local section s of E_1 by

$$b(s) = \mathrm{pr}_{E/E_1}(\nabla(s)).$$

If we choose a splitting of $E \twoheadrightarrow E/E_1$, i.e. we write $E = E_1 \oplus E_2$ with $E_2 \cong E/E_1$ via the projection, then $b(s) = \nabla(s) - \nabla_1(s)$. Using the Leibniz rule (4.2) for ∇ and ∇_1 one proves $b(f \cdot s) = f \cdot b(s)$. Thus, b really defines an element in $\mathcal{A}^1(X, \mathrm{Hom}(E_1, E_2))$. Often we will consider situations where E is the trivial vector bundle together with the trivial connection

If E is endowed with an additional datum, e.g. an hermitian or a holomorphic structure, then one can formulate compatibility conditions for connections on E. Let us first discuss the hermitian case.

Definition 4.2.9 Let (E, h) be an hermitian vector bundle. A connection ∇ on E is an *hermitian connection* with respect to h if for arbitrary local sections s_1, s_2 one has

$$d(h(s_1, s_2)) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2)). \tag{4.3}$$

Here, the exterior differential is applied to the function $h(s_1, s_2)$ and by definition $h(\alpha \otimes s, s') := \alpha h(s, s')$ for a (complex) one-form α and sections s and s'. Analogously, $h(s, \alpha \otimes s') = \bar{\alpha} h(s, s')$. If the bundle E is real and the hermitian product is a real hermitian product, then one speaks of metric connections. See also Exercise 4.2.8 for an alternative description of (4.3).

Let ∇ be an hermitian connection and let $a \in \mathcal{A}^1(M, \mathrm{End}(E))$. By Proposition 4.2.3 one knows that $\nabla' = \nabla + a$ is again a connection. Then, ∇' is hermitian if and only if $h(a(s_1), s_2) + h(s_1, a(s_2)) = 0$ for all section s_1, s_2. Thus, ∇' is hermitian if and only if a can locally be written as $a = \alpha \otimes A$ with $a \in \mathcal{A}^1_M$ and where A at each point is contained in the Lie algebra $\mathfrak{u}(E(x), h(x))$, which, after diagonalization of h, is the Lie algebra of all skew-hermitian matrices.

Definition 4.2.10 Let (E, h) be an hermitian vector bundle. By $\mathrm{End}(E, h)$ we denote the subsheaf of sections a of $\mathrm{End}(E)$ satisfying

$$h(a(s_1), s_2) + h(s_1, a(s_2)) = 0$$

for all local sections s_1, s_2.

Note that $\mathrm{End}(E, h)$ has the structure of a real vector bundle. For line bundles, i.e. $\mathrm{rk}(E) = 1$, the vector bundle $\mathrm{End}(E)$ is the trivial complex vector bundle $\mathbb{C} \times M$ and $\mathrm{End}(E, h)$ is the imaginary part $i \cdot \mathbb{R} \times M$ of it. Then using Corollary 4.2.4 and Exercise 4.2.1 we find

Corollary 4.2.11 *The set of all hermitian connections on an hermitian vector bundle (E, h) is an affine space over the (infinite-dimensional) real vector space $\mathcal{A}^1(M, \mathrm{End}(E, h))$.* □

So far, the underlying manifold M was just a real manifold and the vector bundle E was a differentiable complex (or real) vector bundle. In what follows, we consider a holomorphic vector bundle E over a complex manifold X. Recall from Section 2.6 that in this case there exists the $\bar{\partial}$-operator $\bar{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$.

Using the decomposition $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$ we can decompose any connection ∇ on E in its two components $\nabla^{1,0}$ and $\nabla^{0,1}$, i.e. $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ with

$$\nabla^{1,0} : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{1,0}(E) \ \text{ and } \ \nabla^{0,1} : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E).$$

Note that $\nabla^{0,1}$ satisfies $\nabla^{0,1}(f \cdot s) = \bar{\partial}(f) \otimes s + f \cdot \nabla^{0,1}(s)$, i.e. it behaves similarly to $\bar{\partial}$. (Of course, the decomposition $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ makes sense also when E is not holomorphic.)

Definition 4.2.12 A connection ∇ on a holomorphic vector bundle E is *compatible with the holomorphic structure* if $\nabla^{0,1} = \bar{\partial}$.

Similarly to Corollaries 4.2.4 and 4.2.11 one proves

Corollary 4.2.13 *The space of connections ∇ on a holomorphic vector bundle E compatible with the holomorphic structure forms an affine space over the (infinite-dimensional) complex vector space $\mathcal{A}^{1,0}(X, \mathrm{End}(E))$.* □

The existence of at least one such connection (which is needed for the corollary) can be proved directly or it can be seen as a consequence of the following existence result.

Proposition 4.2.14 *Let (E, h) be a holomorphic vector bundle together with an hermitian structure. Then there exists a unique hermitian connection ∇ that is compatible with the holomorphic structure. This connection is called the* Chern connection *on (E, h).*

Proof. Let us first show the uniqueness. This is is a purely local problem. Thus, we may assume that E is the trivial holomorphic vector bundle, i.e. $E = X \times \mathbb{C}^r$. According to Remark 4.2.5 the connection ∇ is of the form $\nabla = d + A$, where $A = (a_{ij})$ is a matrix valued one-form on X. The hermitian structure on E is given by a function H on X that associates to any $x \in X$ a positive-definite hermitian matrix $H(x) = (h_{ij}(x))$.

Let e_i be the constant i-th unit vector considered as a section of E. Then the assumption that ∇ be compatible with the hermitian structure yields $dh(e_i, e_j) = h(\sum a_{ki}e_k, e_j) + h(e_i, \sum a_{\ell j}e_\ell)$ or, equivalently,

$$dH = A^{\mathrm{t}} \cdot H + H \cdot \bar{A}.$$

Since ∇ is compatible with $\bar{\partial}$, the matrix A is of type $(1, 0)$. A comparison of types of both sides yields $\bar{\partial} H = H \cdot \bar{A}$ and, after complex conjugation

$$A = \bar{H}^{-1}\partial(\bar{H}).$$

Thus, A is uniquely determined by H.

Equivalently, by using Corollaries 4.2.11 and 4.2.13 one could argue that $\mathcal{A}^1(X, \mathrm{End}(E, h)) \cap \mathcal{A}^{1,0}(X, \mathrm{End}(E)) = 0$. Indeed, any endomorphism a in this intersection satisfies $h(a(s_1), s_2) + h(s_1, a(s_2)) = 0$, where the first summand is a $(1, 0)$-form and the second is of type $(0, 1)$. Thus, both have to be trivial and hence $a = 0$.

In any case, describing the connection form A explicitly in terms of the hermitian structure H turns out to be helpful for the existence result as well. On argues as follows: Going the argument backwards, we find that locally one can find connections which are compatible with both structures. Due to the uniqueness, the locally defined connections glue. □

Example 4.2.15 Let E be a holomorphic line bundle. Then an hermitian structure H on E is given by a positive real function and the Chern connection E is locally given as $\nabla = d + \partial \log H$.

The proposition can be applied to the geometric situation. Let (X, g) be an hermitian manifold and let E be the holomorphic (co)tangent bundle. Then the Proposition asserts the existence of a natural hermitian connection $\nabla = \nabla^{1,0} + \bar{\partial}$ on the (co)tangent bundle $\mathcal{T}_X$ (respectively Ω_X). Let us study this in two easy cases.

Examples 4.2.16 i) If we endow the complex torus $\mathbb{C}^n/\Gamma$ with a constant hermitian structure, then the Chern connection on the trivial tangent bundle is the exterior differential.

ii) The second example is slightly more interesting. We study the Fubini–Study metric on $\mathbb{P}^n$ introduced in Examples 3.1.9, i). Recall that on the standard open subset $U_i \subset \mathbb{P}^n$ with coordinates $w_1, \ldots, w_n$ it is given by

$$H = \frac{1}{2\pi}\left(\frac{\delta_{ij}}{1 + \sum |w_k|^2} - \frac{\bar{w}_i w_j}{(1 + \sum |w_k|^2)^2}\right)_{i,j=1,\ldots,n}.$$

Then the distinguished connection we are looking for is locally on U_i given by $\nabla = d + \bar{H}^{-1}(\partial \bar{H})$.

Let us now study the second fundamental form for connections compatible with a given hermitian and/or holomorphic structure.

• Let

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

be a short exact sequence of holomorphic vector bundles. In general, a sequence like this need not split. However, the sequence of the underlying differentiable complex bundles can always be split and hence $E = E_1 \oplus E_2$ as complex bundles. (See Appendix A.)

If $\nabla^{0,1} = \bar{\partial}_E$, then also $\nabla_1^{0,1} = \bar{\partial}_{E_1}$ for the induced connection ∇_1 on E_1, because E_1 is a holomorphic subbundle of E. Thus, the second fundamental form b_1 is of type $(1,0)$, i.e. $b_1 \in \mathcal{A}^{1,0}(X, \mathrm{Hom}(E_1, E_2))$. The analogous statement holds true for ∇_2 on E_2 and the second fundamental form b_2 if and only if the chosen split is holomorphic.

• Let (E, h) be an hermitian vector bundle and assume that $E = E_1 \oplus E_2$ is an orthogonal decomposition, i.e. E_1, E_2 are both endowed with hermitian structures h_1 and h_2, respectively, such that $h = h_1 \oplus h_2$.

Let ∇ be a hermitian connection on (E, h). Then the induced connections ∇_1, ∇_2 are again hermitian (cf. Exercise 4.2.5) and for the fundamental forms b_1 and b_2 one has

$$h_1(s_1, b_2(t_2)) + h_2(b_1(s_1), t_2) = 0$$
$$h_1(b_2(s_2), t_1) + h_2(s_2, b_1(t_1)) = 0$$

for any local sections s_i and t_i of E_1 respectively E_2. This follows easily from

$$\begin{aligned} & dh(s_1 \oplus s_2, t_1 \oplus t_2) \\ = & h((\nabla_1(s_1) + b_2(s_2)) \oplus (\nabla_2(s_2) + b_1(s_1)), t_1 \oplus t_2) \\ & + h(s_1 \oplus s_2, (\nabla_1(t_1) + b_2(t_2)) \oplus (\nabla_2(t_2) + b_1(t_1))) \end{aligned}$$

Usually, one combines both situations. A short exact sequence of holomorphic vector bundles can be splitted as above by choosing the orthogonal complement $E^\perp \cong E_2$ of $E_1 \subset E$ with respect to a chosen hermitian structure on E.

We conclude this section with a brief discussion of the notion of a holomorphic connection, which should not be confused with the notion of a connection compatible with the holomorphic structure. In fact, the notion of a holomorphic connection is much more restrictive, but has the advantage to generalize to the purely algebraic setting.

Definition 4.2.17 Let E be a holomorphic vector bundle on a complex manifold X. A *holomorphic connection* on E is a $\mathbb{C}$-linear map (of sheaves) $D : E \to \Omega_X \otimes E$ with

$$D(f \cdot s) = \partial(f) \otimes s + f \cdot D(s)$$

for any local holomorphic function f on X and any local holomorphic section s of E.

Here, E denotes both, the vector bundle and the sheaf of holomorphic sections of this bundle. Clearly, if f is a holomorphic function, then $\partial(f)$ is a holomorphic section of $\bigwedge^{1,0} X$, i.e. a section of Ω_X (use $\bar\partial\partial(f) = -\partial\bar\partial(f)$). (See Proposition 2.6.11.)

Most of what has been said about ordinary connections holds true for holomorphic connections with suitable modifications. E.g. if D and D' are holomorphic connections on E, then $D - D'$ is a holomorphic section of $\Omega_X \otimes \mathrm{End}(E)$. Locally, any holomorphic connection D is of the form $\partial + A$ where A is a holomorphic section of $\Omega_X \otimes \mathrm{End}(E)$.

Writing a holomorphic connection D locally as $\partial + A$ shows that D also induces a $\mathbb{C}$-linear map $D : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$ which satisfies $D(f \cdot s) = \partial(f) \otimes s + f \cdot D(s)$. Thus, D looks like the $(1,0)$-part of an ordinary connection and, indeed, $\nabla := D + \bar\partial$ defines an ordinary connection on E.

However, the $(1,0)$-part of an arbitrary connection need not be a holomorphic connection in general. It might send holomorphic sections of E to those of $\mathcal{A}^{1,0}(E)$ that are not holomorphic, i.e. not contained in $\Omega_X \otimes E$. In fact, holomorphic connections exist only on very special bundles (see Remark 4.2.20 and Exercise 4.4.12).

We want to introduce a natural cohomology class whose vanishing decides whether a holomorphic connection on a given holomorphic bundle can be found. Let E be a holomorphic vector bundle and let $X = \bigcup U_i$ be an open covering such that there exist holomorphic trivializations $\psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r$.

Definition 4.2.18 The *Atiyah class*

$$A(E) \in H^1(X, \Omega_X \otimes \operatorname{End}(E))$$

of the holomorphic vector bundle E is given by the Čech cocycle

$$A(E) = \left\{U_{ij}, \psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j\right\}.$$

Due to the cocycle condition $\psi_{ij}\psi_{jk}\psi_{ki} = 1$, the collection $\{U_{ij}, \psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j\}$ really defines a cocycle. The definition of $A(E)$ is indeed independent of the cocycle $\{\psi_{ij}\}$. We leave the straightforward proof to the reader (see Exercise 4.2.9).

Proposition 4.2.19 *A holomorphic vector bundle E admits a holomorphic connection if and only if its Atiyah class $A(E) \in H^1(X, \Omega_X \otimes \operatorname{End}(E))$ is trivial.*

Proof. First note that $d\psi_{ij} = \partial\psi_{ij}$, as the ψ_{ij} are holomorphic.

Local holomorphic connections on $U_i \times \mathbb{C}^r$ are of the form $\partial + A_i$. Those can be glued to a connection on the bundle E if and only if

$$\psi_i^{-1} \circ (\partial + A_i) \circ \psi_i = \psi_j^{-1} \circ (\partial + A_j) \circ \psi_j$$

on U_{ij} or, equivalently,

$$\psi_i^{-1} \circ \partial \circ \psi_i - \psi_j^{-1} \circ \partial \circ \psi_j = \psi_j^{-1} A_j \psi_j - \psi_i^{-1} A_i \psi_i. \qquad (4.4)$$

The left hand side of (4.4) can be written as

$$\begin{aligned} &\psi_j^{-1} \circ (\psi_{ij}^{-1} \circ \partial \circ \psi_i \circ \psi_j^{-1}) \circ \psi_j - \psi_j^{-1} \circ \partial \circ \psi_j \\ &= \psi_j^{-1} \circ (\psi_{ij}^{-1} \circ \partial \circ \psi_{ij} - \partial) \circ \psi_j = \psi_j^{-1} \circ (\psi_{ij}^{-1} \partial(\psi_{ij})) \circ \psi_j \end{aligned}$$

The right hand side of (4.4) is the boundary of $\{B_i \in \Gamma(U_i, \Omega \otimes \operatorname{End}(E))\}$ with $B_i = \psi_i^{-1} A_i \psi_i$. Thus, $A(E) = 0$ if and only if there exist local connections on E that can be glued to a global one. □

Remarks 4.2.20 i) Later we will see that $A(E)$ is related to the curvature of E. Roughly, a holomorphic connection on a vector bundle E over a compact manifold exists, i.e. $A(E) = 0$, if and only if the bundle is flat. Moreover, we will see that $A(E)$ encodes all characteristic classes of E.

ii) Note that for vector bundles of rank one, i.e. line bundles, one has $A(L) = \{\partial \log(\psi_{ij})\}$. This gives yet another way of defining a first Chern class of a holomorphic line bundle as $A(E) \in H^1(X, \Omega_X) = H^1(X, \Omega_X \otimes \operatorname{End}(E))$. A comparison of the various possible definitions of the first Chern class encountered so far will be provided in Section 4.4.

Exercises

4.2.1 i) Show that any (hermitian) vector bundle admits a(n hermitian) connection.

ii) Show that a connection ∇ is given by its action on the space of global sections $\mathcal{A}^0(X, E)$.

4.2.2 Let ∇_i be connections on vector bundles E_i, $i = 1, 2$. Change both connections by one-forms $a_i \in \mathcal{A}^1(X, \mathrm{End}(E_i))$ and compute the new connections on the associated bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\mathrm{Hom}(E_1, E_2)$.

4.2.3 Prove that connections on bundles E_i, $i = 1, 2$ which are compatible with given hermitian or holomorphic structures induce compatible connections on the associated bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\mathrm{Hom}(E_1, E_2)$.

4.2.4 Study connections on an hermitian holomorphic vector bundle (E, h) that admits local holomorphic trivialization which are at the same time orthogonal with respect to the hermitian structure.

4.2.5 Let (E, h) be an hermitian vector bundle. If $E = E_1 \oplus E_2$, then E_1 and E_2 inherit natural hermitian structures h_1 and h_2. Are the induced connections ∇_i on E_i again hermitian with respect to these hermitian structures? What can you say about the second fundamental form?

4.2.6 Let ∇ be a connection on E. Describe the induced connections on $\bigwedge^2 E$ and $\det(E)$.

4.2.7 Show that the pull-back of an hermitian connection is hermitian with respect to the pull-back hermitian structure. Analogously, the pull-back of a connection compatible with the holomorphic structure on a holomorphic vector bundle under a holomorphic map is again compatible with the holomorphic structure on the pull-back bundle.

4.2.8 Show that a connection ∇ on an hermitian bundle (E, h) is hermitian if and only if $\nabla(h) = 0$, where by ∇ we also denote the naturally induced connection on the bundle $(E \otimes \bar{E})^*$.

4.2.9 Show that the definition of the Atiyah class does not depend on the chosen trivialization. Proposition 4.3.10 will provide an alternative proof of this fact.

Comments: - To a large extent, it is a matter of taste whether one prefers to work with connections globally or in terms of their local realizations as $d + A$ with the connections matrix A. However, both approaches are useful. Often, an assertion is first established locally or even fibrewise and afterwards, and sometimes more elegantly, put in a global language.

- For this and the next section we recommend Kobayashi's excellent textbook [78].

- The Atiyah class was introduced by Atiyah in [3]. There is a way to define the Atiyah class via the jet-sequence or, equivalently, using the first infinitesimal neighbourhood of the diagonal. See [31].

4.3 Curvature

In the previous paragraph we introduced the notion of a connection generalizing the exterior differential to sections of vector bundles. Indeed, by definition a connection satisfies the Leibniz rule. However, in general a connection ∇ need not satisfy $\nabla^2 = 0$, i.e. ∇ is not a differential. The obstruction for a connection to define a differential is measured by its curvature. This concept will be explained now.

The reader familiar with the basic concepts in Riemannian geometry will find the discussion similar to the treatment of the curvature of a Riemannian manifold. As before, we shall also study what happens in the presence of additional structures, e.g. hermitian and holomorphic ones.

Let E be a vector bundle on a manifold M endowed with a connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$. Then a natural extension

$$\nabla : \mathcal{A}^k(E) \longrightarrow \mathcal{A}^{k+1}(E)$$

is defined as follows: If α is a local k-form on M and s is a local section of E then

$$\nabla(\alpha \otimes s) = d(\alpha) \otimes s + (-1)^k \alpha \wedge \nabla(s).$$

Observe that for $k = 0$ this is just the Leibniz formula (4.2) which also ensures that the extension is well-defined, i.e. $\nabla(\alpha \otimes (f \cdot s)) = \nabla(f\alpha \otimes s)$ for any local function f. Moreover, a generalized Leibniz rule also holds for this extension, i.e. for any section t of $\mathcal{A}^\ell(E)$ and any k-form β one has

$$\nabla(\beta \wedge t) = d(\beta) \wedge t + (-1)^k \beta \wedge \nabla(t).$$

Indeed, if $t = \alpha \otimes s$ then

$$\begin{aligned}\nabla(\beta \wedge t) &= \nabla((\beta \wedge \alpha) \otimes s) = d(\beta \wedge \alpha) \otimes s + (-1)^{k+\ell}(\beta \wedge \alpha) \otimes \nabla(s)\\ &= d(\beta) \wedge t + (-1)^k((\beta \wedge d(\alpha)) \otimes s + (-1)^\ell(\beta \wedge \alpha) \otimes \nabla(s))\\ &= d(\beta) \wedge t + (-1)^k \beta \wedge \nabla(t)\end{aligned}$$

Definition 4.3.1 The *curvature* F_∇ of a connection ∇ on a vector bundle E is the composition

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1(E) \longrightarrow \mathcal{A}^2(E).$$

Usually, the curvature F_∇ will be considered as a global section of $\mathcal{A}^2(\mathrm{End}(E))$, i.e. $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$. This is justified by the following result.

Lemma 4.3.2 *The curvature homomorphism* $F_\nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ *is* $\mathcal{A}^0$-*linear.*

Proof. For a local section s of E and a local function f on M one computes

$$\begin{aligned} F_\nabla(f\cdot s) &= \nabla(\nabla(f\cdot s)) = \nabla\left(df\otimes s + f\cdot\nabla(s)\right)\\ &= \underbrace{d^2(f)}_{0}\otimes s \underbrace{-df\wedge\nabla(s) + df\wedge\nabla(s)}_{0} + f\cdot\nabla(\nabla(s))\\ &= f\cdot F_\nabla(s) \end{aligned}$$

□

Examples 4.3.3 i) Let us compute the curvature of a connection on the trivial bundle $M\times\mathbb{C}^r$. If ∇ is the trivial connection, i.e. $\nabla = d$, then $F_\nabla = 0$.

Any other connection is of the form $\nabla = d + A$, where A is a matrix of one-forms. For a section s one obtains $F_\nabla(s) = (d+A)(d+A)(s) = (d+A)(d(s)+A\cdot s) = d^2(s)+d(A\cdot s)+A\cdot d(s)+A(A(s)) = d(A)(s)+(A\wedge A)(s)$, i.e.

$$F_\nabla = d(A) + A\wedge A.$$

ii) For a line bundle this calculation yields $F_\nabla = d(A)$. In this case, the curvature is an ordinary two-form.

For $a\in\mathcal{A}^1(M,\mathrm{End}(E))$ the two-form $a\wedge a\in\mathcal{A}^2(M,\mathrm{End}(E))$ is given by exterior product in the form part and composition in $\mathrm{End}(E)$. Using this, the example is easily generalized to

Lemma 4.3.4 *Let ∇ be a connection on a vector bundle E and let $a\in\mathcal{A}^1(M,\mathrm{End}(E))$. Then $F_{\nabla+a} = F_\nabla + \nabla(a) + a\wedge a$.* □

Any connection ∇ on a vector bundle E induces a natural connection on $\mathrm{End}(E)$ which will also be called ∇. In particular, this connection can be applied to the curvature $F_\nabla\in\mathcal{A}^2(M,\mathrm{End}(E))$ of the original connection on E. Here is the next remarkable property of the curvature:

Lemma 4.3.5 (Bianchi identity) *If $F_\nabla\in\mathcal{A}^2(M,\mathrm{End}(E))$ is the curvature of a connection ∇ on a vector bundle E, then*

$$0 = \nabla(F_\nabla)\in\mathcal{A}^3(M,\mathrm{End}(E)).$$

Proof. This follows from $\nabla(F_\nabla)(s) = \nabla(F_\nabla(s)) - F_\nabla(\nabla(s)) = \nabla(\nabla^2(s)) - \nabla^2(\nabla(s)) = 0$. Here we use Exercise 4.3.1. □

Example 4.3.6 For the connection $\nabla = d + A$ on the trivial bundle the Bianchi identity becomes $dF_\nabla = F_\nabla\wedge A - A\wedge F_\nabla$.

The curvature of induced connections on associated bundles can usually be expressed in terms of the curvature of the given connections. For the most frequent associated bundles they are given by the following proposition.

Proposition 4.3.7 *Let E_1, E_2 be vector bundles endowed with connections ∇_1 and ∇_2, respectively.*

i) *The curvature of the induced connection on the direct sum $E_1 \oplus E_2$ is given by*

$$F = F_{\nabla_1} \oplus F_{\nabla_2}.$$

ii) *On the tensor product $E_1 \otimes E_2$ the curvature is given by*

$$F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}.$$

iii) *For the induced connection ∇^* on the dual bundle E^* one has*

$$F_{\nabla^*} = -F_{\nabla}^{\rm t}.$$

iv) *The curvature of the pull-back connection $f^*\nabla$ of a connection ∇ under a differentiable map $f : M \to N$ is*

$$F_{f^*\nabla} = f^* F_{\nabla}.$$

Proof. Let us prove ii). This is the following straightforward calculation

$$\begin{aligned}
&F_{\nabla}(s_1 \otimes s_2)\\
&= \nabla(\nabla(s_1 \otimes s_2)) = \nabla(\nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2))\\
&= \nabla_1^2(s_1) \otimes s_2 - \nabla_1(s_1) \otimes \nabla_2(s_2) + \nabla_1(s_1) \otimes \nabla_2(s_2) + s_1 \otimes \nabla_2^2(s_2)\\
&= F_{\nabla_1}(s_1) \otimes s_2 + s_1 \otimes F_{\nabla_2}(s_2).
\end{aligned}$$

The sign appears, because ∇ is applied to the one-form $\nabla_1(s_1) \otimes s_2$. We leave i) and iii) to the reader (cf. Exercise 4.3.2).

iv) follows from the local situation, where ∇ is given as $d + A$. Then $F_{f^*\nabla} = F_{d+f^*A} = d(f^*A) + f^*(A) \wedge f^*(A) = f^*(d(A) + A \wedge A) = f^* F_{\nabla}$. □

Next we will study the curvature of the special connections introduced in Section 4.2.

Proposition 4.3.8 i) *The curvature of an hermitian connection ∇ on an hermitian vector bundle (E, h) satisfies $h(F_{\nabla}(s_1), s_2) + h(s_1, F_{\nabla}(s_2)) = 0$, i.e.*

$$F_{\nabla} \in \mathcal{A}^2(M, \mathrm{End}(E, h)).$$

ii) *The curvature F_{∇} of a connection ∇ on a holomorphic vector bundle E over a complex manifold X with $\nabla^{0,1} = \bar\partial$ has no $(0,2)$-part, i.e.*

$$F_{\nabla} \in (\mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1})(X, \mathrm{End}(E)).$$

iii) *Let E be a holomorphic bundle endowed with an hermitian structure h. The curvature of the Chern connection ∇ is of type $(1,1)$, real, and skew-hermitian, i.e. $F_{\nabla} \in \mathcal{A}^{1,1}_{\mathbb{R}}(X, \mathrm{End}(E, h))$. (Recall that $\mathrm{End}(E, h)$ is only a real vector bundle.)*

Proof. i) A local argument goes as follows: Choose a trivialization such that (E, h) is isomorphic to the trivial bundle $M \times \mathbb{C}^r$ endowed with the constant standard hermitian product. Then, $\nabla = d + A$ with $\bar{A}^{\mathrm{t}} = -A$. For its curvature $F_\nabla = d(A) + A \wedge A$ one obtains

$$\begin{aligned}\bar{F}_\nabla^{\mathrm{t}} &= d(\bar{A}^{\mathrm{t}}) + (\bar{A} \wedge \bar{A})^{\mathrm{t}} = d(\bar{A}^{\mathrm{t}}) - \bar{A}^{\mathrm{t}} \wedge \bar{A}^{\mathrm{t}} \\ &= d(-A) - (-A) \wedge (-A) = -F_\nabla.\end{aligned}$$

For a global argument one first observes that the assumption that ∇ is hermitian yields for form-valued sections $s_i \in \mathcal{A}^{k_i}(E)$ the equation

$$dh(s_1, s_2) = h(\nabla(s_1), s_2) + (-1)^{k_1} h(s_1, \nabla(s_2)).$$

Recall that $h(\alpha_1 \otimes t_1, \alpha_2 \otimes t_2) = (\alpha_1 \wedge \bar{\alpha}_2) h(t_1, t_2)$ for local forms α_1, α_2 and sections t_1, t_2.

Hence, for $s_1, s_2 \in \mathcal{A}^0(E)$ one has

$$\begin{aligned}dh(\nabla(s_1), s_2) &= h(F_\nabla(s_1), s_2) - h(\nabla(s_1), \nabla(s_2)) \\ dh(s_1, \nabla(s_2)) &= h(\nabla(s_1), \nabla(s_2)) + h(s_1, F_\nabla(s_2))\end{aligned}$$

and on the other hand

$$dh(\nabla(s_1), s_2)) + dh(s_1, \nabla(s_2)) = d(dh(s_1, s_2)) = 0.$$

This yields the assertion.

ii) Here, one first observes that the extension $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ splits into a $(1,0)$-part and a $(0,1)$-part, the latter of which is $\bar{\partial}$. Then one computes $\nabla^2 = (\nabla^{1,0})^2 + \nabla^{1,0} \circ \bar{\partial} + \bar{\partial} \circ \nabla^{1,0}$, as $\bar{\partial}^2 = 0$. Hence, $F_\nabla(s) \in (\mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1})(E)$ for all $s \in \mathcal{A}^0(E)$.

Locally one could argue as follows: Since $\nabla = d + A$ with A of type $(1,0)$, the curvature $d(A) + A \wedge A = \bar{\partial}(A) + (\partial A + A \wedge A)$ is a sum of a $(1,1)$-form and a $(2,0)$-form.

iii) Combine i) and ii). By comparing the bidegree of F_∇^{t} and F_∇ in local coordinates we find that F_∇ is of type $(1,1)$.

More globally, due to ii) one knows that $h(F_\nabla(s_1), s_2)$ and $h(s_1, F_\nabla(s_2))$ are of bidegree $(2,0) + (1,1)$ respectively $(1,1) + (0,2)$. Recall the convention $h(s_1, \alpha_2 \otimes s_2) = \bar{\alpha}_2 h(s_1, s_2)$. Using i), i.e. $h(F_\nabla(s_1), s_2) + h(s_1, F_\nabla(s_2)) = 0$, and comparing the bidegree shows that $h(F_\nabla(s_1), s_2)$ has trivial $(2,0)$-part for all sections s_1, s_2. Hence, F_∇ is of type $(1,1)$. □

Examples 4.3.9 i) Suppose E is the trivial vector bundle $M \times \mathbb{C}^r$ with the constant standard hermitian structure. If $\nabla = d + A$ is an hermitian connection, then i) says $d(A + \bar{A}^{\mathrm{t}}) + (A \wedge A + \overline{(A \wedge A)}^{\mathrm{t}}) = 0$. For $r = 1$ this means that the real part of A is constant.

ii) If (L, h) is an hermitian holomorphic line bundle, then the curvature F_∇ of its Chern connection is a section of $\mathcal{A}^{1,1}_{\mathbb{R}}(X, \mathrm{End}(L, h))$, which can be identified with the imaginary $(1,1)$-forms on X. Indeed, $\mathrm{End}(L, h)$ is the purely imaginary line bundle $i \cdot \mathbb{R} \times X$ (cf. page 176).

iii) If the hermitian structure on the holomorphic vector bundle E is locally given by the matrix H, then the Chern connection is of the form $d+\bar{H}^{-1}\partial(\bar{H})$. Hence, the curvature is

$$F = \bar{\partial}(\bar{H}^{-1}\partial(\bar{H})).$$

Indeed, a priori $F = \bar{\partial}(\bar{H}^{-1}\partial(\bar{H})) + \partial(\bar{H}^{-1}\partial(\bar{H}) + (\bar{H}^{-1}\partial(\bar{H})) \wedge (\bar{H}^{-1}\partial(\bar{H}))$, but the sum of the last two terms must vanish, as both are of type $(2,0)$.

If E is a line bundle, then the hermitian matrix is just a positive real function h. In this case one writes $F = \bar{\partial}\partial\log(h)$. Once again, we see that F can be considered as a purely imaginary two-form

iv) Let (X, g) be an hermitian manifold. Then the tangent bundle is naturally endowed with an hermitian structure. The curvature of the Chern connection on $\mathcal{T}_X$ is called the *curvature of the hermitian manifold* (X, g). In Section 4.A we shall see that the curvature of a Kähler manifold (X, g) is nothing but the usual curvature of the underlying Riemannian manifold.

For the Chern connection on a holomorphic hermitian bundle the Bianchi identity yields

$$0 = (\nabla(F_\nabla))^{1,2} = \bar{\partial}(F_\nabla),$$

i.e. F_∇ as an element of $\mathcal{A}^{1,1}(X, \mathrm{End}(E))$ is $\bar{\partial}$-closed. Thus, in this case the curvature yields a natural Dolbeault cohomology class $[F_\nabla] \in H^1(X, \Omega_X \otimes \mathrm{End}(E))$ for any holomorphic vector bundle E. A priori, this cohomology class might depend on the chosen hermitian structure or, equivalently, on the connection. That this is not the case is an immediate consequence of the following description of it as the Atiyah class of E.

Proposition 4.3.10 *For the curvature F_∇ of the Chern connection on an hermitian holomorphic vector bundle (E, h) one has*

$$[F_\nabla] = A(E) \in H^1(X, \Omega_X \otimes \mathrm{End}(E)).$$

Proof. The comparison of Dolbeault and Čech cohomology can be done by chasing through the following commutative diagram of sheaves:

$$\begin{array}{ccccc}
\Omega_X \otimes \mathrm{End}(E) & \longrightarrow & \mathcal{C}^0(\{U_i\}, \Omega_X \otimes \mathrm{End}(E)) & \longrightarrow & \mathcal{C}^1(\{U_i\}, \Omega_X \otimes \mathrm{End}(E)) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\
\mathcal{A}^{1,0}(\mathrm{End}(E)) & \longrightarrow & \mathcal{C}^0(\{U_i\}, \mathcal{A}^{1,0}(\mathrm{End}(E))) & \xrightarrow{\delta_1} & \mathcal{C}^1(\{U_i\}, \mathcal{A}^{1,0}(\mathrm{End}(E))) \\
\downarrow & & \downarrow{\scriptstyle \bar{\partial}} & & \\
\mathcal{A}^{1,1}(\mathrm{End}(E)) & \xrightarrow{\delta_0} & \mathcal{C}^0(\{U_i\}, \mathcal{A}^{1,1}(\mathrm{End}(E))) & &
\end{array}$$

Here, $X = \bigcup U_i$ is an open covering of X trivializing E via $\psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r$. With respect to ψ_i the hermitian structure h on U_i is given by an hermitian matrix H_i. Then the curvature F_∇ of the Chern connection on the holomorphic bundle E on U_i is given by $F_\nabla|_{U_i} = \psi^{-1}(\bar\partial(\bar H_i^{-1}\partial\bar H_i))\psi_i$. Thus,

$$\begin{aligned}\delta_0(F_\nabla) &= \{U_i, \psi_i^{-1}\circ(\bar\partial(\bar H_i^{-1}\partial\bar H_i))\circ\psi_i\}\\ &= \bar\partial\{U_i, \psi_i^{-1}\circ(\bar H_i^{-1}\partial\bar H_i)\circ\psi_i\},\end{aligned}$$

as the maps ψ_i are holomorphic trivializations.

Hence, it suffices to show that

$$i\{U_{ij}, \psi_j^{-1}\circ(\psi_{ij}^{-1}d\psi_{ij})\circ\psi_j\} = \delta_1\{U_i, \psi_i^{-1}\circ(\bar H_i^{-1}\partial\bar H_i)\circ\psi_i\},$$

because the cocycle $\{U_i, \psi_i^{-1}\circ(\psi_{ij}^{-1}d\psi_{ij})\circ\psi_j\}$ represents by definition the Atiyah class of E. Using the definition of δ_1 one shows that the term on the right hand side equals

$$\{U_{ij}, \psi_j^{-1}\circ(\bar H_j^{-1}\partial\bar H_j)\circ\psi_j - \psi_i^{-1}\circ(\bar H_i^{-1}\partial\bar H_i)\circ\psi_i)\}.$$

Since

$$\begin{aligned}&\psi_j^{-1}\circ(\bar H_j^{-1}\partial\bar H_j)\circ\psi_j - \psi_i^{-1}\circ(\bar H_i^{-1}\partial\bar H_i)\circ\psi_i\\ &= \psi_j^{-1}\circ(\bar H_j^{-1}\partial\bar H_j - \psi_{ij}^{-1}\circ(\bar H_i^{-1}\partial\bar H_i)\circ\psi_{ij})\circ\psi_j,\end{aligned}$$

it suffices to prove

$$\bar H_j^{-1}\partial\bar H_j - \psi_{ij}^{-1}\circ(\bar H_i^{-1}\partial\bar H_i)\circ\psi_{ij} = \psi_{ij}^{-1}\partial\psi_{ij}.$$

The latter is a consequence of the compatibility $\psi_{ij}^{\rm t}H_i\bar\psi_{ij} = H_j$ or, equivalently, $\bar\psi_{ij}^{\rm t}\bar H_i\psi_{ij} = \bar H_j$ and the chain rule

$$\begin{aligned}\bar H_j^{-1}\partial\bar H_j &= \psi_{ij}^{-1}\bar H_i^{-1}(\bar\psi_{ij}^{\rm t})^{-1}\left(\bar\psi_{ij}^{\rm t}(\partial\bar H_i)\psi_{ij} + \bar\psi_{ij}^{\rm t}\bar H_i\partial\psi_{ij}\right)\\ &= \psi_{ij}^{-1}(\bar H_i^{-1}\partial\bar H_i)\psi_{ij} + \psi_{ij}^{-1}\partial\psi_{ij},\end{aligned}$$

where we have used $\partial\bar\psi_{ij} = 0$. □

Remark 4.3.11 Here is a more direct argument to see that $[F_\nabla]$ does not depend on the connection ∇. Indeed, any other connection is of the form $\nabla + a$ and $F_{\nabla+a} = F_\nabla + \nabla(a) + a\wedge a$. If both connections ∇ and $\nabla + a$ are Chern connections with respect to certain hermitian structures, then F_∇ and $F_{\nabla+a}$ are $(1,1)$-forms and $a \in \mathcal{A}^{1,0}(\mathrm{End}(E))$. Thus, $a\wedge a \in \mathcal{A}^{2,0}(\mathrm{End}(E))$ and, therefore, $\nabla(a) + a\wedge a = (\nabla(a) + a\wedge a)^{1,1} = \nabla(a)^{1,1} = \bar\partial(a)$, i.e. $F_{\nabla+a} = F_\nabla + \bar\partial(a)$ and hence

$$[F_{\nabla+a}] = [F_\nabla].$$

We have used here that the induced connection on the endomorphism bundle is again compatible with the holomorphic structure.

Example 4.3.12 Let us consider the standard homogeneous linear coordinates $z_0, \ldots, z_n$ on $\mathbb{P}^n$ as sections of $\mathcal{O}(1)$. According to Example 4.1.2 one associates with them a natural hermitian metric $h(t) = |\psi(t)|^2/(\sum |\psi(z_i)|^2)$ on $\mathcal{O}(1)$.

We claim that the curvature F of the Chern connection on the holomorphic line bundle $\mathcal{O}(1)$ endowed with this hermitian metric is

$$\frac{i}{2\pi}F = \omega_{\mathrm{FS}},$$

where ω_{FS} is the Fubini–Study Kähler form (see Example 3.1.9, i)).

This can be verified locally. On a standard open subset $U_i \subset \mathbb{P}^n$ one has $\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\bar\partial \log(1 + \sum |w_i|^2)$.

The hermitian structure of $\mathcal{O}(1)|_{U_i}$ with respect to the natural trivialization is given by the scalar function $h = (1 + \sum |w_i|^2)^{-1}$. By Example iii) in 4.3.9 we have $F = \bar\partial\partial \log(h) = \partial\bar\partial \log(1 + \sum |w_i|^2)$.

Remark 4.3.13 In the general situation, i.e. of a holomorphic line bundle L on a complex manifold X, the hermitian structure $h_{\{s_i\}}$ induced by globally generating sections $s_0, \ldots, s_n$ is the pull-back of the hermitian structure on $\mathcal{O}(1)$ of the previous example under the induced morphism $\varphi : X \to \mathbb{P}^n$. Thus, for the curvature F_∇ of the Chern connection ∇ on L one has: $(i/2\pi)F_\nabla = \varphi^*\omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini–Study form on $\mathbb{P}^n$.

Suppose now that L is not only holomorphic, but also endowed with an hermitian structure h. What is the relation between h and $h_{\{s_i\}}$? In general, they are not related at all, but if the sections $s_0, \ldots, s_n$ are chosen such that they form an orthonormal base of $H^0(X, L)$ then one might hope to approximate h by $h_{\{s_i\}}$. (Here, $H^0(X, L)$ is equipped with the hermitian product defined in 4.1.11.) This circle of questions is intensively studied at the moment and there are many open questions.

Definition 4.3.14 A real $(1,1)$-form α is called *(semi-)positive* if for all holomorphic tangent vectors $0 \neq v \in T^{1,0}X$ one has

$$-i\alpha(v, \bar v) > 0 \text{ (resp. } \geq).$$

At a point $x \in X$ any semi-positive $(1,1)$-form is a positive linear combination of forms of the type $i\beta\wedge\bar\beta$, where β is a $(1,0)$-form. The standard example of a positive form is provided by a Kähler form $\omega = \frac{i}{2}\sum h_{ij} dz_i \wedge d\bar z_j$. In this case (h_{ij}) is a positive hermitian matrix and thus $v^{\mathrm{t}}(h_{ij})\bar v > 0$ for all non-zero holomorphic tangent vectors. Together with Example 4.3.12 this shows that iF_∇ of the Chern connection ∇ on $\mathcal{O}(1)$ (with the standard hermitian structure) is positive.

Clearly, the pull-back of a semi-positive form is again semi-positive. For the curvature of a globally generated line bundle L this implies

$$F_\nabla(v, \bar v) \geq 0.$$

We next wish to generalize this observation to the higher rank case. Let (E,h) be an hermitian vector bundle over a complex manifold X and let ∇ be an hermitian connection that satisfies $F_\nabla \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$.

Definition 4.3.15 The curvature F_∇ is (Griffiths-)*positive* if for any $0 \neq s \in E$ one has

$$h(F_\nabla(s), s)(v, \bar{v}) > 0$$

for all non-trivial holomorphic tangent vectors v. Semi-positivity (negativity, semi-negativity) is defined analogously.

Remark 4.3.16 Let L be an hermitian holomorphic line bundle and let ∇ be the Chern connection. Its curvature F_∇ is an imaginary $(1,1)$-form. Then the curvature F_∇ is positive in the sense of Definition 4.3.15 if and only if the real $(1,1)$-form iF_∇ is positive in the sense of Definition 4.3.14. The extra factor i is likely to cause confusion at certain points, but these two concepts of positivity (and others) are met frequently in the literature.

Before proving the semi-positivity of the curvature of any globally generated vector bundle, we need to relate the curvature and the second fundamental forms of a split vector bundle $E = E_1 \oplus E_2$. Let ∇ be a connection on E. We denote the induced connections by ∇_i and the second fundamental forms by b_i, $i = 1, 2$. Thus,

$$\nabla = \begin{pmatrix} \nabla_1 & b_2 \\ b_1 & \nabla_2 \end{pmatrix}$$

which immediately shows

Lemma 4.3.17 *The curvature of the induced connection ∇_1 on E_1 is given by $F_{\nabla_1} = \mathrm{pr}_1 \circ F_\nabla - b_2 \circ b_1$.* □

Now let E_1 be a holomorphic subbundle of the trivial holomorphic vector bundle $E = \mathcal{O}^{\oplus r}$ endowed with the trivial constant hermitian structure. The curvature of the Chern connection ∇ on E is trivial, as ∇ is just the exterior differential. Hence, $F_{\nabla_1} = -b_2 \circ b_1$. This leads to

Proposition 4.3.18 *The curvature F_{∇_1} of the Chern connection ∇_1 of a subbundle $E_1 \subset E = \mathcal{O}^{\oplus r}$ (with the induced hermitian structure) is semi-negative.*

Proof. By the previous lemma we have $F_{\nabla_1} = -b_2 \circ b_1$. Thus, if h_2 is the induced hermitian structure on the quotient E/E_1 then

$$h_1(F_{\nabla_1}(s), s) = -h_1(b_2(b_1(s)), s) = -h_2(b_1(s), b_1(s)).$$

Here we use properties of the second fundamental form proved in Section 4.2.

More precisely, one has

$$\begin{aligned} h_1(b_2(\beta \otimes t), s) &= h_1(-\beta \otimes b_2(t), s) = -\beta h_1(b_2(t), s) \\ &= \beta h_2(t, b_1(s)) = h_2(\beta \otimes t, b_1(s)) \end{aligned}$$

for any one-form β.

Now, it suffices to show that $h(\alpha, \alpha) \geq 0$ for any $\alpha \in \mathcal{A}^{1,0}(E)$. Fix an orthonormal basis s_i of E and write $\alpha = \sum \alpha_i s_i$. Then $h(\alpha, \alpha) = \sum \alpha_i \wedge \bar{\alpha}_i$. But $\alpha_i \wedge \bar{\alpha}_i$ is semi-positive due to

$$(\alpha_i \wedge \bar{\alpha}_i)(v, \bar{v}) = \alpha_i(v) \cdot \bar{\alpha}_i(\bar{v}) = \alpha_i(v) \cdot \overline{\alpha_i(v)} \geq 0.$$

(Note that we did use that b_1 is of type $(1,0)$.) □

A holomorphic vector bundle E is *globally generated* if there exist global holomorphic sections $s_1, \ldots, s_r \in H^0(X, E)$ such that for any $x \in X$ the values $s_1(x), \ldots, s_r(x)$ generate the fibre $E(x)$. In other words, the sections $s_1, \ldots, s_r$ induce a surjection $\mathcal{O}^{\oplus r} \twoheadrightarrow E$. The standard constant hermitian structure on $\mathcal{O}^{\oplus r}$ induces via this surjection an hermitian structure on E and dualizing this surjection yields an inclusion of vector bundles $E^* \subset \mathcal{O}^{\oplus r}$.

Using Exercise 4.3.3 the proposition yields

Corollary 4.3.19 *The curvature of a globally generated vector bundle is semi-positive.* □

Here, the curvature is the curvature of the Chern connection with respect to a hermitian structure on E defined by the choice of the globally generating holomorphic sections.

Example 4.3.20 The Euler sequence on $\mathbb{P}^n$ twisted by $\mathcal{O}(-1)$ is of the form

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n}(-1) \longrightarrow 0\,.$$

Hence, $\mathcal{T}_{\mathbb{P}^n}(-1)$ admits an hermitian structure such that the curvature of the Chern connection is semi-positive. Twisting by $\mathcal{O}(1)$ yields a connection on $\mathcal{T}_{\mathbb{P}^n}$ with positive curvature. In fact, this is the curvature of the Fubini–Study metric on $\mathbb{P}^n$. See Example 4.1.5.

Remark 4.3.21 As for line bundles, one could have first studied the universal case. Recall that on the Grassmannian $\mathrm{Gr}_r(V)$ there exists a universal short exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O} \otimes V \longrightarrow Q \longrightarrow 0.$$

By definition, the fibre of S over a point of $\mathrm{Gr}(V)$ that corresponds to $W \subset V$ is naturally isomorphic to W. Fixing an hermitian structure on V induces hermitian structures on S and Q (and therefore on the tangent bundle $\mathcal{T}$ of $\mathrm{Gr}_r(V)$, which is $\mathrm{Hom}(S, Q)$).

If E is any holomorphic vector bundle of rank r on a complex manifold X generated by global sections spanning $V \subset H^0(X, E)$, then there exists a morphism $\varphi : X \to \mathrm{Gr}_r(V)$ with $\varphi^* Q = E$ (see Exercise 2.4.11). Again, choosing an hermitian product on V induces an hermitian structure on E which coincides with the pull-back of the hermitian structure on Q under φ. This shows that the curvature of E and Q can be compared, namely $\varphi^* F_Q = F_E$.

Thus, showing the positivity of F_Q proves the semi-positivity for any globally generated vector bundle E. Moreover, if $\varphi : X \to \mathrm{Gr}_r(V)$ is an embedding, then the curvature F_E is positive.

Exercises

4.3.1 Show that $\nabla^2 : \mathcal{A}^k(E) \to \mathcal{A}^{k+2}(E)$ is given by taking the exterior product with the form part of the curvature $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$ and applying its endomorphism part to E.

4.3.2 Prove i) and iii) of Proposition 4.3.7. Compute the connection and the curvature of the determinant bundle.

4.3.3 Let (E_1, h_1) and (E_2, h_2) be two hermitian holomorphic vector bundles endowed with hermitian connections ∇_1, ∇_2 such that the curvature of both is (semi-)positive. Prove the following assertions.

i) The curvature of the induced connection ∇^* on the dual vector bundle E_1^* is (semi-)negative.

ii) The curvature on $E_1 \otimes E_2$ is (semi-)positive and it is positive if at least one of the two connections has positive curvature.

iii) The curvature on $E_1 \oplus E_2$ is (semi-)positive.

4.3.4 Find an example of two connections ∇_1 and ∇_2 on a vector bundle E, such that F_{∇_1} is positive and F_{∇_2} is negative.

4.3.5 Let L be a holomorphic line bundle on a complex manifold. Suppose L admits an hermitian structure whose Chern connection has positive curvature. Show that X is Kähler. If X is in addition compact prove $\int_X A(L)^n > 0$.

4.3.6 Show that the canonical bundle of $\mathbb{P}^n$ comes along with a natural hermitian structure such that the curvature of the Chern connection is negative.

4.3.7 Show that the curvature of a complex torus $\mathbb{C}^n/\Gamma$ endowed with a constant hermitian structure is trivial. Is this true for any hermitian structure on $\mathbb{C}^n/\Gamma$?

4.3.8 Show that the curvature of a the natural metric on the ball quotient introduced in Example 3.1.9, iv) is negative. The one-dimensional case is a rather easy calculation.

4.3.9 Let X be a compact Kähler manifold with $b_1(X) = 0$. Show that there exists a unique flat connection ∇ on the trivial holomorphic line bundle $\mathcal{O}$ with $\nabla^{0,1} = \bar{\partial}$. Moreover, up to isomorphism $\mathcal{O}$ is the only line bundle with trivial Chern class $c_1 \in H^2(X, \mathbb{Z})$.

4.3.10 Let ∇ be a connection on a (complex) line bundle L on a manifold M. Show that L locally admits trivializing parallel sections if and only if $F_\nabla = 0$.

This is the easiest case of the general fact that a connection on a vector bundle is flat if and only if parallel frames can be found locally. (Frobenius integrability.)

Comments: - Various notions for the positivity of vector bundles can be found in the literature. Usually they all coincide for line bundles, but the exact relations between them is not clear for higher rank vector bundles. Positivity is usually exploited to control higher cohomology groups. This will be illustrated in Section 5.2. We shall also see that, at least for line bundles on projective manifolds, an algebraic description of bundles admitting an hermitian structure whose Chern connection has positive curvature can be given. For a more in depth presentation of the material we refer to [35, 100] and the forthcoming book [83].

- The problem alluded to in Remark 4.3.13 is subtle. See [37, 107].

- The curvature of Kähler manifold will be dealt with in subsequent sections, in particular its comparison with the well-known Riemannian curvature shall be explained in detail.

- In [75, Prop. 1.2.2] one finds a cohomological version of the Bianchi identity in terms of the Atiyah class.

4.4 Chern Classes

Connection and curvature are not only objects that are naturally associated with any vector bundle, they also provide an effective tool to define cohomological and numerical invariants, called characteristic classes and numbers. These invariants are in fact topological, but the topological aspects will not be treated here. The goal of this sections is first to discuss the multilinear algebra needed for the definition of Chern forms and classes and then to present the precise definition of Chern classes and Chern and Todd characters. We conclude the section by comparing the various definitions of the first Chern class encountered throughout the text.

Let ∇ be a connection on a complex vector bundle E and let F_∇ denote its curvature. Recall that $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$. We would like to apply certain multilinear operations to the linear part of F_∇, i.e. to the components in $\mathrm{End}(E)$, in order to obtain forms of higher degree. Let us start out with a discussion of the linear algebra behind this approach.

Let V be a complex vector space. A k-multilinear symmetric map

$$P : V \times \ldots \times V \longrightarrow \mathbb{C}$$

corresponds to an element in $\mathrm{S}^k(V)^*$. To each such P one associates its polarized form $\tilde{P} : V \to \mathbb{C}$, which is a homogeneous polynomial of degree k given by $\tilde{P}(B) := P(B, \ldots, B)$. Conversely, any homogeneous polynomial is uniquely obtained in this way. In our situation, we will consider $V = \mathfrak{gl}(r, \mathbb{C})$, the space of complex (r, r)-matrices.

Definition 4.4.1 A symmetric map

$$P : \mathfrak{gl}(r, \mathbb{C}) \times \ldots \times \mathfrak{gl}(r, \mathbb{C}) \longrightarrow \mathbb{C}$$

is called *invariant* if for all $C \in \mathrm{Gl}(r, \mathbb{C})$ and all $B_1, \ldots, B_k \in \mathfrak{gl}(r, \mathbb{C})$ one has

$$P(CB_1C^{-1}, \ldots, CB_kC^{-1}) = P(B_1, \ldots, B_k). \tag{4.5}$$

This condition can also be expressed in terms of the associated homogeneous polynomial as $\tilde{P}(CBC^{-1}) = \tilde{P}(B)$ for all $C \in \mathrm{Gl}(r, \mathbb{C})$ and all $B \in \mathfrak{gl}(r, \mathbb{C})$.

Lemma 4.4.2 *The k-multilinear symmetric map P is invariant if and only if for all $B, B_1, \ldots, B_k \in \mathfrak{gl}(r, \mathbb{C})$ one has*

$$\sum_{j=1}^{k} P(B_1, \ldots, B_{j-1}, [B, B_j], B_{j+1}, \ldots, B_k) = 0.$$

Proof. Use the invertible matrix $C = e^{tB}$ and differentiate the invariance equation (4.5) at $t = 0$. The converse is left as an exercise (cf. Exercise 4.4.8). □

Proposition 4.4.3 *Let P be an invariant k-multilinear symmetric form on $\mathfrak{gl}(r,\mathbb{C})$. Then for any vector bundle E of rank r and any partition $m = i_1 + \ldots + i_k$ there exists a naturally induced k-linear map:*

$$P : \left(\bigwedge^{i_1} M \otimes \mathrm{End}(E)\right) \times \ldots \times \left(\bigwedge^{i_k} M \otimes \mathrm{End}(E)\right) \longrightarrow \bigwedge^m_{\mathbb{C}} M$$

defined by $P(\alpha_1 \otimes t_1, \ldots, \alpha_k \otimes t_k) = (\alpha_1 \wedge \ldots \wedge \alpha_k)P(t_1, \ldots, t_k)$.

Proof. Once a trivialization $E(x) \cong \mathbb{C}^r$ is fixed, the above definition makes sense. Since P is invariant, it is independent of the chosen trivialization. □

Clearly, the k-linear map defined in this way induces also a k-multilinear map on the level of global sections

$$P : \mathcal{A}^{i_1}(M, \mathrm{End}(E)) \times \ldots \times \mathcal{A}^{i_k}(M, \mathrm{End}(E)) \longrightarrow \mathcal{A}^m_{\mathbb{C}}(M).$$

Note that P applied to form-valued endomorphisms is only graded symmetric, but restricted to even forms it is still a k-multilinear symmetric map. In particular, P restricted to $\mathcal{A}^2(M, \mathrm{End}(E)) \times \ldots \times \mathcal{A}^2(M, \mathrm{End}(E))$ can be recovered from its polarized form $\tilde{P}(\alpha \otimes t) = P(\alpha \otimes t, \ldots, \alpha \otimes t)$. In the following, this polarized form shall be applied to the curvature form F_∇ of a connection ∇ on E. We will need the following

Lemma 4.4.4 *For any forms $\gamma_j \in \mathcal{A}^{i_j}(M, \mathrm{End}(E))$ one has*

$$dP(\gamma_1, \ldots, \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \ldots, \nabla(\gamma_j), \ldots, \gamma_k),$$

where ∇ also denotes the induced connection on $\mathrm{End}(E)$.

Proof. This can be seen by a local calculation. We write $\nabla = d + A$, where A is the local connection matrix of ∇. The induced connection on $\mathrm{End}(E)$ is of the form $\nabla = d + A$ with A acting as $\gamma \mapsto [A, \gamma]$. Using the usual Leibniz formula for the exterior differential one finds

$$\begin{aligned} dP(\gamma_1, \ldots, \gamma_k) &= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \ldots, d\gamma_j, \ldots, \gamma_k) \\ &= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \ldots, (\nabla - A)(\gamma_j), \ldots, \gamma_k). \end{aligned}$$

By Lemma 4.4.2 the invariance of P proves the assertion. □

Corollary 4.4.5 *Let F_∇ be the curvature of an arbitrary connection ∇ on a vector bundle E of rank r. Then for any invariant k-multilinear symmetric polynomial P on $\mathfrak{gl}(r,\mathbb{C})$ the induced $2k$-form $\tilde{P}(F_\nabla) \in \mathcal{A}^{2k}_{\mathbb{C}}(M)$ is closed.*

Proof. This is an immediate consequence of the Bianchi identity (Lemma 4.3.5), which says $\nabla(F_\nabla) = 0$, and the previous lemma. $\square$

Thus, to any invariant k-multilinear symmetric map P on $\mathfrak{gl}(r,\mathbb{C})$ and any vector bundle E of rank r one can associate a cohomology class $[\tilde{P}(F_\nabla)] \in H^{2k}(M,\mathbb{C})$. In fact, due to the following lemma, this class is independent of the chosen connection.

Lemma 4.4.6 *If ∇ and ∇' are two connections on the same bundle E, then $[\tilde{P}(F_\nabla)] = [\tilde{P}(F_{\nabla'})]$.*

Proof. The space of all connections is an affine space over $\mathcal{A}^1(M,\mathrm{End}(E))$, i.e. if ∇ is given, then any other connection is of the form $\nabla' = \nabla + A$ for some $A \in \mathcal{A}^1(\mathrm{End}(E))$ (see Corollary 4.2.4). Thus, it suffices to show that the induced map

$$\mathcal{A}^1(M,\mathrm{End}(E)) \longrightarrow H^{2k}(M,\mathbb{C})$$

is constant. We use that $F_{\nabla+A} = F_\nabla + A\wedge A + \nabla(A)$.

The assertion can be proven by an infinitesimal calculation, i.e. in the following calculation we only consider terms of order at most one in t:

$$\tilde{P}(F_{\nabla+tA}) = \tilde{P}(F_\nabla) + ktP(F_\nabla,\ldots,F_\nabla,\nabla(A)).$$

Now the assertion follows from Lemma 4.4.4 and the Bianchi identity:

$$\begin{aligned} P(F_\nabla,\ldots,F_\nabla,\nabla(A)) &= dP(F_\nabla,\ldots,F_\nabla,A) - P(\nabla(F_\nabla),F_\nabla,\ldots,F_\nabla,A) - \ldots \\ &\quad -P(F_\nabla,\ldots,F_\nabla,\nabla(F_\nabla),A) \\ &= dP(F_\nabla,\ldots,F_\nabla,A). \end{aligned}$$

$\square$

Remark 4.4.7 If we denote by $(\mathrm{S}^k\mathfrak{gl}(r,\mathbb{C}))^{\mathrm{Gl}(r)}$ the invariant k-multilinear polynomials, then the above construction induces a canonical homomorphism

$$(\mathrm{S}^k\mathfrak{gl}(r,\mathbb{C}))^{\mathrm{Gl}(r)} \longrightarrow H^{2k}(M,\mathbb{C})$$

for any vector bundle E of rank r. In fact, we actually obtain an algebra homomorphism

$$(\mathrm{S}^*\mathfrak{gl}(r,\mathbb{C}))^{\mathrm{Gl}(r)} \longrightarrow H^{2*}(M,\mathbb{C})$$

which is called the *Chern–Weil homomorphism.*

So far, everything was explained for arbitrary invariant polynomials, but some polynomials are more interesting than others, at least regarding their applications to geometry. In the following we discuss the most frequent ones.

Examples 4.4.8 i) **Chern classes**. Let $\{\tilde{P}_k\}$ be the homogeneous polynomials with $\deg(\tilde{P}_k) = k$ defined by

$$\det(\mathrm{Id} + B) = 1 + \tilde{P}_1(B) + \ldots + \tilde{P}_r(B).$$

Clearly, these $\tilde{P}_k$ are invariant. The *Chern forms* of a vector bundle of rank r endowed with a connection are

$$\mathrm{c}_k(E, \nabla) := \tilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(M).$$

The *k-th Chern class* of the vector bundle E is the induced cohomology class

$$\mathrm{c}_k(E) := [\mathrm{c}_k(E, \nabla)] \in H^{2k}(M, \mathbb{C}).$$

Note that $\mathrm{c}_0(E) = 1$ and $\mathrm{c}_k(E) = 0$ for $k > \mathrm{rk}(E)$. The *total Chern class* is $\mathrm{c}(E) := \mathrm{c}_0(E) + \mathrm{c}_1(E) + \ldots + \mathrm{c}_r(E) \in H^{2*}(M, \mathbb{C})$.

ii) **Chern characters**. In order to define the Chern character of E one uses the invariant homogeneous polynomials $\tilde{P}_k$ of degree k defined by

$$\mathrm{tr}(e^B) = \tilde{P}_0(B) + \tilde{P}_1(B) + \ldots.$$

Then the *k-th Chern character* $\mathrm{ch}_k(E) \in H^{2k}(M, \mathbb{C})$ of E is defined as the cohomology class of

$$\mathrm{ch}_k(E, \nabla) := \tilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(M).$$

Note that $\mathrm{ch}_0(E) = \mathrm{rk}(E)$. The *total Chern character* is $\mathrm{ch}(E) := \mathrm{ch}_0(E) + \ldots + \mathrm{ch}_r(E) + \mathrm{ch}_{r+1}(E) + \ldots$.

iii) **Todd classes**. The homogeneous polynomials used to define the Todd classes are given by

$$\frac{\det(tB)}{\det(\mathrm{Id} - e^{-tB})} = \sum_k \tilde{P}_k(B) t^k.$$

(The additional variable t is purely formal and is supposed to indicate that the left hand side can be developed as a power series in t with coefficients P_k which are of degree k. This could also have been done for the other characteristic classes introduced earlier.)

Then $\mathrm{td}_k(E, \nabla) := \tilde{P}_k((i/2\pi) F_\nabla)$ and

$$\mathrm{td}_k(E) := [\mathrm{td}_k(E, \nabla)] \in H^{2k}(M, \mathbb{C}).$$

The *total Todd class* is $\mathrm{td}(E) := \mathrm{td}_0(E) + \mathrm{td}_1(E) + \ldots$.

Note that the Todd classes are intimately related to the Bernoulli numbers B_k. In fact, by definition

$$\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_k}{(2k)!} t^{2k}.$$

Let us study some of the natural operations for vector bundles and see how the characteristic classes behave in these situations.

• Let $E = E_1 \oplus E_2$ be endowed with the direct sum ∇ of connections ∇_1 and ∇_2 on E_1 and E_2, respectively. The curvature F_∇ is again the direct sum $F_{\nabla_1} \oplus F_{\nabla_2}$ (Proposition 4.3.7) and since $\det\left((\mathrm{Id}_{E_1} + \frac{i}{2\pi}F_{\nabla_1}) \oplus (\mathrm{Id}_{E_2} + \frac{i}{2\pi}F_{\nabla_2})\right) = \det\left(\mathrm{Id}_{E_1} + \frac{i}{2\pi}F_{\nabla_1}\right) \cdot \det\left(\mathrm{Id}_{E_2} + \frac{i}{2\pi}F_{\nabla_2}\right)$, we obtain the *Whitney product formula*:

$$\mathrm{c}(E, \nabla) = \mathrm{c}(E_1, \nabla_1) \cdot \mathrm{c}(E_2, \nabla_2).$$

Of course, this relation then also holds true for the total Chern class. In particular, one has $\mathrm{c}_1(E_1 \oplus E_2) = \mathrm{c}_1(E_1) + \mathrm{c}_1(E_2)$ and $\mathrm{c}_2(E) = \mathrm{c}_2(E_1) + \mathrm{c}_2(E_2) + \mathrm{c}_1(E_1) \cdot \mathrm{c}_1(E_2)$. A similar calculation shows

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2).$$

• Consider two vector bundles E_1 and E_2 endowed with connections ∇_1 and ∇_2, respectively. Let ∇ be the induced connection $\nabla_1 \otimes 1 + 1 \otimes \nabla_2$ on $E = E_1 \otimes E_2$. Then $F_\nabla = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$ (see Proposition 4.3.7) and hence $\mathrm{tr}(e^{(i/2\pi)F_\nabla}) = \mathrm{tr}(e^{(i/2\pi)F_{\nabla_1}} \otimes e^{(i/2\pi)F_{\nabla_2}}) = \mathrm{tr}(e^{(i/2\pi)F_{\nabla_1}}) \cdot \mathrm{tr}(e^{(i/2\pi)F_{\nabla_2}})$. Therefore,

$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \cdot \mathrm{ch}(E_2).$$

If $E_2 = L$ is a line bundle one finds $\mathrm{c}_1(E_1 \otimes L) = \mathrm{c}_1(E_1) + \mathrm{rk}(E_1) \cdot \mathrm{c}_1(L)$ and $\mathrm{c}_2(E_1 \otimes L) = \mathrm{c}_2(E_1) + (\mathrm{rk}(E_1) - 1) \cdot \mathrm{c}_1(E_1) \cdot \mathrm{c}_1(L) + \binom{\mathrm{rk}(E_1)}{2}\mathrm{c}_1^2(L)$. See Exercise 4.4.5 for the first few terms of the Chern character and Exercise 4.4.6 for the general formula for the Chern classes of $E_1 \otimes L$.

• The curvature F_{∇^*} of the naturally associated connection ∇^* on the dual bundle E^* is $F_{\nabla^*} = -F_\nabla^{\mathrm{t}}$ (see Proposition 4.3.7). Thus, $\mathrm{c}(E^*, \nabla^*) = \det(\mathrm{Id} + \frac{i}{2\pi}F_{\nabla^*}) = \det(\mathrm{Id} - \frac{i}{2\pi}F_\nabla^{\mathrm{t}}) = \det(\mathrm{Id} - \frac{i}{2\pi}F_\nabla)$. Hence,

$$\mathrm{c}_k(E^*, \nabla^*) = (-1)^k \mathrm{c}_k(E, \nabla).$$

• Let $f : M \to N$ be a differentiable map and let E be a vector bundle on N endowed with a connection ∇. By Proposition 4.3.7 we know that $F_{f^*\nabla} = f^*F_\nabla$. This readily yields

$$\mathrm{c}_k(f^*E, f^*\nabla) = f^*\mathrm{c}_k(E, \nabla).$$

• The first Chern class of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ satisfies the normalization

$$\int_{\mathbb{P}^1} \mathrm{c}_1(\mathcal{O}(1)) = 1.$$

Indeed, in Example 4.3.12 we have shown that the Chern connection on $\mathcal{O}(1)$ on $\mathbb{P}^n$ with respect to the natural hermitian structure has curvature $F = (2\pi/i)\omega_{\mathrm{FS}}$ and by Example 3.1.9, i) we know $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 1$.

Remarks 4.4.9 i) In fact, all the characteristic classes introduced above are real. This can be seen as follows. Pick an hermitian metric on the vector bundle E and consider an hermitian connection ∇, which always exists. Then locally and with respect to an hermitian trivialization of E the curvature satisfies the equation $F_\nabla^* = \bar{F}_\nabla^{\rm t} = -F_\nabla$. Hence, $\overline{(i/2\pi)F_\nabla} = (i/2\pi)F_\nabla^{\rm t}$ and, therefore, $\mathrm{c}(E,\nabla) = \det(\mathrm{Id} + (i/2\pi)F_\nabla) = \det(\mathrm{Id} + (i/2\pi)F_\nabla^{\rm t}) = \overline{\det(\mathrm{Id} + (i/2\pi)F_\nabla)} = \overline{\mathrm{c}(E,\nabla)}$, i.e. $\mathrm{c}(E,\nabla)$ is a real form. Thus,

$$\mathrm{c}(E) \in H^*(M,\mathbb{R}).$$

The same argument works for $\mathrm{ch}(E)$ and $\mathrm{td}(E)$.

ii) If E is a holomorphic vector bundle over a complex manifold X, then we may use an hermitian connection ∇ that is in addition compatible with the holomorphic structure of E (cf. Proposition 4.2.14). In this case the curvature F_∇ is a $(1,1)$-form, i.e. $F_\nabla \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$. But then also the Chern forms $\mathrm{c}_k(E,\nabla)$ are of type (k,k) for all k.

If X is compact and Kähler, we have the decomposition $H^{2k}(X,\mathbb{C}) = \bigoplus_{p+q=2k} H^{p,q}(X)$ (Corollary 3.2.12) and the Chern classes of any holomorphic bundle are contained in the (k,k)-component, i.e.

$$\mathrm{c}_k(E) \in H^{2k}(X,\mathbb{R}) \cap H^{k,k}(X).$$

iii) We are going to explain an ad hoc version of the *splitting principle*. The geometric splitting principle works on the level of cohomology and tries to construct for a given vector bundle E a ring extension $H^*(M,\mathbb{R}) \subset A^*$ and elements $\gamma_i \in A^2$, $i = 1,\ldots,\mathrm{rk}(E)$, such that $\mathrm{c}(E) = \prod(1+\gamma_i)$. Moreover, A^* is constructed geometrically as the cohomology ring of a manifold N such that the inclusion $H^*(M,\mathbb{R}) \subset A^* = H^*(N,\mathbb{R})$ is induced by a submersion $\pi : N \to M$ (e.g. N can be taken as the full flag manifold associated to E). The map π is constructed such that π^*E is a direct sum $\bigoplus L_i$ of line bundles L_i with $\gamma_i = \mathrm{c}_1(L_i)$.

We propose to study a similar construction on the level of forms. This approach is less geometrical but sufficient for many purposes.

Consider $\mathbb{C}^r$ with the standard hermitian structure and let $B \in \mathfrak{gl}(r,\mathbb{C})$ be a self-adjoint (or, hermitian matrix), i.e. $B^{\rm t} = \bar{B}$. Then, there exists an orthonormal basis with respect to which B takes diagonal form with eigenvalues $\lambda_1,\ldots,\lambda_r$. Clearly, every expression of the form $\tilde{P}(B)$, with P an invariant symmetric map, can be expressed in terms of $\lambda_1,\ldots,\lambda_r$. E.g. $\mathrm{tr}(B) = \lambda_1 + \ldots + \lambda_r$.

Let us now consider the curvature matrix F_∇ of an hermitian connection ∇ on an hermitian vector bundle (E,h) of rank r on a manifold M. At a fixed point $x \in M$ we may trivialize (E,h) such that it becomes isomorphic to $\mathbb{C}^r$ with the standard hermitian structure. Then $i \cdot F_\nabla$ in x corresponds to an hermitian matrix B, but with coefficients not in $\mathbb{C}$ but in $R := \mathbb{C}[\bigwedge_x^2 M]$.

Diagonalizing $B = i \cdot F_\nabla(x)$ can still be achieved, but in general only over a certain ring extension $R \subset R'$. (One has to adjoint certain eigenvalues, to assure that a vector of length one can be completed to an orthonormal basis, etc.) Let us suppose this has been done, i.e. in the new basis one has $B = \mathrm{diag}(\gamma_1, \ldots, \gamma_r)$ with $\gamma_i \in R'$. Then for any invariant symmetric map P one finds $\tilde{P}(iF_\nabla) = \tilde{P}(\mathrm{diag}(\gamma_1, \ldots, \gamma_r))$, where P is extended R'-linearly from $\mathbb{C}^r$ to $\mathbb{C}^r \otimes_\mathbb{C} R'$. The result $\tilde{P}(iF_\nabla)$ is of course contained in $\mathbb{C}[\bigwedge_x^2 M]$ and can thus be projected to $\bigwedge_x^{2*} M$.

In general, this procedure will not work globally, but it often suffices to have at one's disposal the splitting principle in this form. The elements $\gamma_1, \ldots, \gamma_r$ (up to the scalar factor $(1/2\pi)$, which we suppress) are called the *formal Chern roots* of ∇ on E.

The primary use of this construction is to verify various formal identities. As an example, let us show how the rather elementary identity $\mathrm{ch}_2(E) = (1/2)\mathrm{c}_1^2(E) - \mathrm{c}_2(E)$ can be proved. This can be done pointwise and so we may assume that the ring extension $R = \mathbb{C}[\bigwedge_x^2 M] \subset R'$ and the formal Chern roots of E have been found. Hence,

$$\mathrm{ch}_2(E, \nabla)(x) = \frac{1}{8\pi^2}\mathrm{tr}\,(iF_\nabla(x)) = \frac{1}{8\pi^2}\mathrm{tr}\left(\mathrm{diag}(\gamma_1^2, \ldots, \gamma_r^2)\right) = \frac{1}{8\pi^2}\sum_i \gamma_i^2$$

and

$$\mathrm{c}_1^2(E, \nabla)(x) - 2\mathrm{c}_2(E, \nabla)(x) = \frac{1}{4\pi^2}\left(\left(\sum_i \gamma_i\right)^2 - 2\sum_{i<j}\gamma_i\gamma_j\right) = \frac{1}{4\pi^2}\sum_i \gamma_i^2.$$

Another example for this type of argument can be found in Section 5.1. See also Exercises 4.4.5 and 4.4.9.

iv) There is also an axiomatic approach to Chern classes which shows that the Whitney product formula, the compatibility under pull-back, and the normalization $\int \mathrm{c}_1(\mathcal{O}(1)) = 1$ determine the Chern classes uniquely.

Definition 4.4.10 The *Chern classes of a complex manifold* X are

$$\mathrm{c}_k(X) := \mathrm{c}_k(\mathcal{T}_X) \in H^{2k}(X, \mathbb{R}),$$

where $\mathcal{T}_X$ is the holomorphic tangent bundle. Similarly, one defines $\mathrm{ch}_k(X)$ and $\mathrm{td}_k(X)$ by means of $\mathcal{T}_X$.

Note that we actually only need an almost complex structure in order to define the Chern classes of the manifold. Also note that it might very well happen that two different complex structures on the same differentiable manifold yield different Chern classes, but in general counter-examples are not easy to construct. A nice series of examples of complex structures on the product of a K3 surface with S^2 with different Chern classes can be found in the recent paper [84].

Example 4.4.11 Let us compute the characteristic classes of a hypersurface $Y \subset X$. The normal bundle sequence in this case takes the form

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{O}_Y(Y) \longrightarrow 0.$$

Since any short exact sequence of holomorphic vector bundles splits as a sequence of complex vector bundles, the Whitney product formula yields $i^*\mathrm{c}(X) = \mathrm{c}(Y) \cdot i^*\mathrm{c}(\mathcal{O}(Y))$. Therefore,

$$\mathrm{c}(Y) = i^* \left(\mathrm{c}(X) \cdot (1 - \mathrm{c}_1(\mathcal{O}(Y)) + \mathrm{c}_1(\mathcal{O}(Y))^2 \pm \ldots)\right).$$

In particular, $\mathrm{c}_1(Y) = i^*(\mathrm{c}_1(X) - \mathrm{c}_1(\mathcal{O}(Y))$ which reflects the adjunction formula 2.2.17.

For a quartic hypersurface $Y \subset \mathbb{P}^3$ this yields $\mathrm{c}_1(Y) = 0$ and $\mathrm{c}_2(Y) = i^*\mathrm{c}_2(\mathbb{P}^3)$. Hence, $\int_Y \mathrm{c}_2(Y) = \int_{\mathbb{P}^3} \mathrm{c}_2(\mathbb{P}^3)(4\mathrm{c}_1(\mathcal{O}(1))) = 24$. Here we use $\mathrm{c}_2(\mathbb{P}^3) = 6\mathrm{c}_1^2(\mathcal{O}(1))$ which follows from the Euler sequence and the Whitney formula (cf. Exercise 4.4.4).

So far, we have encountered various different ways to define the first Chern class of a complex or holomorphic line bundle. We will now try to summarize and compare these.

For a holomorphic line bundle L on a complex manifold X we have used the following three definitions:

i) Via the curvature as $\mathrm{c}_1(L) = [\mathrm{c}_1(L, \nabla)] \in H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C})$, where ∇ is a connection on L.

ii) Via the Atiyah class $A(L) \in H^1(X, \Omega_X)$. See Remarks 4.2.20 ii).

iii) Via the exponential sequence and the induced boundary operator $\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$. See Definition 2.2.13.

By Proposition 4.3.10 the first two definitions are compatible in the sense that $A(L) = [F_\nabla]$ if ∇ is the Chern connection on L endowed with an hermitian structure. In case that X is a compact Kähler manifold we can naturally embed $H^1(X, \Omega_X) = H^{1,1}(X) \subset H^2(X, \mathbb{C})$ and thus obtain

$$\frac{i}{2\pi} A(L) = \mathrm{c}_1(L).$$

The comparison of i) and iii) will be done more generally for complex line bundles L on a differentiable manifold M.

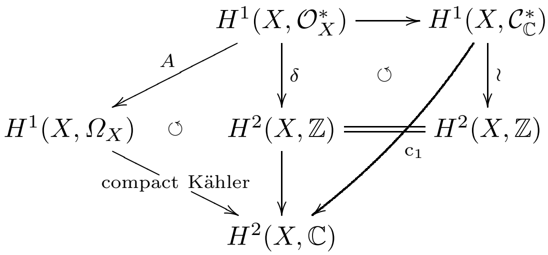
A complex line bundle L on a differentiable manifold M is described by its cocycle $\{U_{ij}, \psi_{ij}\} \in H^1(M, \mathcal{C}_{\mathbb{C}}^*)$ (see Appendix B). The invertible complex valued differentiable functions $\psi_{ij} \in \mathcal{C}_{\mathbb{C}}^*(U_{ij})$ are given as $\psi_{ij} = \psi_i \circ \psi_j^{-1}$, where $\psi_i : L|_{U_i} \cong U_i \times \mathbb{C}$ are trivializations.

In the present context we work with the smooth exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}_{\mathbb{C}} \longrightarrow \mathcal{C}_{\mathbb{C}}^* \longrightarrow 0$$

which induces a boundary isomorphism $\delta : H^1(M, \mathcal{C}_{\mathbb{C}}^*) \cong H^2(M, \mathbb{Z})$, for $\mathcal{C}_{\mathbb{C}}$ is a soft sheaf (Appendix B). In other words, complex line bundles on a manifold M are parametrized by the (discrete) group $H^2(M, \mathbb{Z})$. Since $H^2(M, \mathbb{Z})$ maps naturally to $H^2(M, \mathbb{R}) \subset H^2(M, \mathbb{C})$, one may compare $\delta(L)$ and $c_1(L)$. In Proposition 4.4.12 we will see that they only differ by a sign.

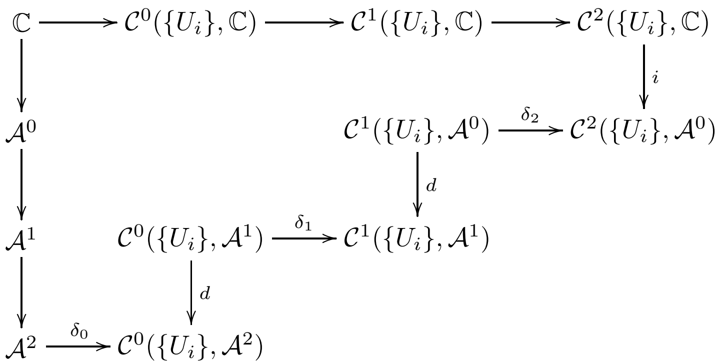
Clearly, the exponential sequence on a complex manifold X and the exponential sequence on the underlying real manifold M are compatible. Thus, we show at the same time that i) and iii) above are compatible. We therefore obtain the following commutative diagram



Proposition 4.4.12 *Let L be a complex line bundle over a differentiable manifold M. Then the image of $\delta(L) \in H^2(M, \mathbb{Z})$ under the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})$ equals $-c_1(L)$. Here, δ is the boundary map of the exponential sequence.*

The annoying sign is due to various conventions, e.g. in the definition of the boundary operator. Often, it is dropped altogether, as it is universal and of no importance.

Proof. In order to prove this, we have to consider the two resolutions of the constant sheaf $\mathbb{C}$ on M given by the de Rham complex and the Čech complex, respectively. They are compared as follows:



Let $M = \bigcup U_i$ be an open covering trivializing L and such that $U_{ij} = U_i \cap U_j$ are simply connected. Choose trivialization $\psi_i : L|_{U_i} \cong U_i \times \mathbb{C}$. Then, $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ are sections of $\mathcal{C}^*_{\mathbb{C}}(U_{ij})$. Furthermore, by choosing a branch of the logarithm for any U_{ij} we find $\varphi_{ij} \in \mathcal{C}_{\mathbb{C}}(U_{ij})$ with $e^{2\pi i \varphi_{ij}} = \psi_{ij}$. The boundary $\delta(L) = \delta\{\psi_{ij}\}$ is given by $\{U_{ijk}, \varphi_{jk} - \varphi_{ik} + \varphi_{ij}\}$ which takes values in the locally constant sheaf $\mathbb{Z}$.

Now choose an arbitrary connection ∇ on L. Locally with respect to the trivialization ψ_i it can be written as $\nabla = d + A_i$, where the connection matrices A_i are one-forms on U_i. The compatibility condition ensures $\psi_{ij}^{-1} d(\psi_{ij}) + \psi_{ij}^{-1} A_i \psi_{ij} = A_j$ (see Remark 4.2.5), i.e.

$$A_j - A_i = \psi_{ij}^{-1} d(\psi_{ij}) = (2\pi i) d(\varphi_{ij}),$$

since in the rank one case one has $\psi_{ij}^{-1} A_i \psi_{ij} = A_i$.

The curvature F_∇ of the line bundle L in terms of the connection forms A_i is given as $F_\nabla = d(A_i)$. With these information we can now easily go through the above diagram:

$$\delta_0\left(\tfrac{i}{2\pi} F_\nabla\right) = \{U_i, \tfrac{i}{2\pi} d(A_i)\} = d\{U_i, \tfrac{i}{2\pi} A_i\}$$

$$\delta_1\{U_i, \tfrac{i}{2\pi} A_i\} = \{U_{ij}, \tfrac{i}{2\pi}(A_j - A_i)\} = -d\{U_{ij}, \varphi_{ij}\}$$

$$-\delta_2\{U_{ij}, \varphi_{ij}\} = -\{U_{ijk}, \varphi_{jk} - \varphi_{ik} + \varphi_{ij}\}.$$

This proves the claim. □

There is yet another way to associate a cohomology class to a line bundle in case the line bundle is given in terms of a divisor. Let X be a compact complex manifold and let $D \subset X$ be an irreducible hypersurface. Its fundamental class $[D] \in H^2(X, \mathbb{R})$ is in fact contained in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ (cf. Remark 2.3.11).

Proposition 4.4.13 *Under the above assumptions one has* $\mathrm{c}_1(\mathcal{O}(D)) = [D]$.

Proof. Let $L = \mathcal{O}(D)$ and let ∇ be the Chern connection on L with respect to a chosen hermitian structure h. In order to prove that $[\frac{i}{2\pi} F_\nabla] = \mathrm{c}_1(L)$ equals $[D]$ one needs to show that for any closed real form α one has

$$\frac{i}{2\pi} \int_X F_\nabla \wedge \alpha = \int_D \alpha.$$

Let us fix an open covering $X = \bigcup U_i$ and holomorphic trivializations $\psi_i : L|_{U_i} \cong U_i \times \mathbb{C}$. On U_i the hermitian structure h shall be given by the function $h_i : U_i \to \mathbb{R}_{>0}$, i.e. $h(s(x)) = h(s(x), s(x)) = h_i(x) \cdot |\psi_i(s(x))|^2$ for any local section s.

If s is holomorphic on U_i vanishing along D one has (Examples 4.3.9, iii)):

$$\bar{\partial}\partial\log(h\circ s)=\bar{\partial}\partial\log(h_i)=F_\nabla|_{U_i}\quad\text{on } U_i\setminus D.$$

Here we have used that $\bar{\partial}\partial\log(\psi_i\circ s)=\bar{\partial}\partial\log(\bar{\psi}_i\circ s)=0$, since ψ_i is holomorphic.

Let now $s\in H^0(X,L)$ be the global holomorphic section defining D and denote by D_ε the tubular neighbourhood $D_\varepsilon:=\{x\in X\mid |h(s(x))|<\varepsilon\}$. Then

$$\begin{aligned}
&\frac{i}{2\pi}\int_X F_\nabla\wedge\alpha=\lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_{X\setminus D_\varepsilon}F_\nabla\wedge\alpha\\
&=\lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_{X\setminus D_\varepsilon}\bar{\partial}\partial\log(h\circ s)\wedge\alpha\\
&=\lim_{\varepsilon\to 0}\frac{i}{4\pi}\int_{X\setminus D_\varepsilon}d(\partial-\bar{\partial})\log(h\circ s)\wedge\alpha\\
&=\lim_{\varepsilon\to 0}\frac{i}{4\pi}\int_{\partial D_\varepsilon}(\partial-\bar{\partial})\log(h\circ s)\wedge\alpha\quad\text{by Stokes and using } d\alpha=0.
\end{aligned}$$

On the open subset U_i we may write

$$\begin{aligned}
&(\partial-\bar{\partial})\log(h\circ s)\\
&=(\partial-\bar{\partial})\log(\psi_i\circ s)+(\partial-\bar{\partial})\log(\bar{\psi}_i\circ s)+(\partial-\bar{\partial})\log(h_i)\\
&=\partial\log(\psi_i\circ s)-\overline{\partial\log(\psi_i\circ s)}+(\partial-\bar{\partial})\log(h_i)\\
&=(2i)\cdot\mathrm{Im}(\partial\log(\psi_i\circ s))+(\partial-\bar{\partial})\log(h_i).
\end{aligned}$$

The second summand does not contribute to the integral for $\varepsilon\to 0$ as h_i is bounded from below by some $\delta>0$. Thus, it suffices to show

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\partial D_\varepsilon\cap U_i}\mathrm{Im}(\partial\log(\psi_i\circ s))\wedge\alpha=-\int_{D\cap U_i}\alpha.$$

This is a purely local statement. In order to prove it, we may assume that D_ε is given by $z_1=0$ in a polydisc B. Moreover, $|h(z_1)|=h_i\cdot|z_i|$ if h on U_i is given by h_i. Hence, $\partial D_\varepsilon=\{z\mid |z_1|=\varepsilon/h_i\}$. Furthermore, $\partial\log(\psi_i\circ s)=\partial\log(z_1)=z_1^{-1}dz_1$ and $\alpha=f(dz_2\wedge\ldots\wedge dz_n)\wedge(d\bar{z}_2\wedge\ldots\wedge d\bar{z}_n)+dz_1\wedge\alpha'+d\bar{z}_1\wedge\bar{\alpha}'$. Notice that $\partial\log(\psi_i\circ s)\wedge(dz_1\wedge\alpha')=0$ and that $\partial\log(\psi_i\circ s)\wedge(d\bar{z}_1\wedge\bar{\alpha}')=(dz_1\wedge d\bar{z}_1)\wedge((1/z_1)\bar{\alpha}')$ does not contribute to the integral over ∂D_ε.

Thus,

$$\int_{z_1=0}\alpha=\int f(0,z_2,\ldots,z_n)(dz_2\wedge\ldots\wedge dz_n)\wedge(d\bar{z}_2\wedge\ldots\wedge d\bar{z}_n)$$

and

$$\int_{\partial D_\varepsilon}\partial\log(\psi_i\circ s)\wedge\alpha=-\int_{|h(z_1)|=\varepsilon}fz_1^{-1}dz_1\wedge(dz_2\wedge\ldots\wedge dz_n)\wedge(d\bar{z}_2\wedge\ldots\wedge d\bar{z}_n).$$

The minus sign appears as we initially integrated over the exterior domain.

Eventually, one applies the Cauchy integral formula (1.4):

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\partial\log(\psi_i\circ s)\wedge\alpha\\
&= -\lim_{\varepsilon\to 0}\int_{|z_1|=\varepsilon/h_i}\left(\int_{z_{i>1}} f\cdot(dz_2\wedge\ldots\wedge dz_n\wedge d\bar z_2\wedge\ldots\wedge d\bar z_n)\right)\frac{dz_1}{z_1}\\
&= (-2\pi i)\cdot\int_{z_1=0} f(0,z_2,\ldots,z_n)(dz_2\wedge\ldots\wedge dz_n\wedge d\bar z_2\wedge\ldots\wedge d\bar z_n)\\
&= -2\pi i\int_{z_1=0}\alpha
\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\partial D_\varepsilon}\mathrm{Im}\left(\partial\log(\psi_i\circ s)\wedge\alpha\right) = -\mathrm{Im}\left(\int_{z_1=0} i\cdot\alpha\right) = -\int_{z_1=0}\alpha.$$

□

Remark 4.4.14 Since taking the first Chern class c_1 and taking the fundamental class are both linear operations, the assertion of Proposition 4.4.13 holds true for arbitrary divisors, i.e. $c_1(\mathcal{O}(\sum n_iD_i)) = \sum n_i[D_i]$.

The reader may have noticed that in the proof above we have, for simplicity, assumed that D is smooth. The argument might easily be adjusted to the general case.

Exercises

4.4.1 Let C be a connected compact curve. Then there is a natural isomorphism $H^2(C,\mathbb{Z})\cong\mathbb{Z}$. Show that with respect to this isomorphism (or, rather, its $\mathbb{R}$-linear extension) one has $c_1(L)=\deg(L)$ for any line bundle L on C.

4.4.2 Show that for a base-point free line bundle L on a compact complex manifold X the integral $\int_X c_1(L)^n$ is non-negative.

4.4.3 Show that $\mathrm{td}(E_1\oplus E_2)=\mathrm{td}(E_1)\cdot\mathrm{td}(E_2)$.

4.4.4 Compute the Chern classes of (the tangent bundle of) $\mathbb{P}^n$ and $\mathbb{P}^n\times\mathbb{P}^m$. Try to interpret $\int_{\mathbb{P}^n} c_n(\mathbb{P}^n)$ and $\int_{\mathbb{P}^n\times\mathbb{P}^m} c_n(\mathbb{P}^n\times\mathbb{P}^m)$.

4.4.5 Prove the following explicit formulae for the first three terms of $\mathrm{ch}(E)$ and $\mathrm{td}(E)$ in terms of $c_i(E)$:

$$\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \frac{c_1^2(E)-2c_2(E)}{2} + \frac{c_1^3(E)-3c_1(E)c_2(E)+3c_3(E)}{6} + \ldots$$

$$\mathrm{td}(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1^2(E)+c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} + \ldots$$

4.4.6 Let E be a vector bundle and L a line bundle. Show

$$\mathrm{c}_i(E \otimes L) = \sum_{j=0}^{i} \binom{\mathrm{rk}(E) - j}{i - j} \mathrm{c}_j(E)\mathrm{c}_1(L)^{i-j}.$$

This generalizes the computation for the first two Chern classes of $E \otimes L$ on page 197.

4.4.7 Show that on $\mathbb{P}^n$ one has $\mathrm{c}_1(\mathcal{O}(1)) = [\omega_{\mathrm{FS}}] \in H^2(\mathbb{P}^n, \mathbb{R})$. Consider first the case of $\mathbb{P}^1$ and then the restriction of $\mathcal{O}(1)$ and the Fubini–Study metric to $\mathbb{P}^1$ under a linear embedding $\mathbb{P}^1 \subset \mathbb{P}^n$.

4.4.8 Prove that a polynomial P of degree k on the space of $r \times r$ matrices is invariant if and only if $\sum P(A_1, \ldots, A_{i-1}, [A, A_i], A_{i+1}, \ldots, A_k) = 0$ for all matrices $A_1, \ldots, A_k, A$ (cf. Lemma 4.4.2).

4.4.9 Show that $\mathrm{c}_1(\mathrm{End}(E)) = 0$ on the form level and compute $\mathrm{c}_2(\mathrm{End}(E))$ in terms of $\mathrm{c}_i(E)$, $i = 1, 2$. Compute $(4\mathrm{c}_2 - \mathrm{c}_1^2)(L \oplus L)$ for a line bundle L. Show that $\mathrm{c}_{2k+1}(E) = 0$, if $E \cong E^*$.

4.4.10 Let L be a holomorphic line bundle on a compact Kähler manifold X. Show that for any closed real $(1,1)$-form α with $[\alpha] = \mathrm{c}_1(L)$ there exists an hermitian structure on L such that the curvature of the Chern connection ∇ on L satisfies $(i/2\pi)F_\nabla = \alpha$. (Hint: Fix an hermitian structure on h_0 on L. Then any other is of the form $e^f \cdot h_0$. Compute the change of the curvature. We will give the complete argument in Remark 4.B.5.

4.4.11 Let X be a compact Kähler manifold. Show that via the natural inclusion $H^k(X, \Omega_X^k) \subset H^{2k}(X, \mathbb{C})$ one has

$$\mathrm{ch}_k(E) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{tr}\left(A(E)^{\otimes k}\right).$$

Here, $A(E)^{\otimes k}$ is obtained as the image of $A(E) \otimes \ldots \otimes A(E)$ under the natural map $H^1(X, \Omega_X \otimes \mathrm{End}(E)) \times \ldots \times H^1(X, \Omega_X \otimes \mathrm{End}(E)) \to H^k(X, \Omega_X^k \otimes \mathrm{End}(E))$ which is induced by composition in $\mathrm{End}(E)$ and exterior product in Ω_X^*.

4.4.12 Let X be a compact Kähler manifold and let E be a holomorphic vector bundle admitting a holomorphic connection $D : E \to \Omega_X \otimes E$. Show that $\mathrm{c}_k(E) = 0$ for all $k > 0$.

Comments: Chern classes were first defined by Chern in [26]. A more topological approach to characteristic classes, also in the real situation, can be found in [91]. Since Chern classes are so universal, adapted versions appear in many different areas, e.g. algebraic geometry, Arakelov theory, etc.

Appendix to Chapter 4

4.A Levi-Civita Connection and Holonomy on Complex Manifolds

In this section we will compare our notion of a connection with the notion used in Riemannian geometry. In particular, we will clarify the relation between the Chern connection on the holomorphic tangent bundle of an hermitian manifold and the Levi-Civita connection on the underlying Riemannian manifold. Very roughly, we will see that these connections coincide if and only if the hermitian manifold is Kähler.

We will end this appendix with a discussion of the holonomy group of a Riemannian manifold and the interpretation of a Kähler structure in terms of the holonomy group of its underlying Riemannian structure.

Let us first review some basic concepts from Riemannian geometry. For this purpose we consider a Riemannian manifold (M, g). A connection on M by definition is a connection on the real tangent bundle TM, i.e. an $\mathbb{R}$-linear map $D : \mathcal{A}^0(TM) \to \mathcal{A}^1(TM)$ satisfying the Leibniz rule 4.2. For any two vector fields u and v we denote by $D_u v$ the one-form Dv with values in TM applied to the vector field u. Note that $D_u v$ is $\mathbb{R}$-linear in v and $\mathcal{A}^0$-linear in u. With this notation, the Leibniz rule reads $D_u(f \cdot v) = f \cdot D_u(v) + (df)(u) \cdot v$.

A connection is *metric* if $dg(u, v) = g(Du, v) + g(u, Dv)$. In other words, D is metric if and only if g is parallel, i.e. $D(g) = 0$, where D is the induced connection on $T^*M \otimes T^*M$ (cf. Exercise 4.2.8).

Before defining the torsion of a connection recall that the Lie bracket is an $\mathbb{R}$-linear skew-symmetric map $[\ ,\ ] : \mathcal{A}^0(TM) \times \mathcal{A}^0(TM) \to \mathcal{A}^0(TM)$ which locally for $u = \sum_i a_i \frac{\partial}{\partial x_i}$ and $v = \sum_i b_i \frac{\partial}{\partial x_i}$ is defined by

$$
\begin{aligned}
[u, v] &= \sum_j \sum_i \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \\
&= \sum_j \left(db_j(u) - da_j(v) \right) \frac{\partial}{\partial x_j}.
\end{aligned}
$$

In particular, one has $[f \cdot u, v] = f \cdot [u, v] - df(v) \cdot u$.

Definition 4.A.1 The *torsion* of a connection D is given by

$$T_D(u, v) := D_u v - D_v u - [u, v]$$

for any two vector fields u and v.

The first thing one observes is that T_D is skew-symmetric, i.e. $T_D : \bigwedge^2 TM \to TM$. Moreover, T_D is $\mathcal{A}^0$-linear and can therefore be considered as an element of $\mathcal{A}^2(TM)$. Indeed, $T_D(f\cdot u, v) = f\cdot D_u v - (f\cdot D_v u + df(v)\cdot u) - (f\cdot[u,v] - df(v)\cdot u) = f\cdot T_D(u,v)$. A connection ∇ is called *torsion free* if $T_D = 0$.

Let us try to describe the torsion in local coordinates. So we may assume that our connection is of the form $D = d + A$. Here, A is a one-form with values in $\mathrm{End}(TM)$. In the following we will write $A(u) \in \mathcal{A}^0(\mathrm{End}(TM))$ for the endomorphism that is obtained by applying the one-form part of A to the vector field u. If u is constant, then $A(u) = D_u$. On the other hand, $A\cdot u \in \mathcal{A}^1(TM)$ is obtained by applying the endomorphism part of A to u. Confusion may arise whenever we use the canonical isomorphism $\mathcal{A}^1(TM) \cong \mathcal{A}^0(\mathrm{End}(TM))$.

Lemma 4.A.2 *If $D = d + A$ then $T_D(u,v) = A(u)\cdot v - A(v)\cdot u$.*

Proof. If $u = \sum_i a_i \frac{\partial}{\partial x_i}$ and $v = \sum_i b_i \frac{\partial}{\partial x_i}$ then

$$\begin{aligned} T_D(u,v) =& \left(\sum db_i(u)\frac{\partial}{\partial x_i} + A(u)\cdot v\right) - \left(\sum da_i(v)\frac{\partial}{\partial x_i} + A(v)\cdot u\right) - [u,v] \\ =& A(u)\cdot v - A(v)\cdot u \end{aligned}$$

□

Classically, one expresses the connection matrix A in terms of the *Christoffel symbols* Γ_{ij}^k as

$$A\left(\frac{\partial}{\partial x_i}\right)\cdot\frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Then

$$T_D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_k \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right)\frac{\partial}{\partial x_k}.$$

In particular, D is torsion free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i,j,k. The following result is one of the fundamental statements in Riemannian geometry and can be found in most textbooks on the subject, see e.g. [79].

Theorem 4.A.3 *Let (M,g) be a Riemannian manifold. Then there exists a unique torsion free metric connection on M; the* Levi-Civita connection. □

Why the notion of a torsion free connection is geometrically important is not evident from the definition. But in any case torsion free connections turn out to behave nicely in many ways. E.g. the exterior differential can be expressed in terms of torsion free connections.

Proposition 4.A.4 *If D is a torsion free connection on M then the induced connection on the space of forms satisfies*

$$(d\alpha)(v_1, \ldots v_{k+1}) = \sum_{i=0}^{k} (-1)^i (D_{v_i}\alpha)(v_1, \ldots, \hat{v}_i, \ldots, v_{k+1})$$

for any k-form α and vector fields $v_1, \ldots v_{k+1}$.

Proof. We leave the complete proof to the reader (cf. Exercise 4.A.1). One has to use the following definition of the exterior differential (see Appendix A):

$$d\alpha(v_1,\ldots,v_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}v_i(\alpha(v_1,\ldots,\hat{v}_i,\ldots,v_{k+1})) + \sum_{1\le i<j\le k+1}(-1)^{i+j}\alpha([v_i,v_j],v_1,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots v_{k+1})$$

□

A form $\alpha \in \mathcal{A}^k(M)$ is parallel if $D(\alpha) = 0$. Thus, Proposition 4.A.4 implies

Corollary 4.A.5 *Let D be a torsion free connection on M. Any D-parallel form is closed.* □

Let us now turn to hermitian manifolds. By definition, an hermitian structure on a complex manifold X is just a Riemannian metric g on the underlying real manifold compatible with the complex structure I defining X (see Definition 3.1.1). Recall that the complexified tangent bundle $T_{\mathbb{C}}X$ decomposes as $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ and that the bundle $T^{1,0}X$ is the complex bundle underlying the holomorphic tangent bundle $\mathcal{T}_X$ (Proposition 2.6.4). Moreover, the hermitian extension $g_{\mathbb{C}}$ of g to $T_{\mathbb{C}}X$ restricted to $T^{1,0}X$ is $\frac{1}{2}(g - i\omega)$, where the complex vector bundles $T^{1,0}X$ and (TX, I) are identified via the isomorphism

$$\xi : TX \longrightarrow T^{1,0}X, \quad u \longmapsto \frac{1}{2}(u - iI(u))$$

and ω is the fundamental form $g(I(\ ),(\ ))$ (cf. Section 1.2).

We will compare hermitian connections ∇ on $(T^{1,0}X, g_{\mathbb{C}})$ with the Levi-Civita connection D on TX via the isomorphism ξ. One first observes the following easy

Lemma 4.A.6 *Under the natural isomorphism ξ any hermitian connection ∇ on $T^{1,0}X$ induces a metric connection D on the Riemannian manifold (X, g).*

Proof. By assumption $dg_{\mathbb{C}}(u,v) = g_{\mathbb{C}}(\nabla u, v) + g_{\mathbb{C}}(u, \nabla v)$. Taking real parts of both sides yields $dg(u,v) = g(Du,v) + g(u,Dv)$, i.e. the induced connection D is metric. □

In general, an hermitian connection ∇ on $(T^{1,0}X, g_{\mathbb{C}})$ will not necessarily induce the Levi-Civita connection on the Riemannian manifold (X, g). In fact, this could hardly be true, as the Levi-Civita connection is unique, but there are many hermitian connections $(T^{1,0}X, g_{\mathbb{C}})$. But even for the Chern connection on the holomorphic tangent bundle $(\mathcal{T}_X, g_{\mathbb{C}})$, which is unique, the induced connection is not the Levi-Civita connection in general.

In order to state the relevant result comparing these two notions, we need to introduce the torsion $T_\nabla \in \mathcal{A}^2(X)$ of an hermitian connection ∇. By definition, T_∇ of ∇ is the torsion of the induced connection D on TX, i.e. for $u, v \in TX$ one has $T_\nabla(u,v) = \xi^{-1}(\nabla_u \xi(v) - \nabla_v \xi(u)) - [u,v] = D_u v - D_v u - [u,v] = T_D(u,v)$. As before we call the hermitian connection torsion free if its torsion is trivial.

Proposition 4.A.7 *Let ∇ be a torsion free hermitian connection on the hermitian bundle $(T^{1,0}X, g_{\mathbb{C}})$.*

i) *Then ∇ is the Chern connection on the holomorphic bundle $\mathcal{T}_X$ endowed with the hermitian structure $g_{\mathbb{C}}$.*

ii) *The induced connection D on the underlying Riemannian manifold is the Levi-Civita connection.*

iii) *The hermitian manifold (X, g) is Kähler.*

Proof. If we write the connection ∇ with respect to a local holomorphic base $\frac{\partial}{\partial z_i}$ as $\nabla = d + A$ then we have to show that the assumption $T_\nabla = 0$ implies $A \in \mathcal{A}^{1,0}(\mathrm{End}(T^{1,0}X))$. By definition the latter condition is equivalent to the vanishing of $A(u + iI(u))$ for any $u \in TX$ or, equivalently, to $A(u + iI(u)) \cdot \xi(v) = 0$ for all v. Using the analogue of Lemma 4.A.2 for the torsion of the connection ∇, one computes

$$\begin{aligned}
&A(u + iI(u)) \cdot \xi(v) \\
&= A(u + iI(u)) \cdot \xi(v) - A(v) \cdot (\xi(u) + iI(\xi(u))), \quad \text{since } \xi = -iI\xi \\
&= (A(u) \cdot \xi(v) - A(v) \cdot \xi(u)) + i\,(A(I(u)) \cdot \xi(v) - A(v) \cdot \xi(I(u))) \\
&= \xi(T(u,v)) + i\xi(T(I(u),v)) = 0,
\end{aligned}$$

as ξ and I are $\mathbb{C}$-linear. This proves i).

ii) is a consequence of what has been said before. In order to show iii), i.e. that the fundamental form ω is closed, one may use Corollary 4.A.5. Thus, it suffices to show that ω is parallel. This is the following straightforward calculation:

$$\begin{aligned}
(D\omega)(u,v) &= d(\omega(u,v)) - \omega(Du, v) - \omega(u, Dv) \\
&= dg(Iu, v) - g(DI(u), v) - g(I(u), D(v)) = 0,
\end{aligned}$$

as the connection is metric. □

Note that in the proof we have tacitly assumed that D commutes with the complex structure I, which is obvious as the hermitian connection ∇ on the complex vector bundle $(T^{1,0}X, g_{\mathbb{C}})$ is in particular $\mathbb{C}$-linear and $T^{1,0}X \cong (TX, I)$ is an isomorphism of complex vector bundles. However, if we try to associate to a connection on the underlying real manifold a connection on the holomorphic tangent bundle, then the compatibility with the complex structure is the crucial point. If D is a connection on the tangent bundle TX. Then $DI = ID$ if and only if I is a parallel section of $\mathrm{End}(TX)$ with respect to the induced connection on the endomorphism bundle.

Proposition 4.A.8 *Let D be the Levi-Civita connection on the Riemannian manifold (X, g) and assume that the complex structure I is parallel.*

i) *Under the isomorphism $\xi : TX \cong T^{1,0}X$ the connection D induces the Chern connection ∇ on the holomorphic tangent bundle $T^{1,0}X$.*

ii) *The manifold is Kähler and, moreover, the Kähler form ω is parallel.*

Proof. Since I is parallel, we do obtain a connection ∇ on the complex vector bundle $T^{1,0}X$. This connection is hermitian if and only if $dg_{\mathbb{C}}(u, v) = g_{\mathbb{C}}(\nabla u, v) + g_{\mathbb{C}}(u, \nabla v)$. Since the Levi-Civita connection is metric, the real parts of both sides are equal. The imaginary parts are (up to a factor) $d\omega(u, v) = dg(I(u), v)$ respectively $g(I(Du), v) + g(I(u), Dv)$. Using $DI = ID$ and again that ∇ is metric, one sees that they also coincide.

The Levi-Civita connection is by definition torsion free and using Proposition 4.A.7 this proves i) and the first assertion of ii). That $D(\omega) = 0$ follows from $D(I) = 0$ and $D(g) = 0$ as in the proof of Proposition 4.A.7. □

As a partial converse of Proposition 4.A.7, one proves.

Proposition 4.A.9 *Let (X, g) be a Kähler manifold. Then under the isomorphism $\xi : TX \cong T^{1,0}X$ the Chern connection ∇ on the holomorphic tangent bundle $\mathcal{T}_X = T^{1,0}X$ corresponds to the Levi-Civita connection D.*

Proof. We have to show that under the assumption that (X, g) is Kähler the Chern connection is torsion free. This is done in local coordinates. Locally we write $\omega = \frac{1}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$ and $A = (\bar{h}_{ji})^{-1}(\partial \bar{h}_{ji})$ for the connection form. The fundamental form is closed if and only if $\frac{\partial h_{ij}}{\partial z_k} = \frac{\partial h_{kj}}{\partial z_i}$. The latter can then be used to prove the required symmetry of the torsion form. We leave the details to the reader (cf. Exercise 4.A.2) □

From this slightly lengthy discussion the reader should only keep in mind that the following four conditions are equivalent:

i) The complex structure is parallel with respect to the Levi-Civita connection.

ii) (X, g) is Kähler.

iii) Levi-Civita connection D and Chern connection ∇ are identified by ξ.

iv) The Chern connection is torsion free.

For this reason we will in the following not distinguish anymore between the Levi-Civita connection D and the Chern connection ∇ provided the manifold is Kähler.

Let us now turn to the curvature tensor of a Riemannian manifold (M, g). Classically it is defined as

$$R(u, v) := D_u D_v - D_v D_u - D_{[u,v]},$$

where D is the Levi-Civita connection. How does this definition compare with the one given in Section 4.3? Not surprisingly they coincide. This is shown by the following direct calculation. We assume for simplicity that $Ds = \alpha \otimes t$:

$$\begin{aligned} R(u,v)(s) &= D_u D_v s - D_v D_u s - D_{[u,v]} s \\ &= D_u(\alpha(v)t) - D_v(\alpha(u)t) - \alpha([u,v])t \\ &= u(\alpha(v))t + \alpha(v)D_u t - v(\alpha(u))t - \alpha(u)D_v t - \alpha([u,v])t \\ &= (d\alpha)(u,v)\cdot t + \alpha(v)D_u t - \alpha(u)D_v t = \left((d\alpha)\cdot t + \alpha \cdot D(t)\right)(u,v) \\ &= D(\alpha\otimes t)(u,v) = F(u,v)(s), \end{aligned}$$

where we have used a special case of the formula that describes the exterior differential in terms of the Lie bracket (cf. Exercise 4.A.1).

In Riemannian geometry one also considers the *Ricci tensor*

$$r(u,v) := \mathrm{tr}(w \mapsto R(w,u)v) = \mathrm{tr}(\ w \longmapsto R(w,v,u)\ )$$

(see [79]). Combined with the complex structure one has

Definition 4.A.10 The *Ricci curvature* $\mathrm{Ric}(X,g)$ of a Kähler manifold (X,g) is the real two-form

$$\mathrm{Ric}(u,v) := r(I(u),v).$$

The Kähler metric is called *Ricci-flat* if $\mathrm{Ric}(X,g) = 0$.

The Ricci curvature can be computed by means of the curvature F_∇ of the Levi-Civita (or, equivalently, the Chern) connection and the Kähler form ω. This goes as follows.

The contraction of the curvature $F_\nabla \in \mathcal{A}^{1,1}(\mathrm{End}(T^{1,0}X))$ with the Kähler form ω yields an element $\Lambda_\omega F \in \mathcal{A}^0(X, \mathrm{End}(T^{1,0}X))$ or, equivalently, an endomorphism $T^{1,0}X \to T^{1,0}X$. Its composition with the isomorphism $T^{1,0}X \to \bigwedge^{0,1} X$ induced by the Kähler form will be denoted

$$\tilde\omega(\Lambda_\omega F) : T^{1,0}X \xrightarrow{\Lambda_\omega F_\nabla} T^{1,0}X \xrightarrow{\ \omega\ } \textstyle\bigwedge^{0,1} X.$$

One easily verifies that $\tilde\omega(\Lambda_\omega F) \in \mathcal{A}^{1,1}(X)$.

Proposition 4.A.11 *Let (X,g) be a Kähler manifold and ∇ the Levi-Civita or, equivalently, the Chern connection. Then the following two identities hold true:*

i) $\mathrm{Ric}(X,g) = i \cdot \tilde\omega\,(\Lambda_\omega F_\nabla)$.

ii) $\mathrm{Ric}(X,g) = i \cdot \mathrm{tr}_{\mathbb{C}}(F_\nabla)$, *where the trace is taken in the endomorphism part of the curvature.*

Proof. We shall use the following well-known identities from Riemannian geometry (see [12, page 42]):

$$g(R(u,v)x,y) = g(R(x,y)u,v) \tag{4.6}$$
$$R(u,v)w + R(v,w)u + R(w,u)v = 0 \tag{4.7}$$
$$g(R(u,v)w,x) + g(w,R(u,v)x) = 0 \tag{4.8}$$

The third one is clearly due to the fact that the Levi-Civita connection is metric. Equation (4.7), which is the algebraic Bianchi identity, and equation (4.6) are more mysterious.

The proof consists of computing all three expressions explicitly in terms of an orthonormal basis of the form $x_1,\ldots,x_n,y_1=I(x_1),\ldots,y_n=I(x_n)$:

$$\begin{aligned}
\mathrm{Ric}(u,v) &= \mathrm{tr}\,(w\mapsto R(w,v)I(u))\\
&= \sum g(R(x_i,v)I(u),x_i) + g(R(y_i,v)I(u),y_i)\\
&\overset{(4.8)}{=} -\sum g(I(u),R(x_i,v)x_i) + g(I(u),R(y_i,v)y_i)\\
&= \sum g(u,R(x_i,v)y_i) - g(u,R(y_i,v)x_i)\\
&\overset{(4.7)}{=} -\sum g(u,R(y_i,x_i)v)\\
&\overset{(4.8)}{=} -\sum g(R(x_i,y_i)u,v),
\end{aligned}$$

where we use twice the compatibility of g and I and the fact that $R(u,v)$ is skew-symmetric.

Furthermore, using the notation and convention of Section 1.2 one calculates:

$$\begin{aligned}
\mathrm{tr}_{\mathbb{C}}(F_\nabla)(u,v) &= \mathrm{tr}_{\mathbb{C}}\,(w\mapsto F_\nabla(u,v)w) \quad \text{with } w\in T^{1,0}\\
&= \mathrm{tr}_{\mathbb{C}}\,(w\mapsto R(u,v)w) \quad \text{with } w\in T\\
&= \sum (R(u,v)x_i,x_i) \quad \text{since } x_1,\ldots,x_n \text{ is an orthonormal basis}\\
&\qquad\qquad \text{of the hermitian vector space } (T,(\ ,\ ))\\
&= \sum g(R(u,v)x_i,x_i) - i\cdot\omega(R(u,v)x_i,x_i)\\
&\qquad\qquad \text{as } (\ ,\ ) = g - i\cdot\omega\\
&= \sum g(R(u,v)x_i,x_i) + i\cdot g(R(u,v)x_i,y_i)\\
&\overset{(4.6)}{=} \sum g(R(x_i,x_i)u,v) + i\cdot g\,(R(x_i,y_i)u,v)\\
&= 0 + i\cdot\sum g(R(x_i,y_i)u,v)
\end{aligned}$$

Both computations together show ii).

In order to see i), let us write $\omega = \sum x^i \wedge y^i$. Then a straightforward computation shows $\Lambda_\omega \alpha = \sum \alpha(x_i, y_i)$ and

$$\begin{aligned}
2i \cdot \tilde{\omega}(\Lambda_\omega F)(u,v) &= i \cdot \omega(\Lambda_\omega F(\xi(u)), \overline{\xi(v)}) = \omega(\Lambda_\omega(F(\xi(I(u)), \overline{\xi(v)})\\
&= \omega\left(\sum F(x_i, y_i)\xi(I(u)), \overline{\xi(v)}\right)\\
&= \omega\left(\xi\left(\sum R(x_i, y_i)I(u)\right), \overline{\xi(v)}\right)\\
&= 2g\left(I\left(\sum R(x_i, y_i)I(u)\right), v\right) = -2\sum g(R(x_i, y_i)u, v).
\end{aligned}$$

□

In the following, we will sketch the relation between the holonomy of a Riemannian manifold and its complex geometry.

To any Riemannian metric g on a manifold M of dimension m there is associated, in a unique way, the Levi-Civita connection D. By means of D one can define the parallel transport of tangent vectors along a path in M. This goes as follows.

Let $\gamma : [0,1] \to M$ be a path connecting two points $x := \gamma(0)$ and $y := \gamma(1)$. The pull-back connection $\gamma^* D$ on $\gamma^*(TM)$ is necessarily flat over the one-dimensional base $[0,1]$ and $\gamma^*(TM)$ can therefore be trivialized by flat sections. In this way, one obtains an isomorphism, the *parallel transport along* the path γ:

$$P_\gamma : T_x M \longrightarrow T_x M \ .$$

In other words, for any $v \in T_x M$ there exists a unique vector field $v(t)$ with $v(t) \in T_{\gamma(t)} M$, $v(0) = v$, and such that $v(t)$ is a flat section of $\gamma^*(TM)$. Then $P_\gamma(v) = v(1)$.

The first observation concerns the compatibility of P_γ with the scalar products on $T_x M$ and $T_y M$ given by the chosen Riemannian metric g.

Lemma 4.A.12 P_γ *is an isometry.* □

In particular, if γ is a closed path, i.e. $\gamma(0) = \gamma(1) = x$, then $P_\gamma \in \mathrm{O}(T_x M, g_x) \cong \mathrm{O}(m)$.

Definition 4.A.13 For any point $x \in M$ of a Riemannian manifold (M, g) the *holonomy group* $\mathrm{Hol}_x(M, g) \subset \mathrm{O}(T_x M)$ is the group of all parallel transports P_γ along closed paths $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = x$.

If two points $x, y \in M$ can be connected at all, e.g. if M is connected, then the holonomy groups $\mathrm{Hol}_x(M, g)$ and $\mathrm{Hol}_y(M, g)$ are conjugate and thus isomorphic. More precisely, if $\gamma : [0,1] \to M$ is a path connecting x with y then $\mathrm{Hol}_y(M, g) = P_\gamma \circ \mathrm{Hol}_x(M, g) \circ P_\gamma^{-1}$.

Hence, if M is connected then one can define the group $\mathrm{Hol}(M, g)$ as a subgroup of $\mathrm{O}(m)$ up to conjugation.

There is a further technical issue if M is not simply connected. Then there is a difference between $\mathrm{Hol}(M,g)$ and the *restricted holonomy group* $\mathrm{Hol}^{\mathrm{o}}(M,g) \subset \mathrm{Hol}(M,g)$ of all parallel transports P_γ along contractible paths γ (i.e. $1 = [\gamma] \in \pi_1(M)$). For simplicity, we will assume throughout that M is simply connected.

One fundamental problem in Riemannian geometry is the classification of holonomy groups. What groups $\mathrm{Hol}(M,g) \subset \mathrm{O}(m)$ can arise?

Firstly, the holonomy of a product $(M,g) = (M_1,g_1) \times (M_2,g_2)$ is the product $\mathrm{Hol}(M,g) = \mathrm{Hol}(M_1,g_1) \times \mathrm{Hol}(M_2,g_2) \subset \mathrm{O}(m_1) \times \mathrm{O}(m_2) \subset \mathrm{O}(m_1 + m_2)$.

Thus, in order to be able to classify all possible holonomy groups we shall assume that (M,g) is *irreducible*, i.e. cannot be written (locally) as a product.

This is indeed reflected by an algebraic property of the holonomy group:

Proposition 4.A.14 *If* (M,g) *is irreducible Riemannian manifold, then the inclusion* $\mathrm{Hol}(M,g) \subset \mathrm{O}(m)$ *defines an irreducible representation on* $\mathbb{R}^m$. □

This proposition is completed by the following theorem ensuring the existence of a decomposition into irreducible factors.

Theorem 4.A.15 (de Rham) *If* (M,g) *is a simply connected complete (e.g. compact) Riemannian manifold then there exists a decomposition* $(M,g) = (M_1,g_1) \times \ldots \times (M_k,g_k)$ *with irreducible factors* (M_i,g_i). □

Secondly, many groups can occur as holonomy groups of symmetric spaces, a special type of homogeneous spaces. For the precise definition and their classification see [12]. If symmetric spaces are excluded then, surprisingly, a finite list of remaining holonomy groups can be given.

Theorem 4.A.16 (Berger) *Let* (M,g) *be a simply connected, irreducible Riemannian manifold of dimension* m *and let us assume that* (M,g) *is irreducible and not locally symmetric. Then the holonomy group* $\mathrm{Hol}(M,g)$ *is isomorphic to one of the following list:*

i) $\mathrm{SO}(m)$.
ii) $\mathrm{U}(n)$ *with* $m = 2n$.
iii) $\mathrm{SU}(n)$ *with* $m = 2n$, $n \geq 3$
iv) $\mathrm{Sp}(n)$ *with* $m = 4n$.
v) $\mathrm{Sp}(n)\mathrm{Sp}(1)$ *with* $m = 4n$, $n \geq 2$.
vi) G_2, *with* $m = 7$.
vii) $\mathrm{Spin}(7)$, *with* $m = 8$. □

We don't go into any detail here, in particular we don't define G_2 or explain the representation of $\mathrm{Spin}(7)$. Very roughly, $\mathrm{SO}(m)$ is the case of a general Riemannian metric and vi) and vii) are very special. In fact compact examples for vi) have been found only recently.

The irreducible holonomy groups that are relevant in complex geometry are ii), iii), iv) and, a little less, v).

In the following we shall discuss the case $\mathrm{Hol}(M,g) = \mathrm{U}(n)$ and its subgroup $\mathrm{Hol}(M,g) = \mathrm{SU}(n)$. We will see that this leads to Kähler respectively Ricci-flat Kähler manifolds. The case $\mathrm{Hol}(M,g) = \mathrm{Sp}(n)$, not discussed here, is related to so-called hyperkähler or, equivalently, holomorphic symplectic manifolds.

How is the holonomy $\mathrm{Hol}(M,g)$ of a Riemannian manifold related to the geometry of M at all?

This is explained by the *holonomy principle* :

Choose a point $x \in M$ and identify $\mathrm{Hol}(M,g) = \mathrm{Hol}_x(M,g) \subset \mathrm{O}(T_xM)$. This representation of $\mathrm{Hol}(M,g)$ induces representations on all tensors associated with the vector space T_xM, e.g. on $\mathrm{End}(T_xM)$. Suppose α_x is a tensor invariant under $\mathrm{Hol}(M,g)$. We will in particular be interested in the case of an almost complex structure $\alpha_x = I_x \in \mathrm{End}(T_xM)$.

One way to obtain such an invariant tensor α_x is by starting out with a parallel tensor field α on M. (Recall that the Levi-Civita connection induces connections on all tensor bundles, e.g. on $\mathrm{End}(TM)$, so that we can speak about parallel tensor fields.) Clearly, any parallel tensor field α yields a tensor α_x which is invariant under the holonomy group. As an example take a Kähler manifold (M,g,I). Then I is a parallel section of $\mathrm{End}(TM)$ and the induced $I_x \in \mathrm{End}(T_xM)$ is thus invariant under the holonomy group. Hence, $\mathrm{Hol}(M,g) \subset \mathrm{O}(2n) \cap \mathrm{Gl}(n,\mathbb{C}) = \mathrm{U}(n)$.

We are more interested in the other direction which works equally well:

Holonomy principle. If α_x is an $\mathrm{Hol}(M,g)$-invariant tensor on T_xM then α_x can be extended to a parallel tensor field α over M.

Let us consider the case of an almost complex structure $I_x \in \mathrm{End}(T_xM)$ invariant under $\mathrm{Hol}(M,g)$. Then there exists a parallel section I of $\mathrm{End}(TM)$. Since $\mathrm{id} \in \mathrm{End}(TM)$ is parallel and $I_x^2 = -\mathrm{id}$, we have $I^2 = -\mathrm{id}$ everywhere, i.e. I is an almost complex structure on M.

Moreover, since g and I are both parallel, also the Kähler form $g(I(\ ),\ )$ is parallel and, in particular, closed. There is an additional argument that shows that I is in fact integrable. (One uses the fact that the Nijenhuis tensor, which we have not defined but which determines whether an almost complex structure is integrable, is a component of $\nabla(I)$ in case the connection is torsion free. But in our case, I is parallel. See [61].) Thus, we obtain an honest Kähler manifold (M,g,I).

This yields the following proposition. The uniqueness statement is left to the reader.

Proposition 4.A.17 *If* $\mathrm{Hol}(M,g) \subset \mathrm{U}(n)$ *with* $m = 2n$, *then there exists a complex structure* I *on* M *with respect to which* g *is Kähler. If* $\mathrm{Hol}(M,g) = \mathrm{U}(n)$, *then* I *is unique.*

In the next proposition we relate $\mathrm{SU}(n)$-holonomy to Ricci-flat Kähler metrics, which will be explained in the next section in more detail. So the reader might prefer to skip the following proposition at first reading.

Proposition 4.A.18 *If* $\mathrm{Hol}(M,g) \subset \mathrm{SU}(n)$ *with* $m = 2n$*, then there exists a complex structure* I *on* M *with respect to which* g *is a Ricci-flat Kähler metric. If* $\mathrm{Hol}(M,g) = \mathrm{SU}(n)$ *with* $n \geq 3$ *then* I *is unique.*

Proof. The existence of the Kähler metric and its uniqueness are proved as in the previous proposition. The assumption that the holonomy group is contained in $\mathrm{SU}(n)$ says that any chosen trivialization of $\bigwedge_x M = \det(T_xM^*)$ is left invariant by parallel transport. Hence, there exists a parallel section Ω of $\det_{\mathbb{C}}(\bigwedge M) = \det(\Omega_X) = K_X$ (where $X = (M,I)$). A parallel section of K_X is holomorphic and, if not trivial, without zeros, as the zero section itself is parallel. Thus, K_X is trivialized by a holomorphic volume form Ω.

Moreover, as Ω and ω^n are both parallel, they differ by a constant. Using Corollary 4.B.23 this shows that the Kähler metric g is Ricci-flat. □

Exercises

4.A.1 Complete the proof of Proposition 4.A.4, i.e. prove that for a torsion free connection on a differentiable manifold M one has for any k-form α

$$(d\alpha)(v_1,\ldots,v_{k+1}) = \sum_{i=0}^{k}(-1)^i(D_{v_i}\alpha)(v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}).$$

4.A.2 Complete the proof of Proposition 4.A.9.

4.A.3 Let (X,g) be a compact Kähler manifold. Show that $i \cdot \mathrm{Ric}(X,g)$ is the curvature of the Chern connection on K_X with respect to the induced hermitian metric.

Comments: - For a thorough discussion of most of this section we refer to [12]. A short introduction to holonomy with special emphasize on the relations to algebraic geometry can be found in [9].

- We have not explained the relation between the curvature and the holonomy group. Roughly, the curvature tensor determines the Lie algebra of Hol. See [12] for more details.

- A detailed account of the more recent results on the holonomy of (compact) manifolds can be found in [61] or [72]. Joyce was also the first one to construct compact G_2 manifolds.

4.B Hermite–Einstein and Kähler–Einstein Metrics

Interesting metrics on compact manifolds are not easy to construct. This appendix discusses two types of metrics which are of importance in Kähler geometry.

If (X, g) is an hermitian manifold, $\omega := g(I(\ ), (\ ))$ is its fundamental form. By definition (X, g) is Kähler if and only if ω is closed, i.e. $d\omega = 0$. The hermitian structure on X can be viewed as an hermitian structure on its holomorphic tangent bundle. So we might look more generally for interesting metrics on an arbitrary holomorphic vector bundle E on X. We will discuss special hermitian metrics on E, so called Hermite–Einstein metrics, by comparing the curvature F_∇ of the Chern connection ∇ on E with the fundamental form ω. In the special case that E is the holomorphic tangent bundle $\mathcal{T}_X$ and the hermitian structure h is induced by g this will lead to the concept of Kähler–Einstein metrics on complex manifolds.

In some of our examples, e.g. the Fubini–Study metric on $\mathbb{P}^n$, we have already encountered this special type of Kähler metrics. However, on other interesting manifolds, like K3 surfaces, concrete examples of Kähler metrics have not been discussed. Of course, if a manifold is projective one can always consider the restriction of the Fubini–Study metric, but this usually does not lead to geometrically interesting structures (at least not directly).

For the time being, we let E be an arbitrary holomorphic vector bundle with an arbitrary hermitian metric h. Recall that the curvature F_∇ of the Chern connection on (E, h) is of type $(1,1)$, i.e. $F_\nabla \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$. The fundamental form ω induces an element of the same type $\omega \cdot \mathrm{id}_E \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$. These two are related to each other by the Hermite–Einstein condition:

Definition 4.B.1 An hermitian structure h on a holomorphic vector bundle E is called *Hermite–Einstein* if

$$i \cdot \Lambda_\omega F_\nabla = \lambda \cdot \mathrm{id}_E$$

for some constant scalar $\lambda \in \mathbb{R}$. Here, Λ_ω is the contraction by ω.

In this case, we will also say that the connection ∇ is Hermite–Einstein or even that the holomorphic bundle E is Hermite–Einstein. Note that the Hermite–Einstein condition strongly depends on the hermitian structure on the manifold X. It may happen that a vector bundle E admits an Hermite–Einstein structure with respect to one hermitian structure g on X, but not with respect to another g'.

Example 4.B.2 The easiest example of an Hermite–Einstein bundle is provided by flat bundles. In this case the curvature F_∇ is trivial and the Hermite–Einstein condition is, therefore, automatically satisfied with $\lambda = 0$.

Let us discuss a few equivalent formulations of the Hermite–Einstein condition. Firstly, one can always write the curvature of the Chern connection ∇ on any bundle E as

$$F_\nabla = \frac{\mathrm{tr}(F_\nabla)}{\mathrm{rk}(E)} \cdot \mathrm{id} + F_\nabla^\circ$$

where F_∇° is the trace free part of the curvature. Let us now assume that g is a Kähler metric, i.e. that ω is closed (and thus, automatically, harmonic). Then the connection is Hermite–Einstein if and only if $\mathrm{tr}(F_\nabla)$ is an harmonic $(1,1)$-form and F_∇° is (locally) a matrix of primitive $(1,1)$-forms. Indeed, if ∇ is Hermite–Einstein, then $i \cdot \mathrm{tr}(F_\nabla) = (\mathrm{rk}(E) \cdot \lambda) \cdot \omega + \alpha$ with α a primitive $(1,1)$-form. Since the trace is closed (Bianchi identity), the form α is closed and hence harmonic (see Exercise 3.1.12). The other assertion and the converse are proved analogously.

Secondly, the Hermite–Einstein condition for the curvature of a Chern connection is equivalent to writing

$$i \cdot F_\nabla = (\lambda/n) \cdot \omega \cdot \mathrm{id}_E + F'_\nabla,$$

where F'_∇ is locally a matrix of ω-primitive $(1,1)$-forms. Here, $n = \dim_{\mathbb{C}} X$. The factor $(1/n)$ is explained by the commutator relation $[L, \Lambda] = H$, which yields $\Lambda L(1) = n$ (cf. Proposition 1.2.26).

Using standard results from Section 1.2 one easily finds that h is Hermite–Einstein if and only if

$$i \cdot F_\nabla \wedge \omega^{n-1} = (\lambda/n) \cdot \omega^n \cdot \mathrm{id}_E.$$

In particular,

$$i \cdot \mathrm{tr}(F_\nabla) \wedge \omega^{n-1} = \frac{\mathrm{rk}(E) \cdot \lambda}{n} \cdot \omega^n. \tag{4.9}$$

If X is compact and Kähler, (4.9) can be used to show that λ depends only on the first Chern class of E and its rank. Indeed, integrating (4.9) yields

$$\lambda \cdot \int_X [\omega]^n = n \cdot \frac{\int_X i \cdot \mathrm{tr}(F_\nabla) \wedge \omega^{n-1}}{\mathrm{rk}(E)}.$$

Hence, $\lambda = (2\pi) \cdot n \cdot \left(\int_X [\omega]^n\right)^{-1} \mu(E)$, where the slope $\mu(E)$ of E is defined as follows:

Definition 4.B.3 The *slope* of a vector bundle E with respect to the Kähler form ω is defined by

$$\mu(E) := \frac{\int_X \mathrm{c}_1(E) \wedge [\omega]^{n-1}}{\mathrm{rk}(E)}.$$

In general, Hermite–Einstein metrics are not easy to describe, but they exist quite frequently and those holomorphic bundles that admit Hermite–Einstein metrics can be described algebraically (see Theorem 4.B.9).

Lemma 4.B.4 *Any holomorphic line bundle L on a compact Kähler manifold X admits an Hermite–Einstein structure.*

Proof. The curvature $i \cdot F_\nabla$ of the Chern connection on the holomorphic line bundle L endowed with an hermitian structure h is a real $(1,1)$-form. Hence, $i \cdot \Lambda_\omega F_\nabla$ is a real function φ, which can be written as $\lambda - \partial^* \partial f$ for some function f and some constant λ. Since $\partial^* \partial f = (1/2) d^* df$, we can assume that the function f is real.

Then define a new hermitian structure h' on L by $h' = e^f \cdot h$. The curvature of the induced connection ∇' is $F_{\nabla'} = F_\nabla + \bar\partial \partial f$ (see iii), Examples 4.3.9). Using the Kähler identity $[\Lambda, \bar\partial] = -i\partial^*$ on $\mathcal{A}^0(X)$ one computes $\Lambda_\omega \bar\partial \partial f = -i\partial^* \partial f$. Hence, $i \cdot \Lambda_\omega(F_{\nabla'}) = (\varphi + \partial^* \partial f) = \lambda$. □

Remark 4.B.5 Clearly, the first Chern class $\mathrm{c}_1(L) \in H^2(X, \mathbb{R})$ can uniquely be represented by an harmonic form. The lemma shows that one actually finds an hermitian structure on L such that the first Chern form $\mathrm{c}_1(L, \nabla)$ of the associated Chern connection ∇ is this harmonic representative. The same argument can be used to solve Exercise 4.4.10.

From here one can go on and construct many more vector bundles admitting Hermite–Einstein structures. E.g. the tensor product $E_1 \otimes E_2$ of two Hermite–Einstein bundles is again Hermite–Einstein, as well as the dual bundle E_1^*. However, the direct sum $E_1 \oplus E_2$ admits an Hermite–Einstein structure if and only if $\mu(E_1) = \mu(E_2)$ (cf. Exercise 4.B.2). Indeed, it is not hard to see that the direct sum of the two Hermite–Einstein connections is Hermite–Einstein under this condition. The other implication is slightly more complicated.

Vector bundles admitting Hermite–Einstein metrics satisfy surprising topological restrictions.

Proposition 4.B.6 *Let E be a holomorphic vector bundle of rank r on a compact hermitian manifold (X, g). If E admits an Hermite–Einstein structure then*

$$\int_X \left(2r\mathrm{c}_2(E) - (r-1)\mathrm{c}_1^2(E)\right) \wedge \omega^{n-2} \geq 0.$$

Proof. The bundle $G := \mathrm{End}(E)$ with the naturally induced connection ∇_G is Hermite–Einstein (cf. Exercise 4.B.2) and has vanishing first Chern form $\mathrm{c}_1(G, \nabla_G)$ (cf. Exercise 4.4.9). In particular, $\Lambda_\omega F_{\nabla_G} = 0$.

For such a bundle G we will show that $\mathrm{ch}_2(G, \nabla_G)\omega^{n-2} \leq 0$ (pointwise!). Since $\mathrm{ch}_2(\mathrm{End}(E), \nabla) = -(2r\mathrm{c}_2(E, \nabla) - (r-1)\mathrm{c}_1^2(E, \nabla))$ (cf. Exercise 4.4.9), this then proves the assertion.

The rest of the proof is pure linear algebra applied to the tangent space at an arbitrary point. By definition

$$\mathrm{ch}_2(G,\nabla_G)=\frac{1}{2}\mathrm{tr}\left(\left(\frac{i}{2\pi}F_{\nabla_G}\right)^2\right)=\mathrm{tr}\left(\frac{i}{2\pi}F_{\nabla_G}\wedge\overline{\left(\frac{i}{2\pi}F_{\nabla_G}\right)}^{\mathrm{t}}\right),$$

where the expression on the right hand side is meant with respect to a local orthonormal basis.

But by the Hodge–Riemann bilinear relation 1.2.36 any matrix of primitive $(1,1)$-forms $A=(a_{ij})$ satisfies $\mathrm{tr}(A\bar{A}^{\mathrm{t}})\wedge\omega^{n-2}=\sum(a_{ij}\bar{a}_{ij})\wedge\omega^{n-2}\leq 0$. □

Remarks 4.B.7 i) In this context the inequality is due to Lübke [87]. It is often called the Bogomolov–Lübke inequality, as its algebraic version was first observed by Bogomolov [14].

ii) Also note that the above Chern class combination is the only natural one (up to scaling) among those that involve only the first two Chern classes, as it is the only one that remains unchanged when passing from E to a line bundle twist $E\otimes L$, which also carries a Hermite–Einstein structure (cf. Exercise 4.B.2).

iii) In the proof we have actually shown the pointwise inequality. Thus, in this sense the assertion holds true also for non-compact manifolds X. Moreover, from the proof one immediately deduces that equality (global or pointwise) implies that the endomorphism bundle has vanishing curvature.

iv) Furthermore, we only used the weak Hermite–Einstein condition where the scalar λ is replaced by a function. Due to the Exercise 4.B.3 this does not really generalize the statement as formulated above.

When does a holomorphic bundle that satisfies the above inequality really admit a Hermite–Einstein metric? This is a difficult question, but a complete answer is known due to the spectacular results of Donaldson, Uhlenbeck, and Yau. It turns out that the question whether E admits an Hermite–Einstein metric can be answered by studying the algebraic geometry of E. In particular, one has to introduce the concept of stability.

Definition 4.B.8 A holomorphic vector bundle E on a compact Kähler manifold X is *stable* if and only if

$$\mu(F)<\mu(E)$$

for any proper non-trivial $\mathcal{O}_X$-subsheaf $F\subset E$.

A few comments are needed here. First of all, the notion depends on the chosen Kähler structure of X or, more precisely, on the Kähler class $[\omega]$. Secondly, the slope was defined only for vector bundles F and not for arbitrary $\mathcal{O}_X$-sheaves $F\subset E$, but it is not difficult to find the correct definition in this more general context. E.g. one could first define the determinant of F and then define the first Chern class of F as the first Chern class of its determinant. Another possibility would be to use the Atiyah class to define Chern classes of coherent sheaves in general (cf. Remark 5.1.5).

Unfortunately, already from dimension three on one really needs to check arbitrary subsheaves of E and not just locally free subsheaves (let alone sub-vector bundles, i.e. locally free subsheaves with locally free quotient). There is however one condition that is slightly stronger than stability and which uses only vector bundles. A holomorphic bundle E is stable if for any $0 < s < \mathrm{rk}(E)$ and any line bundle $L \subset \bigwedge^s E$ one has $\mu(L) < s \cdot \mu(E)$.

Also note that other stability concepts for holomorphic vector bundles exist. The one we use is usually called *slope-stability* or *Mumford–Takemoto stability* .

One can also define polystability for holomorphic vector bundles (not to be confused with semi-stability, which we shall not define). A holomorphic vector bundle E is *polystable* if $E = \bigoplus E_i$ with E_i stable vector bundles all of the same slope $\mu(E) = \mu(E_i)$. The following beautiful result shows that algebraic geometry of a vector bundle determines whether an Hermite–Einstein metric exists. The proof is a pure existence result and it allows to deduce the existence of Hermite–Einstein metrics without ever actually constructing any Hermite–Einstein metric explicitly.

Theorem 4.B.9 (Donaldson, Uhlenbeck, Yau) *A holomorphic vector bundle E on a compact Kähler manifold X admits an Hermite–Einstein metric if and only if E is polystable.* □

Remark 4.B.10 One direction of the theorem is not very hard, any holomorphic bundle endowed with an Hermite–Einstein metric is polystable. Donaldson in [36] proved the converse for algebraic surfaces. This was a generalization of an old result of Narasimhan and Seshadri [94] for vector bundles on curves. On curves, Hermite–Einstein metrics are intimately related to unitary representation of the fundamental group of the curve. Uhlenbeck and Yau [112] generalized Donaldson's result to arbitrary compact Kähler manifolds and Buchdahl managed to adjust the proof to the case of compact hermitian manifolds. This kind of result is nowadays known as Kobayashi–Hitchin correspondence (cf. [88]).

Next, we shall study the case that E is the holomorphic tangent bundle $\mathcal{T}_X$. This leads to a much more restrictive notion. Any hermitian structure g on the complex manifold X induces an hermitian structure on the holomorphic tangent bundle $\mathcal{T}_X$. So, it would not be very natural to look for another unrelated hermitian structure on $\mathcal{T}_X$. Recall that the Hermite–Einstein condition intertwines the hermitian structure on X with the hermitian structure on the vector bundle in question.

Definition 4.B.11 An hermitian manifold (X, g) is called *Kähler–Einstein* if (X, g) is Kähler and the naturally induced hermitian structure on the holomorphic tangent bundle is Hermite–Einstein. In this case the metric g is called

Kähler–Einstein. If a Kähler–Einstein metric g on X exists, the complex manifold X is also called a Kähler–Einstein manifold.

Explicitly, this means that the curvature of the Levi-Civita connection satisfies

$$i \cdot \Lambda_\omega F_\nabla = \lambda \cdot \mathrm{id}_{\mathcal{T}_X} \tag{4.10}$$

for some constant scalar factor λ.

One may try to avoid the condition that the manifold is Kähler, but as we explained in Section 4.A the Chern connection on the holomorphic tangent bundle coincides with the Levi-Civita connection if and only if the manifold (X, g) is Kähler. If not, the Hermite–Einstein condition would not seem very natural for the Riemannian metric g.

Note that the holomorphic tangent bundle on an hermitian (or Kähler) manifold (X, g) might very well admit a Hermite–Einstein metric without X being Kähler–Einstein.

Usually the Kähler–Einstein condition is introduced via the Ricci curvature (cf. Definition 4.A.10). Let us begin with the Riemannian version.

Definition 4.B.12 A Riemannian metric g on a differentiable manifold M is *Einstein* if its Ricci tensor $r(M, g)$ satisfies

$$r(M, g) = \lambda \cdot g$$

for some constant scalar factor λ.

If g is a Kähler metric on the complex manifold $X = (M, I)$, then the Ricci curvature $\mathrm{Ric}(X, g)$ is defined by $\mathrm{Ric}(u, v) = r(I(u), v)$ (see Definition 4.A.10). This is in complete analogy to the definition of the Kähler form ω as $\omega(u, v) = g(I(u), v)$. Thus, a metric g on a complex manifold $X = (M, g)$ is an Einstein metric on M if and only if $\mathrm{Ric}(X, g) = \lambda \cdot \omega$ for some constant scalar λ.

Recall that Proposition 4.A.11 shows $i \cdot \tilde{\omega}(\Lambda_\omega F_\nabla) = \mathrm{Ric}(X, g)$. This leads to:

Corollary 4.B.13 *Let g be a Kähler metric on the complex manifold $X = (M, I)$. Then g is an Einstein metric on M if and only if g is a Kähler–Einstein metric on X.*

Proof. Indeed, if g is a Kähler–Einstein metric, then for the curvature F of the Levi-Civita connection one has $i \cdot \Lambda_\omega F = \lambda \cdot \mathrm{id}$ and hence $\mathrm{Ric}(X, g) = i \cdot \tilde{\omega}(\Lambda_\omega F) = \lambda \cdot \omega$. Thus, $r(M, g) = \lambda \cdot g$.

For the converse, go the argument backwards. □

Applying the argument explained before for Hermite–Einstein metrics (see page 218) and using $c_1(X) = (i/2\pi)[\mathrm{tr}(F)]$, one finds that the scalar factor λ in the Kähler–Einstein condition (4.10) can be computed as

$$\lambda = \frac{i \cdot \int \mathrm{tr}(F) \wedge \omega^{n-1}}{\int \omega^n} = \frac{(2\pi) \int c_1(X) \wedge \omega^{n-1}}{\int \omega^n}.$$

In other words, $c_1(X) \in H^2(X,\mathbb{R})$ and $[\omega] \in H^2(X,\mathbb{R})$ satisfy the linear equation $(c_1(X) - (\lambda/2\pi)[\omega])[\omega]^{n-1} = 0$. In fact, one can prove more. Namely, if g is a Kähler–Einstein metric on X, then for its associated Kähler form ω one has

$$c_1(X) = \frac{\lambda}{2\pi} \cdot [\omega]$$

with λ the scalar factor occurring in the Kähler–Einstein condition (4.10). Indeed, $c_1(X) = c_1(\mathcal{T}_X) = [(i/2\pi)\mathrm{tr}(F_\nabla)]$ and by Proposition 4.A.11 and the Kähler–Einstein condition one has $(i/2\pi)\mathrm{tr}(F_\nabla) = (1/2\pi)\mathrm{Ric} = (\lambda/2\pi)\omega$.

Also note that in the decomposition $F_\nabla = (\lambda/n) \cdot \omega \cdot \mathrm{id} + F'$ the primitive part F' is traceless.

Corollary 4.B.14 *If (X,g) is a Kähler–Einstein manifold, then one of the three conditions holds true:*

i) $c_1(X) = 0$,
ii) $c_1(X)$ *is a Kähler class,*
iii) $-c_1(X)$ *is a Kähler class.* □

In other words, the first Chern class of the canonical bundle K_X of a compact Kähler–Einstein manifold is either trivial, negative, or positive.

If $c_1(X) = 0$, e.g. if the canonical bundle K_X is trivial, and g is a Kähler–Einstein metric then $\mathrm{Ric}(X,g) = 0$. Indeed, in this case the scalar factor λ is necessarily trivial and hence $\mathrm{Ric}(X,g) = \lambda \cdot \omega = 0$, i.e. the Kähler metric g is Ricci-flat.

Remark 4.B.15 Let us emphasize that there are two types of symmetries satisfied by the curvature of the Levi-Civita connection of a Kähler manifold, both stated in Proposition 4.A.11.

The first one allows to show that the primitive part of the curvature is traceless and hence $c_1(X,g) = (i/2\pi)\mathrm{tr}(F) = (\lambda/2\pi)\omega$ if g is a Kähler–Einstein metric. (Recall that $c_1(X,g) = (1/2\pi)\mathrm{Ric}(X,g)$ holds for any Kähler metric g.)

The second relation, which is ii) in Proposition 4.A.11, was used to prove the equivalence of the Einstein condition for the Kähler metric g and the Hermite–Einstein condition for the induced hermitian structure on $\mathcal{T}_X$ (Corollary 4.B.13).

Examples 4.B.16 In the following we will give one example of a Kähler–Einstein manifold in each of the three classes in Corollary 4.B.14.

i) **Projective space.** It turns out that the Fubini–Study metric, the only Kähler metric on $\mathbb{P}^n$ that has been introduced, is indeed Kähler–Einstein. By Exercise 4.4.7 we know that $c_1(\mathcal{O}(1)) = [\omega_{\rm FS}] \in H^2(\mathbb{P}^n, \mathbb{R})$. Thus, if the Fubini–Study metric $g_{\rm FS}$ is indeed Kähler–Einstein then the scalar factor can be computed by $c_1(\mathbb{P}^n) = (n+1) \cdot c_1(\mathcal{O}(1)) = \lambda \cdot [\omega_{\rm FS}]$, i.e. $\lambda = n+1$.

In order to see that $g_{\rm FS}$ is Kähler–Einstein, recall that the induced hermitian structure on $\det(\mathcal{T}_{\mathbb{P}^n}) \cong \mathcal{O}(n+1)$ is $h^{\otimes n+1}$, where h is the standard hermitian structure on $\mathcal{O}(1)$ determined by the choice of the basis $z_0, \ldots, z_n \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ (see Example 4.1.5). For the latter we have computed in Example 4.3.12 that the Chern connection ∇ satisfies $c_1(\mathcal{O}(1), \nabla) = \omega_{\rm FS}$. Hence, $c_1(\mathbb{P}^n, g_{\rm FS}) = (i/2\pi){\rm tr}(F_{\nabla_{\rm FS}}) = (i/2\pi)(n+1)\omega_{\rm FS}$.

Clearly, any other Fubini–Study metric obtained by applying a linear coordinate change is Kähler–Einstein as well.

ii) **Complex tori.** The holomorphic tangent bundle of a complex torus $X = \mathbb{C}^n/\Gamma$ is trivial. The Chern connection for any constant Kähler structure on X is flat. Thus, the Kähler–Einstein condition is satisfied with the choice of the scalar $\lambda = 0$.

Complex tori are trivial examples of Ricci-flat manifolds. Any other example is much harder to come by.

iii) **Ball quotients.** The standard Kähler structure $\omega = (i/2)\partial\bar\partial(1 - \|z\|^2)$ on the unit disc $D^n \subset \mathbb{C}^n$ is Kähler–Einstein with $\lambda < 0$. For simplicity we consider only the one-dimensional case. Then $\omega = (i/2)(1-|z|^2)^{-2} dz \wedge d\bar z$ and, hence, the hermitian metric is $h = (1+|z|^2)^{-2}$. The curvature of its Chern connection is thus given by $F = -\partial(h^{-1}\bar\partial h) = \frac{-2}{(1-|z|^2)^2} dz \wedge d\bar z$. Therefore, $i \cdot F = -4 \cdot \omega$.

This way, one obtains negative Kähler–Einstein structures on all ball quotients.

As the holomorphic tangent bundle of a Kähler–Einstein manifold is in particular Hermite–Einstein, any Kähler–Einstein manifold satisfies the Bogomolov–Lübke inequality 4.B.6. In fact, a stronger inequality can be proved by using the additional symmetries of the curvature of a Kähler manifold. As for the Bogomolov–Lübke inequality, there are algebraic and analytic proofs of this inequality. The first proof, using the Kähler–Einstein condition was given by Chen and Oguie [24]. An algebraic version of it was proved by Miyaoka. The inequality is usually called the *Miyaoka–Yau inequality.*

Proposition 4.B.17 *Let X be a Kähler–Einstein manifold of dimension n and let ω be a Kähler–Einstein form. Then*

$$\int_X \left(2(n+1)c_2(X) - nc_1^2(X)\right) \wedge \omega^{n-2} \geq 0. \tag{4.11}$$

Remark 4.B.18 It might be instructive to consider the Miyaoka–Yau inequality in the case of a compact surface. Here it says $3c_2(X) \geq c_1^2(X)$. Since $c_1^2(X) \geq 0$

for a Kähler–Einstein surface, this inequality is stronger than the Bogomolov–Lübke inequality. It is noteworthy that for the projective plane and a complex torus the inequality becomes an equality.

In fact, one can prove that equality in (4.11) for a Kähler–Enstein manifold X implies that the universal cover of X is isomorphic to $\mathbb{P}^n$, $\mathbb{C}^n$, or a ball. For a proof of this result see e.g. Tian's lecture notes [106].

In many examples it can easily be checked whether the canonical bundle is negative, trivial, or positive and whether the Miyaoka–Yau inequality is satisfied. In fact, often this can be done without even constructing any Kähler metric on X just by using an embedding of X in a projective space and pulling-back the Fubini–Study metric on $\mathcal{O}(1)$. But even when this necessary condition holds, we still don't know whether X admits a Kähler–Einstein structure and if how many.

The key result that is behind many others in this area is the following fundamental theorem of Calabi and Yau. Perhaps, it is worth emphasizing that this result works for arbitrary compact Kähler manifolds without any condition on the canonical bundle. It will also lead to the fundamental result that any form representing $c_1(X)$ is the Ricci curvature of a unique Kähler metric with given Kähler class (cf. Proposition 4.B.21).

Theorem 4.B.19 (Calabi–Yau) *Let (X, g_0) be a compact Kähler manifold of dimension n and let ω_0 be its Kähler form. For any real differentiable function f on X with*

$$\int_X e^f \cdot \omega_0^n = \int_X \omega_0^n$$

there exists a unique Kähler metric g with associated Kähler form ω such that

$$[\omega] = [\omega_0] \quad \text{and} \quad \omega^n = e^f \cdot \omega_0^n.$$

Proof. The proof of the existence is beyond the scope of these notes (see [61]), but for the uniqueness an easy argument, due to Calabi, goes as follows:

Suppose ω_1 and ω_2 are two Kähler forms with $\omega_1^n = \omega_2^n$. If they are cohomologous, there exists a real function f on M with $\omega_2 = \omega_1 + i\partial\bar{\partial}f$ (see Exercise 3.2.16). Hence, $0 = \omega_2^n - \omega_1^n = \gamma \wedge (\omega_2 - \omega_1) = \gamma \wedge (i\partial\bar{\partial}f)$ with $\gamma = \omega_2^{n-1} + \omega_2^{n-2} \wedge \omega_1 + \ldots + \omega_2 \wedge \omega_1^{n-2} + \omega_1^{n-1}$.

The form γ is a positive linear combination of positive forms $\omega_2^k \wedge \omega_1^{n-1-k}$ and, hence, itself positive. The equation $0 = \gamma \wedge \partial\bar{\partial}f$ together with the maximum principle imply that f is constant and hence $\omega_1 = \omega_2$.

For the convenience of the reader we spell out how the maximum principle is applied here. Since M is compact, there exists a point $x \in M$ where f attains its maximum. For simplicity we will assume that the Hessian of f in x is negative definite. (If not one has to perturb by a quadratic function as in the proof of the maximum principle for harmonic functions.)

Now, let us choose local coordinates $(z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$ around $x \in M$ such that ω_1 and ω_2 are simultaneously diagonalized in $x \in M$. We may assume $\omega_1(x) = (i/2)\sum dz_i \wedge d\bar{z}_i = \sum dx_i \wedge y_i$ and $\omega_2(x) = (i/2)\sum \lambda_i dz_i \wedge d\bar{z}_i = \sum \lambda_i dx_i \wedge dy_i$. Since ω_2 is positive, $\lambda_i > 0$. Thus, $0 = \gamma \wedge \partial\bar{\partial} f$ in $x \in M$ yields an equation of the form $0 = \sum \mu_i \left(\frac{\partial^2 f}{\partial x_i^2}(x) + \frac{\partial^2 f}{\partial y_i^2}(x) \right)$, where the coefficients μ_i are positive linear combinations of terms of the form $\lambda_{i_1} \ldots \lambda_{i_k}$. But this contradicts the fact that the Hessian of f is negative definite. $\square$

The theorem can be rephrased as follows: If X is a compact Kähler manifold with a given volume form vol which is compatible with the natural orientation, then there exists a unique Kähler metric g on X with $\omega^n = \mathrm{vol}$ and prescribed $[\omega] \in H^2(X, \mathbb{R})$.

Yet another way to say this uses the Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ of all Kähler classes (see Definition 3.2.14) and the set $\widetilde{\mathcal{K}}_X$ of all Kähler forms ω with $\omega^n = \lambda \cdot \mathrm{vol}$ for some $\lambda \in \mathbb{R}_{>0}$. Then the natural map that projects a closed form to its cohomology class induces the following diagram:

$$\begin{array}{ccc} \mathcal{A}^{1,1}(X)_{\mathrm{cl}} & \twoheadrightarrow & H^{1,1}(X) \\ \uparrow & & \uparrow \\ \widetilde{\mathcal{K}}_X & \xrightarrow{\ \sim\ } & \mathcal{K}_X \end{array}$$

Lemma 4.B.20 *Let ω and ω' be two Kähler forms on a compact Kähler manifold. If $\omega^n = e^f \cdot \omega'^n$ for some real function f, then* $\mathrm{Ric}(X, \omega) = \mathrm{Ric}(X, \omega') + i\bar{\partial}\partial f$.

Proof. The two Kähler forms correspond to Kähler metrics g and g', respectively, which are locally given by matrices (g_{ij}) and $\left(g'_{ij}\right)$. The induced volume forms are thus given by the functions $\det(g_{ij})$ respectively $\det\left(g'_{ij}\right)$. Hence, $\det(g_{ij}) = e^f \cdot \det\left(g'_{ij}\right)$.

On the other hand, the two metrics induce hermitian structures h respectively h' on $\mathcal{T}_X$ and thus on $\det(\mathcal{T}_X)$. The curvature forms of the latter are $\bar{\partial}\partial \log(\det(h))$ and $\bar{\partial}\partial \log(\det(h'))$, respectively. Since $\det(h)$ and $\det(h')$ differ again by the scalar factor e^f, this yields $\mathrm{Ric}(X, \omega) = \mathrm{Ric}(X, \omega') + i\bar{\partial}\partial \log(e^f)$, as the Ricci curvature is the curvature of the induced connection on $\det(\mathcal{T}_X)$ (see Proposition 4.A.11). $\square$

This lemma together with the Calabi–Yau theorem 4.B.19 yields:

Proposition 4.B.21 *Let X be a compact Kähler manifold and let $\alpha \in \mathcal{K}_X$ be a Kähler class. Assume β is a closed real* $(1,1)$*-form with* $[\beta] = \mathrm{c}_1(X)$. *Then there exists a unique Kähler structure g on X such that*

i) $\mathrm{Ric}(X, g) = (2\pi) \cdot \beta$ *and*

ii) $[\omega] = \alpha$ *for the Kähler form ω of the Kähler metric g.*

Proof. Let ω_0 be an arbitrary Kähler form on X. Then $\mathrm{Ric}(X, \omega_0)$ represents $(2\pi) \cdot \mathrm{c}_1(X)$ and hence is cohomologous to $(2\pi) \cdot \beta$. Thus, since X is Kähler, one finds a real function f with $(2\pi) \cdot \beta = \mathrm{Ric}(X, \omega_0) + i\bar{\partial}\partial f$.

By the Calabi–Yau theorem 4.B.19 there exists a unique Kähler metric with associated Kähler form ω such that $[\omega] = \alpha$ and $\omega^n = e^{f+c} \cdot \omega_0^n$, where the constant c is chosen such that $\int_X \alpha^n = e^c \int_X e^f \cdot \omega_0^n$.

Using Lemma 4.B.20, we find that the Ricci curvature of g is given by

$$\mathrm{Ric}(X, \omega) = \mathrm{Ric}(X, \omega_0) + i\bar{\partial}\partial f = (2\pi) \cdot \beta.$$

Using again the lemma and the uniqueness part of the Calabi–Yau theorem, we find that ω is unique. □

Corollary 4.B.22 *If X is a compact Kähler manifold with $\mathrm{c}_1(X) = 0$ then there exists a unique Ricci-flat Kähler structure g on X with given Kähler class $[\omega]$. The volume form up to a scalar does not depend on the chosen Ricci-flat metric or the Kähler class $[\omega]$.*

Proof. Choosing $\beta = 0$ in Proposition 4.B.21 yields a unique Kähler structure in each Kähler class with vanishing Ricci curvature. The uniqueness of the volume is easily deduced from Lemma 4.B.20. □

Thus, any compact Kähler manifold X with $\mathrm{c}_1(X) = 0$ is Ricci-flat. Clearly, any compact Kähler manifold with trivial canonical bundle $K_X \cong \mathcal{O}_X$ has $\mathrm{c}_1(X) = 0$. For this type of manifold, the Ricci-flatness of a Kähler form can be determined by the following criterion

Corollary 4.B.23 *Let X be a compact Kähler manifold of dimension n with trivial canonical bundle K_X. Fix a* holomorphic volume form*, i.e. a trivializing section $\Omega \in H^0(X, K_X)$. Then, a Kähler form ω is Ricci-flat if and only if*

$$\omega^n = \lambda \cdot (\Omega \wedge \overline{\Omega})$$

for some constant $\lambda \in \mathbb{C}^*$.

Proof. Suppose $\omega^n = \lambda \cdot (\Omega \wedge \overline{\Omega})$. Since ω is parallel, i.e. $\nabla(\omega) = 0$ for the Levi-Civita connection ∇ (see Proposition 4.A.8), also $\nabla(\omega^n) = 0$ and hence $\nabla(\Omega \wedge \overline{\Omega}) = 0$.

On the other hand, the Levi-Civita connection on a Kähler manifold is compatible with the complex structure. Since $\bar{\partial}\Omega = 0$, this shows that $\nabla(\Omega) = \alpha \otimes \Omega$ with $\alpha \in \mathcal{A}^{1,0}(X)$. Therefore, using the bidegree decomposition the equality $0 = \nabla(\Omega \wedge \overline{\Omega}) = \nabla(\Omega) \wedge \overline{\Omega} + \Omega \wedge \nabla(\overline{\Omega}) = (\alpha + \bar{\alpha})(\Omega \wedge \overline{\Omega})$ implies $\alpha = 0$. Thus, Ω is a parallel section of K_X and, in particular, the curvature of the Levi-Civita connection on K_X, which is the Ricci curvature, vanishes. Thus, ω is Ricci-flat.

Conversely, if a Ricci-flat Kähler form ω is given there exists a unique Kähler form ω' in the same cohomology class with $\omega'^n = \lambda \cdot (\Omega \wedge \overline{\Omega})$ for some

$\lambda \in \mathbb{C}^*$. By what has been said before, this yields that also ω' is Ricci-flat and the uniqueness of the Ricci-flat representative of a Kähler class proves $\omega = \omega'$.

Clearly, the constant λ is actually real and positive. □

The other two cases for which Kähler–Einstein metrics could *a priori* exist are much harder. If $\mathrm{c}_1(X)$ is negative, i.e. $-\mathrm{c}_1(X)$ can be represented by a Kähler form, the question is completely settled by the following theorem, due to Aubin and Yau.

Theorem 4.B.24 (Aubin, Yau) *Let X be a compact Kähler manifold such that $\mathrm{c}_1(X)$ is negative. Then X admits a unique Kähler–Einstein metric up to scalar factors.*

Proof. The uniqueness is again rather elementary. See [5, 12] for more comments. □

Thus, Theorem 4.B.24 and Corollary 4.B.22 can be seen as the non-linear analogue of the Donaldson–Uhlenbeck–Yau description of Hermite–Einstein metrics, but clearly the situation here is more subtle. E.g. for $\mathrm{c}_1(X)$ positive the situation is, for the time being, not fully understood. One knows that in this case a Kähler–Einstein metric need not exist. E.g. the Fubini–Study metric on $\mathbb{P}^2$ is Kähler–Einstein, but the blow-up of $\mathbb{P}^2$ in two points for which K_X^* is still ample does not admit any Kähler–Einstein metric. In order to ensure the existence of a Kähler–Einstein metric, a certain stability condition on X has to be added . There has been done a lot of work on this problem recently. See the survey articles [19] or [105].

Exercises

4.B.1 Verify that the only stable vector bundles on $\mathbb{P}^1$ are line bundles. Find a semi-stable vector bundle of rank two on an elliptic curve. (A semi-stable bundle satisfies only the weaker stability condition $\mu(F) \leq \mu(E)$ for all sub-bundles $F \subset E$.)

4.B.2 Let E_1, E_2 be holomorphic vector bundles endowed with Hermite–Einstein metrics h_1 and h_2, respectively. Show that the naturally induced metrics on $E_1 \otimes E_2$, $\mathrm{Hom}(E_1, E_2)$, and E_i^* are all Hermite–Einstein. If $\mu(E_1) = \mu(E_2)$, then also $h_1 \oplus h_2$ is Hermite–Einstein on $E_1 \oplus E_2$.

4.B.3 Let (E, h) be an hermitian holomorphic vector bundle on a compact Kähler manifold such that $i \cdot \Lambda_\omega F_\nabla = \lambda \cdot \mathrm{id}_E$ for the Chern connection ∇ and a function λ. Show that by changing h to $e^f \cdot h$ for some real function f, one finds an hermitian metric on E the Chern connection of which satisfies the Hermite–Einstein condition with constant factor λ.

4.B.4 Let E be a holomorphic vector bundle on a compact Kähler manifold X with a chosen Kähler structure ω. Without using Theorem 4.B.9, show that if E admits an Hermite–Einstein metric with respect to ω then E admits an Hermite–Einstein metric with respect to any other Kähler form ω' with $[\omega] = \lambda[\omega']$ for any $\lambda \in \mathbb{R}_{>0}$.

(This corresponds to the easy observation that stability only depends on the Kähler class (and not on the particular Kähler form) and that scaling by a constant does not affect the stability condition.)

4.B.5 Give an algebraic argument for the stability of the tangent bundle of $\mathbb{P}^n$.

Comments: - The Hermite–Einstein condition for holomorphic vector bundles is discussed in detail in [78].

- For the algebraic theory of stable vector bundles and their moduli see [70] and the references therein.

Springer

http://www.springer.com/978-3-540-21290-4

Complex Geometry
An Introduction
Huybrechts, D.
2005, XII, 309 p., Softcover
ISBN: 978-3-540-21290-4