Preface

The geometric approach to the algebraic theory of quadratic forms is the study of projective quadrics over arbitrary fields. Function fields of quadrics were a basic ingredient already in the proof of the Arason–Pfister Hauptsatz of 1971 (or even in Pfister's 1965 construction of fields with prescribed level); they are central in the investigation of deep properties of quadratic forms, such as their splitting pattern, but also in the construction of fields which exhibit particular properties, such as a given u-invariant. Recently, finer geometric tools have been brought to bear on problems from the algebraic theory of quadratic forms: results on Chow groups of quadrics led to an efficient use of motives, and ultimately to Voevodsky's proof of the Milnor conjecture.

The goal of the June 2000 summer school at Université d'Artois in Lens (France), organized locally by J. Burési, N. Karpenko and P. Mammone, was to survey three aspects of the algebraic theory of quadratic forms where geometric methods had led to spectacular advances. Bruno Kahn was invited to talk on the unramified cohomology of quadrics, Alexander Vishik on motives of quadrics, and Oleg Izhboldin on his construction of fields whose u-invariant is 9. However, Izhboldin passed away unexpectedly on April 17, 2000. His work was surveyed by Karpenko, who had collaborated with Izhboldin on several papers.

The closely related texts collected in this volume were written from somewhat different perspectives. The reader will find below:

1. The notes from the lectures of B. Kahn [K], A. Vishik [V] and N. Karpenko [K1] prepared and updated by the authors. Additional material has been included, in particular in Vishik's notes.
2. Two papers left unfinished by O. Izhboldin, and edited by N. Karpenko. The first paper [I1] was essentially complete and formed the basis for the first part of Karpenko's lectures. The second [I2] is only a sketch, listing properties and examples that Izhboldin intended to develop in subsequent work.

3. A paper by N. Karpenko [K2] which provides complete proofs for the statements that Izhboldin listed in [I2].

To give a more precise overview, we introduce some notation. Let F be an arbitrary field of characteristic different from 2. To every quadratic form[1] q in at least two variables over F corresponds the projective quadric Q with equation $q = 0$ (which has an F-rational point if and only if q is isotropic). The quadric Q is a smooth variety if q is nonsingular (which we always assume in the sequel); its dimension is $\dim Q = \dim q - 2$, and it is irreducible if q is not the hyperbolic plane $\mathbb{H}$. We may then consider its function field $F(Q)$, which is also referred to as the *function field of q* and denoted $F(q)$.

The field extension $F(Q)/F$ is of particular interest. Much insight into quadratic forms could be obtained if we knew which quadratic forms over F become isotropic over $F(Q)$. This question can be readily rephrased into geometric terms: a quadratic form q' over F becomes isotropic over $F(q)$ if and only if there is a rational map $Q-->Q'$ between the corresponding quadrics. If there are rational maps in both directions $Q \rightleftharpoons Q'$, the quadrics are stably birationally equivalent, and the quadratic forms q and q' are also called *stably birationally equivalent*. By the preceding observation, this relation, denoted $q \overset{st}{\sim} q'$, holds if and only if the forms $q_{F(q')}$ and $q'_{F(q)}$ are both isotropic.

A very useful geometric construction is to view the quadric Q as an object in a category where the maps are given by Chow correspondences. We thus get the (Chow-) motive $M(Q)$ of the quadric, whose structure carries a lot of information on the form q. For example, Vishik has shown[2] that the motives $M(Q), M(Q')$ associated with quadratic forms q, q' are isomorphic if and only if every field extension E of F produces the same amount of splitting in q and q', i.e., the quadratic forms q_E and q'_E have the same *Witt index*, a notion which is spelled out next.

Recall from [Sch, Corollary 5.11 of Chap. 1] that every quadratic form q has a (Witt) decomposition into an orthogonal sum of an anisotropic quadratic form q_{an}, called an *anisotropic kernel* of q, and a certain number of hyperbolic planes $\mathbb{H}$,

$$q \simeq q_{\mathrm{an}} \perp \underbrace{\mathbb{H} \perp \ldots \perp \mathbb{H}}_{i_W(q)}.$$

The number $i_W(q)$ of hyperbolic planes in this decomposition (which is unique up to isomorphism) is called the *Witt index* of q. Even if q is anisotropic (i.e., $i_W(q) = 0$), it obviously becomes isotropic over $F(q)$, and we have a Witt decomposition over $F(q)$,

$$q_{F(q)} \simeq q_1 \perp \mathbb{H} \perp \ldots \perp \mathbb{H}$$

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[1] With the usual abuse of terminology, a quadratic form is sometimes viewed as a quadratic polynomial, sometimes as a quadratic map on a vector space or a quadratic space.

[2] See [I2, Sect. 1].

where q_1 is an anisotropic form over $F(q)$. Letting $F_1 = F(q)$, we may iterate this construction. The process terminates in a finite number of steps since $\dim q > \dim q_1 > \cdots$. We thus obtain the *generic splitting tower* of q, first constructed by M. Knebusch [Kn],

$$F \subset F_1 \subset \cdots \subset F_h.$$

Clearly, $0 < i_W(q_{F_1}) < i_W(q_{F_2}) < \cdots < i_W(q_{F_h})$. It turns out that for any field extension E/F, the Witt index $i_W(q_E)$ is equal to one of the indices $i_W(q_{F_j})$. The *splitting pattern* of q is the set

$$\{i_W(q_E) \mid E \text{ a field extension of } F\} = \{i_W(q_{F_1}), \ldots, i_W(q_{F_h})\}.$$

Variants of this notion appear in [V] and [I1]: Vishik calls *(incremental) splitting pattern*[3] of q the sequence

$$\mathbf{i}(q) = \big(i_1(q), \ldots, i_h(q)\big)$$

defined by $i_1(q) = i_W(q_{F_1})$ and $i_j(q) = i_W(q_{F_j}) - i_W(q_{F_{j-1}})$ for $j > 1$. The integer $i_j(q)$ indeed measures the Witt index increment resulting from the field extension F_j/F_{j-1}; it is called a *higher Witt index* of q. On the other hand, Izhboldin concentrates on the dimension of the anisotropic kernels and sets

$$\mathrm{Dim}(q) = \{\dim(q_E)_{\mathrm{an}} \mid E \text{ a field extension of } F\}.$$

By counting dimensions in the Witt decomposition of q_E, we obtain

$$\dim q = \dim q_E = \dim(q_E)_{\mathrm{an}} + 2i_W(q_E),$$

hence the set $\mathrm{Dim}(q)$ and the splitting pattern of q carry the same information.

Vishik's contribution [V] to this volume is intended as a general introduction to the state-of-the-art in the theory of motives of quadrics. After setting up the basic principles, he proves the main structure theorems on motives of quadrics. The study of direct sum decompositions of these motives is a powerful tool for investigating the dimensions of anisotropic forms in the powers of the fundamental ideal of the Witt ring, the stable equivalence of quadrics and splitting patterns of quadratic forms. This last application is particularly developed in the last section of [V], where all the possible splitting patterns of odd-dimensional forms of dimension at most 21 and of even-dimensional forms of dimension at most 12 are determined.

The papers of Karpenko [K1, K2] and Izhboldin [I1, I2] are closely intertwined. They also rely less on motives and more on elementary arguments. As mentioned above, [K1] is an exposition of Izhboldin's results in [I1] and on

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[3] No confusion should arise since Vishik's splitting patterns are sequences, whereas the "usual" splitting patterns are sets.

the u-invariant. Recall from [Sch, Sect. 16 of Chap. 2] that the u-invariant of a field F is

$$u(F) = \sup\{\dim q \mid q \text{ anisotropic quadratic form over } F\}.$$

Quadratically closed fields have u-invariant 1, but no other field with odd u-invariant was known before Izhboldin's construction of a field with u-invariant 9. In the second part of [K1], Karpenko discusses the strategy of this construction and provides alternative proofs for the main results on which it is based. In the first part, he gives a simple proof of a theorem of Izhboldin on the first Witt index $i_1(q)$ of quadratic forms of dimension $2^n + 3$. Izhboldin's original proof is given in [I1], while [I2] classifies the pairs of quadratic forms of dimension at most 9 which are stably equivalent and lists without proofs assorted isotropy criteria for quadratic forms over function fields of quadrics. The proofs of Izhboldin's claims are given in [K2], which also contains an extensive discussion of correspondences on odd-dimensional quadrics.

In [K], Kahn studies the field extension $F(Q)/F$ from a different angle. The induced scalar extension map in Galois cohomology with coefficients $\mu_2 = \{\pm 1\}$, called the *restriction* map

$$\text{Res}\colon H^n(F, \mu_2) \to H^n(F(Q), \mu_2)$$

is a typical case of the maps he considers. For every closed point x of Q of codimension 1, the image of this map lies in the kernel of the residue map

$$\partial_x\colon H^n(F(Q), \mu_2) \to H^{n-1}(F(x), \mu_2).$$

Therefore, we may restrict the target of Res to the *unramified cohomology group*

$$H^n_{\text{nr}}(F(Q), \mu_2) = \bigcap_{x \in Q^{(1)}} \text{Ker}\,\partial_x.$$

The kernel and cokernel of the restriction map $H^n(F, \mu_2) \to H^n_{\text{nr}}(F(Q), \mu_2)$ were studied by Kahn–Rost–Sujatha [KRS] and Kahn–Sujatha [KS1, KS2] for $n \leq 4$. In his contribution to this volume, Kahn develops a vast generalization which applies to various cohomology theories besides Galois cohomology with μ_2 coefficients, and to arbitrary smooth projective varieties besides quadrics. If X is a smooth projective variety which is also geometrically cellular, there are two spectral sequences converging to the motivic cohomology of X. Results on the restriction map are obtained by comparing these two sequences, since one of them contains the unramified cohomology of X in its E_2-term. The unramified cohomology of quadrics occurs as a crucial ingredient in the other papers collected here, see [K1, Sect. 2.3], [K2, Lemma 7.5], [V, Lemmas 6.14 and 7.12].

The untimely death of Oleg Izhboldin was felt as a great loss by all the contributors to this volume, who decided to dedicate it to his memory. A

tribute to his work, written by his former thesis advisor Alexandr Merkurjev, and posted on the web site `www-math.univ-mlv.fr/~abakumov/oleg/`, is included as an appendix. We are grateful to A. Merkurjev and E. Abakumov for the permission to reproduce it.

References

[K]   Kahn, B.: Cohomologie non ramifiée des quadriques. This volume.

[KRS]   Kahn, B., Rost, M., Sujatha, R.: Unramified cohomology of quadrics, I. Amer. J. Math., **120**, 841–891 (1998)

[KS1]   Kahn, B., Sujatha, R.: Unramified cohomology of quadrics, II. Duke Math. J. **106**, 449–484 (2001)

[KS2]   Kahn, B., Sujatha, R.: Motivic cohomology and unramified cohomology of quadrics. J. Eur. Math. Soc. **2**, 145–177 (2000)

[K1]   Karpenko, N.A.: Motives and Chow groups of quadrics with application to the u-invariant (after Oleg Izhboldin). This volume.

[K2]   Karpenko, N.A.: Izhboldin's results on stably birational equivalence of quadrics. This volume.

[Kn]   Knebusch, M: Generic splitting of quadratic forms, I. Proc. London Math. Soc., **33**, 65–93 (1976)

[I1]   Izhboldin, O.T.: Virtual Pfister neighbors and first Witt index. This volume.

[I2]   Izhboldin, O.T.: Some new results concerning isotropy of low-dimensional forms. (List of examples and results (without proofs)). This volume.

[Sch]   Scharlau, W.: Quadratic and Hermitian Forms. Springer, Berlin Heidelberg New York Tokyo (1985)

[V]   Vishik, A.: Motives of quadrics with applications to the theory of quadratic forms. This volume.

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