
Contents

Part I Problems and Tricks

1	Elementary Number Theory	9
1.1	Problems About Primes. Divisibility and Primality	9
1.1.1	Arithmetical Notation	9
1.1.2	Primes and composite numbers	10
1.1.3	The Factorization Theorem and the Euclidean Algorithm	12
1.1.4	Calculations with Residue Classes	13
1.1.5	The Quadratic Reciprocity Law and Its Use	15
1.1.6	The Distribution of Primes	17
1.2	Diophantine Equations of Degree One and Two	22
1.2.1	The Equation $ax + by = c$	22
1.2.2	Linear Diophantine Systems	22
1.2.3	Equations of Degree Two	24
1.2.4	The Minkowski–Hasse Principle for Quadratic Forms	26
1.2.5	Pell’s Equation	28
1.2.6	Representation of Integers and Quadratic Forms by Quadratic Forms	29
1.2.7	Analytic Methods	33
1.2.8	Equivalence of Binary Quadratic Forms	35
1.3	Cubic Diophantine Equations	38
1.3.1	The Problem of the Existence of a Solution	38
1.3.2	Addition of Points on a Cubic Curve	38
1.3.3	The Structure of the Group of Rational Points of a Non–Singular Cubic Curve	40
1.3.4	Cubic Congruences Modulo a Prime	47
1.4	Approximations and Continued Fractions	50
1.4.1	Best Approximations to Irrational Numbers	50
1.4.2	Farey Series	50
1.4.3	Continued Fractions	51

1.4.4	SL_2 -Equivalence	53
1.4.5	Periodic Continued Fractions and Pell's Equation	53
1.5	Diophantine Approximation and the Irrationality	55
1.5.1	Ideas in the Proof that $\zeta(3)$ is Irrational	55
1.5.2	The Measure of Irrationality of a Number	56
1.5.3	The Thue–Siegel–Roth Theorem, Transcendental Numbers, and Diophantine Equations	57
1.5.4	Proofs of the Identities (1.5.1) and (1.5.2)	58
1.5.5	The Recurrent Sequences a_n and b_n	59
1.5.6	Transcendental Numbers and the Seventh Hilbert Problem	61
1.5.7	Work of Yu.V. Nesterenko on e^π , [Nes99]	61
2	Some Applications of Elementary Number Theory	63
2.1	Factorization and Public Key Cryptosystems	63
2.1.1	Factorization is Time-Consuming	63
2.1.2	One-Way Functions and Public Key Encryption	63
2.1.3	A Public Key Cryptosystem	64
2.1.4	Statistics and Mass Production of Primes	66
2.1.5	Probabilistic Primality Tests	66
2.1.6	The Discrete Logarithm Problem and The Diffie-Hellman Key Exchange Protocol	67
2.1.7	Computing of the Discrete Logarithm on Elliptic Curves over Finite Fields (ECDLP)	68
2.2	Deterministic Primality Tests	69
2.2.1	Adleman–Pomerance–Rumely Primality Test: Basic Ideas	69
2.2.2	Gauss Sums and Their Use in Primality Testing	71
2.2.3	Detailed Description of the Primality Test	75
2.2.4	Primes is in P	78
2.2.5	The algorithm of M. Agrawal, N. Kayal and N. Saxena	81
2.2.6	Practical and Theoretical Primality Proving. The ECPP (Elliptic Curve Primality Proving by F.Morain, see [AtMo93b])	81
2.2.7	Primes in Arithmetic Progression	82
2.3	Factorization of Large Integers	84
2.3.1	Comparative Difficulty of Primality Testing and Factorization	84
2.3.2	Factorization and Quadratic Forms	84
2.3.3	The Probabilistic Algorithm CLASNO	85
2.3.4	The Continued Fractions Method (CFRAC) and Real Quadratic Fields	87
2.3.5	The Use of Elliptic Curves	90

Part II Ideas and Theories

3	Induction and Recursion	95
3.1	Elementary Number Theory From the Point of View of Logic .	95
3.1.1	Elementary Number Theory	95
3.1.2	Logic	96
3.2	Diophantine Sets	98
3.2.1	Enumerability and Diophantine Sets	98
3.2.2	Diophantineness of enumerable sets	98
3.2.3	First properties of Diophantine sets	98
3.2.4	Diophantineness and Pell's Equation	99
3.2.5	The Graph of the Exponent is Diophantine	100
3.2.6	Diophantineness and Binomial coefficients	100
3.2.7	Binomial coefficients as remainders	101
3.2.8	Diophantineness of the Factorial	101
3.2.9	Factorial and Euclidean Division	101
3.2.10	Supplementary Results	102
3.3	Partially Recursive Functions and Enumerable Sets	103
3.3.1	Partial Functions and Computable Functions	103
3.3.2	The Simple Functions	103
3.3.3	Elementary Operations on Partial functions.....	103
3.3.4	Partially Recursive Description of a Function	104
3.3.5	Other Recursive Functions	106
3.3.6	Further Properties of Recursive Functions	108
3.3.7	Link with Level Sets	108
3.3.8	Link with Projections of Level Sets	108
3.3.9	Matiyasevich's Theorem	109
3.3.10	The existence of certain bijections	109
3.3.11	Operations on primitively enumerable sets	111
3.3.12	Gödel's function.....	111
3.3.13	Discussion of the Properties of Enumerable Sets	112
3.4	Diophantineness of a Set and algorithmic Undecidability.....	113
3.4.1	Algorithmic undecidability and unsolvability	113
3.4.2	Sketch Proof of the Matiyasevich Theorem	113
4	Arithmetic of algebraic numbers	115
4.1	Algebraic Numbers: Their Realizations and Geometry	115
4.1.1	Adjoining Roots of Polynomials	115
4.1.2	Galois Extensions and Frobenius Elements.....	117
4.1.3	Tensor Products of Fields and Geometric Realizations of Algebraic Numbers	119
4.1.4	Units, the Logarithmic Map, and the Regulator	121
4.1.5	Lattice Points in a Convex Body	123

4.1.6	Deduction of Dirichlet's Theorem From Minkowski's Lemma	125
4.2	Decomposition of Prime Ideals, Dedekind Domains, and Valuations	126
4.2.1	Prime Ideals and the Unique Factorization Property . . .	126
4.2.2	Finiteness of the Class Number	128
4.2.3	Decomposition of Prime Ideals in Extensions	129
4.2.4	Decomposition of primes in cyclotomic fields	131
4.2.5	Prime Ideals, Valuations and Absolute Values	132
4.3	Local and Global Methods	134
4.3.1	p -adic Numbers	134
4.3.2	Applications of p -adic Numbers to Solving Congruences	138
4.3.3	The Hilbert Symbol	139
4.3.4	Algebraic Extensions of \mathbb{Q}_p , and the Tate Field	142
4.3.5	Normalized Absolute Values	143
4.3.6	Places of Number Fields and the Product Formula	145
4.3.7	Adeles and Ideles	146
	The Ring of Adeles.	146
	The Idele Group	149
4.3.8	The Geometry of Adeles and Ideles	149
4.4	Class Field Theory	155
4.4.1	Abelian Extensions of the Field of Rational Numbers . .	155
4.4.2	Frobenius Automorphisms of Number Fields and Artin's Reciprocity Map	157
4.4.3	The Chebotarev Density Theorem	159
4.4.4	The Decomposition Law and the Artin Reciprocity Map	159
4.4.5	The Kernel of the Reciprocity Map	160
4.4.6	The Artin Symbol	161
4.4.7	Global Properties of the Artin Symbol	162
4.4.8	A Link Between the Artin Symbol and Local Symbols . .	163
4.4.9	Properties of the Local Symbol	164
4.4.10	An Explicit Construction of Abelian Extensions of a Local Field, and a Calculation of the Local Symbol	165
4.4.11	Abelian Extensions of Number Fields	168
4.5	Galois Group in Arithetical Problems	172
4.5.1	Dividing a circle into n equal parts	172
4.5.2	Kummer Extensions and the Power Residue Symbol . . .	175
4.5.3	Galois Cohomology	178
4.5.4	A Cohomological Definition of the Local Symbol	182
4.5.5	The Brauer Group, the Reciprocity Law and the Minkowski-Hasse Principle	184

5 Arithmetic of algebraic varieties 191

5.1 Arithmetic Varieties and Basic Notions of Algebraic Geometry 191

5.1.1 Equations and Rings 191

5.1.2 The set of solutions of a system 191

5.1.3 Example: The Language of Congruences 192

5.1.4 Equivalence of Systems of Equations 192

5.1.5 Solutions as K -algebra Homomorphisms 192

5.1.6 The Spectrum of A Ring 193

5.1.7 Regular Functions 193

5.1.8 A Topology on $\text{Spec}(A)$ 193

5.1.9 Schemes 196

5.1.10 Ring-Valued Points of Schemes 197

5.1.11 Solutions to Equations and Points of Schemes 198

5.1.12 Chevalley’s Theorem 199

5.1.13 Some Geometric Notions 199

5.2 Geometric Notions in the Study of Diophantine equations 202

5.2.1 Basic Questions 202

5.2.2 Geometric classification 203

5.2.3 Existence of Rational Points and Obstructions to the
Hasse Principle 204

5.2.4 Finite and Infinite Sets of Solutions 206

5.2.5 Number of points of bounded height 208

5.2.6 Height and Arakelov Geometry 211

5.3 Elliptic curves, Abelian Varieties, and Linear Groups 213

5.3.1 Algebraic Curves and Riemann Surfaces 213

5.3.2 Elliptic Curves 213

5.3.3 Tate Curve and Its Points of Finite Order 219

5.3.4 The Mordell – Weil Theorem and Galois Cohomology 221

5.3.5 Abelian Varieties and Jacobians 226

5.3.6 The Jacobian of an Algebraic Curve 228

5.3.7 Siegel’s Formula and Tamagawa Measure 231

5.4 Diophantine Equations and Galois Representations 238

5.4.1 The Tate Module of an Elliptic Curve 238

5.4.2 The Theory of Complex Multiplication 240

5.4.3 Characters of l -adic Representations 242

5.4.4 Representations in Positive Characteristic 243

5.4.5 The Tate Module of a Number Field 244

5.5 The Theorem of Faltings and Finiteness Problems in
Diophantine Geometry 247

5.5.1 Reduction of the Mordell Conjecture to the finiteness
Conjecture 247

5.5.2 The Theorem of Shafarevich on Finiteness for Elliptic
Curves 249

5.5.3 Passage to Abelian varieties 250

5.5.4 Finiteness Problems and Tate’s Conjecture 252

5.5.5	Reduction of the conjectures of Tate to the finiteness properties for isogenies	253
5.5.6	The Faltings–Arakelov Height	255
5.5.7	Heights under isogenies and Conjecture T	257
6	Zeta Functions and Modular Forms	261
6.1	Zeta Functions of Arithmetic Schemes	261
6.1.1	Zeta Functions of Arithmetic Schemes	261
6.1.2	Analytic Continuation of the Zeta Functions	263
6.1.3	Schemes over Finite Fields and Deligne’s Theorem	263
6.1.4	Zeta Functions and Exponential Sums	267
6.2	L -Functions, the Theory of Tate and Explicit Formulae	272
6.2.1	L -Functions of Rational Galois Representations	272
6.2.2	The Formalism of Artin	274
6.2.3	Example: The Dedekind Zeta Function	276
6.2.4	Hecke Characters and the Theory of Tate	278
6.2.5	Explicit Formulae	285
6.2.6	The Weil Group and its Representations	288
6.2.7	Zeta Functions, L -Functions and Motives	290
6.3	Modular Forms and Euler Products	296
6.3.1	A Link Between Algebraic Varieties and L -Functions.	296
6.3.2	Classical modular forms	296
6.3.3	Application: Tate Curve and Semistable Elliptic Curves	299
6.3.4	Analytic families of elliptic curves and congruence subgroups	301
6.3.5	Modular forms for congruence subgroups	302
6.3.6	Hecke Theory	304
6.3.7	Primitive Forms	310
6.3.8	Weil’s Inverse Theorem	312
6.4	Modular Forms and Galois Representations	317
6.4.1	Ramanujan’s congruence and Galois Representations.	317
6.4.2	A Link with Eichler–Shimura’s Construction	319
6.4.3	The Shimura–Taniyama–Weil Conjecture	320
6.4.4	The Conjecture of Birch and Swinnerton–Dyer	321
6.4.5	The Artin Conjecture and Cusp Forms	327
	The Artin conductor	329
6.4.6	Modular Representations over Finite Fields	330
6.5	Automorphic Forms and The Langlands Program	332
6.5.1	A Relation Between Classical Modular Forms and Representation Theory	332
6.5.2	Automorphic L -Functions	335
	Further analytic properties of automorphic L -functions.	338
6.5.3	The Langlands Functoriality Principle	338
6.5.4	Automorphic Forms and Langlands Conjectures	339

7 Fermat’s Last Theorem and Families of Modular Forms 341

7.1 Shimura–Taniyama–Weil Conjecture and Reciprocity Laws . . . 341

7.1.1 Problem of Pierre de Fermat (1601–1665) 341

7.1.2 G.Lamé’s Mistake 342

7.1.3 A short overview of Wiles’ Marvelous Proof 343

7.1.4 The STW Conjecture 344

7.1.5 A connection with the Quadratic Reciprocity Law 345

7.1.6 A complete proof of the STW conjecture 345

7.1.7 Modularity of semistable elliptic curves 348

7.1.8 Structure of the proof of theorem 7.13 (Semistable STW Conjecture) 349

7.2 Theorem of Langlands–Tunnell and Modularity Modulo 3 352

7.2.1 Galois representations: preparation 352

7.2.2 Modularity modulo p 354

7.2.3 Passage from cusp forms of weight one to cusp forms of weight two 355

7.2.4 Preliminary review of the stages of the proof of Theorem 7.13 on modularity 356

7.3 Modularity of Galois representations and Universal Deformation Rings 357

7.3.1 Galois Representations over local Noetherian algebras . . 357

7.3.2 Deformations of Galois Representations 357

7.3.3 Modular Galois representations 359

7.3.4 Admissible Deformations and Modular Deformations . . 361

7.3.5 Universal Deformation Rings 363

7.4 Wiles’ Main Theorem and Isomorphism Criteria for Local Rings 365

7.4.1 Strategy of the proof of the Main Theorem 7.33 365

7.4.2 Surjectivity of φ_Σ 365

7.4.3 Constructions of the universal deformation ring R_Σ . . . 367

7.4.4 A sketch of a construction of the universal modular deformation ring \mathbb{T}_Σ 368

7.4.5 Universality and the Chebotarev density theorem 369

7.4.6 Isomorphism Criteria for local rings 370

7.4.7 J -structures and the second criterion of isomorphism of local rings 371

7.5 Wiles’ Induction Step: Application of the Criteria and Galois Cohomology 373

7.5.1 Wiles’ induction step in the proof of Main Theorem 7.33 373

7.5.2 A formula relating $\#\Phi_{R_\Sigma}$ and $\#\Phi_{R_{\Sigma'}}$: preparation 374

7.5.3 The Selmer group and Φ_{R_Σ} 375

7.5.4 Infinitesimal deformations 375

7.5.5 Deformations of type \mathcal{D}_Σ 377

- 7.6 The Relative Invariant, the Main Inequality and The Minimal Case 382
 - 7.6.1 The Relative invariant 382
 - 7.6.2 The Main Inequality 383
 - 7.6.3 The Minimal Case 386
- 7.7 End of Wiles' Proof and Theorem on Absolute Irreducibility .. 388
 - 7.7.1 Theorem on Absolute Irreducibility 388
 - 7.7.2 From $p = 3$ to $p = 5$ 390
 - 7.7.3 Families of elliptic curves with fixed $\bar{\rho}_{5,E}$ 391
 - 7.7.4 The end of the proof 392
 - The most important insights. 393

Part III Analogies and Visions

III-0 Introductory survey to part III: motivations and description 397

- III.1 Analogies and differences between numbers and functions:
 - ∞ -point, Archimedean properties etc. 397
 - III.1.1 Cauchy residue formula and the product formula 397
 - III.1.2 Arithmetic varieties 398
 - III.1.3 Infinitesimal neighborhoods of fibers 398
- III.2 Arakelov geometry, fiber over ∞ , cycles, Green functions (d'après Gillet-Soulé) 399
 - III.2.1 Arithmetic Chow groups 400
 - III.2.2 Arithmetic intersection theory and arithmetic Riemann-Roch theorem 401
 - III.2.3 Geometric description of the closed fibers at infinity ... 402
- III.3 ζ -functions, local factors at ∞ , Serre's L -factors 404
 - III.3.1 Archimedean L -factors 405
 - III.3.2 Deninger's formulae 406
- III.4 A guess that the missing geometric objects are noncommutative spaces 407
 - III.4.1 Types and examples of noncommutative spaces, and how to work with them. Noncommutative geometry and arithmetic 407
 - Isomorphism of noncommutative spaces and Morita equivalence 409
 - The tools of noncommutative geometry 410
 - III.4.2 Generalities on spectral triples 411
 - III.4.3 Contents of Part III: description of parts of this program 412

8 Arakelov Geometry and Noncommutative Geometry 415

8.1 Schottky Uniformization and Arakelov Geometry 415

8.1.1 Motivations and the context of the work of
Consani-Marcolli 415

8.1.2 Analytic construction of degenerating curves over
complete local fields 416

8.1.3 Schottky groups and new perspectives in Arakelov
geometry 420
Schottky uniformization and Schottky groups 421
Fuchsian and Schottky uniformization. 424

8.1.4 Hyperbolic handlebodies 425
Geodesics in \mathfrak{X}_Γ 427

8.1.5 Arakelov geometry and hyperbolic geometry 427
Arakelov Green function 427
Cross ratio and geodesics 428
Differentials and Schottky uniformization 428
Green function and geodesics 430

8.2 Cohomological Constructions 431

8.2.1 Archimedean cohomology 431
Operators 433
 $SL(2, \mathbb{R})$ representations 434

8.2.2 Local factor and Archimedean cohomology 435

8.2.3 Cohomological constructions 436

8.2.4 Zeta function of the special fiber and Reidemeister
torsion 437

8.3 Spectral Triples, Dynamics and Zeta Functions 440

8.3.1 A dynamical theory at infinity 442

8.3.2 Homotopy quotion 443

8.3.3 Filtration 444

8.3.4 Hilbert space and grading 446

8.3.5 Cuntz–Krieger algebra 446
Spectral triples for Schottky groups 448

8.3.6 Arithmetic surfaces: homology and cohomology 449

8.3.7 Archimedean factors from dynamics 450

8.3.8 A Dynamical theory for Mumford curves 451
Genus two example 452

8.3.9 Cohomology of $\mathcal{W}(\Delta/\Gamma)_T$ 454

8.3.10 Spectral triples and Mumford curves 456

8.4 Reduction mod ∞ 458

8.4.1 Homotopy quotients and “reduction mod infinity” 458

8.4.2 Baum-Connes map 460

References 461

Index 503



<http://www.springer.com/978-3-540-20364-3>

Introduction to Modern Number Theory
Fundamental Problems, Ideas and Theories
Manin, Y.I.; Panchishkin, A.A.
2005, XVI, 514 p., Hardcover
ISBN: 978-3-540-20364-3