

**Exercise 4.8.4.** Interpret the constant  $c_{-1;2}$  and compute it.

[Answer:  $\frac{1}{72} \text{tr}(\Lambda^{-1})^6$ ].

In the examples above, the contribution  $c_{g;n}$  of all genus  $g$  gluings of  $2n$  copies of 3-stars is a monomial in  $\text{tr}(\Lambda^{-(2k+1)})$  with rational coefficients. In the next section, we will show that this is the case for all  $g$  and  $n$ .

## 4.9 A Sketch of Kontsevich's Proof of Witten's Conjecture

In this section we discuss very briefly exciting connections of the Kontsevich model with the one-matrix model on the one hand, and with the intersection theory model on the other hand.

### 4.9.1 The Generating Function for the Kontsevich Model

The sample calculations in the previous section show that the contribution  $c_{g;n}$  of all genus  $g$  gluings of  $2n$  copies of 3-stars is a monomial in  $\text{tr}(\Lambda^{-1})$ ,  $\text{tr}(\Lambda^{-3})$ ,  $\dots$ . It is more convenient, however, to use a slightly different normalized infinite sequence of independent variables:  $t_0 = -\text{tr}(\Lambda^{-1})$ ,  $t_1 = -1!! \text{tr}(\Lambda^{-3})$ ,  $t_2 = -3!! \text{tr}(\Lambda^{-5})$ ,  $\dots$ ,  $t_i = -(2i-1)!! \text{tr}(\Lambda^{-2i-1})$ ,  $\dots$ .

**Theorem 4.9.1 ([178]).** *The integral (4.6) is a formal power series in the variables  $t_0, t_1, \dots$  with rational coefficients.*

Let us denote this series by  $K(t_0, t_1, \dots)$ . We have already computed the first few terms:

$$K(t_0, t_1, \dots) = \log \left( 1 + \frac{1}{3!} t_0^3 + \frac{1}{24} t_1 + \frac{25}{144} t_0^3 t_1 + \frac{1}{72} t_0^6 + \dots \right). \quad (4.10)$$

**Remark 4.9.2.** The integral (4.7) can be easily interpreted as a formal power series since each monomial in  $\text{tr}(\Lambda^{-1})$ ,  $\text{tr}(\Lambda^{-3})$ ,  $\dots$  appears in the integral evaluation only a finite number of times. Equation (4.10) is valid for arbitrary value of the dimension  $N$ . However, if we want to compute a specific coefficient in this expansion, the value of  $N$  must be chosen sufficiently large.

Kontsevich's proof of the Witten conjecture consists of two parts. First, he shows that the coefficient of  $t_0^{l_0} \dots t_s^{l_s} / (l_0! \dots l_s!)$  in the expansion of his integral  $K(t_0, t_1, \dots)$  in the variables  $t_i$  coincides with the intersection number  $\langle \tau_0^{l_0} \dots \tau_s^{l_s} \rangle$ . This part of the proof is based on the study of the combinatorial model for the moduli space of curves. The second part consists in verification that the integral is a  $\tau$ -function for the KdV hierarchy. This means, essentially, that the second derivative  $\partial^2 K / \partial t_0^2$  is a solution to the KdV equation. The proof of this statement is achieved by treating the function  $K$  as a matrix Airy function.

### 4.9.2 The Kontsevich Model and Intersection Theory

A formal justification of the argument in this section requires the construction of a “minimal compactification” of the moduli space of smooth marked curves (elaborated by Looijenga in [203]) and an analysis of circle bundles over this compactification. The latter part is accurately written in the Ph.D. thesis of D. Zvonkine [313]. Below, we simply outline the original Kontsevich’s argument.

Consider the projection

$$\pi : \mathcal{M}_{g;n}^{\text{comb}} \cong \mathcal{M}_{g;n} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$$

of the combinatorial model to the second factor. This projection takes a marked graph with a metric to the  $n$ -tuple of the lengths of the perimeters of the marked points. Introduce the real 2-forms  $\omega_i$  defined only on open strata of  $\mathcal{M}_{g;n}^{\text{comb}}$  by the following formulas:

$$\omega_i = \sum d(l_{e'}/p_i) \wedge d(l_{e''}/p_i),$$

where  $p_i$  is the perimeter of the  $i$ th face, and  $e', e''$  run over all pairs of distinct edges of the  $i$ th face,  $e'$  preceding  $e''$  in some fixed order with a chosen starting vertex. The 2-form  $\omega_i$  represents the class  $\psi_i$ . Indeed, fix a smooth curve  $(X; x_1, \dots, x_n)$  and take the canonical Jenkins–Strebel quadratic differential associated to the  $n$ -tuple  $p_1, \dots, p_n$ . Then vertical trajectories of this quadratic differential through  $x_i$  identify the perimeter of the  $i$ th face of the corresponding embedded graph with the “spherized” cotangent line  $L_i$  considered as a real plane (that is, the fiber punctured at the origin is projected to the unit circle along the half-lines passing through the origin) at the  $i$ th point. Now it is possible to represent the intersection numbers  $\langle \tau_{m_1} \dots \tau_{m_n} \rangle$  in terms of integrals of very explicit differential forms:

$$\langle \tau_{m_1} \dots \tau_{m_n} \rangle = \int_{\pi^{-1}(\bar{p})} \prod_{i=1}^n \omega_i^{m_i}$$

over any generic point  $\bar{p} \in \mathbb{R}_+^n$ .

From now on we use the notation  $d$  for the complex dimension of  $\mathcal{M}_{g;n}$ ,  $d = 3g - 3 + n$ . Introduce the volume form on (the open strata of)  $\mathcal{M}_{g;n}^{\text{comb}}$ :

$$\text{Vol}(\lambda_1, \dots, \lambda_n) = \frac{1}{d!} \Omega^d \times \prod_{i=1}^n e^{-\lambda_i p_i} dp_i,$$

where  $\Omega = p_1^2 \omega_1 + \dots + p_n^2 \omega_n$  and  $\lambda_i$  are real positive parameters.

Then the volume of  $\mathcal{M}_{g;n}^{\text{comb}}$  with respect to this volume form can be computed in two ways: directly, under the projection to  $\mathbb{R}_+^n$ , and summing the volumes of all open cells. The first computation gives

$$\begin{aligned}
 \int_{\mathcal{M}_{g;n}^{\text{comb}}} \text{Vol}(\lambda_1, \dots, \lambda_n) &= \frac{1}{d!} \int_{\mathbb{R}_+^n} \left( \int_{\pi^{-1}(\bar{p})} \Omega^d \right) e^{-\sum \lambda_i p_i} dp_1 \wedge \dots \wedge dp_n \\
 &= \sum_{m_1 + \dots + m_n = d} \frac{\langle \tau_{m_1} \dots \tau_{m_n} \rangle}{m_1! \dots m_n!} \prod_i \int_0^\infty p_i^{2m_i} e^{-\lambda_i p_i} dp_i \\
 &= \sum_{m_1 + \dots + m_n = d} \langle \tau_{m_1} \dots \tau_{m_n} \rangle \prod_{i=1}^n \frac{(2m_i)!}{m_i!} \lambda_i^{-(2m_i+1)} \\
 &= 2^d \sum_{m_1 + \dots + m_n = d} \langle \tau_{m_1} \dots \tau_{m_n} \rangle \prod_{i=1}^n \frac{(2m_i - 1)!!}{\lambda_i^{(2m_i+1)}}.
 \end{aligned}$$

The first computation is completed, and we start the second one. Consider the open cell in  $\mathcal{M}_{g;n}^{\text{comb}}$  corresponding to a 3-valent embedded graph  $\Gamma$ . The lengths  $l_1, \dots, l_{|E(\Gamma)|}$  of the edges of  $\Gamma$  form a set of coordinates on this cell. In these coordinates, the volume form  $\text{Vol}(\lambda_1, \dots, \lambda_n)$  can be rewritten as

$$\text{Vol}_\Gamma(\lambda_1, \dots, \lambda_n) = 2^{d+|E(\Gamma)|-|V(\Gamma)|} e^{-\sum_j l_j \tilde{\lambda}_j} dl_1 \wedge \dots \wedge dl_{|E(\Gamma)|}.$$

Here  $j$  runs over the set of all edges of  $\Gamma$ , and  $\tilde{\lambda}_j$  is the sum

$$\tilde{\lambda}_j = \lambda_- + \lambda_+$$

of the two  $\lambda$ 's corresponding to the two faces of  $\Gamma$  adjacent to the  $j$ th edge. Note that the two faces neighboring to an edge may coincide, and in this case  $\lambda_- = \lambda_+$ . Obtaining the correct power of 2 in the last formula (and hence showing that it is independent of the chosen cell) is a rather cumbersome task, and we refer the reader to [178] for details. An immediate calculation gives

$$\text{Vol}_\Gamma(\lambda_1, \dots, \lambda_n) = \prod_{j=1}^{|E(\Gamma)|} \frac{1}{\tilde{\lambda}_j}.$$

The contribution of a marked embedded graph to the total volume is proportional to the inverse cardinality of the automorphism group of the graph, whence summing over all 3-valent marked genus  $g$  embedded graphs with  $n$  marked faces and multiplying by  $2^{-d}$  we obtain the *main combinatorial identity*

$$\sum_{m_1 + \dots + m_n = d} \langle \tau_{m_1} \dots \tau_{m_n} \rangle \prod_{i=1}^n \frac{(2m_i - 1)!!}{\lambda_i^{2m_i+1}} = \sum_\Gamma \frac{2^{-|V(\Gamma)|}}{|\text{Aut}(\Gamma)|} \prod_{j=1}^{|E(\Gamma)|} \frac{2}{\tilde{\lambda}_j}. \quad (4.11)$$

The main combinatorial identity is an identity between two rational functions in variables  $\lambda_i$ . Making an arbitrary substitution of the form  $\lambda_i = \Lambda_{k_i}$ ,  $1 \leq k_i \leq N$  and summing the resulting identities over all such substitutions one gets

$$\begin{aligned} & \sum_{m_1+\dots+m_n=d} \langle \tau_{m_1} \dots \tau_{m_n} \rangle \prod_{i=1}^n (2m_i - 1)!! \operatorname{tr}(\Lambda^{-(2m_i-1)}) \\ &= \sum_{\Gamma} \frac{2^{-|V(\Gamma)|}}{|\operatorname{Aut}(\Gamma)|} \prod_{j=1}^{|E(\Gamma)|} \frac{2}{\tilde{\Lambda}_j}. \end{aligned} \tag{4.12}$$

Here  $\tilde{\Lambda}_j = \Lambda_- + \Lambda_+$  and the sum on the right-hand side is taken over all possible ways to color the faces of the graph  $\Gamma$  in  $N$  colors  $\Lambda_1, \dots, \Lambda_N$ . Recall that  $\Lambda$  denotes the diagonal  $N \times N$  matrix with positive entries  $\Lambda_1, \dots, \Lambda_N$ .

The right-hand side of the last equation coincides with the matrix integral expansion in the Kontsevich model, and we obtain the first part of the Kontsevich theorem: the generating function  $K$  of the Kontsevich model coincides with the generating function  $F$  of the intersection model.

### 4.9.3 The Kontsevich Model and the KdV Equation

The second part of the proof consists in showing that the integral of the Kontsevich model is a  $\tau$ -function for the KdV-hierarchy, in other words, that it obeys the Korteweg–de Vries equation.

Let

$$a(y) = \int_{-\infty}^{\infty} e^{i(\frac{1}{3}x^3 - yx)} dx$$

be the classical Airy function, i.e., the unique (up to a scalar factor) bounded solution to the linear differential equation

$$a''(y) + ya(y) = 0.$$

We are interested in the “asymptotic behavior” of this function as  $y \rightarrow \infty$ . An application of the stationary phase method (which must be justified in this case) gives

$$a(y) \sim e^{-\frac{2i}{3}y^{3/2}} \int_{U(y^{1/2})} e^{i(\frac{1}{3}x^3 + y^{1/2}x^2)} dx + e^{\frac{2i}{3}y^{3/2}} \int_{U(-y^{1/2})} e^{i(\frac{1}{3}x^3 - y^{1/2}x^2)} dx,$$

where the integration is carried out over arbitrary neighborhoods of the points  $\pm y^{1/2}$ .

Similar constructions are valid for the case of the *matrix Airy function*

$$A(Y) = \int_{\mathcal{H}_N} e^{i(\frac{1}{3} \operatorname{tr} H^3 - HY)} d\mu(H),$$

for a positive diagonal matrix  $Y$ . This function obeys the *matrix Airy equation*

$$\Delta A(Y) + \operatorname{tr} Y \cdot A(Y) = 0,$$

where  $\Delta$  denotes the Laplace operator. Similarly to the 1-dimensional Airy function, the matrix Airy function admits an asymptotic expansion as a sum

of  $2^N$  expressions of the form

$$e^{-i\frac{2}{3}\text{tr}Y^{3/2}} \int e^{i\text{tr}(\frac{1}{3}H^3 - H^2Y^{1/2})} d\mu(H) = e^{-i\frac{2}{3}\text{tr}Y^{3/2}} \int e^{i\text{tr}\frac{1}{3}H^3} d\mu_{Y^{1/2}}(H).$$

The sum is taken over all  $2^N$  quadratic roots  $Y^{1/2}$  of the matrix  $Y$ , and the integral is taken over a neighborhood of the origin in  $\mathcal{H}_N$ . As  $Y \rightarrow \infty$ , the integral can be replaced with that over the entire space  $\mathcal{H}_N$ , i.e., it becomes the integral of the Kontsevich model for  $\Lambda = Y^{1/2}$ . The asymptotic expansion of the latter we already know.

Another way to compute the matrix Airy function consists in the application of formulas borrowed from [146] and [216]:

$$\begin{aligned} A(Y) &= c_N \Delta(Y_i)^{-1} \int_{\mathbb{R}^N} \prod_{i=1}^n \Delta(X_i) e^{i(\frac{1}{3}X_i^3 - X_i Y_i)} dX_i \\ &= c_N \frac{\det(a^{(j-1)}(Y_i))}{\det(Y_i^{j-1})}, \end{aligned}$$

where this time  $\Delta$  denotes the Vandermonde determinant. Here we made use of the obvious identity

$$\int e^{i(x^3/3 - xy)} x^{j-1} dx = (ia(y))^{(j-1)}.$$

The derivatives of the Airy function admit natural asymptotic expansions

$$a^{(j-1)}(y) \sim \sum_{y^{1/2}} \text{const} \cdot y^{-3/4} e^{-\frac{2i}{3}y^{3/2}} \cdot f_j(y^{-1/2})$$

for some Laurent series  $f_j(z) = z^{-j} + \dots \in \mathbb{Q}((z))$ . Substituting the last formula into the expression for the matrix Airy function we obtain

$$A(Y) = \sum_{Y^{1/2}} \text{const} \times e^{-\frac{2i}{3}\text{tr}Y^{3/2}} \prod_{i=1}^N Y_i^{-3/4} \cdot \frac{\det(f_j(Y_i^{-1/2}))}{\det(Y_i^{j-1})}.$$

The last expression relates the matrix Airy function to the  $\tau$ -function corresponding to the subspace  $\langle f_1, f_2, \dots \rangle \subset \mathbb{C}((z^{-1}))$ , see Sec. 3.6.4. The proposition and the argument in the end of that section complete the proof of Witten's conjecture.

\* \* \*

The main theorem established in this chapter permits to compute the intersection indices for certain classes; but the structure of the cohomology ring of the moduli spaces remains unknown. There also remains one more Witten's conjecture (it is discussed, in particular, in [178] and [202]), and

though it is not apparent from its formulation, it is also related to embedded graphs.

The general idea behind the notion of a moduli space is that of “the space of parameters”. In this chapter we parametrized algebraic curves. It is no less interesting to parametrize the pairs  $(X, f)$  where  $X$  is a curve and  $f$  is a meromorphic function on  $X$ . The corresponding parameter spaces are called *Hurwitz spaces*. The reader will find an introduction to this theory – from the point of view of embedded graphs, to be sure – in the next chapter.



<http://www.springer.com/978-3-540-00203-1>

Graphs on Surfaces and Their Applications

Lando, S.K.; Zvonkin, A.K. - Gamkrelidze, R.V.; Vassiliev,  
V.A. (Eds.)

2004, XV, 455 p. 3 illus., Hardcover

ISBN: 978-3-540-00203-1