# Chapter 2
# Standard Replacement Policies

To begin with, this chapter gives three standard replacement models that have been obtained as basic replacement policies for an operating unit with shock and damage [2]. Here, the so-called standard models are formulated under the classical approach of *whichever triggering event occurs first* [22] in reliability theory. That is, the unit is replaced preventively at some thresholds or planned measurements such as operating time, usage number, damage level, repair cost, number of faults or repairs, etc., or at failure, *whichever occurs first*, which is named as *replacement first*.

Replacement first is absolutely reasonable when a single preventive replacement scenario is planned to avoid catastrophic failure, and models based on this approach have been surveyed and extended for decades. However, it may typically cause frequent and unnecessary replacement actions when several compound preventive replacement scenarios are scheduled. The features of models for replacement policies acting on the approach of first have been observed in [7, 9, 12, 19]. We name the models in this chapter as replacement first, as it will be used to compare with *replacement last* in Chap. 3, and *replacement middle* and *replacement overtime* in Chap. 4, which are based on the newly proposed approaches of *whichever triggering event occurs last*, *whichever triggering event occurs middle*, and *replacing over a planned measure*, respectively.

In this chapter, we consider an operating unit suffered for cumulative damage due to random shocks should operate over an infinite time span, in which it is of great importance to make suitable replacement plans to avoid catastrophic failure when the total damage has exceeded a failure threshold $K$. That is, we focus mainly on replacement policies for an operating unit with the failure mode of cumulative damage. As discussed in [2], we give the following three preventive replacement actions: The most easy way is to replace an operating unit at planned ages without monitoring its shock and damage; however, using monitoring equipment, we could count the number of shocks and investigate the amount of total damage at shock times, and more precise replacement plans can be done at a pre-specified number of shocks



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or damage level before failure, despite the relatively rough. So that the operating unit is supposed to be replaced correctively after failure $K$ and preventively before failure at planned time $T$, at shock number $N$, or at damage level $Z$, whichever occurs first [2].

The planned replacement actions can be optimized in order to balance failure losses and replacement costs. It was shown from numerical examples [2] that optimum preventive replacement policies acting on shock and damage conditions show more superiority than the policy acting on a planned time, that is, from the cost saving point, replacement done at $Z$ is better than those at $T$ and $N$, and replacement done at $N$ is better than that at $T$ in most cases. However, replacement at $T$ is much easier to perform but additional monitoring equipment should be provided for those at $N$ and $Z$, of which monitoring costs may not be neglected in practice. Additional comparisons of replacement policies for cumulative damage models have been studied in [26]. Further, it seems to be a waste of cost for replacement first, as we know that, replacement should be done as soon as possible before failure when any policy at $T$, $N$, or $Z$ is first triggered. However, we will find the cases when replacement first saves cost for a long run, by comparing to the proposed replacement policies in the following chapters.

In Sect. 2.1, we obtain the expected cost rate of three combined preventive replacement policies planned at time $T$, at shock $N$, and at damage $Z$. In Sects. 2.1.1 and 2.1.2, we derive analytically optimum policies which minimize the expected cost rates for each of three policies and for two combinations of three ones. It would be of great interest to show theoretically that when all preventive replacement costs of three policies are the same, the best policy among three ones is replacement with damage $Z$, the next one is replacement with shock $N$, and the last one is replacement with time $T$. In Sect. 2.1.3, when shocks occur at a Poisson process and each amount of damage due to shocks is exponential, optimum policies in Sects. 2.1.1 and 2.1.2 are computed numerically and compared with each other. Sections 2.2 and 2.3 consider several extended replacement models, of which the level of failure threshold $K$ is a random variable with an estimated probability distribution, and the unit also fails when the total number of shocks reaches to a certain value of $N$.

## 2.1 Three Replacement Policies

A new unit with damage level 0 begins to operate at time 0 and degrades with damage produced by shocks. It is assumed that shocks occur at a renewal process according to an identical distribution $F(t)$ with a density function $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$ and finite mean $\mu \equiv \int_0^\infty \overline{F}(t)\mathrm{d}t$, where $\overline{F}(t) \equiv 1 - F(t)$. When $F(t)$ has a density function $f(t)$, $h(t) \equiv f(t)/\overline{F}(t)$ is assumed to increase from $h(0) \equiv \lim_{t\to 0} h(t)$ to $h(\infty) \equiv \lim_{t\to\infty} h(t)$. Clearly, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, $h(t) = \lambda$ for any $t \geq 0$. An amount $W_j$ $(j = 1, 2, \cdots)$ of damage due to the $j$th shock has an identical distribution $G(x) \equiv \Pr\{W_j \leq x\}$ with finite mean $1/\omega \equiv \int_0^\infty \overline{G}(x)\mathrm{d}x$ and is additive to the current damage level.

Let $N(t)$ denote the number of shocks during the interval $[0, t]$. Then, the probability that $j$ shocks occur exactly in $[0, t]$ is

$$\Pr\{N(t) = j\} = F^{(j)}(t) - F^{(j+1)}(t) \quad (j = 0, 1, 2, \cdots),$$

and the distribution of the total damage $W(t)$ at time $t$ is

$$\Pr\{W(t) \le x\} = \sum_{j=0}^{\infty}[F^{(j)}(t) - F^{(j+1)}(t)]G^{(j)}(x),$$

where $\Phi^{(j)}(t)$ denotes the $j$-fold Stieltjes convolution of any function $\Phi(t)$ with itself, i.e., $\Phi^{(j)}(t) = \int_0^t \Phi^{(j-1)}(t-u)\mathrm{d}\Phi(u)$ $(j = 1, 2, \cdots)$ and $\Phi^{(0)}(t) \equiv 1$ for $t \ge 0$.

The unit fails when the total damage has exceeded a failure threshold $K$ $(0 < K < \infty)$ at some shock, and its failure is detected and *corrective replacement* (CR) is done immediately. *Preventive replacement* (PR) times are scheduled before failure at planned time $T$ $(0 < T \le \infty)$, at shock number $N$ $(N = 1, 2, \cdots)$, or at damage level $Z$ $(0 < Z \le K)$, whichever occurs first, which is called *replacement first* (RF). In addition, the unit is supposed to be replaced at damage $K$ or $Z$ rather than at shock $N$, when the total damage has exceeded $K$ or $Z$ at shock $N$. Furthermore, it is assumed that both CR and PR remove all damage perfectly, and the unit becomes as good as new after any replacement.

It can be understood for RF that the replacement should be done as soon as possible before CR when any PR is first triggered. Then, the probability that the unit is replaced at time $T$ is

$$\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(Z), \tag{2.1}$$

the probability that it is replaced at shock $N$ is

$$F^{(N)}(T)G^{(N)}(Z), \tag{2.2}$$

the probability that it is replaced at damage $Z$ is

$$\sum_{j=0}^{N-1} F^{(j+1)}(T) \int_0^Z [G(K-x) - G(Z-x)]\mathrm{d}G^{(j)}(x), \tag{2.3}$$

and the probability that it is replaced at failure $K$ is

$$\sum_{j=0}^{N-1} F^{(j+1)}(T) \int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x), \tag{2.4}$$

where note that (2.1) + (2.2) + (2.3) + (2.4) = 1. Thus, the mean time to replacement is (Problem 2.1)

$$T\sum_{j=0}^{N-1}[F^{(j)}(T)-F^{(j+1)}(T)]G^{(j)}(Z)+G^{(N)}(Z)\int_0^T t\mathrm{d}F^{(N)}(t)$$
$$+\sum_{j=0}^{N-1}\int_0^T t\mathrm{d}F^{(j+1)}(t)\int_0^Z[G(K-x)-G(Z-x)]\mathrm{d}G^{(j)}(x)$$
$$+\sum_{j=0}^{N-1}\int_0^T t\mathrm{d}F^{(j+1)}(t)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)$$
$$=\sum_{j=0}^{N-1}G^{(j)}(Z)\int_0^T[F^{(j)}(t)-F^{(j+1)}(t)]\mathrm{d}t. \tag{2.5}$$

Therefore, the expected replacement cost rate is

$$C_F(T,N,Z)=\frac{\begin{array}{l}c_K-(c_K-c_T)\sum_{j=0}^{N-1}[F^{(j)}(T)-F^{(j+1)}(T)]G^{(j)}(Z)\\ -(c_K-c_N)F^{(N)}(T)G^{(N)}(Z)\\ -(c_K-c_Z)\sum_{j=0}^{N-1}F^{(j+1)}(T)\int_0^Z[G(K-x)-G(Z-x)]\mathrm{d}G^{(j)}(x)\end{array}}{\sum_{j=0}^{N-1}G^{(j)}(Z)\int_0^T[F^{(j)}(t)-F^{(j+1)}(t)]\mathrm{d}t}, \tag{2.6}$$

where $c_T$ = replacement cost at time $T$, $c_N$ = replacement cost at shock $N$, $c_Z$ = replacement cost at damage $Z$, and $c_K$ = replacement cost at failure $K$, where $c_K > c_T$, $c_K > c_N$, and $c_K > c_Z$.

In particular, when the unit is replaced only after failure $K$, i.e., when $T \to \infty$, $N \to \infty$ and $Z \to K$,

$$C \equiv \lim_{\substack{T\to\infty\\ N\to\infty\\ Z\to K}} C_F(T,N,Z)=\frac{c_K}{\mu[1+M_G(K)]}, \tag{2.7}$$

where $M_G(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$ is a renewal function of $G(x)$ and represents the expected number of shocks at damage level $x$. Further, if $Z \to 0$, then $N$ is equal to 1, and

$$\lim_{\substack{T\to\infty\\ Z\to 0}} C_F(T,N,Z)=\lim_{\substack{T\to\infty\\ Z\to 0}} C_F(T,1,Z)=\frac{1}{\mu}[c_K-(c_K-c_Z)G(K)]. \tag{2.8}$$

### 2.1.1 Optimum Policies with One Variable

We should avoid the high failure cost without any PR in (2.7). Meanwhile, it would be unreasonable to make frequent replacement actions to waste PR cost, such as an extreme case in (2.8). In order to minimize the respective cost rates of the above three replacement policies, we next obtain analytically optimum time $T^*$, shock $N^*$, and damage $Z^*$, respectively.

**(1) Optimum $T^*$**

Suppose that the unit is replaced preventively only at time $T$ $(0 < T \le \infty)$. Then, putting that $N \to \infty$ and $Z \to K$ in (2.6),

$$C(T) \equiv \lim_{\substack{N\to\infty \\ Z\to K}} C_F(T, N, Z)$$
$$= \frac{c_K - (c_K - c_T)\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}. \tag{2.9}$$

It can be easily seen that $\lim_{T\to 0} C(T) = \infty$ and $\lim_{T\to\infty} C(T) = C$ in (2.7). We find optimum $T^*$ to minimize $C(T)$. Differentiating $C(T)$ with respect to $T$ and setting it equal to zero,

$$Q_1(T)\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t$$
$$-\sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] = \frac{c_T}{c_K - c_T}, \tag{2.10}$$

where

$$Q_1(T, N) \equiv \frac{\sum_{j=0}^{N-1} f^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}, \tag{2.11}$$

$$Q_1(T) \equiv \lim_{N\to\infty} Q_1(T, N) = \frac{\sum_{j=0}^{\infty} f^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)},$$

and $f^{(j+1)}(t) \equiv \mathrm{d}F^{(j+1)}(t)/\mathrm{d}t$. Note that if $h(t)$ increases with $t$, i.e., $F(t)$ has an IFR (Increasing Failure Rate) property, its convolution is also IFR, and so that, $f^{(j+1)}(t)/[F^{(j)}(t) - F^{(j+1)}(t)]$ increases with $t$ [22]. In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, $f^{(j+1)}(t)/[F^{(j)}(t) - F^{(j+1)}(t)] = \lambda$ for any $t \ge 0$.

If $Q_1(T)$ increases strictly with $T$ to $Q_1(\infty) \equiv \lim_{T\to\infty} Q_1(T)$, then the left-hand side of (2.10) increases strictly to (Problem 2.2)

$$\mu Q_1(\infty)[1 + M_G(K)] - 1.$$

Therefore, if

$$Q_1(\infty)[1 + M_G(K)] > \frac{c_K}{\mu(c_K - c_T)},$$

then there exists a finite and unique $T^*$ $(0 < T^* < \infty)$ which satisfies (2.10), and the resulting cost rate is

$$C(T^*) = (c_K - c_T)Q_1(T^*). \tag{2.12}$$

Conversely, if

$$Q_1(\infty)[1 + M_G(K)] \leq \frac{c_K}{\mu(c_K - c_T)},$$

then $T^* = \infty$, i.e., the unit is replaced only at failure, and the resulting cost rate is given in (2.7).

**(2) Optimum $N^*$**

Suppose that the unit is replaced preventively only at shock $N$ $(N = 1, 2, \cdots)$. Then, putting that $T \to \infty$ and $Z \to K$ in (2.6),

$$C(N) \equiv \lim_{\substack{T\to\infty \\ Z\to K}} C_F(T, N, Z) = \frac{c_K - (c_K - c_N)G^{(N)}(K)}{\mu \sum_{j=0}^{N-1} G^{(j)}(K)} \quad (N = 1, 2, \cdots). \tag{2.13}$$

In particular, when $N = 1$, i.e., the unit is always replaced at the first shock, the expected cost rate is given in (2.8) when $c_N = c_Z$.

We find optimum $N^*$ to minimize $C(N)$. Forming the inequality $C(N+1) - C(N) \geq 0$ (Problem 2.3),

$$r_{N+1}(K)\sum_{j=0}^{N-1} G^{(j)}(K) - [1 - G^{(N)}(K)] \geq \frac{c_N}{c_K - c_N}, \tag{2.14}$$

where for $0 < x \leq K$,

$$r_{N+1}(x) \equiv \frac{G^{(N)}(x) - G^{(N+1)}(x)}{G^{(N)}(x)} \quad (N = 1, 2, \cdots).$$

If $r_{N+1}(x)$ increases strictly with $N$, i.e., $G^{(N+1)}(x)/G^{(N)}(x)$ decreases strictly with $N$, then the left-hand side of (2.14) increases strictly with $N$ to $r_\infty(K)[1 + M_G(K)] - 1$ (Problem 2.4), where $r_\infty(K) \equiv \lim_{N\to\infty} r_{N+1}(K) \leq 1$. Therefore, if

$$r_\infty(K)[1+M_G(K)] > \frac{c_K}{c_K - c_N},$$

then there exists a finite and unique minimum $N^*$ $(1 \leq N^* < \infty)$ which satisfies (2.14), and the resulting cost rate is

$$(c_K - c_N)r_{N^*}(K) < \mu C(N^*) \leq (c_K - c_N)r_{N^*+1}(K), \tag{2.15}$$

whose cost rate is given in (2.8) when $N^* = 1$. Conversely, if

$$r_\infty(K)[1+M_G(K)] \leq \frac{c_K}{c_K - c_N},$$

then $N^* = \infty$, i.e., the unit is replaced only at failure, and the expected cost rate is given in (2.7).

Note that $r_{N+1}(K)$ represents the probability that the unit surviving at the $N$th shock will fail at the next shock, which might increase with $N$ to 1.

**(3) Optimum $Z^*$**

Suppose that the unit is replaced preventively only at damage $Z$ $(0 < Z \leq K)$. Then, putting that $T \to \infty$ and $N \to \infty$ in (2.6),

$$\begin{aligned} C(Z) &\equiv \lim_{\substack{T\to\infty \\ N\to\infty}} C_F(T,N,Z) \\ &= \frac{c_K - (c_K - c_Z)[G(K) - \int_0^Z \overline{G}(K-x)\mathrm{d}M_G(x)]}{\mu[1+M_G(Z)]}. \end{aligned} \tag{2.16}$$

When $Z \to 0$, (2.16) agrees with (2.8).

We find optimum $Z^*$ to minimize $C(Z)$. Differentiating $C(Z)$ with respect to $Z$ and setting it equal to zero,

$$\int_{K-Z}^{K} [1+M_G(K-x)]\mathrm{d}G(x) = \frac{c_Z}{c_K - c_Z}, \tag{2.17}$$

whose left-hand side increases strictly with $Z$ from 0 to $M_G(K)$ (Problem 2.5). Therefore, if $M_G(K) > c_Z/(c_K - c_Z)$, then there exists a finite and unique $Z^*$ $(0 < Z^* < K)$ which satisfies (2.17), and the resulting cost rate is

$$\mu C(Z^*) = (c_K - c_Z)\overline{G}(K - Z^*). \tag{2.18}$$

Conversely, if $M_G(K) \leq c_Z/(c_K - c_Z)$, then $Z^* = K$, i.e., the unit is replaced only at failure, and the resulting cost rate is given in (2.7).

### 2.1.2 Optimum Policies with Two Variables

Our next concerns are (i) how to compare respective replacement policies done at time $T$, at shock $N$, and at damage $Z$, and (ii) what are optimum policies with two variables to minimize their expected cost rates in (2.6). We answer for (i) and (ii) in the following sections, and further discussions for (i) will be addressed in Chap. 3.

**(1) Optimum $T_F^*$ and $N_F^*$**

Suppose that the unit is replaced preventively at time $T$ $(0 < T \leq \infty)$ or at shock $N$ $(N = 1, 2, \cdots)$, whichever occurs first. When $c_T = c_N$, putting that $Z \to K$ in (2.6), the expected cost rate is

$$C_F(T, N) = \frac{c_T + (c_K - c_T)\sum_{j=0}^{N-1} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}. \tag{2.19}$$

We find optimum $(T_F^*, N_F^*)$ to minimize $C_F(T, N)$. Forming the inequality $C_F(T, N-1) - C_F(T, N) > 0$,

$$\begin{aligned}&\frac{r_N(K)F^{(N)}(T)}{\int_0^T [F^{(N-1)}(t) - F^{(N)}(t)]\mathrm{d}t}\sum_{j=0}^{N-1} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t \\ &-\sum_{j=0}^{N-1} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] < \frac{c_T}{c_K - c_T},\end{aligned} \tag{2.20}$$

where $r_N(K)$ is given in (2.14).

Differentiating $C_F(T, N)$ with respect to $T$ and setting it equal to zero,

$$\begin{aligned}&Q_1(T, N)\sum_{j=0}^{N-1} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t \\ &-\sum_{j=0}^{N-1} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] = \frac{c_T}{c_K - c_T},\end{aligned} \tag{2.21}$$

where $Q_1(T, N)$ is given in (2.11).

In addition, because both $T_F^*$ and $N_F^*$ have to satisfy (2.20) and (2.21), (2.21) is rewritten as, by substituting (2.20) for (2.21),

$$Q_1(T, N) > \frac{r_N(K)F^{(N)}(T)}{\int_0^T [F^{(N-1)}(t) - F^{(N)}(t)]\mathrm{d}t}. \tag{2.22}$$

Thus, if the inequality (2.22) does not hold for any $N$, there does not exist any finite $T_F^*$ which satisfies (2.21), i.e., the optimum policy is $(T_F^* = \infty, N_F^* = N^*)$, where $N^*$ is given in (2.14).

For example, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $r_N(x)$ increases strictly with $N$, (2.22) is

$$\frac{\sum_{j=0}^{N-1}[(\lambda T)^j/j!][G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j)}(K)} > r_N(K),$$

whose left-hand side increases with $T$ from $\overline{G}(K)$ to $r_N(K)$ as $T \to \infty$ (Problem 2.2). That is, the above inequality does not hold for $0 < T \leq \infty$. This means that when optimum $N_F^*$ satisfies (2.20), the left-hand side of (2.21) is less than $c_T/(c_K - c_T)$, i.e., $C_F(T, N_F^*)$ decreases with $T$, and hence, $T_F^* = \infty$. This concludes that if the inequality (2.22) does not hold, then the optimum policy which minimizes $C_F(T, N)$ is $(T_F^* = \infty, N_F^* = N^*)$.

In other words, when $c_T \geq c_N$ is supposed for the replacement policies done at time $T$ and at shock $N$, its optimum policy degrades into the case in which only finite $N^*$ for $T = \infty$ can be found in (2.14).

Next, we obtain optimum $T_F^*$ for given $N$ when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $c_T = c_N$. In this case, (2.21) is rewritten as

$$\widetilde{Q}_1(T, N)\sum_{j=0}^{N-1} F^{(j+1)}(T)G^{(j)}(K) - \sum_{j=0}^{N-1} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] = \frac{c_T}{c_K - c_T}, \tag{2.23}$$

where

$$\widetilde{Q}_1(T, N) \equiv \frac{\sum_{j=0}^{N-1}[(\lambda T)^j/j!][G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j)}(K)}.$$

When $r_N(x)$ increases strictly with $N$, it is approved that $\widetilde{Q}_1(T, N) = \overline{G}(K)$ for $N = 1$ and increases strictly with $T$ for $N \geq 2$ from $\overline{G}(K)$ to $r_N(K)$ (Problem 2.2). Thus, the left-hand side of (2.23) increases strictly with $T$ from 0 to

$$r_N(K)\sum_{j=0}^{N-1} G^{(j)}(K) - [1 - G^{(N)}(K)] < r_{N+1}(K)\sum_{j=0}^{N} G^{(j)}(K) - [1 - G^{(N+1)}(K)]$$

$$= r_{N+1}(K)\sum_{j=0}^{N-1} G^{(j)}(K) - [1 - G^{(N)}(K)],$$

which agrees with that of (2.14). Thus, if $N > N^*$ in (2.14), then there exists a finite and unique $T_F^*$ $(0 < T_F^* < \infty)$ which satisfies (2.23), and the resulting cost rate is

$$\frac{C_F(T_F^*, N)}{\lambda} = (c_K - c_T)\widetilde{Q}_1(T_F^*, N). \tag{2.24}$$

Conversely, if $N \le N^*$, then $T_F^* = \infty$.

**(2) Optimum $T_F^*$ and $Z_F^*$**

Suppose that the unit is replaced preventively at time $T$ $(0 < T \le \infty)$ or at damage $Z$ $(0 < Z \le K)$, whichever occurs first. When $c_T = c_Z$, putting that $N \to \infty$ in (2.6), the expected cost rate is

$$C_F(T, Z) = \frac{c_T + (c_K - c_T)\sum_{j=0}^{\infty} F^{(j+1)}(T)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{\infty} G^{(j)}(Z)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}. \tag{2.25}$$

We find optimum $(T_F^*, Z_F^*)$ to minimize $C_F(T, Z)$. Differentiating $C_F(T, Z)$ with respect to $Z$ and setting it equal to zero,

$$Q_2(T, Z)\sum_{j=0}^{\infty} G^{(j)}(Z)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t$$
$$- \sum_{j=0}^{\infty} F^{(j+1)}(T)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x) = \frac{c_T}{c_K - c_T}, \tag{2.26}$$

where

$$Q_2(T, Z) \equiv \frac{\overline{G}(K-Z)\sum_{j=0}^{\infty} g^{(j)}(Z)F^{(j+1)}(T)}{\sum_{j=0}^{\infty} g^{(j)}(Z)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t},$$

and $g^{(j)}(x) \equiv \mathrm{d}G^{(j)}(x)/\mathrm{d}x$ $(j = 1, 2, \cdots)$ and $g^{(0)}(x) \equiv 0$.

Differentiating $C_F(T, Z)$ with respect to $T$ and setting it equal to zero,

$$Q_3(T, Z)\sum_{j=0}^{\infty} G^{(j)}(Z)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t$$
$$- \sum_{j=0}^{\infty} F^{(j+1)}(T)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x) = \frac{c_T}{c_K - c_T}, \tag{2.27}$$

where

$$Q_3(T,Z) \equiv \frac{\sum_{j=0}^{\infty} f^{(j+1)}(T)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{\infty}[F^{(j)}(T)-F^{(j+1)}(T)]G^{(j)}(Z)}.$$

Substituting (2.26) for (2.27),

$$Q_3(T,Z) = Q_2(T,Z). \tag{2.28}$$

Thus, if $Q_3(T,Z) < Q_2(T,Z)$ for any $Z$, then there dose not exist any finite $T_F^*$ which satisfies (2.27), i.e., the optimum policy is ($T_F^* = \infty$, $Z_F^* = Z^*$), where $Z^*$ is given in (2.17).

For example, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, it is proved that $Q_2(T,Z) = \lambda\overline{G}(K-Z)$, and (Problem 2.6)

$$\frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j)}(Z)} < \overline{G}(K-Z)$$

for any $Z$. That is, (2.28) does not hold for $0 < T \leq \infty$. This means that when optimum $Z_F^*$ satisfies (2.26), the left-hand side of (2.27) is less than $c_T/(c_K - c_T)$, i.e., $C_F(T, Z_F^*)$ decreases with $T$, and hence, $T_F^* = \infty$. This concludes that if $Q_3(T,Z) < Q_2(T,Z)$ and $c_T \geq c_Z$, then the optimum policy which minimizes $C_F(T,Z)$ is ($T_F^* = \infty$, $Z_F^* = Z^*$).

In other words, when $c_T \geq c_Z$ is supposed for the replacement policies done at time $T$ and at damage $Z$, its optimum policy degrades into the case in which only $Z^*$ for $T = \infty$ can be found in (2.17).

Next, we obtain optimum $T_F^*$ for given $Z$ when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $c_T = c_Z$. In this case, (2.27) is rewritten as

$$\widetilde{Q}_3(T,Z)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(Z) - \sum_{j=0}^{\infty} F^{(j+1)}(T)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x) = \frac{c_T}{c_K - c_T}, \tag{2.29}$$

where

$$\widetilde{Q}_3(T,Z) \equiv \frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j)}(Z)}.$$

If $\widetilde{Q}_3(T,Z)$ increases strictly with $T$ to $\overline{G}(K-Z)$ (Problem 2.6), then the left-hand side of (2.29) increases strictly with $T$ from 0 to

$$\int_{K-Z}^{K}[1+M_G(K-x)]\mathrm{d}G(x),$$

which agrees with that of (2.17). Thus, if $Z > Z^*$ in (2.17), then there exists a finite and unique $T_F^*$ $(0 < T_F^* < \infty)$ which satisfies (2.29). Conversely, if $Z \le Z^*$, then $T_F^* = \infty$.

**(3) Optimum $N_F^*$ and $Z_F^*$**

Suppose that the unit is replaced preventively at shock $N$ $(N = 1, 2, \cdots)$ or at damage $Z$ $(0 < Z \le K)$, whichever occurs first. When $c_N = c_Z$, putting that $T \to \infty$ in (2.6), the expected cost rate is

$$C_F(N,Z) = \frac{c_N + (c_K - c_N)\sum_{j=0}^{N-1}\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\mu\sum_{j=0}^{N-1}G^{(j)}(Z)}. \tag{2.30}$$

We find optimum $(N_F^*, Z_F^*)$ to minimize $C_F(N,Z)$. Differentiating $C_F(N,Z)$ with respect to $Z$ and setting it equal to zero,

$$\sum_{j=0}^{N-1}\int_0^Z[\overline{G}(K-Z)-\overline{G}(K-x)]\mathrm{d}G^{(j)}(x) = \frac{c_N}{c_K - c_N}. \tag{2.31}$$

Forming the inequality $C_F(N+1,Z) - C_F(N,Z) \ge 0$,

$$\frac{\sum_{j=0}^{N-1}G^{(j)}(Z)}{G^{(N)}(Z)}\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(N)}(x) - \sum_{j=0}^{N-1}\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x) \ge \frac{c_N}{c_K - c_N}. \tag{2.32}$$

Substituting (2.31) for (2.32),

$$\int_0^Z[\overline{G}(K-Z)-\overline{G}(K-x)]\mathrm{d}G^{(N)}(x) \le 0, \tag{2.33}$$

which dose not hold for any $Z$ (Problem 2.7). Thus, there does not exist any finite $N_F^*$ which satisfies (2.32).

When optimum $Z_F^*$ satisfies (2.31), the left-hand side of (2.32) is less than $c_N/(c_K - c_N)$, which means $C_F(N, Z_F^*)$ decreases strictly with $N$, and hence, $N_F^* = \infty$. This concludes that if $c_N \ge c_Z$, then the optimum policy which minimizes $C_F(N,Z)$ is $(N_F^* = \infty, Z_F^* = Z^*)$, where $Z^*$ is given in (2.17).

Next, we obtain optimum $N_F^*$ for given $Z$ when $G(x) = 1 - \mathrm{e}^{-\omega x}$ and $c_N = c_Z$. In this case, (2.32) is

$$e^{-\omega(K-Z)}\left[r_{N+1}(Z)\sum_{j=0}^{N-1}G^{(j)}(Z)+G^{(N)}(Z)-1\right]\geq\frac{c_Z}{c_K-c_Z},\tag{2.34}$$

whose left-hand side increases strictly with $N$ to (Problem 2.4)

$$\omega Z e^{-\omega(K-Z)},$$

which agrees with that of (2.17). Thus, if $Z > Z^*$ in (2.17), then there exists a finite and unique minimum $N_F^*$ $(1 \leq N_F^* < \infty)$ which satisfies (2.34). Conversely, if $Z \leq Z^*$, then $N_F^* = \infty$.

The above optimum results show under the suitable conditions, e.g., different PR costs for $c_T \geq c_N \geq c_Z$, the policy with damage $Z$ is the best among three ones, the next one is the policy with shock $N$, and the third one is the policy with time $T$. However, the order of working load for each preventive replacement is usually damage $Z$, shock $N$ and time $T$, because we have to investigate the amount of total damage at each shock for damage $Z$, count the number of shocks for shock $N$, and record only the passed time for time $T$ which is the easiest work load among three policies. Therefore, the case of $c_T < c_N < c_Z$ should be investigated to compare the above three policies, which will be discussed numerically in **(2)** of Sect. 2.1.3.

### *2.1.3 Poisson Shock Times*

When shocks occur at a Poisson process with rate $\lambda$, and each amount of damage due to shocks is exponential with parameter $\omega$, i.e., $F(t) = 1 - e^{-\lambda t}$ and $G(x) = 1 - e^{-\omega x}$,

$$F^{(j)}(t)=\sum_{i=j}^{\infty}\frac{(\lambda t)^i}{i!}e^{-\lambda t},\quad G^{(j)}(x)=\sum_{i=j}^{\infty}\frac{(\omega x)^i}{i!}e^{-\omega x}\quad (j=0,1,2,\cdots).$$

The expected cost rate in (2.6) is rewritten as

$$\frac{C_F(T,N,Z)}{\lambda}=\frac{\begin{array}{l}c_K-(c_K-c_T)\sum_{j=0}^{N-1}[(\lambda T)^j/j!]e^{-\lambda T}G^{(j)}(Z)\\-(c_K-c_N)F^{(N)}(T)G^{(N)}(Z)\\-(c_K-c_Z)(e^{-\omega Z}-e^{-\omega K})\sum_{j=0}^{N-1}F^{(j+1)}(T)[(\omega Z)^j/j!]\end{array}}{\sum_{j=0}^{N-1}F^{(j+1)}(T)G^{(j)}(Z)}.\tag{2.35}$$

We survey again optimum policies with one variable $T^*$, $N^*$ and $Z^*$ discussed in Sect. 2.1.1 and with two variables $T_F^*$, $N_F^*$ and $Z_F^*$ discussed in Sect. 2.1.2 for different PR costs $c_T$, $c_N$ and $c_Z$.

**(1) Optimum Policies with One Variable**

We obtain optimum $T^*$, $N^*$ and $Z^*$ to minimize $C(T)$, $C(N)$ and $C(Z)$, respectively.

**(a) Optimum $T^*$**

From (2.9) and (2.35),

$$\frac{C(T)}{\lambda} = \frac{c_T + (c_K - c_T)\sum_{j=0}^{\infty} F^{(j+1)}(T)[(\omega K)^j/j!]\mathrm{e}^{-\omega K}}{\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K)}. \tag{2.36}$$

Differentiating $C(T)$ with respect to $T$ and setting it equal to zero,

$$\widetilde{Q}_1(T)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K) - \sum_{j=0}^{\infty} F^{(j+1)}(T)\frac{(\omega K)^j}{j!}\mathrm{e}^{-\omega K} = \frac{c_T}{c_K - c_T}, \tag{2.37}$$

where $\widetilde{Q}_1(T) \equiv \lim_{N\to\infty} \widetilde{Q}_1(T, N)$ is given in (2.23). The left-hand side of (2.37) increases strictly from 0 to $\omega K$ because $\widetilde{Q}_1(T)$ increase strictly with $T$ from $\mathrm{e}^{-\omega K}$ to 1. Therefore, if $\omega K > c_T/(c_K - c_T)$, then there exists a finite and unique $T^*$ $(0 < T^* < \infty)$ which satisfies (2.37), and the resulting cost rate is

$$\frac{C(T^*)}{\lambda} = (c_K - c_T)\widetilde{Q}_1(T^*). \tag{2.38}$$

**(b) Optimum $N^*$**

From (2.13) and (2.35),

$$\frac{C(N)}{\lambda} = \frac{c_N + (c_K - c_N)[1 - G^{(N)}(K)]}{\sum_{j=0}^{N-1} G^{(j)}(K)} \quad (N = 1, 2, \cdots). \tag{2.39}$$

Forming the inequality $C(N + 1) - C(N) \geq 0$,

$$r_{N+1}(K)\sum_{j=0}^{N-1} G^{(j)}(K) - 1 + G^{(N)}(K) \geq \frac{c_N}{c_K - c_N}, \tag{2.40}$$

where

$$r_{N+1}(K) \equiv \frac{(\omega K)^N/N!}{\sum_{j=N}^{\infty}[(\omega K)^j/j!]} \quad (N = 0, 1, 2, \cdots)$$

increases strictly with $N$ from $\mathrm{e}^{-\omega K}$ to 1 (Problem 2.4). Thus, the left-hand side of (2.40) increases strictly with $N$ to $\omega K$. Therefore, if $\omega K > c_N/(c_K - c_N)$, then there exists a finite and unique minimum $N^*$ $(1 \leq N^* < \infty)$ which satisfies (2.40), and the resulting cost rate is

$$(c_K - c_N)r_{N^*}(K) < \frac{C(N^*)}{\lambda} \leq (c_K - c_N)r_{N^*+1}(K). \tag{2.41}$$

**(c) Optimum $Z^*$**

From (2.16) and (2.35),

$$\frac{C(Z)}{\lambda} = \frac{c_Z + (c_K - c_Z)\mathrm{e}^{-\omega(K-Z)}}{1+\omega Z}. \tag{2.42}$$

Differentiating $C(Z)$ with respect to $Z$ and setting it equal to zero,

$$\omega Z\mathrm{e}^{-\omega(K-Z)} = \frac{c_Z}{c_K - c_Z}, \tag{2.43}$$

whose left-hand side increases strictly with $Z$ from 0 to $\omega K$. Therefore, if $\omega K > c_Z/(c_K - c_Z)$, then there exists a finite and unique $Z^*$ $(0 < Z^* < K)$ which satisfies (2.43), and the resulting cost rate is

$$\frac{C(Z^*)}{\lambda} = (c_K - c_Z)\mathrm{e}^{-\omega(K-Z^*)}. \tag{2.44}$$

It is of great interest that if $\omega K > c_i/(c_K - c_i)$ $(i = T, N, Z)$, then finite $T^*$, $N^*$ and $Z^*$ always exist uniquely.

**(2) Optimum Policies with Two Variables**

We obtain optimum $(T_F^*, N_F^*)$, $(T_F^*, Z_F^*)$ and $(N_F^*, Z_F^*)$ to minimize $C_F(T,N)$, $C_F(T,Z)$ and $C_F(N,Z)$ for $c_T < c_N < c_Z$, respectively.

**(a) Optimum $T_F^*$ and $N_F^*$**

Putting that $Z = K$ in (2.35),

$$\frac{C_F(T,N)}{\lambda} = \frac{\begin{array}{l} c_T + (c_K - c_T)\sum_{j=0}^{N-1} F^{(j+1)}(T)[(\omega K)^j/j!]\mathrm{e}^{-\omega K} \\ +(c_N - c_T)F^{(N)}(T)G^{(N)}(K) \end{array}}{\sum_{j=0}^{N-1} F^{(j+1)}(T)G^{(j)}(K)}. \tag{2.45}$$

In particular, when $N = 1$, $T_F^* = \infty$, and

$$\frac{C_F(\infty,1)}{\lambda} = c_N + (c_K - c_N)\mathrm{e}^{-\omega K}.$$

We find optimum $T_F^*$ for $N \geq 2$ to minimize $C_F(T,N)$. Differentiating $C_F(T,N)$ with respect to $T$ and setting it equal to zero,

$$
\begin{aligned}
&(c_K - c_T)\left[\widetilde{Q}_1(T,N)\sum_{j=0}^{N-1}F^{(j+1)}(T)G^{(j)}(K) - \sum_{j=0}^{N-1}F^{(j+1)}(T)\frac{(\omega K)^j}{j!}\mathrm{e}^{-\omega K}\right] \\
&+ (c_N - c_T)\left[Q_4(T,N)\sum_{j=0}^{N-1}F^{(j+1)}(T)G^{(j)}(K) - F^{(N)}(T)G^{(N)}(K)\right] = c_T,
\end{aligned}
\tag{2.46}
$$

where $\widetilde{Q}_1(T,N)$ is given in (2.23) and increases strictly with $T$ to $r_N(K)$ in (2.40), and

$$
Q_4(T,N) \equiv \frac{[(\lambda T)^{N-1}/(N-1)!]G^{(N)}(K)}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j)}(K)},
$$

which increases strictly with $T$ from 0 to $G^{(N)}(K)/G^{(N-1)}(K)$ (Problem 2.8). Thus, the left-hand side of (2.46) increases strictly with $T$ from 0 to

$$
\begin{aligned}
&(c_K - c_T)\left[r_N(K)\sum_{j=0}^{N-1}G^{(j)}(K) + G^{(N)}(K) - 1\right] \\
&+ (c_N - c_T)\left\{[1 - r_N(K)]\sum_{j=0}^{N-1}G^{(j)}(K) - G^{(N)}(K)\right\}.
\end{aligned}
$$

Therefore, if

$$
\begin{aligned}
&(c_K - c_N)\left[r_N(K)\sum_{j=0}^{N-1}G^{(j)}(K) + G^{(N)}(K)\right] \\
&+ (c_N - c_T)\sum_{j=0}^{N-1}G^{(j)}(K) > c_K,
\end{aligned}
\tag{2.47}
$$

then there exists a finite $T_F^*$ $(0 < T_F^* < \infty)$ which satisfies (2.46).

In addition, letting $L(N)$ be the left-hand side of (2.47),

$$
L(N+1) - L(N) > (c_K - c_N)[r_{N+1}(K) - r_N(K)]\sum_{j=0}^{N-1}G^{(j)}(K) > 0,
$$

$$
\lim_{N\to\infty} L(N) = (c_K - c_T)(1 + \omega K).
$$

Therefore, if $\omega K \le c_T/(c_K - c_T)$, then $T_F^* = \infty$. If $\omega K > c_T/(c_K - c_T)$, then a finite $T_F^*$ might exist. On the other hand, if

$$L(2) = (c_K - c_T)G(K) - (c_K - c_N)\frac{G^{(2)}(K)}{G(K)} > c_T,$$

then a finite $T_F^*$ $(0 < T_F^* < \infty)$ always exists for $N \geq 2$, and the resulting cost rate is

$$\frac{C_F(T_F^*, N)}{\lambda} = (c_K - c_T)\widetilde{Q}_1(T_F^*, N) + (c_N - c_T)Q_4(T_F^*, N). \tag{2.48}$$

Because $L(2) > (c_N - c_T)G(K)$, if $G(K) \geq c_T/(c_N - c_T)$, then a finite $T_F^*$ exists.

In general, it is very difficult to derive optimum $N_F^*$ for given $T$ analytically (Problem 2.9). So that, we compute numerically $T_F^*$ which satisfies (2.46) and the resulting cost rate $C_F(T_F^*, N)$ in (2.48) for $N$. Comparisons for different $N$ will be shown in numerical examples.

**(b) Optimum $T_F^*$ and $Z_F^*$**

Putting that $N \to \infty$ in (2.35),

$$\frac{C_F(T, Z)}{\lambda} = \frac{\begin{array}{c} c_T + (c_K - c_Z)\sum_{j=0}^{\infty} F^{(j+1)}(T)[(\omega Z)^j/j!]\mathrm{e}^{-\omega K} \\ +(c_Z - c_T)\sum_{j=0}^{\infty} F^{(j+1)}(T)[(\omega Z)^j/j!]\mathrm{e}^{-\omega Z} \end{array}}{\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(Z)}. \tag{2.49}$$

We find optimum $T_F^*$ and $Z_F^*$ to minimize $C_F(T, Z)$. Differentiating $C_F(T, Z)$ with respect to $T$ and setting it equal to zero,

$$[(c_K - c_Z)\mathrm{e}^{-\omega(K-Z)} + (c_Z - c_T)]\left[Q_5(T, Z)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(Z) - \sum_{j=0}^{\infty} F^{(j+1)}(T)\frac{(\omega Z)^j}{j!}\mathrm{e}^{-\omega Z}\right] = c_T, \tag{2.50}$$

where

$$Q_5(T, Z) \equiv \frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][(\omega Z)^j/j!]\mathrm{e}^{-\omega Z}}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j)}(Z)}.$$

Note that $Q_5(T, K) = \widetilde{Q}_1(T)$ in (2.37), and $Q_5(T, Z)$ increases strictly with $T$ from $\mathrm{e}^{-\omega Z}$ to 1. Thus, the left-hand side of (2.50) increases strictly with $T$ from 0 to

$$\omega Z[(c_K - c_Z)\mathrm{e}^{-\omega(K-Z)} + (c_Z - c_T)].$$

Therefore, if

$$\omega Z[(c_K - c_Z)e^{-\omega(K-Z)} + (c_Z - c_T)] > c_T, \tag{2.51}$$

then there exists a finite and unique $T_F^*$ $(0 < T_F^* < \infty)$ which satisfies (2.50) for given $Z$, and the resulting cost rate is

$$\frac{C_F(T_F^*, Z)}{\lambda} = [(c_K - c_Z)e^{-\omega(K-Z)} + (c_Z - c_T)]Q_5(T_F^*, Z). \tag{2.52}$$

Clearly, if $\omega K > c_T/(c_K - c_T)$, then a finite $T_F^*$ might exist.

Next, differentiating $C_F(T, Z)$ with respect to $Z$ and setting it equal to zero,

$$\begin{aligned}[(c_K - c_Z)e^{-\omega(K-Z)} &+ (c_Z - c_T)]\sum_{j=1}^{\infty} F^{(j)}(T)G^{(j)}(Z) \\ &= c_T + (c_Z - c_T)Q_6(T, Z)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(Z),\end{aligned} \tag{2.53}$$

where

$$Q_6(T, Z) \equiv \frac{\sum_{j=0}^{\infty} F^{(j+1)}(T)[(\omega Z)^j/j!]}{\sum_{j=0}^{\infty} F^{(j+2)}(T)[(\omega Z)^j/j!]} \geq 1,$$

which increases strictly with $Z$ from $F(T)/F^{(2)}(T)$ to $Q_6(T, K)$ (Problem 2.10). Thus, the left-hand side of (2.53) increases strictly with $Z$ from 0 to

$$(c_K - c_T)\sum_{j=1}^{\infty} F^{(j)}(T)G^{(j)}(K),$$

and its right-hand side also increase strictly with $Z$ from

$$c_T + (c_Z - c_T)\frac{F(T)^2}{F^{(2)}(T)}$$

to

$$c_T + (c_Z - c_T)Q_6(T, K)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K).$$

Therefore, if

$$(c_K - c_T)\sum_{j=1}^{\infty} F^{(j)}(T)G^{(j)}(K)$$
$$> c_T + (c_Z - c_T)Q_6(T, K)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K),$$

then a finite and unique $Z_F^*$ $(0 < Z_F^* < K)$ to satisfy (2.53) for given $T$ exists. Clearly, if $\omega K > c_Z/(c_K - c_Z)$, then a finite $Z_F^*$ might exist, because $Q_6(T, K)$ decreases with $T$ to 1.

**(c) Optimum $N_F^*$ and $Z_F^*$**

Putting that $T \to \infty$ in (2.35),

$$\frac{C_F(N, Z)}{\lambda} = \frac{c_Z - (c_Z - c_N)G^{(N)}(Z) + (c_K - c_Z)\mathrm{e}^{-\omega(K-Z)}[1 - G^{(N)}(Z)]}{\sum_{j=0}^{N-1} G^{(j)}(Z)}. \tag{2.54}$$

In particular, when $N = 1$, $Z_F^* = K$, and

$$\frac{C_F(1, K)}{\lambda} = c_N + (c_K - c_N)\mathrm{e}^{-\omega K}.$$

We find optimum $N_F^*$ and $Z_F^*$ to minimize $C_F(N, Z)$. Differentiating $C_F(N, Z)$ with respect to $Z$ and setting it equal to zero for $N \geq 2$,

$$(c_K - c_Z)\mathrm{e}^{-\omega(K-Z)}\sum_{j=1}^{N} G^{(j)}(Z)$$
$$= c_Z + (c_Z - c_N)\left[\widetilde{r}_N(Z)\sum_{j=0}^{N-1} G^{(j)}(Z) - G^{(N)}(Z)\right], \tag{2.55}$$

where

$$\widetilde{r}_{N+1}(x) \equiv \frac{(\omega x)^N/N!}{\sum_{j=0}^{N-1}[(\omega x)^j/j!} \quad (N = 1, 2, \cdots).$$

The left-hand side of (2.55) increases strictly with $Z$ from 0 to

$$(c_K - c_Z)\sum_{j=1}^{N} G^{(j)}(K),$$

and its right-hand side increases strictly with $Z$ from $c_Z$ to

$$c_Z + (c_Z - c_N)\left[\widetilde{r}_N(K)\sum_{j=0}^{N-1} G^{(j)}(K) - G^{(N)}(K)\right],$$

because $\widetilde{r}_N(Z)$ increases strictly with $Z$ from 0 and decreases strictly with $N$ to 0 (Problem 2.4). Thus, if

$$(c_K - c_Z)\sum_{j=1}^{N} G^{(j)}(K) > c_Z + (c_Z - c_N)\left[\widetilde{r}_N(K)\sum_{j=0}^{N-1} G^{(j)}(K) - G^{(N)}(K)\right],$$

then there exists a unique $Z_F^*$ $(0 < Z_F^* < K)$ to satisfy (2.55), and the expected cost rate is

$$\frac{C_F(N, Z_F^*)}{\lambda} = (c_K - c_Z)\mathrm{e}^{-\omega(K - Z_F^*)} - (c_Z - c_N)\widetilde{r}_N(Z_F^*). \tag{2.56}$$

Clearly, the left-hand side of (2.55) goes to $(c_K - c_Z)\omega K$ as $Z \to K$ and $N \to \infty$, and its right-hand side goes to $c_Z$ as $N \to \infty$. Thus, if $\omega K > c_Z/(c_K - c_Z)$, then a finite $Z_F^*$ $(0 < Z_F^* < K)$ might exist.

Forming the inequality $C_F(N+1, Z) - C_F(N, Z) \geq 0$,

$$[(c_K - c_Z)\mathrm{e}^{-\omega(K-Z)} + (c_Z - c_N)]\left[r_{N+1}(Z)\sum_{j=0}^{N-1} G^{(j)}(Z) + G^{(N)}(Z) - 1\right] \geq c_N, \tag{2.57}$$

whose left-hand side increases strictly with $N$. Thus, if

$$\omega Z[(c_K - c_Z)\mathrm{e}^{-\omega(K-Z)} + (c_Z - c_N)] > c_N,$$

then there exists a finite and unique minimum $N_F^*$ $(1 \leq N_F^* < \infty)$ to satisfy (2.57) for given $Z$. Clearly, if $\omega K > c_N/(c_K - c_N)$, then a finite $N_F^*$ $(1 \leq N_F^* < \infty)$ might exist.

**(3) Numerical Examples**

When $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $G(x) = 1 - \mathrm{e}^{-\omega x}$, i.e., $F^{(j)}(t) = \sum_{i=j}^{\infty}[(\lambda t)^i/i!]\mathrm{e}^{-\lambda t}$ and $G^{(j)}(x) = \sum_{i=j}^{\infty}[(\omega x)^i/i!]\mathrm{e}^{-\omega x}$ $(j = 0, 1, 2, \cdots)$, we compute optimum $\lambda T^*$, $N^*$, $\omega Z^*$ and $(\lambda T_F^*, N_F^*)$, $(\lambda T_F^*, \omega Z_F^*)$, $(N_F^*, \omega Z_F^*)$, which minimize the expected cost rates $C(T)$ in (2.36), $C(N)$ in (2.39), $C(Z)$ in (2.42), and $C_F(T, N)$ in (2.45), $C_F(T, Z)$ in (2.49), $C_F(N, Z)$ in (2.54), respectively.

From (2.37), optimum $T^*$ satisfies

$$\frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][(\omega K)^j/j!]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]\sum_{i=j}^{\infty}[(\omega K)^i/i!]}\sum_{j=0}^{\infty}\left[\sum_{i=j}^{\infty}\frac{(\lambda T)^{i+1}}{(i+1)!}\mathrm{e}^{-\lambda T}\sum_{i=j}^{\infty}\frac{(\omega K)^i}{i!}\mathrm{e}^{-\omega K}\right]$$
$$-\sum_{j=0}^{\infty}\left[\sum_{i=j}^{\infty}\frac{(\lambda T)^{i+1}}{(i+1)!}\mathrm{e}^{-\lambda T}\right]\frac{(\omega K)^j}{j!}\mathrm{e}^{-\omega K}=\frac{c_T}{c_K-c_T},$$

and from (2.38), the resulting cost rate is

$$\frac{C(T^*)}{\lambda}=(c_K-c_T)\frac{\sum_{j=0}^{\infty}[(\lambda T^*)^j/j!][(\omega K)^j/j!]}{\sum_{j=0}^{\infty}[(\lambda T^*)^j/j!]\sum_{i=j}^{\infty}[(\omega K)^i/i!]}.$$

From (2.40), optimum $N^*$ satisfies

$$\frac{(\omega K)^N/N!}{\sum_{j=N}^{\infty}[(\omega K)^j/j!]}\sum_{j=0}^{N-1}\sum_{i=j}^{\infty}\frac{(\omega K)^i}{i!}\mathrm{e}^{-\omega K}-\sum_{j=0}^{N-1}\frac{(\omega K)^j}{j!}\mathrm{e}^{-\omega K}\geq\frac{c_N}{c_K-c_N},$$

and from (2.39), the resulting cost rate is

$$\frac{C(N^*)}{\lambda}=\frac{c_K-(c_K-c_N)\sum_{j=N^*}^{\infty}[(\omega K)^j/j!]\mathrm{e}^{-\omega K}}{\sum_{j=0}^{N^*-1}\sum_{i=j}^{\infty}[(\omega K)^i/i!]\mathrm{e}^{-\omega K}}.$$

Optimum $\omega Z^*$ and its resulting cost rate $C(Z^*)/\lambda$ satisfy (2.43) and (2.44), respectively.

Tables 2.1, 2.2 and 2.3 presents optimum $\lambda T^*$, $N^*$ and $\omega Z^*$, and their cost rates $C(T^*)/(\lambda c_T)$, $C(N^*)/(\lambda c_N)$ and $C(Z^*)/(\lambda c_Z)$ for $\omega K$ and $c_K/c_T$, $c_K/c_N$ and $c_K/c_Z$. All of optimum values $\lambda T^*$, $N^*$ and $\omega Z^*$ increase with $\omega K$ and decrease with $c_K/c_i$ $(i=T,N,Z)$, and their cost rates decrease with $\omega K$ and increase with $c_K/c_i$. When $\omega K=5.0, 10.0$, optimum $\lambda T^*$, $N^*$ and $\omega Z^*$ are almost the same, and when $\omega K=20.0$, $\lambda T^*<N^*<\omega Z^*$. It can be understood that when a failure level $K$ becomes smaller or cost $c_K$ for failure becomes larger, early replacement should be decided and it's replacement cost rate becomes higher.

When the same values $c_K/c_i$ $(i=T,N,Z)$ are assigned, it can be found from Tables 2.1, 2.2 and 2.3 that $C(T^*)>C(N^*)>C(Z^*)$. As is known, unit failure is caused by total cumulative damage, that is, if more precise damage or shock could be observed and monitored, more effective replacement actions can be done. However, monitoring costs should not be neglected in practice, it is more reasonable to suppose $c_T<c_N<c_Z$ when replacement polices are valued, which will be shown in Tables 2.4, 2.5 and 2.6.

**Table 2.1** Optimum $\lambda T^*$ and its cost rate $C(T^*)/(\lambda c_T)$

| $c_K/c_T$ | $\omega K = 5.0$ | | $\omega K = 10.0$ | | $\omega K = 20.0$ | |
|---|---|---|---|---|---|---|
| | $\lambda T^*$ | $C(T^*)/(\lambda c_T)$ | $\lambda T^*$ | $C(T^*)/(\lambda c_T)$ | $\lambda T^*$ | $C(T^*)/(\lambda c_T)$ |
| 5 | 3.328 | 0.617 | 5.750 | 0.260 | 11.835 | 0.106 |
| 10 | 2.221 | 0.910 | 4.378 | 0.334 | 9.971 | 0.124 |
| 15 | 1.804 | 1.130 | 3.805 | 0.382 | 9.137 | 0.134 |
| 20 | 1.567 | 1.317 | 3.460 | 0.420 | 8.617 | 0.142 |
| 30 | 1.292 | 1.637 | 3.040 | 0.479 | 7.962 | 0.153 |
| 50 | 1.019 | 2.162 | 2.594 | 0.564 | 7.235 | 0.168 |

**Table 2.2** Optimum $N^*$ and its cost rate $C(N^*)/(\lambda c_N)$

| $c_K/c_N$ | $\omega K = 5.0$ | | $\omega K = 10.0$ | | $\omega K = 20.0$ | |
|---|---|---|---|---|---|---|
| | $N^*$ | $C(N^*)/(\lambda c_N)$ | $N^*$ | $C(N^*)/(\lambda c_N)$ | $N^*$ | $C(N^*)/(\lambda c_N)$ |
| 5 | 3 | 0.508 | 6 | 0.213 | 13 | 0.089 |
| 10 | 2 | 0.684 | 5 | 0.253 | 12 | 0.100 |
| 15 | 2 | 0.786 | 5 | 0.283 | 11 | 0.105 |
| 20 | 2 | 0.887 | 4 | 0.299 | 10 | 0.110 |
| 30 | 2 | 1.090 | 4 | 0.325 | 10 | 0.115 |
| 50 | 1 | 1.330 | 4 | 0.377 | 9 | 0.122 |

**Table 2.3** Optimum $\omega Z^*$ and its cost rate $C(Z^*)/(\lambda c_Z)$

| $c_K/c_Z$ | $\omega K = 5.0$ | | $\omega K = 10.0$ | | $\omega K = 20.0$ | |
|---|---|---|---|---|---|---|
| | $\omega Z^*$ | $C(Z^*)/(\lambda c_Z)$ | $\omega Z^*$ | $C(Z^*)/(\lambda c_Z)$ | $\omega Z^*$ | $C(Z^*)/(\lambda c_Z)$ |
| 5 | 2.642 | 0.378 | 6.710 | 0.149 | 15.850 | 0.063 |
| 10 | 2.073 | 0.482 | 6.009 | 0.166 | 15.089 | 0.066 |
| 15 | 1.783 | 0.561 | 5.632 | 0.178 | 14.675 | 0.068 |
| 20 | 1.591 | 0.628 | 5.374 | 0.186 | 14.389 | 0.069 |
| 30 | 1.340 | 0.746 | 5.019 | 0.199 | 13.994 | 0.071 |
| 50 | 1.055 | 0.948 | 4.585 | 0.218 | 13.505 | 0.074 |

Tables 2.4, 2.5 and 2.6 presents optimum $(\lambda T_F^*, N_F^*)$, $(\lambda T_F^*, \omega Z_F^*)$ and $(N_F^*, \omega Z_F^*)$, and their cost rates $C_F(T_F^*, N_F^*)/\lambda c_Z$, $C_F(T_F^*, Z_F^*)/\lambda c_Z$ and $C_F(N_F^*, \omega Z_F^*)/\lambda c_Z$ for $\omega K$ and $c_K/c_Z$ when $\omega K = 10$ and $c_T : c_N : c_Z = 1 : 2 : 3$.

For optimum $(\lambda T_F^*, N_F^*)$ and $C_F(T_F^*, N_F^*)/\lambda c_Z$:

1. When $N = 1$, $T_F^* = \infty$,

$$\frac{C_F(\infty, 1)}{\lambda c_Z} = \frac{c_N}{c_Z} + \left(\frac{c_K}{c_Z} - \frac{c_N}{c_Z}\right) \mathrm{e}^{-\omega K}.$$

**Table 2.4** Optimum $(\lambda T_F^*, N_F^*)$ and its cost rate $C_F(T_F^*, N_F^*)/(\lambda c_Z)$ when $c_T : c_N : c_Z = 1 : 2 : 3$

| $c_K/c_Z$ | $\omega K = 10.0$ | | | $\omega K = 20.0$ | | |
|---|---|---|---|---|---|---|
| | $\lambda T_F^*$ | $N_F^*$ | $C_F(T_F^*, N_F^*)/(\lambda c_Z)$ | $\lambda T_F^*$ | $N_F^*$ | $C_F(T_F^*, N_F^*)/(\lambda c_Z)$ |
| 5 | 3.887 | 8 | 0.127 | 9.199 | 17 | 0.045 |
| 10 | 3.331 | 6 | 0.157 | 8.204 | 14 | 0.051 |
| 15 | 2.933 | 6 | 0.177 | 7.722 | 13 | 0.054 |
| 20 | 2.912 | 5 | 0.191 | 7.337 | 13 | 0.057 |
| 30 | 2.553 | 5 | 0.216 | 6.959 | 12 | 0.061 |
| 50 | 2.527 | 4 | 0.247 | 6.523 | 11 | 0.065 |

**Table 2.5** Optimum $(\lambda T_F^*, \omega Z_F^*)$ and its cost rate $C_F(T_F^*, Z_F^*)/(\lambda c_Z)$ when $c_T : c_N : c_Z = 1 : 2 : 3$

| $c_K/c_Z$ | $\omega K = 10.0$ | | | $\omega K = 20.0$ | | |
|---|---|---|---|---|---|---|
| | $\lambda T_F^*$ | $\omega Z_F^*$ | $C_F(T_F^*, Z_F^*)/(\lambda c_Z)$ | $\lambda T_F^*$ | $\omega Z_F^*$ | $C_F(T_F^*, Z_F^*)/(\lambda c_Z)$ |
| 5 | 4.932 | 7.878 | 0.111 | 11.008 | 17.471 | 0.040 |
| 10 | 4.502 | 7.136 | 0.127 | 10.474 | 16.692 | 0.042 |
| 15 | 4.274 | 6.735 | 0.137 | 10.185 | 16.268 | 0.043 |
| 20 | 4.119 | 6.459 | 0.145 | 9.986 | 15.976 | 0.044 |
| 30 | 3.909 | 6.079 | 0.158 | 9.712 | 15.571 | 0.046 |
| 50 | 3.656 | 5.611 | 0.176 | 9.374 | 15.070 | 0.048 |

**Table 2.6** Optimum $(N_F^*, \omega Z_F^*)$ and its cost rate $C_F(N_F^*, Z_F^*)/(\lambda c_Z)$ when $c_T : c_N : c_Z = 1 : 2 : 3$

| $c_K/c_Z$ | $\omega K = 10.0$ | | | $\omega K = 20.0$ | | |
|---|---|---|---|---|---|---|
| | $N_F^*$ | $\omega Z_F^*$ | $C_F(N_F^*, Z_F^*)/(\lambda c_Z)$ | $N_F^*$ | $\omega Z_F^*$ | $C_F(N_F^*, Z_F^*)/(\lambda c_Z)$ |
| 5 | 7 | 7.435 | 0.140 | 15 | 16.920 | 0.056 |
| 10 | 7 | 6.522 | 0.158 | 14 | 16.195 | 0.059 |
| 15 | 6 | 6.309 | 0.168 | 14 | 15.706 | 0.061 |
| 20 | 6 | 5.962 | 0.177 | 14 | 15.367 | 0.063 |
| 30 | 6 | 5.486 | 0.190 | 13 | 15.078 | 0.064 |
| 50 | 5 | 5.217 | 0.209 | 13 | 14.496 | 0.067 |

2. For $N = 2, 3, 4, \cdots$, compute $\lambda T_N$ to satisfy (2.46).
3. Compute and compare $C_F(\infty, 1)$ and $C_F(T_N, N)$ $(N = 2, 3, \cdots)$ to determine optimum $T_F^* = T_N$ and $N_F^* = N$, where

$$\frac{C_F(T_F^*, N_F^*)}{\lambda c_Z} = \left(\frac{c_K}{c_Z} - \frac{c_T}{c_Z}\right) \widetilde{Q}_1(T_F^*, N_F^*) + \left(\frac{c_N}{c_Z} - \frac{c_T}{c_Z}\right) Q_4(T_F^*, N_F^*).$$

For optimum $(\lambda T_F^*, \omega Z_F^*)$ and $C_F(T_F^*, Z_F^*)/\lambda c_Z$:

1. Compute $T_Z$ for $0 < Z \le K$ to satisfy (2.50), and compute $Z_T$ for $0 < T \le \infty$ to satisfy (2.53).
2. Let $Z = K$ and compute $T_{Z1}$, compute $Z_{T1}$ for $T_{Z1}$, compute $T_{Z2}$ for $Z_{T1}, \cdots$, to determine $\lambda T_F^* = T_{Zi}$ and $\omega Z_F^* = Z_{Ti}$ $(i = 1, 2, \cdots)$, where

$$\frac{C_F(T_F^*, Z_F^*)}{\lambda c_Z} = \left[\left(\frac{c_K}{c_Z} - 1\right) e^{-\omega(K - Z_F^*)} + \left(1 - \frac{c_T}{c_Z}\right)\right] Q_5(T_F^*, Z_F^*).$$

For optimum $(N_F^*, \omega Z_F^*)$ and $C_F(N_F^*, Z_F^*)/\lambda c_Z$:

1. When $N = 1$, $Z_F^* = K$, and

$$\frac{C_F(1, K)}{\lambda c_Z} = \frac{c_N}{c_Z} + \left(\frac{c_K}{c_Z} - \frac{c_N}{c_Z}\right) e^{-\omega K}.$$

2. For $N = 2, 3, \cdots$, compute $Z_N$ to satisfy (2.55).
3. Compute and compare $C_F(1, K)$ and $C_F(N, Z_N)$ $(N = 2, 3, \cdots)$ to determine optimum $Z_F^* = Z_N$ and $N_F^* = N$, where

$$\frac{C_F(N_F^*, Z_N)}{\lambda c_Z} = \left(\frac{c_K}{c_Z} - 1\right) e^{-\omega(K - Z_F^*)} - \left(1 - \frac{c_N}{c_Z}\right) \frac{(\omega Z_F^*)^{N_F^* - 1}/(N_F^* - 1)!}{\sum_{j=0}^{N_F^* - 2} [(\omega Z_F^*)^j / j!]}.$$

These tables indicate that when $c_T : c_N : c_Z = 1 : 2 : 3$, $\lambda T_F^* < N_F^*$ and $\lambda T_F^* < \omega Z_F^*$, but $N_F^*$ and $\omega Z_F^*$ are almost the same. As the scale of $c_T$, $c_N$ and $c_Z$ is roughly given, we cannot compare optimum policies $(\lambda T_F^*, N_F^*)$, $(\lambda T_F^*, \omega Z_F^*)$ and $(N_F^*, \omega Z_F^*)$ exactly. However, we may conclude: (i) Optimum policy $(\lambda T_F^*, N_F^*)$ saves more cost rate than $(\lambda T_F^*, \omega Z_F^*)$ does, though $c_Z$ is greater than $c_N$. (ii) Optimum policy $(\lambda T_F^*, \omega Z_F^*)$ is better than $(N_F^*, \omega Z_F^*)$, which may be due to $c_T < c_N$. (iii) Either $(\lambda T_F^*, N_F^*)$ or $(N_F^*, \omega Z_F^*)$ would be better, though $c_T$ is much less than $c_Z$. The reasons of (i)–(iii) should be explored further, and different scales of costs $c_T$, $c_N$ and $c_Z$ will be discussed in Chap. 4.

## 2.2 Random Failure Levels

Units operating in different degradation environments exhibit varying levels of toughness, in which case, the above defined failure threshold $K$ could not be determined as constant values; however, its probability distribution can be estimated statistically from historical failure data. In this section, a level of failure threshold $K$ is supposed to be a random variable and has a general distribution $L(x) \equiv \Pr\{K \le x\}$ with finite mean $1/\theta$ and $L(0) = 0$ [2].

When the unit is replaced preventively at time $T$ $(0 < T \le \infty)$ or at shock $N$ $(N = 1, 2, \cdots)$, whichever occurs first, the expected cost rate is [2]

$$C_R(T,N;L)=\frac{\begin{array}{c}c_K-(c_K-c_N)F^{(N)}(T)\int_0^\infty G^{(N)}(x)\mathrm{d}L(x)\\ -(c_K-c_T)\sum_{j=0}^{N-1}[F^{(j)}(T)-F^{(j+1)}(T)]\int_0^\infty G^{(j)}(x)\mathrm{d}L(x)\end{array}}{\sum_{j=0}^{N-1}\int_0^T[F^{(j)}(t)-F^{(j+1)}(t)]\mathrm{d}t\int_0^\infty G^{(j)}(x)\mathrm{d}L(x)}. \tag{2.58}$$

When $L(x)=1-\mathrm{e}^{-\theta x}$ and the unit is replaced preventively only at time $T$, the expected cost rate is

$$\begin{aligned}C_R(T;\theta)&\equiv\lim_{N\to\infty}C_R(T,N;L)\\&=\frac{c_K-(c_K-c_T)\sum_{j=0}^{\infty}[G^*(\theta)]^j[F^{(j)}(T)-F^{(j+1)}(T)]}{\sum_{j=0}^{\infty}[G^*(\theta)]^j\int_0^T[F^{(j)}(t)-F^{(j+1)}(t)]\mathrm{d}t},\end{aligned} \tag{2.59}$$

where $G^*(\theta)\equiv\int_0^\infty \mathrm{e}^{-\theta x}\mathrm{d}G(x)$, which represents the Laplace-Stieltjes transform of $G(x)$.

Furthermore, when shocks occur at a nonhomogeneous Poisson process with a cumulative hazard rate $H(t)\equiv\int_0^t h(u)\mathrm{d}u$, i.e., $h(t)\equiv \mathrm{d}H(t)/\mathrm{d}t$ and $F^{(j)}(t)=\sum_{i=j}^{\infty}[H(t)^i/i!]\mathrm{e}^{-H(t)}$ $(j=0,1,2,\cdots)$ from (1.6), the expected cost rate in (2.59) is

$$C_R(T;\theta)=\frac{c_K-(c_K-c_T)\exp\{-[1-G^*(\theta)]H(T)\}}{\int_0^T\exp\{-[1-G^*(\theta)]H(t)\}\mathrm{d}t}. \tag{2.60}$$

If $h(t)$ increases strictly with $t$ to $h(\infty)$, and

$$[1-G^*(\theta)]h(\infty)\int_0^\infty\exp\left\{-[1-G^*(\theta)]H(t)\right\}\mathrm{d}t>\frac{c_K}{c_K-c_T},$$

then there exists a finite and unique $T_R^*$ $(0<T_R^*<\infty)$ which satisfies (Problem 2.11)

$$\begin{aligned}&[1-G^*(\theta)]h(T)\int_0^T\exp\left\{-[1-G^*(\theta)]H(t)\right\}\mathrm{d}t\\&+\exp\left\{-[1-G^*(\theta)]H(T)\right\}=\frac{c_K}{c_K-c_T},\end{aligned} \tag{2.61}$$

and the resulting cost rate is

$$C_R(T_R^*;\theta)=(c_K-c_T)[1-G^*(\theta)]h(T_R^*). \tag{2.62}$$

When the unit is replaced preventively only at shock $N$, the expected cost rate is

$$C_R(N;L) \equiv \lim_{T\to\infty} C_R(T,N;L)$$
$$= \frac{c_K - (c_K - c_N)\int_0^\infty G^{(N)}(x)\mathrm{d}L(x)}{\mu\sum_{j=0}^{N-1}\int_0^\infty G^{(j)}(x)\mathrm{d}L(x)} \quad (N=1,2,\cdots). \tag{2.63}$$

Forming the inequality $C_R(N+1;L) - C_R(N;L) \geq 0$,

$$Q_7(N)\sum_{j=0}^{N-1}\int_0^\infty G^{(j)}(x)\mathrm{d}L(x) + \int_0^\infty G^{(N)}(x)\mathrm{d}L(x) \geq \frac{c_K}{c_K - c_N}, \tag{2.64}$$

where

$$Q_7(N) \equiv \frac{\int_0^\infty [G^{(N)}(x) - G^{(N+1)}(x)]\mathrm{d}L(x)}{\int_0^\infty G^{(N)}(x)\mathrm{d}L(x)} \leq 1.$$

If $Q_7(N)$ increases strictly with $N$ to 1, then the left-hand side of (2.64) increases strictly with $N$ to $\int_0^\infty [1 + M_G(x)]\mathrm{d}L(x)$, where $M_G(x)$ is given in (2.7). Therefore, if

$$\int_0^\infty M_G(x)\mathrm{d}L(x) > \frac{c_N}{c_K - c_N},$$

then there exists a finite and unique minimum $N_R^*$ $(1 \leq N_R^* < \infty)$ which satisfies (2.64), and the resulting cost rate is (Problem 2.12)

$$(c_K - c_N)Q_7(N_R^* - 1) < \mu C_R(N_R^*;L) \leq (c_K - c_N)Q_7(N_R^*). \tag{2.65}$$

In particular, when $L(x) = 1 - \mathrm{e}^{-\theta x}$, $Q_7(N) = 1 - G^*(\theta)$, and $N_R^* = \infty$. Table 2.7 presents optimum $T_R^*$ and its cost rate $C_R(T_R^*;\theta)/c_T$ for $G^*(\theta)$ and $c_K/c_T$ when $H(t) = t^{2.0}$. Obviously, optimum $T_R^*$ increases with $G^*(\theta)$ and decreases with $c_K/c_T$, and its cost rate decreases with $G^*(\theta)$ and increases with $c_K/c_T$.

**Table 2.7** Optimum $T_R^*$ and its cost rate $C_R(T_R^*;\theta)/c_T$ when $H(t) = t^2$

| $c_K/c_T$ | $G^*(\theta) = 0.1$ | | $G^*(\theta) = 0.5$ | | $G^*(\theta) = 0.9$ | |
|---|---|---|---|---|---|---|
| | $T_R^*$ | $C_R(T_R^*;\theta)/c_T$ | $T_R^*$ | $C_R(T_R^*;\theta)/c_T$ | $T_R^*$ | $C_R(T_R^*;\theta)/c_T$ |
| 5 | 0.538 | 3.875 | 0.772 | 2.888 | 1.615 | 1.292 |
| 10 | 0.355 | 5.746 | 0.476 | 4.283 | 1.064 | 1.915 |
| 15 | 0.283 | 7.142 | 0.380 | 5.322 | 0.850 | 2.380 |
| 20 | 0.243 | 8.305 | 0.326 | 6.191 | 0.729 | 2.769 |
| 30 | 0.196 | 10.247 | 0.263 | 7.640 | 0.589 | 3.416 |
| 50 | 0.151 | 13.303 | 0.202 | 9.918 | 0.452 | 4.434 |

## 2.3 Double Failure Modes

In crack growth models [27, 28] for aircrafts, it has been well-known that the unit would be failure when the size of one crack exceeds a threshold level, or when the total sizes of all cracks attain to a certain level, e.g., multi-site damage, which is defined as the simultaneous occurrence of many tiny fatigue cracks at multiple locations in the same structural element, and has become recently a major issue of aging aircrafts since the Aloha Airlines affair in 1988 [29].

This section takes up one extended model where the unit fails when the total damage has exceeded a failure threshold $K$ $(0 < K < \infty)$ in Sect. 2.1, and also fails when the total number of shocks reaches to a certain value of $N$ $(N = 1, 2, \cdots)$, whichever occurs first, and corrective replacement is done immediately when the failure is detected. As preventive replacement policies, the unit is replaced preventively at time $T$ $(0 < T \leq \infty)$, or at damage $Z$ $(0 < Z \leq K)$, whichever occurs first.

The probability that the unit is replaced at time $T$, at shock $N$, at damage $Z$, and at failure $K$ are respectively given in (2.1)–(2.4), and the mean time to replacement is given in (2.5). Then, the expected replacement cost rate is

$$C_N(T,Z) = \frac{\begin{array}{c} c_K - (c_K - c_T)\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(Z) \\ -(c_K - c_Z)\sum_{j=0}^{N-1} F^{(j+1)}(T)\int_0^Z [G(K-x) - G(Z-x)]\mathrm{d}G^{(j)}(x) \end{array}}{\sum_{j=0}^{N-1} G^{(j)}(Z)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \quad (2.66)$$

where $c_T$ and $c_Z$ are defined in (2.6), and $c_K$ is replacement cost at shock $N$ and at damage $K$ where $c_K > c_T$ and $c_K > c_Z$.

When the unit is replaced preventively only at time $T$ $(0 < T \leq \infty)$,

$$C_N(T) \equiv \lim_{Z \to K} C_N(T,Z) = \frac{c_K - (c_K - c_T)\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{N-1} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \quad (2.67)$$

and when the unit is replaced preventively only at damage $Z$ $(0 < Z \leq K)$,

$$C_N(Z) \equiv \lim_{T \to \infty} C_N(T,Z) = \frac{c_K - (c_K - c_Z)\sum_{j=0}^{N-1}\int_0^Z [G(K-x) - G(Z-x)]\mathrm{d}G^{(j)}(x)}{\mu \sum_{j=0}^{N-1} G^{(j)}(Z)}. \quad (2.68)$$

### (1) Optimum $T_N^*$

We find optimum $T_N^*$ to minimize $C_N(T)$ in (2.67) for given $N$ and $K$. In particular, when $N = 1$,

$$C_1(T) = \frac{c_K - (c_K - c_T)\overline{F}(T)}{\int_0^T \overline{F}(t)\mathrm{d}t}, \tag{2.69}$$

which corresponds to the expected cost rate of the standard age replacement policy [1]. When shocks occur at a nonhomogeneous Poisson process with a cumulative hazard rate $H(t)$, i.e., $F(t) = 1 - \mathrm{e}^{-H(t)}$, and $h(t) \equiv \mathrm{d}H(t)/\mathrm{d}t$ increases strictly with $t$ to $h(\infty) \equiv \lim_{t\to\infty} h(t)$ and $\mu h(\infty) > c_K/(c_K - c_T)$, optimum $T_1^*$ $(0 < T_1^* < \infty)$ satisfies

$$h(T)\int_0^T \overline{F}(t)\mathrm{d}t + \overline{F}(T) = \frac{c_K}{c_K - c_T}, \tag{2.70}$$

and the resulting cost rate is

$$C_1(T_1^*) = (c_K - c_T)h(T_1^*). \tag{2.71}$$

From (2.67),

$$\begin{aligned} C_N(0) &\equiv \lim_{T\to 0} C_N(T) = \infty, \\ C_N(\infty) &\equiv \lim_{T\to\infty} C_N(T) = \frac{c_K}{\mu\sum_{j=0}^{N-1} G^{(j)}(K)}. \end{aligned} \tag{2.72}$$

Thus, there exist a positive $T_N^*$ $(0 < T_N^* \le \infty)$ to minimize $C_N(T)$.

Differentiating $C_N(T)$ with respect to $T$ and setting it equal to zero,

$$\begin{aligned} &Q_8(T,N)\sum_{j=0}^{N-1} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t \\ &+ \sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K) = \frac{c_K}{c_K - c_T}, \end{aligned} \tag{2.73}$$

where

$$Q_8(T,N) \equiv \frac{-\sum_{j=0}^{N-1}[f^{(j)}(T) - f^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)},$$

and $f^{(j)}(t) \equiv \mathrm{d}F^{(j)}(t)/\mathrm{d}t$. If $Q_8(T,N)$ increases strictly with $T$ to $Q_8(\infty,N)$, then left-hand side of (2.73) also increases strictly from 1 to $\mu Q_8(\infty,N)\sum_{j=0}^{N-1} G^{(j)}(K)$. Therefore, if

$$\mu Q_8(\infty,N)\sum_{j=0}^{N-1} G^{(j)}(K) > \frac{c_K}{c_K - c_T},$$

then there exists a finite and unique $T_N^*$ $(0 < T_N^* < \infty)$ which satisfies (2.73), and the resulting cost rate is

$$C_N(T_N^*) = (c_K - c_T)Q_8(T_N^*, N). \tag{2.74}$$

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, the expected cost rate in (2.67) is

$$\frac{C_N(T)}{\lambda} = \frac{c_K - (c_K - c_T)\sum_{j=0}^{N-1} F^{(j)}(T)G^{(j)}(K)}{\sum_{j=0}^{N-1} F^{(j+1)}(T)G^{(j)}(K)} + (c_K - c_T), \tag{2.75}$$

which decreases strictly with $T$. Thus, $T_N^* = \infty$, and the resulting cost rate is given in (2.72).

### (2) Optimum $Z_N^*$

We find optimum $Z_N^*$ to minimize $C_N(Z)$ in (2.68) for given $N$ and $K$. In particular, when $N = 1$, $C_1(Z)$ increases with $Z$, and hence, $Z_1^* = 0$, and when $N \to \infty$, $C_N(Z)$ is given in (2.16).

Differentiating $C_N(Z)$ with respect to $Z$ and setting it equal to zero for $N \geq 2$,

$$Q_9(Z)\sum_{j=0}^{N-1} G^{(j)}(Z) + \sum_{j=0}^{N-1}\int_0^Z [\overline{G}(K-Z) + \overline{G}(Z-x) - \overline{G}(K-x)]\mathrm{d}G^{(j)}(x)$$
$$= \frac{c_K}{c_K - c_Z}, \tag{2.76}$$

where

$$Q_9(Z) \equiv \frac{g^{(N)}(Z)}{\sum_{j=1}^{N-1} g^{(j)}(Z)},$$

and $g^{(N)}(x) \equiv \mathrm{d}G^{(N)}(x)/\mathrm{d}x$.

If $Q_9(Z)$ increases strictly with $Z$ from 0 to $Q_9(K)$, then the left-hand side of (2.76) also increases strictly with $Z$ from 1 to $[Q_9(K) + 1]\sum_{j=0}^{N-1} G^{(j)}(K)$. Therefore, if $[Q_9(K) + 1]\sum_{j=0}^{N-1} G^{(j)}(K) > c_K/(c_K - c_Z)$, then there exists a finite and unique $Z_N^*$ $(0 < Z_N^* < K)$ which satisfies (2.76), and the resulting cost rate is

$$\mu C_N(Z_N^*) = (c_K - c_Z)[Q_9(Z_N^*) + \mathrm{e}^{-\omega(K - Z_N^*)}]. \tag{2.77}$$

Conversely, if $[Q_9(K) + 1]\sum_{j=0}^{N-1} G^{(j)}(K) \leq c_K/(c_K - c_Z)$, then $Z_N^* = K$, and the resulting cost rate is given in (2.72).

When $G(x) = 1 - \mathrm{e}^{-\omega x}$,

$$Q_9(Z) = \frac{(\omega Z)^{N-1}/(N-1)!}{\sum_{j=0}^{N-2}[(\omega Z)^j/j!]} \quad (N = 2, 3, \cdots),$$

which agrees with $\widetilde{r}_N(Z)$ in (2.55), and increases strictly with $Z$ from 0 and decreases strictly with $N$ to 0 (Problem 2.4). Thus, (2.76) becomes

$$[\widetilde{r}_N(Z) + 1] \sum_{j=0}^{N-1} G^{(j)}(Z) - [1 - \mathrm{e}^{-\omega(K-Z)}] \sum_{j=1}^{N} G^{(j)}(Z) = \frac{c_K}{c_K - c_Z}, \tag{2.78}$$

whose left-hand side increases strictly with $Z$ from 1. Therefore, if

$$[\widetilde{r}_N(K) + 1] \sum_{j=0}^{N-1} G^{(j)}(K) > \frac{c_K}{c_K - c_Z},$$

then there exists a finite and unique $Z_N^*$ $(0 < Z_N^* < K)$ which satisfies (2.78). If $\omega K > c_Z/(c_K - c_Z)$, then a finite $Z_N^*$ might exist.

## 2.4 Problem 2

2.1 Prove that (2.1)+(2.2)+(2.3)+(2.4)=1 and derive (2.5).

2.2 Prove that if $Q_1(T)$ increases strictly with $T$ to $Q_1(\infty)$, then the left-hand side of (2.10) increases strictly to $Q_1(\infty)\mu[1 + M_G(K)] - 1$. Furthermore, prove in (2.23) that when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $r_{j+1}(x)$ increases strictly with $j$,

$$\widetilde{Q}_1(T, N) = \frac{\sum_{j=0}^{N-1}[(\lambda T)^j/j!][G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j)}(K)}$$

increases strictly with $T$ from $\overline{G}(K)$ to $r_N(K)$ for $N \geq 2$, and increases strictly with $N$ from $\overline{G}(K)$ to $Q_1(T)/\lambda$.

2.3 Show why the inequality $C(N+1) - C(N) \geq 0$ should be formulated.

2.4 Prove that if $r_{N+1}(K)$ increases strictly with $N$, then the left-hand side of (2.14) increases with $N$ to $r_\infty(K)[1 + M_G(K)] - 1$. Furthermore, prove that when $G(x) = 1 - \mathrm{e}^{-\omega x}$ for $0 < x \leq K$,

$$r_{N+1}(x) = \frac{(\omega x)^N/N!}{\sum_{j=N}^{\infty}(\omega x)^j/j!} \quad (N = 0, 1, 2, \cdots)$$

increases strictly with $N$ from $\mathrm{e}^{-\omega K}$ to 1, and decreases strictly with $x$ from 1 to $r_{N+1}(K)$. Prove that

$$\widetilde{r}_{N+1}(x) = \frac{(\omega x)^N/N!}{\sum_{j=0}^{N-1}(\omega x)^j/j!} \quad (N = 1, 2, \cdots)$$

decreases strictly with $N$ from $\omega x$ to 0 and increases strictly with $x$ from 0 to $\widetilde{r}_{N+1}(K)$.

2.5 Derive (2.17) and prove that its left-hand side increases strictly with $Z$ from 0 to $M_G(K)$.

2.6 Prove that

$$\frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j)}(Z)}$$

increases with $T$ from $\overline{G}(K)$ and is less than $\overline{G}(K-Z)$.

2.7 Prove that (2.33) does not hold for any $Z$, and show that there does not exist any finite $N_F^*$ which satisfies (2.32).

2.8 Prove that $Q_4(T, N)$ increases strictly with $T$ from 0 to $G^{(N)}(K)/G^{(N-1)}(K)$.

2.9 Challenge to derive optimum $N_F^*$ for given $T$ for $c_N > c_T$.

2.10 Prove that $Q_6(T, Z)$ increases strictly with $Z$ from $F(T)/F^{(2)}(T)$ to $Q_6(T, K)$, and decreases strictly with $T$ from $\infty$ to 1.

2.11 Derive (2.61) and prove its result.

2.12 Derive (2.64) and prove its result.

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