# Chapter 2
# Dynamics of Service Collapse

## 2.1 Introduction

To reiterate a central point, unlike aircraft, that can be constructed to be inherently stable in linear flight by placing the aerodynamic center of pressure sufficiently behind the mechanical center of gravity, the complex nature of road geometry and the local dynamics of vehicular traffic ensure that V2V/V2I systems will be inherently unstable, requiring constant input of control information to prevent crashes, traffic jams, and other tie-ups.

The Data Rate Theorem (Nair et al. 2007) establishes the minimum rate at which externally-supplied control information must be provided for an inherently unstable system to maintain stability. Given the linear expansion near a nonequilibrium steady state, an n-dimensional vector of system parameters at time $t$, $x_t$, determines the state at time $t+1$ according to the model of Fig. 2.1, so that

$$x_{t+1} = \mathbf{A}x_t + \mathbf{B}u_t + W_t \tag{2.1}$$

where $\mathbf{A}, \mathbf{B}$ are fixed $n \times n$ matrices, $u_t$ is the vector of control information, and $W_t$ is an n-dimensional vector of white noise. The Data Rate Theorem (DRT) under such conditions states that the minimum control information rate $\mathscr{H}$ is determined by the relation

$$\mathscr{H} > \log[|\det[\mathbf{A}^m|] \equiv a_0 \tag{2.2}$$

where, for $m \leq n$, $\mathbf{A}^m$ is the subcomponent of $\mathbf{A}$ having eigenvalues $\geq 1$. The right hand side of Eq. (2.2) is interpreted as the rate at which the system generates 'topological information'. The proof of Eq. (2.2) is not particularly straightforward (Nair et al. 2007), and the Mathematical Appendix uses the Rate Distortion Theorem (RDT) to derive a more general version of the DRT.



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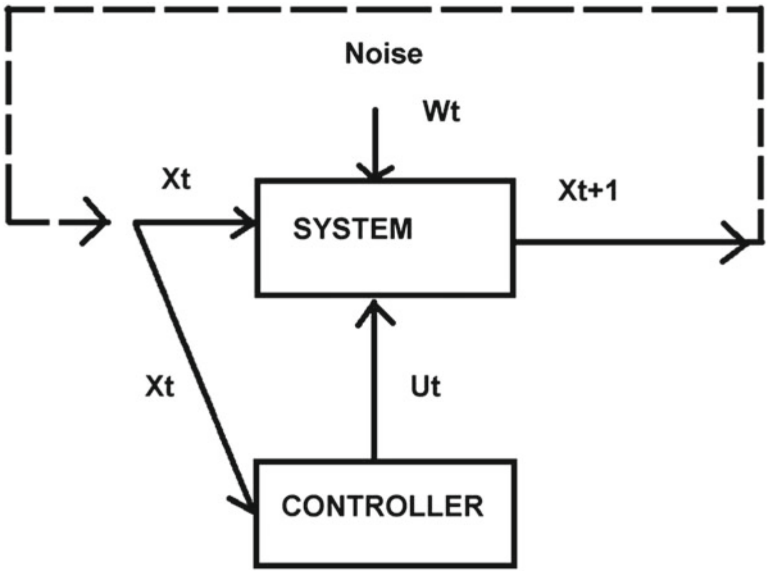


**Fig. 2.1** A linear expansion near a nonequilibrium steady state of an inherently unstable control system, for which $x_{t+1} = \mathbf{A}x_t + \mathbf{B}u_t + W_t$. $\mathbf{A}, \mathbf{B}$ are square matrices, $x_t$ the vector of system parameters at time $t$, $u_t$ the control vector at time $t$, and $W_t$ a white noise vector. The Data Rate Theorem states that the minimum rate at which control information must be provided for system stability is $\mathscr{H} > \log[|\det[\mathbf{A}^m|]$, where $\mathbf{A}^m$ is the subcomponent of $\mathbf{A}$ having eigenvalues $\geq 1$

For a simple traffic flow system on a fixed highway network, the source of 'topological information' is the linear vehicle density $\rho$. The 'fundamental diagram' of traffic flow studies relates the total vehicle flow to the linear vehicle density, shown in Fig. 1.1. A similar pattern can be expected from 'macroscopic fundamental diagrams' that examine multimodal travel networks (Geroliminis et al. 2014; Chiabaut 2015).

Given $\rho$ as the critical traffic density parameter, we can extend Eq. (2.2) as

$$\mathscr{H}(\rho) > f(\rho)a_0 \tag{2.3}$$

where $a_0$ is a road network constant and $f(\rho)$ is a positive, monotonically increasing function. The Mathematical Appendix uses a Black-Scholes model to approximate the 'cost' of $\mathscr{H}$ as a function of the 'investment' $\rho$. The first approximation is linear, so that $\mathscr{H} \approx \kappa_1\rho + \kappa_2$. Expanding $f(\rho)$ to similar order,

$$f(\rho) \approx \kappa_3\rho + \kappa_4 \tag{2.4}$$

the limit condition for stability becomes

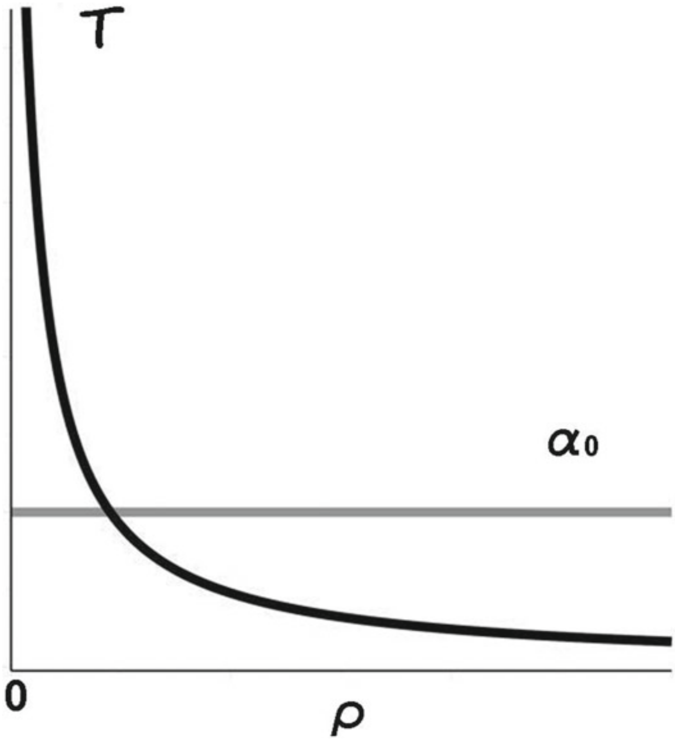


**Fig. 2.2** The horizontal line represents the critical limit $a_0$. If $\kappa_2/\kappa_4 \gg \kappa_1/\kappa_3$, at some intermediate value of linear traffic density $\rho$, the temperature analog $\mathscr{T} \equiv (\kappa_1\rho + \kappa_2)/(\kappa_3\rho + \kappa_4)$ falls below that limit, traffic flow becomes 'supercooled', and traffic jams become increasingly probable

$$\mathscr{T} \equiv \frac{\kappa_1\rho + \kappa_2}{\kappa_3\rho + \kappa_4} > a_0 \tag{2.5}$$

For $\rho = 0$, the stability condition is $\kappa_2/\kappa_4 > a_0$. At large $\rho$ this becomes $\kappa_1/\kappa_3 > a_0$. If $\kappa_2/\kappa_4 \gg \kappa_1/\kappa_3$, the stability condition may be violated at high traffic densities, and instability becomes manifest, as at the higher ranges of Fig. 1.1. See Fig. 2.2.

## 2.2 Multimodal Traffic on Bad Roads

For vehicles embedded in a larger traffic stream there are many other possible critical densities that must interact: different kinds of vehicles per linear mile, V2V/V2I communications bandwidth crowding, and an inverse index of roadway quality that one might call 'potholes per mile', and so on. There is not, then, a simple 'density' index, but rather a possibly large non-symmetric density matrix $\hat{\rho}$ having interacting components with $\rho_{i,j} \neq \rho_{j,i}$.

Can there still be some scalar '$\rho$' under such complex circumstances so that the conditions of Fig. 2.2 apply? An $n \times n$ matrix $\hat{\rho}$ has $n$ invariants $r_i$, $i = 1...n$, that remain fixed when 'principal component analysis' transformations are applied to data, and these can be used to construct an invariant scalar measure, using the polynomial relation

$$p(\lambda) = \det(\hat{\rho} - \lambda I) = \lambda^n + r_1\lambda^{n-1} + ... + r_{n-1}\lambda + r_n \quad (2.6)$$

det is the determinant, $\lambda$ is a parameter and $I$ the $n \times n$ identity matrix. The invariants are the coefficients of $\lambda$ in $p(\lambda)$, normalized so that the coefficient of $\lambda^n$ is 1. Typically, the first invariant will be the matrix trace and the last $\pm$ the matrix determinant.

For an $n \times n$ matrix it then becomes possible to define a composite scalar index $\Gamma$ as a monotonic increasing function of these invariants

$$\Gamma = f(r_1, ..., r_n) \quad (2.7)$$

The simplest example, for a $2 \times 2$ matrix, would be

$$\Gamma = m_1 Tr[\hat{\rho}] + m_2 |\det[\hat{\rho}]|$$

for positive $m_i$. Recall that, for $n = 2$, $\mathrm{Tr}[\hat{\rho}] = \rho_{11} + \rho_{22}$ and $\det[\hat{\rho}] = \rho_{11}\rho_{22} - \rho_{12}\rho_{21}$. In terms of the two possible eigenvalues $\alpha_1, \alpha_2$, $\mathrm{Tr}[\hat{\rho}] = \alpha_1 + \alpha_2$, $\det[\hat{\rho}] = \alpha_1\alpha_2$.

Again, an $n \times n$ matrix will have $n$ such invariants from which a scalar index $\Gamma$ can be constructed.

In Eq. (2.5) defining $\mathscr{T}$, $\rho$ is then replaced by the composite density index $\Gamma$,

$$\mathscr{T} = \frac{\kappa_1\Gamma + \kappa_2}{\kappa_3\Gamma + \kappa_4} \quad (2.8)$$

The method is a variant of the 'Rate Distortion Manifold' of Glazebrook and Wallace (2009) or the 'Generalized Retina' of Wallace and Wallace (2013, Sect. 10.1) in which high dimensional data flows can be projected down onto lower dimensional, shifting, tunable 'tangent spaces' with minimal loss of essential information.

## 2.3 The Dynamic Model

We next examine the dynamics of $\mathscr{T}(\Gamma)$ itself under stochastic circumstances. We begin by asking how a control signal $u_t$ in Fig. 2.1 is expressed in the system response $x_{t+1}$. We suppose it possible to deterministically retranslate an observed sequence of system outputs $X^i = x_1^i, x_2^i, ...$ into a sequence of possible control signals $\hat{U}^i = \hat{u}_0^i, \hat{u}_1^i, ...$ and to compare that sequence with the original control sequence $U^i = u_0^i, u_1^i, ...$, with the difference between them having a particular value under some chosen distortion measure and hence having an average distortion

$$< d >= \sum_i p(U^i) d(U^i, \hat{U}^i) \tag{2.9}$$

where $p(U^i)$ is the probability of the sequence $U^i$ and $d(U^i, \hat{U}^i)$ is the distortion between $U^i$ and the sequence of control signals that has been deterministically reconstructed from the system output.

We can then apply a classic Rate Distortion argument. According to the Rate Distortion Theorem, there exists a Rate Distortion Function, $R(D)$, that determines the minimum channel capacity necessary to keep the average distortion below some fixed limit $D$ (Cover and Thomas 2006). Based on Feynman's (2000) interpretation of information as a form of free energy, it becomes possible to construct a Boltzmann-like pseudoprobability in the 'temperature' $\mathscr{T}$ as

$$dP(R, \mathscr{T}) = \frac{\exp[-R/\mathscr{T}]dR}{\int_0^\infty \exp[-R/\mathscr{T}]dR} \tag{2.10}$$

since higher $\mathscr{T}$ must necessarily be associated with greater channel capacity.

The denominator can be interpreted as a statistical mechanical partition function, and it becomes possible to define a 'free energy' Morse Function (Pettini 2007) $\mathscr{F}$ as

$$\exp[-\mathscr{F}/\mathscr{T}] \equiv \int_0^\infty \exp[-R/\mathscr{T}]dR = \mathscr{T} \tag{2.11}$$

so that $\mathscr{F}(\mathscr{T}) = -\mathscr{T}\log[\mathscr{T}]$.

See the Mathematical Appendix for a brief introduction to Morse Theory.

Then an 'entropy' can also be defined as the Legendre transform of $\mathscr{F}$,

$$\mathscr{S} \equiv \mathscr{F}(\mathscr{T}) - \mathscr{T} d\mathscr{F}/d\mathscr{T} = \mathscr{T} \tag{2.12}$$

The Onsager treatment of nonequilibrium thermodynamics (de Groot and Mazur 1984), can now be invoked, based on the gradient of $\mathscr{S}$ in $\mathscr{T}$, so that a stochastic Onsager equation can be written as

$$\begin{aligned} d\mathscr{T}_t = (\mu d\mathscr{S}/d\mathscr{T})dt + \beta\mathscr{T}_t dW_t = \\ \mu dt + \beta\mathscr{T}_t dW_t \end{aligned} \tag{2.13}$$

where $\mu$ is a diffusion coefficient and $\beta$ is the magnitude of the impinging white noise $dW_t$. Although at first sight the mean for $\mathscr{T}$ would appear to increase at the rate $\mu$, simulations at high noise show this is simply not true. The stochastic self-stabilization theorem (e.g., Mao 1996, 2007) indicates that unstable differential equations of the form $dx(t)/dt = f(x(t), t)$—for which $x(t)$ 'explodes' as $t \to \infty$—can be stabilized in a stochastic differential equation model $dx_t = f(x_t, t)dt + \sigma x_t dW_t$ if $\sigma$ is sufficiently large and $|f(x, t)| \leq |x|\omega$ for some $\omega > 0$. Then

$$\limsup_{t\to\infty} \frac{1}{t} \log[x(t)] \leq -\frac{\sigma^2}{2} + \omega \tag{2.14}$$

As a consequence, if $\sigma^2/2 > \omega$, then $x(t) \to 0$, according to this model.

Indeed, there is a far more general result for multidimensional systems whose implications we will explore below (Appleby et al. 2008).

Something akin to the Doleans-Dade exponential (Protter 1990) of the Mathematical Appendix emerges by applying the Ito chain rule to $\log(\mathscr{T})$ in Eq. (2.13) (Protter 1990). Via Jensen's inequality for a concave function, the nonequilibrium steady state (nss) expectation of $\mathscr{T}$ then has the lower limit

$$E(\mathscr{T}_t) \geq \frac{\mu}{\beta^2/2} \tag{2.15}$$

In the V2V/V2I context, $\mu$ represents attempts by the system to keep traffic flowing well by raising $\mathscr{T}$, and $\beta$ is the magnitude of a traffic flow/roadway state 'white noise' $dW_t$ contrary to those attempts.

Recall that, in the multimodal extension of the model, the condition for stability is

$$\mathscr{T} \approx \frac{\kappa_1 \Gamma + \kappa_2}{\kappa_3 \Gamma + \kappa_4} > a_0 \tag{2.16}$$

The inference is that sufficient system noise, $\beta$, can drive $\mathscr{T}$ below critical values in Fig. 2.2, triggering a system collapse analogous to a large, propagating traffic jam. Under real world conditions, adequate service will simultaneously raise $\mu$ and lower $\beta$. Nonetheless, Eq. (2.15) is an expectation, *and there will always be some probability that* $\mathscr{T} < a_0$, i.e., that the condition for stability is violated. The system then becomes 'supercooled' and subject to a raised likelihood of sudden, rapidly propagating, traffic jam-like condensations in the sense of Kerner et al. (2015).

It is of some significance that, if the second term in Eq. (2.13) has the plausible form

$$\beta\sqrt{\mathscr{T}_t^2 + \alpha^2} dW_t, \ \alpha > 0 \tag{2.17}$$

so there is intrinsic volatility independent of $\mathscr{T}$, then, applying the Ito expansion to $\log[\mathscr{T}]$, there are two nonequilibrium steady state lower limits:

$$E(\mathscr{T}_t) \geq \frac{\mu \pm \sqrt{\mu^2 - \alpha^2\beta^4}}{\beta^2} \tag{2.18}$$

suggesting, first, the onset of systemic instability if $\alpha\beta^2 \geq \mu$, where $\mu$ incorporates the ability of the system to meet demand. This condition would seem to be independent of, and in addition to, the DRT stability requirement that $\mathscr{T} > a_0$. But further study shows only the larger value solution is actually stable. The lower level either rises to the higher or crashes out to zero in a total system collapse: gridlock. This is probably an observable effect to which we will return below in a section on the 'macroscopic fundamental diagram' describing traffic flow on a network rather than a single road segment.

## 2.4 Multiple Phases of Dysfunction

The DRT argument implies a raised probability of a transition between stable and unstable behavior if the temperature analog $\mathscr{T}(\Gamma)$ falls below a critical value. Kerner et al. (2015), however, argue that traffic flow can be subject to more than two phases. We can recover something similar via a 'cognitive paradigm' like that used by Atlan and Cohen (1998) in their study of the immune system. They view a system as cognitive if it must compare incoming signals with a learned or inherited picture of the world, then actively choosing a response from a larger set of those possible to it. V2V/V2I systems are clearly cognitive in that sense. Such choice, however, implies the existence of an information source, since it reduces uncertainty in a formal way. See Wallace (2012, 2015a, b) for details of the argument.

Given the 'dual' information source associated with the inherently unstable cognitive V2V/V2I system, an equivalence class algebra can be constructed by choosing different system origin states and defining the equivalence of subsequent states at a later time by the existence of a high probability path connecting them to the same origin state. Disjoint partition by equivalence class, analogous to orbit equivalence classes in dynamical systems, defines a symmetry groupoid associated with the cognitive process (Wallace 2012). Again, groupoids are generalizations of group symmetries in which there is not necessarily a product defined for each possible element pair (Weinstein 1996), for example in the disjoint union of different groups.

The equivalence classes across possible origin states define a set of information sources dual to different cognitive states available to the inherently unstable V2V/V2I system. These create a large groupoid, with each orbit corresponding to a transitive groupoid whose disjoint union is the full groupoid. Each subgroupoid is associated with its own dual information source, and larger groupoids must have richer dual information sources than smaller.

Let $X_{G_i}$ be the system's dual information source associated with groupoid element $G_i$. Given the argument leading to Eqs. (2.5–2.7), we construct another Morse Function (Pettini 2007) as follows.

Let $H(X_{G_i}) \equiv H_{G_i}$ be the Shannon uncertainty of the information source associated with the groupoid element $G_i$. We define another pseudoprobability as

$$P[H_{G_i}] \equiv \frac{\exp[-H_{G_i}/\mathscr{T}]}{\sum_j \exp[-H_{G_j}/\mathscr{T}]} \tag{2.19}$$

where the sum is over the different possible cognitive modes of the full system.

Another, more complicated, 'free energy' Morse Function $F$ can then be defined as

$$\exp[-F/\mathscr{T}] \equiv \sum_j \exp[-H_{G_j}/\mathscr{T}] \tag{2.20}$$

or, more explicitly,

$$F = -\mathscr{T} \log[\sum_j \exp[-H_{G_j}/\mathscr{T}]] \tag{2.21}$$

As a consequence of the groupoid structures associated with complicated cognition, as opposed to a 'simple' stable-unstable control system, we can now apply an extension of Landau's version of phase transition (Pettini 2007). Landau saw spontaneous symmetry breaking as representing phase change in physical systems, with the higher energies available at higher temperatures being more symmetric. The shift between symmetries is highly punctuated in the temperature index, here the 'temperature' analog of Eq. (2.5), in terms of the scalar construct $\Gamma$, but in the context of groupoid rather than group symmetries. Usually, for physical systems, there are only a few phases possible. Kerner et al. (2015) recognize three phases in ordinary traffic flow, but V2V/V2I systems may have relatively complex stages of dysfunction, with highly punctuated transitions between them as various density indicies change and interact.

Later we will explore sufficient conditions for the pathological ground state to 'lock-in' and become highly resistant to managerial intervention, that is, for a highly persistent large-scale traffic jam. Such 'lock-in' may help explain often-observed hysteresis effects in traffic flow, and the general intractability of the Macroscopic Fundamental Diagram for certain road networks.

In this context, Birkhoff's (1960, p.146) perspective on the central role of groups in fluid mechanics is of considerable interest:

> [Group symmetry] underlies the entire theories of dimensional analysis and modeling. In the form of 'inspectional analysis' it greatly generalizes these theories... [R]ecognition of groups... often makes possible reductions in the number of independent variables involved in partial differential equations... [E]ven after the number of independent variables is reduced to one... the resulting system of ordinary differential equations can often be integrated most easily by the use of group-theoretic considerations.

We argue here that, for 'cognitive fluids' like vehicle traffic flows, groupoid generalizations of group theory become central.

Decline in the richness of control information, or in the ability of that information to influence the system as measured by the 'temperature' index $\mathscr{T}(\Gamma)$, can lead to punctuated decline in the complexity of cognitive process possible within the $C^3$ system, driving it into a ground state collapse that may not be actual 'instability' but rather a kind of dead zone in which, using the armed drone example, ‘all possible targets are enemies’. This condition represents a dysfunctionally simple cognitive groupoid structure roughly akin to certain individual human psychopathologies (Wallace 2015a).

It appears that, for large-scale autonomous vehicle/intelligent infrastructure systems, the ground state dead zone involves massive, propagating tie-ups that far more resemble power network blackouts than traditional traffic jams. Again, the essential feature is the role of composite system ‘temperature’ $\mathscr{T}(\Gamma)$. Most of the topology of the inherently unstable vehicles/roads system will be ‘factored out’ via the construction of geodesics in a topological quotient space, so that $\mathscr{T}(\Gamma)$ inversely indexes the rate of topological information generation for an extended DRT.

Lowering the ‘temperature’ $\mathscr{T}$ forces the system to pass from high symmetry ‘free flow’ to different forms of ‘crystalline’ structure—broken symmetries representing platoons, shock fronts, traffic jams, and more complicated system-wide patterns of breakdown such as hysteresis.

In the next chapter the underlying dynamics are treated in finer detail from different perspectives, viewing the initial phase transition as a transition from free flow to ‘flock’ structures like those studied in ‘active matter’ physics. Indeed, the traffic engineering perspective is quite precisely the inverse of mainstream active matter studies, which Ramaswamy (2010) describes as follows:

> It is natural for a condensed matter physicist to regard a coherently moving flock of birds, beasts, or bacteria as an orientationally ordered phase of living matter. ...[M]odels showed a nonequilibrium phase transition from a disordered state to a flock with long-range order... in the particle velocities as the noise strength was decreased or the concentration of particles was raised.

In traffic engineering, the appearance of such ‘long range order’ is the first stage of a traffic jam (Kerner and Klenov 2009; Kerner et al. 2015), a relation made explicit by Helbing (2001, Sect. 6) in his comprehensive review of traffic and related self-driven many-particle systems.

While flocking and schooling have obvious survival value against predation for animals in three-dimensional venues, long-range order—aggregation—among blood cells flowing along arteries is a blood clot and can be rapidly fatal.

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