Chapter 2
Quantum Fluctuations in Linear Systems

Here are some words which have no place in a formulation with any pretension to physical precision: system, apparatus, environment, microscopic, macroscopic, reversible, irreversible, observable, information, measurement.

John Bell

Despite John Bell's eloquent tirade against the arbitrary division of the universe into a *system* surrounded by an *environment*, the experimental physicist, due to his limited means of enquiry, is forced to subscribe to a well-defined notion of what is considered the *system* under study. In the example relevant to this thesis, it is a macroscopic mechanical oscillator. Everything beyond, is the *environment*. In this sense, the measuring device—the *meter*—is a form of environment, the crucial difference being that the experimenter has the ability to prepare it in well-characterised (quantum) states. In this thesis, the meter is an electromagnetic field that interacts with the oscillator.

The purpose of this chapter is to provide a reasonably self-contained presentation of a few basic results pertaining to a certain class of interactions—linear interactions—between a system and its environment. When the environment in question is a thermal bath at finite temperature, classical (thermal) fluctuations drive the system; at zero temperature, quantum (vacuum) fluctuations remain. When the environment is a meter, fluctuations from the meter excite the system, an effect called *measurement back-action*; when the meter is prepared in a pure quantum state—measurement back-action is due to quantum fluctuations in its degrees of freedom, i.e. *quantum back-action*. Ultimately, for an ideal measurement chain—a system in contact with a zero-temperature thermal environment measured by a meter limited by quantum fluctuations—the output of the meter will feature an additional source of

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quantum fluctuations, called *measurement imprecision*, which, together with quantum back-action, constraints the precision with which the system can be measured. The rest of the chapter systematically unravels this tale.

2.1 Kinematics of Fluctuations in Quantum Mechanics

In the following we adopt the standard mathematical formalism of quantum mechanics [1, 2]: to every system (not necessarily *the* system) is associated a Hilbert space[1]; the states of the system are the positive, unit-trace operators. The relation between this abstract construct and the outcomes of experiments is through a set of distinguished operators called *observables*, defined as follows.

Definition 2.1 (*Observable*) The observables are the self-adjoint operators in the Hilbert space of the system.

If the system is repeatedly prepared in a definite state, say $\hat{\rho}$, and one of the observables, say $\hat{X}$, is measured per preparation, the outcomes will be random real numbers drawn from the eigenspectrum of the observable (self-adjointedness guarantees that the eigenspectrum is real [1, 2, 5]). This random variable is drawn according to a probability distribution.[2] The fluctuations of the random variable can be associated with the operator,

$$\delta\hat{X} := \hat{X} - \left\langle \hat{X} \right\rangle, \quad \text{where,} \quad \left\langle \hat{X} \right\rangle = \mathrm{Tr}[\hat{X}\hat{\rho}].$$

In a large variety of cases, the statistical dispersion in the random variable may be quantified by the variance,

$$\mathrm{Var}\left[\hat{X}\right] := \left\langle \delta\hat{X}^{\dagger}\delta\hat{X} \right\rangle = \left\langle \delta\hat{X}^{2} \right\rangle. \tag{2.1.1}$$

In light of the following fact, the variance is positive.

Lemma 2.1 (Operator positivity) *For any operator $\hat{A}$, not necessarily self-adjoint, it is true that $\left\langle \hat{A}^{\dagger}\hat{A} \right\rangle \geq 0$; i.e. $\hat{A}^{\dagger}\hat{A}$ is a positive operator.*

Proof Consider that the state $\hat{\rho}$ over which the expectation is taken, is represented as,

[1] It turns out that, on technical grounds, the framework of Hilbert space is too restrictive to realize the flexibility of Dirac's formulation of quantum mechanics [3, 4]; here, we will however be satisfied with using the Dirac formalism rather than justifying each step of the usage rigorously.

[2] A peculiarity of quantum mechanics is that although the value taken by each observable, for a fixed state, can be assumed to be drawn from a classical probability distribution (exhibited in Appendix A), there is generally no joint probability distribution for the values of a set of operators [6].

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

where, $p_i \geq 0$, and, $\langle\psi_i|\psi_j\rangle = \delta_{ij}$; such a representation is always possible [2]. Evaluating the expectation value gives,

$$\left\langle \hat{A}^\dagger \hat{A} \right\rangle = \mathrm{Tr}[\hat{A}^\dagger \hat{A}\hat{\rho}] = \mathrm{Tr}\left[\hat{A}^\dagger \hat{A} \sum_i p_i |\psi_i\rangle\langle\psi_i|\right] = \sum_i p_i \langle\psi_i|\hat{A}^\dagger \hat{A}|\psi_i\rangle = \sum_i p_i \|\hat{A}|\psi_i\rangle\|^2,$$

which is always positive by the property of norms on Hilbert space. □

The mathematical structure of quantum mechanics dictates that the variances of a pair of observables, say $\hat{X}, \hat{Y}$, satisfies the inequality (see Appendix A for a proof and further discussion),

$$\mathrm{Var}\left[\hat{X}\right] \mathrm{Var}\left[\hat{Y}\right] \geq \frac{1}{4}\left|\left\langle\left\{\delta\hat{X}, \delta\hat{Y}\right\}\right\rangle\right|^2 + \frac{1}{4}\left|\left\langle\left[\delta\hat{X}, \delta\hat{Y}\right]\right\rangle\right|^2 \geq \frac{1}{4}\left|\left\langle\left[\delta\hat{X}, \delta\hat{Y}\right]\right\rangle\right|^2, \tag{2.1.2}$$

conventionally called the *uncertainty principle*. The first inequality (due to Robertson [7] and Schrödinger [8] for pure states) is saturated for pure states defined as eigenstates of the operator $\alpha_X \delta\hat{X} + \alpha_Y \delta\hat{Y}$, with the constants $\alpha_{X,Y}$ chosen to maximize the correlation term $\langle\{\delta\hat{X}, \delta\hat{Y}\}\rangle$ [9]. In contrast, the second (looser) inequality (due to Heisenberg [10], Kennard [11], and Weyl [12]), obtained by omitting the correlation term, is saturated by the same eigenstates for any value of the constants $\alpha_{X,Y}$.

The physical content of the uncertainty principle (Eq. 2.1.2) is that the measurement outcomes of the pair of observables $\hat{X}, \hat{Y}$, on *identical and independent* preparations of the system, have a fundamental statistical dispersion. It is thus a purely kinematic statement devoid of any a priori relevance to the notion of "simultaneous", or "sequential" measurements.[3] It is best interpreted to mean that there exists no state which is jointly dispersion-free for certain pairs of observables—a distinctly quantum mechanical feature [6, 17].

Before describing an approach to treating outcomes of continuous measurements, and in an act of foresight, we generalize the definition of the variance of an observable, given in Eq. (2.1.1), to the case of a general operator. Following [18], the variance of an operator $\hat{A}$, not necessarily self-adjoint, is defined by,

$$\mathrm{Var}\left[\hat{A}\right] := \frac{1}{2}\left\langle\left\{\delta\hat{A}^\dagger, \delta\hat{A}\right\}\right\rangle. \tag{2.1.3}$$

When $\hat{A}$ is self-adjoint, this reduces to the standard definition in Eq. (2.1.1); but when it isn't, Lemma 2.1 still ensures that,

[3] Attempts to formulate inequalities applicable to *sequential* measurements [13–16] give results very different from Eq. (2.1.2).

$$\mathrm{Var}\left[\hat{A}\right] \geq 0.$$

Since any non-self-adjoint operator has a Cartesian decomposition in terms of two self-adjoint operators, i.e. $\hat{A} = \hat{X} + i\hat{Y}$, for $\hat{X}$, $\hat{Y}$ self-adjoint, the uncertainty inequality in Eq. (2.1.2) satisfied by the Cartesian components implies a bound for the variance of the corresponding non-self-adjoint operator. The following lemma codifies the resulting inequality.

Lemma 2.2 *For any (not necessarily self-adjoint) operator $\hat{A}$, the following inequality holds [18],*

$$\mathrm{Var}[\hat{A}] \geq \frac{1}{2}\left|\left\langle\left[\hat{A}^{\dagger}, \hat{A}\right]\right\rangle\right|. \tag{2.1.4}$$

Proof Denoting the Cartesian decomposition, $\hat{A} = \hat{X} + i\hat{Y}$, direct computation shows that the variance, defined by Eq. (2.1.3), takes the form,

$$\mathrm{Var}\left[\hat{A}\right] = \mathrm{Var}\left[\hat{X}\right] + \mathrm{Var}\left[\hat{Y}\right].$$

The sum on the right-hand side can be bounded by the arithmetic-geometric mean inequality,[4] and subsequently the Heisenberg form of the inequality in Eq. (2.1.2), leading to,

$$\mathrm{Var}\left[\hat{A}\right] \geq 2\sqrt{\mathrm{Var}\left[\hat{X}\right]\mathrm{Var}\left[\hat{X}\right]} \geq \left|\left\langle\left[\delta\hat{X}, \delta\hat{Y}\right]\right\rangle\right| = \left|\left\langle\frac{1}{2i}\left[\delta\hat{A}^{\dagger}, \delta\hat{A}\right]\right\rangle\right|.$$

$\square$

2.1.1 Operational Description of Fluctuations in Time

In order to treat system observables varying in time, the Heisenberg picture is most convenient: the system is in some time-independent state $\hat{\rho}$, while its observables undergo fluctuations due to the pervasive *environment* that the system is in contact with. These fluctuations are reflected in the observables as deviations from their mean values, viz.

$$\delta\hat{X}(t) = \hat{X}(t) - \mathrm{Tr}\left[\hat{\rho}\,\hat{X}(t)\right].$$

The fluctuating part, $\delta\hat{X}(t)$, represents a continuous random variable—stochastic process—taking values in the set of observables.

[4] For positive real numbers x, y, it is true that $x + y \geq 2\sqrt{xy}$; this follows from the identity, $(\sqrt{x} - \sqrt{y})^2 \geq 0$.

In the following, we employ an operational description of the statistical properties of the operator-valued stochastic process.[5] In order to resolve the variance of the process over the different time scales over which the fluctuations happen, we consider the windowed Fourier transform,[6]

$$\delta\hat{X}^{(T)}[\Omega] := \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2}\delta\hat{X}(t)e^{i\Omega t}\,\mathrm{d}t, \tag{2.1.5}$$

which is in general non-hermitian. The definition of the variance of a non-hermitian operator in Eq. (2.1.3) then implies,

$$\begin{aligned}
\mathrm{Var}\left[\delta\hat{X}^{(T)}[\Omega]\right] &= \frac{1}{2}\left\langle\left\{\delta\hat{X}^{(T)}[\Omega]^{\dagger}, \delta\hat{X}^{(T)}[\Omega]\right\}\right\rangle \\
&= \frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{2}\left\langle\left\{\delta\hat{X}(t), \delta\hat{X}_i(t')\right\}\right\rangle e^{i\Omega(t-t')}\,\mathrm{d}t\,\mathrm{d}t' \\
&= \frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{2}\left\langle\left\{\delta\hat{X}(t-t'), \delta\hat{X}(0)\right\}\right\rangle e^{i\Omega(t-t')}\,\mathrm{d}t\,\mathrm{d}t' \\
&= \int_{-T/2}^{T/2}\frac{1}{2}\left\langle\left\{\delta\hat{X}(\tau), \delta\hat{X}(0)\right\}\right\rangle e^{i\Omega\tau}\left(1-\frac{|\tau|}{T}\right)\mathrm{d}\tau.
\end{aligned}$$

Note that here and henceforth we assume processes are weak-stationary, i.e. that their first and second moments are time-translation invariant. In the limit $T \to \infty$ (i.e. the limit of infinite resolution in frequency), this variance defines the function,

$$\bar{S}_{XX}[\Omega] := \lim_{T\to\infty}\mathrm{Var}\left[\delta\hat{X}^{(T)}[\Omega]\right] = \int_{-\infty}^{\infty}\left\langle\frac{1}{2}\left\{\delta\hat{X}(t), \delta\hat{X}(0)\right\}\right\rangle e^{i\Omega t}\,\mathrm{d}t, \tag{2.1.6}$$

characterising the distribution of the variance of the process about each frequency. The second equality, giving the value of the limit, is the analogue of the Wiener-Khinchine theorem [25, 26].

[5] There exists two, progressively finer, levels of description of the evolution of a quantum system in contact with an environment. The coarse description concerns itself with the time evolution of observables, and some of its statistics. The finer description addresses the question of how the quantum state itself changes. The former is subsumed by the latter in a variety of equivalent ways [19–23].

[6] The normalisation warrants clarification: if the integrand were a classical Brownian process, its root-mean-square diverges as the square root of the observation window, i.e. as $T^{1/2}$, which is checked by the normalisation. For a wide class of classical stochastic processes, a theorem due to Donsker [24] guarantees that the integral limits to a Brownian process (a "functional central limit theorem")—the $T^{-1/2}$ normalisation is necessary. This result from classical probability theory suffices to justify the normalisation.

The function is reminiscent of the classical notion of a power spectral density. Firstly, being a variance, $\bar{S}_{XX}[\Omega] \geq 0$, at all frequencies and for any operator-valued process. Secondly, being a distribution (obtained by applying the Fourier inversion theorem [25] to Eq. (2.1.6)),

$$\mathrm{Var}\left[\delta\hat{X}(t)\right] = \left\langle \delta\hat{X}(0)^2 \right\rangle = \int_{-\infty}^{\infty} \bar{S}_{XX}[\Omega]\, \frac{\mathrm{d}\Omega}{2\pi}, \tag{2.1.7}$$

which exhibits the complementary aspect that the integral of the power spectral density is the variance of the process $\delta\hat{X}(t)$. Equations (2.1.6) and (2.1.7) are fundamental properties of the *symmetrised spectrum*[7] so defined, that render it useful (irrespective of whether it is generically measured in an experiment [26–28]).

2.1.2 Spectral Densities and Uncertainty Relations

A formal hierarchy of spectral distributions generalise the above concept of the symmetrized spectrum of an observable. For a general (i.e. not necessarily hermitian) operator $\hat{A}$, define its Fourier transform,

$$\hat{A}[\Omega] = \int_{-\infty}^{+\infty} \hat{A}(t)e^{i\Omega t}\, \mathrm{d}t, \tag{2.1.8}$$

and its inverse,

$$\hat{A}(t) = \int_{-\infty}^{+\infty} \hat{A}[\Omega]e^{-i\Omega t}\, \frac{\mathrm{d}\Omega}{2\pi}. \tag{2.1.9}$$

We shall denote by, $\hat{A}^\dagger[\Omega]$, the Fourier transform of $\hat{A}^\dagger(t)$; and by, $\hat{A}[\Omega]^\dagger$, the hermitian conjugate of $\hat{A}[\Omega]$. With this convention, $\hat{A}^\dagger[\Omega] = \hat{A}[-\Omega]^\dagger$. For an observable, say $\hat{X}(t)$, it is further true that, $\hat{X}[\Omega]^\dagger = \hat{X}[-\Omega]$, and so $\hat{X}^\dagger[\Omega] = \hat{X}[\Omega]$.

The *unsymmetrized (cross-)spectrum* of two operators $\hat{A}$, $\hat{B}$ (not necessarily equal) is defined as the Fourier transform of their unsymmetrized two-time correlation function, i.e.,

$$S_{AB}[\Omega] := \int_{-\infty}^{+\infty} \left\langle \delta\hat{A}^\dagger(t)\delta\hat{B}(0)\right\rangle e^{i\Omega t}\, \mathrm{d}t = \int_{-\infty}^{+\infty} \left\langle \delta\hat{A}^\dagger[\Omega]\delta\hat{B}[\Omega']\right\rangle \frac{\mathrm{d}\Omega'}{2\pi}, \tag{2.1.10}$$

which is in general a complex number at each Fourier frequency Ω; here, the second equality follows from replacing the operators with their Fourier transforms (i.e. Eq. (2.1.9)). When the operators involved are weak-stationary, i.e.,

$$\left\langle \hat{A}^\dagger(t)\hat{B}(t')\right\rangle = \left\langle \hat{A}^\dagger(t-t')\hat{B}(0)\right\rangle,$$

[7] Short for "symmetrised power spectral density", by abuse of terminology.

their unsymmetrized spectrum is directly related to their two-point correlation in the frequency domain; specifically,

$$\begin{aligned} S_{AB}[\Omega] \cdot 2\pi\,\delta[\Omega+\Omega'] &= \left\langle \delta\hat{A}^\dagger[\Omega]\delta\hat{B}[\Omega'] \right\rangle \\ \text{i.e.,} \qquad S_{AB}[\Omega] \cdot 2\pi\,\delta[0] &= \left\langle \delta\hat{A}[-\Omega]^\dagger \delta\hat{B}[-\Omega] \right\rangle. \end{aligned}$$

The last form makes explicit the symmetry,

$$S_{AB}[\Omega]^* = S_{BA}[\Omega]. \tag{2.1.11}$$

The other algebraic property that is practically useful is bilinearity, which can be expressed as follows: consider an operator which is a linear superposition of another pair, i.e. $\hat{A}[\Omega] = \alpha_1[\Omega]\hat{B}_1[\Omega] + \alpha_2[\Omega]\hat{B}_2[\Omega]$, then,

$$S_{AA}[\Omega] = \begin{cases} \alpha_1[-\Omega]S_{AB_1}[\Omega] + \alpha_2[-\Omega]S_{AB_2}[\Omega] \\ \alpha_1^*[-\Omega]S_{B_1A}[\Omega] + \alpha_2^*[-\Omega]S_{B_2A}[\Omega]. \end{cases} \tag{2.1.12}$$

These two properties allow for the practical computation of the spectra of operators defined as linear superpositions of other operators. Concretely, if a set of operators (arranged into a column vector) $\hat{\mathbf{A}} = [\hat{A}_i]^T$ are related to another set, $\hat{\mathbf{B}} = [\hat{B}_i]^T$, as,

$$\begin{aligned} \hat{A}_i[\Omega] &= \sum_k \alpha_{ik}[\Omega]\hat{B}_k[\Omega], \\ \text{equivalently,} \quad \hat{\mathbf{A}}[\Omega] &= \boldsymbol{\alpha}[\Omega]\hat{\mathbf{B}}[\Omega], \end{aligned}$$

for some (matrix $\boldsymbol{\alpha}$ of) coefficients α_{ik}, then,

$$\begin{aligned} S_{A_iA_j}[\Omega] &= \sum_{k,l} \alpha_{ik}^*[-\Omega]S_{B_kB_l}[\Omega]\alpha_{jl}[-\Omega] \\ \text{equivalently,} \quad S_{\mathbf{AA}}[\Omega] &= \boldsymbol{\alpha}[-\Omega]^* S_{\mathbf{BB}}[\Omega]\boldsymbol{\alpha}[-\Omega]^T. \end{aligned}$$

The second form expresses the first as a matrix equation, where $S_{\mathbf{AA}}$ denotes the matrix whose elements are $S_{A_iA_j}$, i.e. it is the unsymmetrised covariance matrix of $\hat{\mathbf{A}}$ in the frequency domain.

The physical motivation for the definition of the unsymmetrized spectrum becomes obvious when considering the properties of the spectrum of a single operator, viz.,

$$S_{AA}[\Omega] \cdot 2\pi\,\delta[0] = \left\langle \delta\hat{A}[-\Omega]^\dagger \delta\hat{A}[-\Omega] \right\rangle \geq 0; \tag{2.1.13}$$

specifically, S_{AA} is real, and positive. Mathematically, the positivity follows from lemma 2.1; its physical content is that S_{AA} can be interpreted as (being proportional to) the transition probability of a process mediated by an interaction that couples the system to its environment via the operator $\hat{A}[-\Omega]$ [29–31]. In classic examples in quantum optics [30, 31], the operator may be the destruction operator of a photon in which case the spectrum is the output spectrum of a photodetector [32], or, it may be the raising/lowering operator for an atomic level in which case the spectrum is the absorption/emission spectrum of that level [33].

Pairs of spectra of operators, in analogy with the variances of pairs of observables, satisfy an inequality reminiscent of the (Robertson-Schrodinger) uncertainty principle (Eq. (2.1.2)).

Proposition 2.1 (Spectral uncertainty relation I) *The spectra of any pair of operator-valued stochastic processes, $\hat{A}(t), \hat{B}(t)$, that are weak-stationary, satisfies the inequality,*

$$S_{AA}[\Omega]S_{BB}[\Omega] - |S_{AB}[\Omega]|^2 \geq 0. \tag{2.1.14}$$

Proof A slick proof follows by a direct adaptation of the one used by Roberston to originally establish the uncertainty relation in Eq. (2.1.2) (see Appendix A), as done for example in [20]. For a pair of observables, a simpler method is as follows: define, $\hat{M}_\lambda(t) = \hat{A}(t) + \lambda\hat{B}(t)$, for some complex λ. From Eq. (2.1.13), it must be that, $S_{M_\lambda M_\lambda}[\Omega] \geq 0$ for all λ. Writing this out explicitly using the bilinearity (Eq. (2.1.12)) and symmetry (Eq. (2.1.11)):

$$S_{M_\lambda M_\lambda} = S_{AA} + |\lambda|^2\, S_{BB} + 2\text{Re}\, \lambda S_{AB} \geq 0.$$

This trivial inequality can be tightened by replacing $S_{M_\lambda M_\lambda}$ with $\min_\lambda S_{M_\lambda M_\lambda}$. A straightforward exercise shows that the minimum is achieved for,

$$\lambda = \lambda_{\min} := \frac{|S_{AB}|}{S_{BB}} \exp\left(i \ \arg S^*_{AB}\right),$$

for which the inequality reduces to the required result. □

Returning back to physics, it may happen that in some situations, distinguishing between an emission and an absorption event may not be possible. To model the outcomes of such cases, we introduce the *symmetrised spectrum*,

$$\bar{S}_{AB}[\Omega] := \int_{-\infty}^{+\infty} \left\langle \frac{1}{2}\left\{\delta\hat{A}^\dagger(t), \delta\hat{B}(0)\right\}\right\rangle e^{i\Omega t}\, \mathrm{d}t = \frac{1}{2}\left(S_{AB}[\Omega] + S_{B^\dagger A^\dagger}[-\Omega]\right),$$

which is a complex quantity in general. For the case of an observable, say $\hat{X}$, with $\hat{X}^\dagger = \hat{X}$, we have,

$$\bar{S}_{XX}[\Omega] = \frac{1}{2}(S_{XX}[\Omega] + S_{XX}[-\Omega]), \tag{2.1.15}$$

i.e. symmetrisation in ordering is equivalent to symmetrisation in frequency. Note that this formally-motivated definition is equivalent to the physically-motivated one given in Eq. (2.1.6), allowing the symmetrized spectra of observables to be interpreted as the variance of the observable process.

The frequency symmetry,

$$\bar{S}_{XX}[\Omega] = \bar{S}_{XX}[-\Omega], \tag{2.1.16}$$

suggests that the *single-sided* spectrum defined by,

$$\bar{S}_X[\Omega] := 2\ \bar{S}_{XX}[\Omega], \quad \text{for} \quad \Omega \geq 0,$$

encodes the full information contained in the double-sided symmetrised spectrum $\bar{S}_{XX}[\Omega]$ of an observable $\hat{X}$. In terms of the single-sided spectrum, the variance of the process is,

$$\mathrm{Var}\left[\delta\hat{X}(t)\right] = \int_0^\infty \bar{S}_X[\Omega]\,\frac{\mathrm{d}\Omega}{2\pi}.$$

The single-sided spectrum thus defined is apparently equivalent to the conventional definition of the spectral density of a real-valued classical stochastic process [34].

Despite similarities to classical spectral densities at the level of definition, the lack of commutativity of time-dependent observables amongst each other, and even amongst the same observable at different times, leads to certain basic quantum mechanical conditions on the symmetrized spectra. Firstly, any observable will feature a fundamental level of statistical dispersion, preventing it from saturating the naive lower bound in $\bar{S}_{XX}[\Omega] \geq 0$; secondly, two (or more) observables will never be jointly dispersion-free at all frequencies. These constraints, expressed respectively in Propostitions 2.2 and 2.3 that follow, may be viewed as the irreducible content of quantum mechanics expressed at the level of spectra.

Proposition 2.2 (Spectral minimum) *Any observable $\hat{X}$ of a quantum mechanical system satisfies the inequality,*

$$\bar{S}_{XX}[\Omega] \geq \frac{1}{2}\left|\int_{-\infty}^{\infty}\left\langle\left[\delta\hat{X}(t),\,\delta\hat{X}(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t\right|. \tag{2.1.17}$$

Proof Using the definition of the spectral density Eq. (2.1.6), together with the uncertainty relation Eq. (2.1.4) gives the crux of the inequality, viz.

$$\bar{S}_{XX}[\Omega] = \lim_{T\to\infty} \mathrm{Var}\left[\delta\hat{X}^{(T)}[\Omega]\right] \geq \lim_{T\to\infty}\frac{1}{2}\left|\left\langle\left[\delta\hat{X}^{(T)}[\Omega],\,\delta\hat{X}^{(T)}[\Omega]^\dagger\right]\right\rangle\right|.$$

Expressing the windowed Fourier transform in terms of the time domain operator, and employing the weak-stationary property results in,

$$\begin{aligned}\bar{S}_{XX}[\Omega] &\geq \frac{1}{2}\left|\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left\langle\left[\delta\hat{X}(t-t'),\delta\hat{X}(0)\right]\right\rangle e^{i\Omega(t-t')}\,\mathrm{d}t\,\mathrm{d}t'\right|\\ &= \frac{1}{2}\left|\lim_{T\to\infty}\int_{-\infty}^{\infty}\left\langle\left[\delta\hat{X}(\tau),\delta\hat{X}(0)\right]\right\rangle\left(1-\frac{|\tau|}{T}\right)e^{i\Omega\tau}\,\mathrm{d}\tau\right|\\ &= \frac{1}{2}\left|\int_{-\infty}^{\infty}\left\langle\left[\delta\hat{X}(\tau),\delta\hat{X}(0)\right]\right\rangle e^{i\Omega\tau}\,\mathrm{d}\tau\right|;\end{aligned}$$

the second equality is obtained from a change of variables, while the third follows from evaluating the limit inside the integral. □

If $\hat{X}(t)$ were classical, its spectrum $\bar{S}_{XX}$ could in principle exhibit no statistical dispersion—when X is deterministic—in which case, $\bar{S}_{XX}[\Omega] = 0$. However, as per Propositition 2.2, such a dispersion-free situation is untenable for a quantum mechanical process, unless $[\delta\hat{X}(t), \delta\hat{X}(0)] = 0$. This motivates the following definition of a *continuous observable*.[8]

Definition 2.2 (*Continuous observable*) An observable $\hat{X}(t)$ is said to be a continuous observable iff.

$$[\hat{X}(t), \hat{X}(t')] = 0. \tag{2.1.18}$$

They are also called "quantum non-demolition" observables [37, 38], to emphasize the fact that generic observables do not satisfy this constraint, and therefore cannot be measured without causing disturbance.

Given a pair of continuous observables—i.e. which individually feature no statistical dispersion—they may still exhibit a joint statistical dispersion; a feature that is classically impossible. The following proposition encodes this idea and it may be viewed as a generalization of the Robertson-Schrodinger inequality given in Eq. (2.1.2) to the case of continuous observables.

Proposition 2.3 (Spectral uncertainty relation II) *A pair of continuous observables $\hat{X}$, $\hat{Y}$ of a quantum mechanical system satisfying the (cross-)commutation relation,*

$$\left[\hat{X}(t), \hat{Y}(t')\right] = i\hat{C}_{XY}(t-t'), \quad \textit{or,} \quad \left[\hat{X}[\Omega], \hat{Y}[\Omega']\right] = i\hat{C}_{XY}[\Omega]\cdot 2\pi\ \delta[\Omega+\Omega']$$

satisfy the following inequality for their symmetrised spectra:

$$\bar{S}_{XX}[\Omega]\bar{S}_{YY}[\Omega] \geq \left|\bar{S}_{XY}[\Omega]\right|^2 + \frac{1}{4}\left|\left\langle\hat{C}_{XY}[\Omega]\right\rangle\right|^2 \tag{2.1.19}$$

[8]Note that the notion of a *continuous observable*, as introduced here, is very different from that of a *continuous variable* used in the context of quantum information [35, 36]. The latter refers to hermitian operators (i.e. observables) whose eigenspectrum is continuous. The former, as used here, refers to time-dependent observables (with a continuous, or discrete, eigenspectrum) which can (in principle) be continuously monitored in time.

Proof The strategy is to specialise the uncertainty relation for unsymmetrized spectra, given in Proposition 2.1, to the case of continuous observables. The uncertainty relation gives,

$$S_{XX}[\Omega]S_{YY}[\Omega] - |S_{XY}[\Omega]|^2 \geq 0.$$

To translate it into symmetrized spectra, note the following identity,

$$\begin{aligned} S_{XY}[\Omega] &= \int \left\langle \delta\hat{X}(t)\delta\hat{Y}(0) \right\rangle e^{i\Omega t}\,\mathrm{d}t \\ &= \int \left\langle \tfrac{1}{2}\left\{\delta\hat{X}(t),\delta\hat{Y}(0)\right\} + \tfrac{1}{2}\left[\delta\hat{X}(t),\delta\hat{Y}(0)\right] \right\rangle e^{i\Omega t}\,\mathrm{d}t \\ &= \bar{S}_{XY}[\Omega] + \frac{i}{2}\left\langle \hat{C}_{XY}[\Omega] \right\rangle. \end{aligned}$$

Inserting this expression for S_{XY} into the inequality and simplifying gives the result. □

Having developed the theoretical apparatus to deal with statistical properties of operator-valued stochastic processes, the next two sections will apply them to the case where a system is coupled to a thermal environment, and a meter, respectively.

2.2 Dynamics Due to a Thermal Environment

Consider a system, with a prescribed average energy, in equilibrium with an environment. The state of the system—described by a single parameter, the temperature T—is the one with the maximal entropy compatible with the average energy [2]. This unique state is the canonical thermal state,

$$\hat{\rho}_\beta = \frac{e^{-\beta\hat{H}_0}}{Z}, \tag{2.2.1}$$

where $\beta = (k_B T)^{-1}$ is the inverse temperature, $\hat{H}_0$ is the free hamiltonian of the system, and $Z = \mathrm{Tr}\, e^{-\beta\hat{H}_0}$ is the partition function that ensures the normalisation of the state, i.e. $\mathrm{Tr}\,\hat{\rho}_\beta = 1$. At zero temperature ($\beta \to \infty$) the canonical thermal state picks out the ground state of the hamiltonian, i.e., $\hat{\rho}_{\beta\to\infty} = |0\rangle\langle 0|$.

In a thermal state, the observables of the system, $\{\hat{X}_i\}$, are weak-stationary; i.e., their mean values are constant, while their second moments are time-translation invariant:

$$\begin{aligned} \left\langle \hat{X}_i(t) \right\rangle &= \left\langle \hat{X}_i(0) \right\rangle \\ \left\langle \hat{X}_i(t)\hat{X}_j(t') \right\rangle &= \left\langle \hat{X}_i(t-t')\hat{X}_j(0) \right\rangle. \end{aligned} \tag{2.2.2}$$

These identities follow from the observation that the hamiltonian $\hat{H}_0$, and hence the propagator $\hat{U}_t = \exp(-\frac{i}{\hbar}\hat{H}_0 t)$, commutes with the thermal state. Such states—*stationary states*—feature weak-stationarity of observables. However, the thermal states are distinguished among the stationary states by the following property, which is a generalisation of the principle of detailed balance.

Lemma 2.3 (Kubo-Martin-Schwinger [39, 40]) *The observables, $\{\hat{X}_i\}$, of a system in a thermal state at inverse temperature β obey the identity,*

$$\left\langle \hat{X}_i(t)\hat{X}_j(0)\right\rangle = \left\langle \hat{X}_j(0)\hat{X}_i(t+i\hbar\beta)\right\rangle \tag{2.2.3}$$

or equivalently,

$$S_{X_iX_j}[\Omega] = e^{\beta\hbar\Omega}\, S_{X_jX_i}[-\Omega]. \tag{2.2.4}$$

Proof Ignoring questions of rigour (see [41] for a remedy), the proof follows through a straightforward algebraic manipulation viz.,

$$\begin{aligned}
\left\langle \hat{X}_i(t)\hat{X}_j(0)\right\rangle &= \mathrm{Tr}\left[\hat{\rho}_\beta\, \hat{X}_i(t)\hat{X}_j(0)\right] \\
&= \mathrm{Tr}\left[e^{-\beta\hat{H}_0}\cdot \hat{U}_t^\dagger \hat{X}_i(0)\hat{U}_t \cdot e^{\beta\hat{H}_0}e^{-\beta\hat{H}_0}\cdot \hat{X}_j(0)\right] Z^{-1} \\
&= \mathrm{Tr}\left[e^{-\beta\hat{H}_0}\hat{X}_j(0)\cdot (e^{-\beta\hat{H}_0}\hat{U}_t^\dagger)\hat{X}_i(0)(\hat{U}_t e^{\beta\hat{H}_0})\right] Z^{-1} \\
&= \mathrm{Tr}\left[\hat{\rho}_\beta\, \hat{X}_j(0)\cdot \hat{U}_{t+i\hbar\beta}^\dagger \hat{X}_i(0)\hat{U}_{t+i\hbar\beta}\right] \\
&= \left\langle \hat{X}_j(0)\hat{X}_i(t+i\hbar\beta)\right\rangle.
\end{aligned}$$

The frequency domain form, in Eq. (2.2.4), can be proven starting by taking the Fourier transform of both sides and using the stationarity property, viz.,

$$\begin{aligned}
S_{X_iX_j}[\Omega] &:= \int \left\langle \hat{X}_i(t)\hat{X}_j(0)\right\rangle e^{i\Omega t}\,\mathrm{d}t = \int \left\langle \hat{X}_j(0)\hat{X}_i(t+i\hbar\beta)\right\rangle e^{i\Omega t}\,\mathrm{d}t \\
&= \int \left\langle \hat{X}_j(-t-i\hbar\beta)\hat{X}_i(0)\right\rangle e^{i\Omega t}\,\mathrm{d}t = \int \left\langle \hat{X}_j(t')\hat{X}_i(0)\right\rangle e^{i\Omega(-t'-i\hbar\beta)}\,\mathrm{d}t' \\
&= e^{\beta\hbar\Omega}\int \left\langle \hat{X}_j(t')\hat{X}_i(0)\right\rangle e^{i(-\Omega)t'}\,\mathrm{d}t' = e^{\beta\hbar\Omega} S_{X_jX_i}[-\Omega].
\end{aligned}$$

□

On the one hand, the KMS identity (Eq. (2.2.3)) may be seen as controlling the commutativity of observables in a thermal state: in the high temperature limit ($\beta \to 0$), it implies that all observables commute—evocative of classical behaviour. On the other hand, its frequency domain form (Eq. (2.2.4)) may be interpreted as a detailed balance principle: the ratio of the forward and reverse transition probabilities, represented by the ratio of the unsymmetrised spectra, is given by the thermal exponent. Perhaps more profoundly, it can be shown that[9] for a time-translation invariant system, if *all* operators satisfy the KMS identity pairwise, then the system is in the canonical thermal state $\hat{\rho}_\beta$. Thus the properties expressed in Eqs. (2.2.3) and (2.2.2) (almost completely) characterize the kinematics of thermal equilibrium.

Having thus described the essential structure of a thermal state, in the following, fluctuations due to a system being in such a state will be analysed. Essentially, this requires a model that describes the interaction between the system and its thermal environment, and then a procedure to calculate the dynamics of the system variables. We suppose that the interaction is mediated by a *linear* coupling between the system and the environment, modelled by an interaction hamiltonian,

$$\hat{H}_{\text{int}} = \sum_i \hat{X}_i \hat{F}_i, \tag{2.2.5}$$

that couples the observable $\hat{X}_i$ to a *generalized force* $\hat{F}_i$ which is a self-adjoint environment operator. Once the interaction is fixed, multiple approaches exist to treat the dynamics of the system [23, 30, 42]; we adopt the *linear response* formalism [39, 43, 44], enshrined in the following celebrated result.

[9] An outline of a proof is as follows (see [41] for the setup required to justify some of the steps). Assume then that there is some state $\hat{\rho}$ for which all operators (not just observables) of the system satisfy Eq. (2.2.3); i.e. $\langle \hat{A}(t)\hat{B}(0)\rangle = \langle \hat{B}(0)\hat{A}(t+i\hbar\beta)\rangle$, for all operators $\hat{A}$, $\hat{B}$. Time-translation invariance means that only the case $t=0$ need to be considered, i.e., $\langle \hat{A}(0)\hat{B}(0)\rangle = \langle \hat{B}(0)\hat{A}(i\hbar\beta)\rangle \ \forall\ \hat{A}, \hat{B}$. Dropping the time argument and writing this out with the unknown state $\hat{\rho}$ explicitly,

$$\mathrm{Tr}[\hat{\rho}\hat{A}\hat{B}] = \mathrm{Tr}[\hat{\rho}\hat{B}e^{-\beta\hat{H}_0}\hat{A}e^{\beta\hat{H}_0}] \qquad \forall\, \hat{A}, \hat{B}.$$

This can be expressed in two different ways. Firstly, since it applies for any $\hat{A}$, it must also apply for $\hat{A} = e^{\beta\hat{H}_0}$; in this case, $\mathrm{Tr}[\hat{\rho}e^{\beta\hat{H}_0}\hat{B}] = \mathrm{Tr}[e^{\beta\hat{H}_0}\hat{\rho}\hat{B}]\ \forall\ \hat{B}$, implying that,

$$\hat{\rho}e^{\beta\hat{H}_0} = e^{\beta\hat{H}_0}\hat{\rho}.$$

Secondly, permuting within the trace gives the alternate form, $\mathrm{Tr}[\hat{B}\hat{\rho}\hat{A}] = \mathrm{Tr}[e^{\beta\hat{H}_0}\hat{\rho}\hat{B}e^{-\beta\hat{H}_0}\hat{A}]\ \forall\ \hat{A}, \hat{B}$, implying that,

$$\hat{B}\hat{\rho} = e^{\beta\hat{H}_0}\hat{\rho}\hat{B}e^{-\beta\hat{H}_0}$$

$$\text{i.e.,} \quad \hat{B}\hat{\rho}e^{\beta\hat{H}_0} = e^{\beta\hat{H}_0}\hat{\rho}\hat{B} \qquad \forall\, \hat{B}.$$

Combining the results from the two forms gives,

$$\hat{B}e^{\beta\hat{H}_0}\hat{\rho} = e^{\beta\hat{H}_0}\hat{\rho}\hat{B} \qquad \forall\, \hat{B},$$

i.e. the operator $e^{\beta\hat{H}_0}\hat{\rho}$ commutes with every operator in the Hilbert space. This means that it must be proportional to the identity operator, i.e. $e^{\beta\hat{H}_0}\hat{\rho} \propto 1$, or, $\hat{\rho} \propto e^{-\beta\hat{H}_0}$. The normalization of the state fixes the proportionality constant.

Lemma 2.4 (Kubo) *If a system is presumed to be maintained in a thermal state by a linear coupling to the environment, i.e. by a hamiltonian of the form,*

$$\hat{H}_F(t) = \hat{H}_0 + \sum_i \hat{X}_i \hat{F}_i(t), \tag{2.2.6}$$

where $\hat{F}_i$ is the generalised force corresponding to $\hat{X}_i$, then fluctuations in the system observables are given by,

$$\begin{aligned} \delta\hat{X}_j(t) &= \int_{-\infty}^{\infty} \sum_k \chi_{jk}(t-t')\,\delta\hat{F}_k(t')\,\mathrm{d}t', \\ \textit{or,} \qquad \delta\hat{X}_j[\Omega] &= \sum_k \chi_{jk}[\Omega]\,\delta\hat{F}_k[\Omega], \end{aligned} \tag{2.2.7}$$

where, the "susceptibilities" χ_{jk} are (here $\Theta(t)$ is the Heaviside step function),

$$\chi_{jk}(t) = -\frac{i}{\hbar}\Theta(t)\left\langle [\hat{X}_j(t), \hat{X}_k(0)]\right\rangle. \tag{2.2.8}$$

Proof Standard time-dependent perturbation theory as for example in [45]. □

Formally, the power of the Kubo formula in Eq. (2.2.8), is that it relates the response of the system to an external influence in terms of expectation values of the system operators taken on the equilibrium (thermal) state of the system. Practically, the great advantage of the linear response formalism is that by relating the fluctuations in the system's observables to the fluctuations of a generalised force, it suggests an avenue to probe the system: coherent response measurements—harmonically driving $\hat{F}_k$ and observing its effect in $\hat{X}_j$—give access to $\chi_{jk}[\Omega]$, which then predict the incoherent behaviour of the system in the absence of an explicit stimulus. Within the regime of its validity, the linear response formalism is pervasive in physics [26, 46–49].

For the set of observables that are assumed to directly couple to the environment—those in the interaction hamiltonian in Eq. (2.2.5)—the spectral uncertainty relation (Eq. (2.1.4)), and the Kubo formula, imply a couple of general properties.

2.2.1 *Effect of Fluctuations from a Thermal Environment*

Proposition 2.4 (Fundamental fluctuations) *Observables of the system that directly couple to the environment exhibit fluctuations, whose spectra $\bar{S}_{X_i X_i}[\Omega]$ have a minimum positive value,*

$$\bar{S}_{X_i X_i}[\Omega] \geq \hbar\,|\mathrm{Im}\;\chi_{ii}[\Omega]|\,. \tag{2.2.9}$$

Proof Proposition 2.2 already states that spectra of operator-valued stochastic processes have a minimum value dictated by the expectation value of its commutator (in any state), i.e.,

$$\bar{S}_{X_i X_i}[\Omega] \geq \frac{1}{2}\left|\int_{-\infty}^{\infty}\left\langle\left[\delta\hat{X}_i(t), \delta\hat{X}_i(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t\right|.$$

When the state is the thermal state, the Kubo formula Eq. (2.2.7) relates the expectation value of the commutator to the susceptibility. This can be incorporated by splitting the integral, and using the symmetries of the susceptibility, viz.,

$$\begin{aligned}
\bar{S}_{X_i X_i}[\Omega] &\geq \frac{1}{2}\left|\int_{-\infty}^{0}\left\langle\left[\delta\hat{X}_i(t), \delta\hat{X}_i(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t + \int_{0}^{\infty}\left\langle\left[\delta\hat{X}_i(t), \delta\hat{X}_i(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t\right| \\
&= \frac{1}{2}\left|\int_{0}^{\infty}\left\langle\left[\delta\hat{X}_i(-t), \delta\hat{X}_i(0)\right]\right\rangle e^{-i\Omega t}\,\mathrm{d}t + \int_{0}^{\infty}\left\langle\left[\delta\hat{X}_i(t), \delta\hat{X}_i(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t\right| \\
&= \frac{1}{2}\left|\int_{0}^{\infty}\left\langle\left[\delta\hat{X}_i(t), \delta\hat{X}(0)\right]\right\rangle\left(e^{i\Omega t} - e^{-i\Omega t}\right)\mathrm{d}t\right| \\
&= \hbar\left|\int_{0}^{\infty}\chi_{ii}(t)\sin(\Omega t)\,\mathrm{d}t\right|
\end{aligned}$$

The third equality follows from the odd property of the average of the commutator of a weak-stationary operator, i.e. $\langle[\delta\hat{X}_i(-t), \delta\hat{X}_i(0)]\rangle = -\langle[\delta\hat{X}_i(t), \delta\hat{X}_i(0)]\rangle$, while the fourth employs Eq. (2.2.8). Since the sine-transform is the imaginary part of the Fourier transform, the right-hand side becomes $\hbar\,|\mathrm{Im}\ \chi_{ii}|$. □

Proposition 2.4 signifies that once a system is coupled to a thermal environment via a linear coupling through its observable $\hat{X}_i$, then that observable exhibits a fundamental fluctuation that depends on the details of the coupling (i.e. the susceptibility), but not the temperature. Tentatively, and with foresight, the minimum value of the spectrum may be identified as the spectrum of *vacuum fluctuations* of that observable.

Clearly the imaginary part of the susceptibility plays a prominent role in determining the fluctuations in system observables in a thermal state. From the expression for the imaginary part of the susceptibility,

$$\mathrm{Im}\ \chi_{ij}[\Omega] = -\frac{i}{2}\int\left(\chi_{ij}(t) - \chi_{ij}(-t)\right)e^{i\Omega t}\,\mathrm{d}t.$$

is is clear that it characterises the lack of invariance to time-reversal $t \to -t$, and thus captures the irreversible character of the system once it is coupled to the environment. On the other hand, the coupling to the environment leads to fluctuations in the system's observables, characterised by the bound Eq. (2.2.9). It is therefore natural to enquire whether a precise equality exists between the imaginary part of the susceptibility and the spectrum of observables that codifies the shared origin of fluctuations and dissipation.

Proposition 2.5 (Fluctuation-Dissipation) *For a system maintained in a thermal state through its contact with an environment, the fluctuations in the observables that couple to the environment are characterised by the relation,*

$$\bar{S}_{X_i X_i}[\Omega] = \hbar(2n_\beta(\Omega) + 1)\mathrm{Im}\ \chi_{ii}[\Omega], \tag{2.2.10}$$

where $n_\beta(\Omega)$ is the Bose occupation at frequency Ω and inverse temperature β,

$$n_\beta(\Omega) := (e^{\beta\hbar\Omega} - 1)^{-1}. \tag{2.2.11}$$

Proof First we prove a slightly general result and then specify to the case at hand. Starting from the left-hand side of Eq. (2.2.10) in the time domain:

$$\mathrm{Im}\ \chi_{ij}(t) = -\frac{i}{2}\left(\chi_{ij}(t) - \chi_{ij}(t)^*\right) = -\frac{i}{2}\left(\chi_{ij}(t) - \chi_{ji}(-t)\right).$$

Using the Kubo formula (Eq. (2.2.7)), and employing time-translation invariance, the susceptibilities can be expressed in terms of correlators (which are the inverse Fourier transforms of the unsymmetrised spectra),

$$\begin{aligned}\chi_{ij}(t) &= -\frac{i}{\hbar}\Theta(t)\left(S_{X_i X_j}(t) - S_{X_j X_i}(-t)\right)\\ \chi_{ji}(-t) &= -\frac{i}{\hbar}\Theta(-t)\left(S_{X_j X_i}(-t) - S_{X_i X_j}(t)\right),\end{aligned}$$

which gives,

$$\mathrm{Im}\ \chi_{ij}(t) = \frac{-1}{2\hbar}\left(S_{X_i X_j}(t) - S_{X_j X_i}(-t)\right).$$

Now using the KMS condition (Eq. (2.2.3)), the order of observables in the second correlator can be reversed, i.e. $S_{X_j X_i}(-t) = S_{X_i X_j}(t - i\hbar\beta)$. Inserting this back gives,

$$\mathrm{Im}\ \chi_{ij}(t) = \frac{-1}{2\hbar}\left(S_{X_i X_j}(t) - S_{X_i X_j}(t - i\hbar\beta)\right).$$

Fourier transforming each side and re-arranging results in

$$S_{X_i X_j}[\Omega] = \frac{2\hbar}{1 - e^{-\beta\hbar\Omega}}\mathrm{Im}\ \chi_{ij}[\Omega], \tag{2.2.12}$$

which relates the unsymmetrised cross-spectra with the susceptibility. For the required result, we consider the case $\hat{X}_j = \hat{X}_i$, and the symmetrised spectral density,

$$\begin{aligned}\bar{S}_{X_iX_i}[\Omega] &= \frac{1}{2}\left(S_{X_iX_i}[\Omega] + S_{X_iX_i}[-\Omega]\right)\\ &= \frac{1}{2}(1 + e^{-\beta\hbar\Omega})S_{X_iX_i}[\Omega]\\ &= \hbar\left(\frac{e^{\beta\hbar\Omega}+1}{e^{\beta\hbar\Omega}-1}\right)\mathrm{Im}\ \chi_{ii}[\Omega];\end{aligned}$$

here, the first equality is the symmetric property of the spectral density (Eq. (2.1.15)), the second follows from the detailed balance condition (Eq. (2.2.4)), and the third from Eq. (2.2.12). Replacing the exponentials in terms of the Bose occupation (Eq. (2.2.11)) gives the result. □

The fluctuation-dissipation theorem (Eq. (2.2.10)) relates the fluctuations in the system to the system-environment coupling, and the environment state (determined by the single parameter, temperature). The bound in Proposition 2.4 (Eq. (2.2.9)), on the other hand, follows from the non-commutativity of the observable and not on the properties of the environment, and is therefore a more general statement. Notably, the zero-temperature limit ($\beta \to \infty$, for which $n_\beta[\Omega] \to 0$) of Eq. (2.2.10) gives Eq. (2.2.9), motivating the interpretation that the lower-bound in the latter is due to intrinsic—vacuum—fluctuations in the system.

An important corollary of the fluctuation-dissipation theorem is that the spectrum of fluctuations of the system observable can be referred to an effective spectrum of the generalised force. In the case where only one observable, $\hat{X}$, is coupled to its generalised force $\hat{F}$, the respective spectra are given by,

$$\begin{aligned}&\bar{S}_{XX}[\Omega] = \hbar\left(2n_\beta(\Omega)+1\right)\mathrm{Im}\ \chi[\Omega]\\ \Rightarrow\quad &\bar{S}_{FF}[\Omega] = |\chi[\Omega]|^{-2}\,\bar{S}_{XX}[\Omega] = \hbar\left(2n_\beta(\Omega)+1\right)\mathrm{Im}\ \chi[\Omega]^{-1},\end{aligned}$$

where $\chi[\Omega]$ is the sole susceptibility involved.

2.3 Dynamics Due to a Meter

Quantum mechanically, a *meter*—a measuring device—is a specific form of environment from the perspective of the system. During an act of measurement, the system is coupled to a meter. The meter, being a quantum mechanical system itself, has intrinsic fluctuations in its variables. Via the measurement interaction, this leads to additional fluctuations in the system variables. Such fluctuations are called *measurement back-action*. Unlike a thermal environment however, the meter needs to be ideally prepared in some non-equilibrium state,[10] meaning that the fluctuations

[10]This is because the meter is expected to output a classical record of the system observable being measured; this can only be arranged for if the states of the meter corresponding to the various values taken by the system observable are macroscopically distinguishable [50].

imparted by it are not determined by the fluctuation-dissipation theorem. However, bounds for the imparted fluctuations can be derived under the minimal assumption of the system-meter coupling being linear and weak.

Very generally, a continuous linear measurement of the observable $\hat{X}$ may be described by an operator $\hat{Y}$ corresponding to the output of a detector. Linearity means that[11] $\hat{Y}(t) \propto \hat{X}(t)$. Since we assume that $\hat{Y}(t)$ is the output of a detector—i.e. the *measurement record*—it must certainly be a continuous observable, i.e.,

$$\left[\hat{Y}(t), \hat{Y}(t')\right] = 0. \tag{2.3.1}$$

However, in general, the system observable, $\hat{X}$, is not a continuous observable. For $\hat{Y}$ to commute with itself, while $\hat{X}$ does not, it is necessary that the record be contaminated by some additional process $\hat{X}_n(t)$, arising from the meter,[12] so that the combination,

$$\hat{Y}(t) = \hat{X}(t) + \hat{X}_n(t) \tag{2.3.2}$$

is a continuous observable. The two equations above operationally characterise the class of so-called *continuous linear measurements* [20, 26].

2.3.1 Effect of Fluctuations from a Meter

Proposition 2.6 (Standard Quantum Limit) *When a meter provides a continuous linear record $\hat{Y}(t)$, of the observable $\hat{X}(t)$ of a system, the spectrum of the output is,*

$$\bar{S}_{YY}[\Omega] \geq 2 \cdot min\, \bar{S}_{XX}[\Omega] = 2\hbar \left|\mathrm{Im}\ \chi_{XX}[\Omega]\right|, \tag{2.3.3}$$

when no correlations exist between the system and meter. In other words, the measurement record contains at least twice the minimum noise in the observable being measured.

Proof The spectrum of the output $\hat{Y}$ (in Eq. (2.3.2)) is,

$$\bar{S}_{YY}[\Omega] = \bar{S}_{XX}[\Omega] + \bar{S}_{X_n X_n}[\Omega] + 2\mathrm{Re}\ \bar{S}_{XX_n}[\Omega].$$

Assuming no correlations between the system and meter, the last term can be neglected, and so,

[11] The most general linear relationship is of the form $\hat{Y}(t) = \int f(t)\hat{X}(t-t')\,\mathrm{d}t'$, corresponding to a filtered version of the observable. However, without loss of generality, the filtering may be considered as happening on the classical measurement record, after the detector.

[12] On the other hand, if it can be arranged that the observable $\hat{X}$ already satisfies $[\hat{X}(t), \hat{X}(t')] = 0$, i.e. it is a continuous observable in the sense defined in Eq. (2.1.18), then there is in principle no additional contamination.

$$\bar{S}_{YY}[\Omega] = \bar{S}_{XX}[\Omega] + \bar{S}_{X_n X_n}[\Omega].$$

The bound set by vacuum fluctuations (Eq. (2.2.9)) implies a lower bound for the spectrum of the system observable $\bar{S}_{XX}$: i.e. $\bar{S}_{XX} \geq \hbar \; |\text{Im} \; \chi_{XX}|$. The remaining task is therefore to lower bound $\bar{S}_{X_n X_n}$. The continuous observability condition (Eq. (2.3.1)) implies that the commutators of $\hat{X}_n$ and $\hat{X}$ are related, viz.,

$$[\hat{X}_n(t), \hat{X}_n(t')] = -[\hat{X}(t), \hat{X}(t')].$$

Now applying the minimum noise bound, in Proposition 2.2, to $\hat{X}_n$ gives,

$$\begin{aligned}\bar{S}_{X_n X_n}[\Omega] &\geq \frac{1}{2}\left|\int_{-\infty}^{\infty}\left\langle\left[\delta\hat{X}_n(t), \delta\hat{X}_n(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t\right| \\ &= \frac{1}{2}\left|\int_{-\infty}^{\infty}\left\langle\left[\delta\hat{X}(t), \delta\hat{X}(0)\right]\right\rangle e^{i\Omega t}\,\mathrm{d}t\right| \\ &\geq \hbar \; |\text{Im} \; \chi_{XX}[\Omega]| \,.\end{aligned}$$

Here, the last inequality follows from arguments given in the proof of Eq. (2.2.9). Ultimately,

$$\bar{S}_{YY}[\Omega] \geq 2 \cdot \hbar \; |\text{Im} \; \chi_{XX}[\Omega]| = 2 \cdot \min \bar{S}_{XX}[\Omega].$$

□

Conceptually, the standard quantum limit (Eq. (2.3.3)) states that quantum mechanics extorts a penalty twice: once in the form of the vacuum fluctuations of the observable (as in Eq. (2.2.9)), and once more, the same price, in the form of unavoidable fluctuations in the linear measurement process. This factor of two may also be understood if the action of the meter is considered to be that of an abstract linear amplifier [18, 51] whose role is to amplify the values taken by the system observable into a classically recordable signal. This perspective sheds light on the relationship between the standard quantum limit derived here for a general scenario, and the specific example of the vacuum-equivalent noise that is added when simultaneously measuring the canonically conjugate variables of a harmonic oscillator [52–56].

The standard quantum limit rests on the validity of the assumptions basic to its existence being fulfilled in a given situation: (1) the system-meter coupling is linear, (2) continuous, (3) stationary, and, (4) the system and meter states are uncorrelated. (Presumably, the adjective "standard" refers to this *standard* configuration.) A violation of one or more of these assumptions can beat the bound in Eq. (2.3.3). In the context of interferometric position measurement [57]—a prototypical example of a continuous linear measurement—all these loop holes have been exploited as a means to improve measurement sensitivity beyond the standard quantum limit.

For example, quantum non-demolition techniques to measure position rely on a time-dependent coupling between the system and meter [37, 38], violating the continuity and/or stationarity assumptions. Injection of squeezed light into the interferometer [58–60], or the use of squeezing generated within the interferometer [60, 61], relies on harnessing system-meter correlations.

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