Lecture 2
\(\mathbb{R}\) - and \(\mathbb{C}\) - Differentiability

Let \(z_0 = x_0 + iy_0 = (x_0, y_0)\) be a point in \(\mathbb{C}\) and \(f\) a function defined on a neighbourhood of \(z_0\) (e.g., on an open disk \(\Delta(z_0, r)\) for some \(r > 0\)) with values in \(\mathbb{C}\). Write \(f(z) = \text{Re} f(z) + i\text{Im} f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)\).

**Definition 2.1.** The function \(f\) is called \(\mathbb{R}\)-differentiable at \(z_0\) if there exist real numbers \(a, b, c, d\) such that
\[
\begin{align*}
u(x, y) &= v(x_0, y_0) + c\Delta x + d\Delta y + o(\Delta x, \Delta y),
\end{align*}
\]
where \(\Delta x := x - x_0, \Delta y := y - y_0\) and \(o(\Delta x, \Delta y)\) denotes any real-valued function with the property
\[
\frac{o(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to 0 \text{ as } (\Delta x)^2 + (\Delta y)^2 \to 0.
\]

Here \(a, b, c, d\) are determined uniquely and are in fact the corresponding first-order partial derivatives of \(u\) and \(v\) at \((x_0, y_0)\):
\[
\begin{align*}
a &= \frac{\partial u}{\partial x}(x_0, y_0), & b &= \frac{\partial u}{\partial y}(x_0, y_0), & c &= \frac{\partial v}{\partial x}(x_0, y_0), & d &= \frac{\partial v}{\partial y}(x_0, y_0).
\end{align*}
\]

We will now introduce the concept of complex differentiability and compare it with that of real differentiability as defined above.

**Definition 2.2.** The function \(f\) is said to be \(\mathbb{C}\)-differentiable at \(z_0\) if there exists the limit
\[
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},
\]
that is, one can find \(A \in \mathbb{C}\) such that for any \(\varepsilon > 0\) there is a sufficiently small \(\delta > 0\) with the property
\[
\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - A \right| < \varepsilon.
\]
if $0 < |\Delta z| < \delta$. In this case, the complex number $A$, i.e., the value of the limit, is called the derivative of $f$ at $z_0$ and is denoted by $f'(z_0)$.

The usual rules from real analysis for computing the derivatives of sums, products, quotients and compositions of $\mathbb{C}$-differentiable functions work here (check!).

Definition 2.2 is equivalent to saying that for some $A \in \mathbb{C}$ the function $f$ is represented as

$$f(z) = f(z_0) + A\Delta z + o(\Delta z),$$

where $\Delta z := z - z_0$ and $o(\Delta z)$ denotes any complex-valued function with the property

$$\frac{o(\Delta z)}{\Delta z} \to 0 \text{ as } \Delta z \to 0$$

(note that one can write $o(\Delta z)$ instead of $o(\Delta x, \Delta y)$ in Definition 2.1). By separating the real and imaginary parts in identity (2.1), we see that it is equivalent to the following pair of real identities:

$$u(x, y) = u(x_0, y_0) + \text{Re} \, A \Delta x - \text{Im} \, A \Delta y + o(\Delta x, \Delta y),$$

$$v(x, y) = v(x_0, y_0) + \text{Im} \, A \Delta x + \text{Re} \, A \Delta y + o(\Delta x, \Delta y).$$

Comparing these identities with Definition 2.1 and taking into account that the constants $a, b, c, d$ are chosen uniquely, we obtain:

**Theorem 2.1.** The function $f$ is $\mathbb{C}$-differentiable at $z_0 = x_0 + iy_0$ if and only if it is $\mathbb{R}$-differentiable at $z_0$ and the first-order partial derivatives of $u$ and $v$ at $(x_0, y_0)$ satisfy the relations

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

(2.2)

Relations (2.2) are called the Cauchy-Riemann equations or simply the CR-equations. We will now rewrite them in a different form.

**Definition 2.3.** Let $f$ be $\mathbb{R}$-differentiable at $z_0$. Set

$$\frac{\partial f}{\partial z}(z_0) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0, y_0) - i \frac{\partial f}{\partial y}(x_0, y_0) \right), \quad \frac{\partial f}{\partial \bar{z}}(z_0) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0, y_0) + i \frac{\partial f}{\partial y}(x_0, y_0) \right),$$

where

$$\frac{\partial f}{\partial x}(x_0, y_0) := \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0), \quad \frac{\partial f}{\partial y}(x_0, y_0) := \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0).$$

Now, multiplying the second identity from Definition 2.1 by $i$, adding it to the first one and substituting

$$\Delta x = \frac{\Delta z + \Delta \bar{z}}{2}, \quad \Delta y = \frac{\Delta z - \Delta \bar{z}}{2i},$$

we obtain:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$
we obtain
\[ f(z) = f(z_0) + \left( \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \right) \frac{\Delta z + \overline{\Delta z}}{2} + \]
\[ \left( \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \right) \Delta z + \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0, y_0) + i \frac{\partial f}{\partial y}(x_0, y_0) \right) \Delta z \]
\[ f(z) = f(z_0) + \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0, y_0) - i \frac{\partial f}{\partial y}(x_0, y_0) \right) \Delta z + \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0, y_0) + i \frac{\partial f}{\partial y}(x_0, y_0) \right) \overline{\Delta z} + o(\Delta z) = \]
\[ f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0) \Delta z + \frac{\partial f}{\partial \overline{z}}(z_0) \overline{\Delta z} + o(\Delta z). \]

Comparing the above formula with (2.1), we see:

**Theorem 2.2.** The function \( f \) is \( \mathbb{C} \)-differentiable at \( z_0 \) if and only if it is \( \mathbb{R} \)-differentiable at \( z_0 \) and \( \frac{\partial f}{\partial \overline{z}}(z_0) = 0 \). In this case \( f'(z_0) = \frac{\partial f}{\partial z}(z_0) \).

**Proof.** Homework. (Hint: use, e.g., Exercise 2.4.) \( \square \)

**Remark 2.1.** The fact that the CR-equations from Theorem 2.1 are equivalent to the condition \( \frac{\partial f}{\partial \overline{z}}(z_0) = 0 \) can be also seen directly from Definition 2.3 by separating the real and imaginary parts of \( \frac{\partial f}{\partial \overline{z}}(z_0) \). Indeed, we have
\[
\frac{\partial f}{\partial \overline{z}}(z_0) = \frac{1}{2} \left( \frac{\partial u}{\partial x}(x_0, y_0) - \frac{\partial v}{\partial y}(x_0, y_0) \right) + i \frac{1}{2} \left( \frac{\partial v}{\partial x}(x_0, y_0) + \frac{\partial u}{\partial y}(x_0, y_0) \right),
\]
hence the complex identity \( \frac{\partial f}{\partial \overline{z}}(z_0) = 0 \) is equivalent to the pair of real identities in (2.2).

The “formal derivatives” \( \frac{\partial f}{\partial z} \) and \( \frac{\partial f}{\partial \overline{z}} \) can be often treated as “usual” derivatives. For example, consider a complex-valued polynomial in \( x, y \)
\[ P(x, y) = \sum_{0 \leq \ell, m \leq K} a_{\ell m} x^{\ell} y^{m}, \ a_{\ell m} \in \mathbb{C}. \]
Substituting
\[ x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}, \]
we express \( P \) via \( z, \overline{z} \):
\[ P = \sum_{0 \leq \ell, m \leq K} b_{\ell m} z^{\ell} \overline{z}^{m}, \]
for some \( b_{\ell m} \in \mathbb{C} \).
Proposition 2.1. For all $z_0 \in \mathbb{C}$ we have
\[
\frac{\partial P}{\partial z}(z_0) = \sum_{0 \leq \ell, m \leq K} \ell b_{\ell m} z_0^{\ell-1} z_0^{m}, \quad \frac{\partial P}{\partial \bar{z}}(z_0) = \sum_{0 \leq \ell, m \leq K} mb_{\ell m} z_0^{\ell} z_0^{m-1}.
\]

Proof. It suffices to prove the proposition for any monomial $Q := z^\ell z^m$. Since $Q = (x + iy)^\ell (x - iy)^m$, for $z_0 = x_0 + iy_0$ we calculate
\[
\frac{\partial Q}{\partial x}(x_0, y_0) = \ell (x_0 + iy_0)^{\ell-1} (x_0 - iy_0)^m + m (x_0 + iy_0)^\ell (x_0 - iy_0)^{m-1} =
\ell z_0^{\ell-1} z_0^{m} + mz_0^\ell z_0^{m-1},
\]
\[
\frac{\partial Q}{\partial y}(x_0, y_0) = i\ell (x_0 + iy_0)^{\ell-1} (x_0 - iy_0)^m - im (x_0 + iy_0)^\ell (x_0 - iy_0)^{m-1} =
i\ell z_0^{\ell-1} z_0^{m} - imz_0^\ell z_0^{m-1}
\]
(check!). Hence,
\[
\frac{\partial Q}{\partial z}(z_0) = \frac{1}{2} \left( \frac{\partial Q}{\partial x}(x_0, y_0) - i \frac{\partial Q}{\partial y}(x_0, y_0) \right) = \ell z_0^{\ell-1} z_0^{m},
\]
\[
\frac{\partial Q}{\partial \bar{z}}(z_0) = \frac{1}{2} \left( \frac{\partial Q}{\partial x}(x_0, y_0) + i \frac{\partial Q}{\partial y}(x_0, y_0) \right) = mz_0^\ell z_0^{m-1},
\]
which completes the proof. \(\square\)

We will be mostly interested in the \(\mathbb{C}\)-differentiability of functions on open subsets of the complex plane.

Definition 2.4. Let \(D \subset \mathbb{C}\) be an open subset. A function \(f : D \rightarrow \mathbb{C}\) is said to be holomorphic on \(D\) if \(f\) is \(\mathbb{C}\)-differentiable at every point of \(D\). All functions holomorphic on \(D\) form a vector space over \(\mathbb{C}\), which we denote by \(H(D)\). Functions in \(H(\mathbb{C})\) (i.e., those holomorphic on all of \(\mathbb{C}\)) are called entire.

We have the usual facts:

1. if \(f, g \in H(D)\), then \(f + g \in H(D)\) and \(fg \in H(D)\);
2. if \(f, g \in H(D)\) and \(g\) does not vanish at any point of \(D\), then \(f/g \in H(D)\);
3. if \(f \in H(D)\), \(g \in H(G)\), \(f(D) \subset G\), then \(g \circ f \in H(D)\).

The proofs are identical to those for functions differentiable on open subsets of \(\mathbb{R}\) from real analysis (check!).

Usually, in what follows \(D\) will be a domain, i.e., an open connected subset of \(\mathbb{C}\), where connectedness is understood as the existence, for any \(z, w \in D\), of a path in \(D\) joining \(z\) and \(w\), that is, of a continuous map \(\gamma : [0, 1] \rightarrow D\) with \(\gamma(0) = z\), \(\gamma(1) = w\) (in this case, we sometimes say that \(z\) is the initial point of \(\gamma\) and \(w\) is its terminal point). Strictly speaking, the above property is called path-connectedness but for open subsets of \(\mathbb{R}^n\) (in fact, for open subsets of any locally path-connected topological space) it is equivalent to connectedness, i.e., to the non-existence of a non-trivial subset that is both open and closed (check!).
In what follows, we will be often interested in extending a function \( f \in H(D) \) holomorphically beyond the domain \( D \). Namely, for a domain \( G \supset D \) we say that \( f \) \{holomorphically\} extends \{to\} \( G \), or that \( f \) can be \{holomorphically\} extended \{to\} \( G \), if there exists \( F \in H(G) \) such that \( F(z) = f(z) \) for all \( z \in D \). In this case \( F \) is called a holomorphic extension of \( f \) \{to\} \( G \) and we also say that \( f \) extends to a function holomorphic on \( G \), or that \( f \) can be extended to a function holomorphic on \( G \).

**Example 2.1.** Let \( f(z) := z = x + iy \). Here \( u = x \), \( v = y \), hence

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 1, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \equiv 0.
\]

Thus, we see that \( f \) is an entire function. It then follows that every polynomial \( P(z) = a_kz^k + a_{k-1}z^{k-1} + \cdots + a_0 \) in \( z \) is an entire function.

**Example 2.2.** Let \( f(z) := e^z = e^x \cos y + i e^x \sin y \). Here \( u = e^x \cos y \), \( v = e^x \sin y \), hence

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x \sin y.
\]

Therefore, \( f \) is an entire function. Also, for any \( z_0 \in \mathbb{C} \) we have

\[
f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0, y_0) - i \frac{\partial f}{\partial y}(x_0, y_0) \right) = \frac{1}{2} (e^{z_0} - i(e^{z_0})) = e^{z_0}
\]

as expected. It then follows that the basic trigonometric functions

\[
\cos z := \frac{e^z + e^{-iz}}{2}, \quad \sin z := \frac{e^z - e^{-iz}}{2i}
\]

are entire as well.

**Example 2.3.** Let \( f(z) := \bar{z} = x - iy \). Here \( u = x \), \( v = -y \), hence

\[
\frac{\partial u}{\partial x} \equiv 1, \quad \frac{\partial v}{\partial y} \equiv -1, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \equiv 0.
\]

Therefore, \( f \) is not \( \mathbb{C} \)-differentiable at any point of \( \mathbb{C} \).

**Example 2.4.** Let \( f(z) := \bar{z}^2 = x^2 - y^2 - 2ixy \). Here \( u = x^2 - y^2 \), \( v = -2xy \), hence

\[
\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = -2y.
\]

Therefore, \( f \) is only \( \mathbb{C} \)-differentiable at the origin. In particular, \( f \) is not holomorphic on any domain in \( \mathbb{C} \).

Notice also that as a consequence of Proposition 2.1 we have:
Corollary 2.1. The polynomial $P$ from Proposition 2.1 is an entire function if and only if $b_{t,m} = 0$ for all $m > 0$. In this case the derivative $P' = \frac{\partial P}{\partial z}$ is given by formal differentiation with respect to $z$.

Proof. Homework. (Hint: use exercise 2.10.) □

Exercises

2.1. Let $f$ be $\mathbb{R}$-differentiable at $z_0$. Prove that the Jacobian of $f$ at $z_0$ is equal to

$$\left| \frac{\partial f}{\partial z}(z_0) \right|^2 - \left| \frac{\partial f}{\partial \bar{z}}(z_0) \right|^2.$$

2.2. Let $f$ be $\mathbb{R}$-differentiable at $z_0$. Prove that the limit set of the ratio

$$g(z) := \frac{f(z) - f(z_0)}{z - z_0}, \quad z \neq z_0,$$

at $z_0$ is the circle centred at the point $\frac{\partial f}{\partial z}(z_0)$ of radius $\left| \frac{\partial f}{\partial \bar{z}}(z_0) \right|$. Here the limit set consists of all limit points of $g$ at $z_0$, i.e., of all complex numbers $A$ for which there is a sequence $\{z_n\}$ not including the point $z_0$, with $|z_n - z_0| \to 0$ as $n \to \infty$, such that $|g(z_n) - A| \to 0$.

2.3. Let $f$ be continuously differentiable on $\mathbb{C}$. Suppose that $f$ preserves distances, i.e.,

$$|f(z_1) - f(z_2)| = |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C}.$$

Prove that either $f(z) = e^{i\alpha}z + a$, or $f(z) = e^{i\alpha}\bar{z} + a$ for some $a \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

(Hint: use Exercise 2.2.)

2.4. Assume that $f$ is defined on a neighbourhood of a point $z_0$ and in this neighbourhood for some $A, B \in \mathbb{C}$ the following holds:

$$f(z) = f(z_0) + A\Delta z + B\overline{\Delta z} + o(\Delta z),$$

where $\Delta z := z - z_0$. Prove that $f$ is $\mathbb{R}$-differentiable at $z_0$ with $A = \frac{\partial f}{\partial z}(z_0)$ and $B = \frac{\partial f}{\partial \bar{z}}(z_0)$.

2.5. Suppose that $f = u + iv$ is defined on a neighbourhood of 0 and is continuous at 0. Assume that all first-order partial derivatives of $u$ and $v$ exist at 0 and satisfy the CR-equations at 0. Does it follow that $f$ is $\mathbb{C}$-differentiable at the origin? Prove your conclusion.
2.6. For each of the following functions, find all points at which it is \( C \)-differentiable:

(i) \( z^2 |z|^4 \),

(ii) \((\text{Re } z)^4\),

(iii) \(\sin(\text{Im } z)\).

2.7. Let \( f = u + iv \) be \( C \)-differentiable at \( z_0 \), with \( f'(z_0) \neq 0 \), and continuously differentiable on a neighbourhood of \( z_0 \). Prove that the angle between the level sets of \( u \) and \( v \) at \( z_0 \) (that is, any of the four angles between the tangent lines to the level sets at \( z_0 \)) is equal to \( \pi/2 \). (Hint: write the tangent lines to the level sets at \( z_0 \) via the first-order partial derivatives of \( u \) and \( v \) at \( z_0 \).)

2.8. Suppose that a function \( f \) is \( C \)-differentiable at a point \( z_0 \). Prove that the function

\[ g(z) := f(\bar{z}) \]

is \( C \)-differentiable at the point \( \bar{z}_0 \) and \( g'(\bar{z}_0) = f'(z_0) \).

2.9. Suppose that a function \( f \) is \( C \)-differentiable at a point \( z_0 \) and \( f'(z_0) \neq 0 \). Show that for any disk \( \Delta(z_0, r) \) on which \( f \) is defined the set \( f(\Delta(z_0, r)) \) cannot lie in a half-plane on either side of any line passing through \( f(z_0) \).

2.10. Show that for a polynomial

\[ P(z, \bar{z}) = \sum_{0 \leq \ell, m \leq K} b_{\ell m} z^\ell \bar{z}^m, \ b_{\ell m} \in \mathbb{C}, \]

in \( z, \bar{z} \) one has \( P \equiv 0 \) on an open subset of \( \mathbb{C} \) if and only if \( b_{\ell m} = 0 \) for all \( \ell, m \in \{0, \ldots, K\} \). (Hint: argue by induction using Proposition 2.1.)

2.11. Suppose that for a polynomial

\[ P(z, \bar{z}) = \sum_{0 \leq \ell, m \leq K} b_{\ell m} z^\ell \bar{z}^m, \ b_{\ell m} \in \mathbb{C}, \]

in \( z, \bar{z} \) we have \( b_{\ell m} \neq 0 \) for some \( 0 \leq \ell \leq K \) and some \( 0 < m \leq K \). Prove that the set of points at which \( P \) is \( C \)-differentiable is nowhere dense in \( \mathbb{C} \). (Hint: use Exercise 2.10.)

2.12. Find the derivatives of \( \sin z \) and \( \cos z \) at an arbitrary point of \( \mathbb{C} \).

2.13. Prove that \( |\cos z| \) and \( |\sin z| \) are not bounded on \( \mathbb{C} \).

2.14. How many zeroes does the entire function \( 2 + \sin z \) have in \( \mathbb{C} \)? Find all the zeroes.

2.15. Using the power series expansions of \( e^x \), \( \cos y \), \( \sin y \), find a power series expansion with centre 0 for each of \( e^z \), \( \cos z \), \( \sin z \), i.e., represent each of these functions in the form
\[
\sum_{n=0}^{\infty} c_n z^n \quad \forall z \in \mathbb{C},
\]
where \( c_n \in \mathbb{C}, \ n = 0, 1, 2, \ldots \) (Hint: substitute the corresponding series into the expression \( e^x \cos y + i e^x \sin y \) using arithmetic operations with absolutely convergent series and the possibility of re-arranging their terms.)

2.16. Prove that the function \( \frac{\sin z}{z} \) is holomorphic on \( \mathbb{C} \setminus \{0\} \) and that it holomorphically extends to \( \mathbb{C} \).

2.17. Let
\[
f(z) := \begin{cases} 
  z^2 \sin \frac{1}{z} & \text{if } z \neq 0, \\
  0 & \text{if } z = 0.
\end{cases}
\]
Is this function \( \mathbb{C} \)-differentiable at 0? Prove your conclusion.

2.18. Let \( f(z) = u(x) + iv(y) \) be an entire function. Prove that \( f(z) = az + b \), where \( a \in \mathbb{R}, b \in \mathbb{C} \).

2.19. Let \( a, b, c \in \mathbb{C} \). Write the quadratic polynomial \( P(x, y) = ax^2 + bxy + cy^2 \) in \( x, y \) via \( z \) and \( \bar{z} \) and find necessary and sufficient conditions on \( a, b, c \) for \( P \) to be an entire function.

2.20. Find all entire functions \( f \) such that \( \operatorname{Re} f(z) = x^2 - y^2 \).

2.21. Find an entire functions \( f \) such that \( \operatorname{Re} f(z) = x^2 - y^2 + xy, f(0) = 0 \).

2.22. Find an entire functions \( f \) such that \( \operatorname{Re} f(z) = e^x (x \cos y - y \sin y), f(0) = 0 \).

2.23. Let \( g : [0, 1] \to \mathbb{C} \) be a continuous function. For all \( z \in \mathbb{C} \setminus [0, 1] \) define
\[
f(z) := \frac{1}{2\pi i} \int_{0}^{1} \frac{g(t)}{t - z} \, dt,
\]
where the integral is understood by separating the real and imaginary parts of the integrand. Prove that \( f \in H(\mathbb{C} \setminus [0, 1]) \).

2.24. Construct an example showing that the Mean Value Theorem for \( \mathbb{C} \)-valued functions does not hold. Namely, find a differentiable function \( f : [0, 1] \to \mathbb{C} \) such that \( f'(t) \neq f(1) - f(0) \) for all \( t \in (0, 1) \).

2.25. Prove the following variant of the Mean Value Theorem for \( \mathbb{C} \)-valued functions: if \( f : [0, 1] \to \mathbb{C} \) is a continuously differentiable function, then the difference \( f(1) - f(0) \) lies in the closure of the convex hull of the set \( \{f'(t) : t \in [0, 1]\} \). (Hint: use the Newton-Leibniz formula for \( \operatorname{Re} f \) and \( \operatorname{Im} f \).)

2.26. Let \( P \) be a polynomial in \( z \) of positive degree and \( S \) the smallest convex polygon containing the roots of \( P \). Prove that all roots of \( P' \) lie in \( S \). (Hint: use Exercise 1.7.)
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