Preface

Many classical partial differential equations, arising in modeling of various physical and biological phenomena, are only taking into consideration local spatial or time variables, ignoring any possible spatial dependence on the neighboring points as well as neglecting any feasible memory effects. For instance, the classical (local) nonlinear heat equation provides the rate of change of temperature at every (localized) point and for any time where the source term overlooking any possible effects from any nearby points and ignoring the history of the heating process. In contrast, *non-local* models consider any possible reliance of the involved physical quantities on the evolution of the inspected process all over the nearby spatial points or any dependence on preceding times. Such a *non-local dependence* usually stems from a distance interaction or from several *conservation laws*. Some of the first *non-local equations* appeared in the literature are encountered in the field of phase transition and are related to theories due to van der Waals, Ginzburg & Landau and Cahn & Hilliard [5]. Lately, a wide variety of *non-local equations* emerged in the literature with significant applications in engineering, astrophysics, and biology. For instance such models with *non-local* spatial terms are encountered in the Ohmic heating production [29, 30], in the shear banding formation in metals being deformed under high strain rates [1, 2], in the theory of gravitational equilibrium of polytropic stars [27], in the investigation of the fully turbulent behavior of real flows, using invariant measures for the Euler equation [3], in population dynamics [8], in modeling aggregation of cells via interaction with a chemical substance (chemotaxis) [32], just to mention a few of them. Some more applications of *non-local equations* with memory (integral) terms can be found in [31]. Therefore, *non-locality* is not a technical obstruction to scientific research but it actually provides an essence of what happens in reality, and for that purpose, its mathematical study can provide very useful predictions in many areas of applications. In general, *non-local models* provide more accurate predictions compared to their local counterparts since they actually use all the available information regarding the evolution of the inspected process. On the other hand, the presence of the *non-local* terms might be responsible for the lack of some fundamental features, that share the local analogue problems, like the maximum principle [30, 31]. Additionally, most of the *non-local* problems exhibit quite rich dynamics, which is usually more complicated to the dynamics of their local counterparts. In particular, the long-time behavior of *non-local* parabolic equations might be more complex than the one of their local complements, see for example [4, 6, 7, 8]. Another very intriguing phenomenon which is basically due to the presence of the *nonlinearity*, but whose impact grows when a *non-local term* is also present, is the occurrence of *finite-time blow-up* or *finite-time quenching*, when solutions of nonlinear equations cannot be extended after a finite time. Notably, blow-up and quenching fight against the *well-posedness* of nonlinear evolution equations, since both, under some circumstances, they rule out the possibility of existence of global in time solutions. Detailed profiles of the *blowing up* and *quenching* solutions, however, are heavily associated with the form of the *nonlinearity*, and the whole mechanisms of these phenomena have not yet been clarified in many equations as it happens with the *blow-up* of the solutions for the famous example of the *Navier–Stokes equation*. Still a lot of efforts have been paid since the pioneering work of Fujita [9, 10] on the *blow-up* of semilinear parabolic equations, and many important outcomes have been produced. At the earliest stage, studies on *blow-up* and *quenching* were thought to be related only within the field of pure mathematics where several toy models were investigated. However, recently it has been recognized that many realistic models are reduced to semilinear equations with *non-local terms* and many of them exhibit the phenomena of finite *blow-up* and *quenching*. This monograph is devoted to this type of *nonlinear* partial differential equations, investigating both their mathematical modeling and their mathematical analysis.

Part I of the current monograph is devoted to the investigation of some non-local models linked with applications from engineering. Chapter 1 focuses on the study of the following *non-local* model associated with electrostatic MEMS control

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1-u)^2\left(1+\alpha\int_\Omega \frac{1}{1-u}\right)^2} \quad \text{in} \quad \Omega\times(0,T) \tag{1}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \tag{2}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{3}$$

where $u(x,t)$ denotes the deformation of an elastic membrane which is part of the MEMS device. Here and henceforth, λ stands for a positive parameter. In the first place, the construction of the above model is presented. To this end, we describe the two main physical problems which build up the operation of an idealized MEMS device: the elastic and the electric problem . In the second place, we proceed with its mathematical analysis. First, the structure of the set of radially symmetric steady-state solutions is investigated together with their stability. Then, the circumstances under which *finite-time quenching*, or otherwise called *touching down* in the MEMS context, occurs are investigated following the approach developed in [11, 16]. Finally Chap. 1 closes up with the investigation of a hyperbolic variation of Eqs. (1)–(3) along the lines of the approach introduced in [14].

Chapter 2 discusses some *non-local models* describing Ohmic heat production in various industrial processes. In the first part of the chapter, the process of food sterilization by using Ohmic heating is considered and the following one-dimensional *non-local model* is formulated

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\lambda f(u)}{\left(\int_{-1}^{1} f(u)dx\right)^2} \quad \text{in} \quad (0,1)\times(0,T), \tag{4}$$

$$u(0,t) = u_x(1,t) = 0 \quad \text{in} \quad (0,T), \tag{5}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad (0,1), \tag{6}$$

along the lines of [24, 29]. Here, $u(x,t)$ stands for the temperature of the sterilized food, while the nonlinearity $f(u)$ represents either electrical conductivity or resistivity and is taken to be positive. Next, and under different circumstances, a hyperbolic variation of Eqs. (4)–(6) with a *non-local* convection velocity is built up following the approach introduced in [25]. Both of these *non-local* models are inspected in terms of their stability and the occurrence of finite-time *blow-up*, where the latter in the current context means food burning. Different approaches should be followed though depending on the monotonicity of the nonlinearity f, since no maximum principle is available for the *non-local* parabolic problem (Eqs. (4)–(6)) when f is increasing. In the case where f is decreasing, some useful estimates of the blow-up (burning) time are given via the method developed in [19, 28]. Finally, the hyperbolic problem is treated via the method of characteristics. The second part of Chap. 2 deals with another application of Ohmic heating process in the thermistor device which is modeled by the following

$$\frac{\partial u}{\partial t} = \nabla(k(u)\nabla u) + \frac{\lambda f(u)}{\left(\int_{\Omega} f(u)\right)^2} \quad \text{in} \quad \Omega\times(0,T), \tag{7}$$

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{8}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{9}$$

where $u(x,t)$ is the temperature across the thermistor device, while $\frac{\partial}{\partial \nu}$ stands for the normal outward derivative to $\partial\Omega$ and $0 \le \beta \le \infty$, $k(u) > 0$. The finite-time blow-up, which suggests either the destruction of the thermistor device or the failure of the model, is investigated via the methods developed in [18].

Chapter 3 debates an application arising in the process of linear friction welding applied in metallurgy. Initially, the following one-dimensional *non-local* model is constructed

$$u_t = u_{xx} + \frac{e^u}{\left(\int_0^\infty e^u\,dx\right)^{1+\alpha}}, \quad \text{in} \quad (0,\infty)\times(0,T), \quad \alpha > 0, \tag{10}$$

$$u_x(0,t), = 0, \quad \lim_{x\to\infty} u_x(x,t) = -1, \quad \text{in} \quad (0,T), \tag{11}$$

$$u(x,0) = u_0(x), \quad \text{in} \quad (0,\infty), \tag{12}$$

for the soft-material case, where $u(x,t)$ serves as the temperature across the welding region. Next, a similar *non-local* model is derived for the hard-material case where the exponential nonlinearity is replaced by $f(u) = (-u)^{-p}$ for $p = \frac{1}{\alpha}$. The stability of Eqs. (10)–(12) is investigated using the analytical approach developed in [13], which actually proves convergence to the unique steady state. On the other hand, the stability in the case of the power-law nonlinearity is investigated by means of a numerical approach [13].

The following degenerate *non-local* model

$$u_t = \Delta\beta(u) + \frac{\lambda f(\beta(u))}{\left(\int_\Omega f(\beta(u))\,dx\right)^2}, \quad \text{in} \quad \Omega\times(0,T),\ T > 0, \tag{13}$$

$$\beta(u) + k(x)\frac{\partial\beta(u)}{\partial\nu} = 0, \quad \text{on} \quad \partial\Omega\times(0,T), \tag{14}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega, \tag{15}$$

is produced in Chap. 4, where $0 \le k(x) \le \infty$ and $\beta(u) \ge 0$ are continuous functions with $\beta(0) = 0$. *Non-local* model (Eqs. (13)–(15)) is associated with the industrial process of resistance spot welding and $u(x,t)$ serves as the temperature in the welding area. By using a numerical scheme developed in [26], the occurrence of an emerging interface (free boundary), stemming from the degeneracy due to the condition $\beta(0) = 0$, is revealed.

Part II is handling some applications of *non-local* models coming for the field of biology. In Chap. 5, the following *non-local* problem is derived

$$u_t = \Delta u - u + \frac{u^p}{\left(\fint_\Omega u^r\,dx\right)^\gamma}, \quad \text{in} \quad \Omega\times(0,T), \tag{16}$$

$$\frac{\partial u}{\partial\nu} = 0, \quad \text{on} \quad \partial\Omega\times(0,T), \tag{17}$$

$$u(x,0) = u_0(x) > 0, \quad \text{in} \quad \Omega, \tag{18}$$

by the shadow system of Gierer–Meinhardt system, an inhibitor–activator system arising in cell biology, when the inhibitor diffuses much faster than the activator does. The longtime behavior of $u(x,t)$, representing the concentration of the activator, is examined according to the values of parameters p, r and γ. Among other interesting results, and following the approach in [23], a diffusion-driven blow-up (a sort of *Turing instability* result) for the solution of Eqs. (16)–(18) is proven under the *Turing instability* condition $p - 1 < r\gamma$.

In Chap. 6, we deal with an application arising in evolutionary game dynamics and in particular in its subarea known as replicator dynamics. Considering an infinite continuous strategy space, corresponding, for example, to a continuously varying trait of a biological population, as well as payoff functions of Gaussian type, we end up with the following *non-local* degenerate model

$$\frac{\partial u}{\partial t} = u(\Delta u + \int_\Omega |\nabla u|^2 \, dx), \qquad \text{in} \quad \Omega \times (0,T), \tag{19}$$

$$u(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,T), \tag{20}$$

$$u(x,0) = u_0(x), \quad \text{in} \quad \Omega, \tag{21}$$

where $u(x,t)$ denotes the probability density of the probability measure providing the state of the biological population (population of players). As it is appropriate for degenerate problems, a regularized approximation of Eqs. (19)–(21) is used and then some a priori estimates for its solutions are derived. Afterward, by adopting the arguments introduced in [17], global-in-time existence and blow-up results are obtained according to the value of the initial mass $||u_0||_{L^1(\Omega)}$.

Chapter 7 debates the biological phenomenon of chemotaxis. Initially, a version of Keller-Segel system is considered which describes the movement of some cell population toward a chemo-attractant produced by the population itself. Next, it is shown, using the approach of [20, 32], that in the case where the chemo-attractant diffuses much faster than the cell population, then Keller-Segel reduces to the following *non-local* problem

$$u_t = \Delta u + \frac{\lambda e^u}{\int_\Omega e^u} \qquad \text{in} \qquad \Omega \times (0,T), \tag{22}$$

$$u(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \tag{23}$$

$$u(x,0) = u_0(x) \qquad \text{in} \quad \Omega, \tag{24}$$

where $u(x,t)$ stands for the concentration of the chemo-attractant. For $\lambda < 8\pi$ global-in-time existence is derived whereas for $\lambda > 8\pi$ and for radially symmetric solutions, by using the approach developed in [20], the occurrence of blow-up is proven.

Chapter 8 introduces the following *non-local* reaction–diffusion system

$$u_t = d_1 \Delta u - ku \cdot \fint_{B(\cdot,R)\cap\Omega} v \quad \text{in} \quad \Omega \times (0,T), \tag{25}$$

$$v_t = d_2 \Delta v - kv \cdot \fint_{B(\cdot,R)\cap\Omega} u \quad \text{in} \quad \Omega \times (0,T), \tag{26}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{27}$$

$$u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{28}$$

where

$$\fint = \frac{1}{\omega_N R^N} \int .$$

In the first place, system (Eqs. (25)–(28)) is built up as a mathematical model in cell biology to describe the evolution of protein dimers within human cells. Indeed, system (Eqs. (25)–(28)) inspects the situation when chemical reactions occur only when two chemicals within cells and with concentrations u and v are in distance R, called the reaction radius [12, 21]. Next, the longtime behavior of the solutions of Eqs. (25)–(28) is investigated as well as the phase separation phenomenon develops on extremely fast reaction rates ($k \to +\infty$) is also examined [21, 22].

In the Appendix, some more *non-local* models are presented arising in further applications ranged from point vortices theory to differential geometry.

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