Chapter 2
Ohmic Heating Phenomena

Abstract The current chapter considers two main applications associated with Ohmic heating phenomena. Initially we deal with an application from food industry, building up two one-dimensional non-local problems illustrating the evolution of the temperature of the sterilized food. The former model consists of a diffusion-convection equation while the latter of a convection equation with non-local convection velocity. Both of these non-local models are investigated in terms of their stability and the occurrence of finite-time blow-up, where the latter in the current context indicates food burning. Different approaches should be followed though depending on the monotonicity of the nonlinearity appearing in the non-local term, since no maximum principle is available for the non-local parabolic problem when this nonlinearity is increasing. The second part of the chapter is devoted to the study of a non-local parabolic model illustrating the operation of the thermistor device. Notably, conditions under which finite-time blow-up, which here indicates the destruction of the thermistor device, occurs are investigated by using both energy and comparison methods.

2.1 Ohmic Heating of Foods

2.1.1 Derivation of the Basic Model and Its Variations

One of the methods developed in recent years for sterilizing food is to heat it rapidly by means of an electric current. The food is passed through a conduit, part of which lies between two electrodes. A high electric current flowing between the electrodes results in Ohmic heating of the food which quickly gets hot. We next present the derivation of the mathematical model describing the above process. A more detailed background on this type of process can be found in [8, 12, 16, 38, 41, 44].

The electric potential φ and the current density $\overrightarrow{j}$ are related by Ohm's law,

$$\overrightarrow{j} = -\sigma \nabla \varphi,$$

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where σ is the electrical conductivity, which is assumed to vary with temperature. Then for conservation of charge, we have

$$\nabla \cdot (\sigma \nabla \varphi) = 0, \tag{2.1}$$

where it is assumed that on electro-magnetic time scales the situation is only slowly varying and the change density is small.

For food (or other substance) with density ρ, specific heat c, and velocity $\overrightarrow{v}$, all assumed constant here, the temperature T satisfies

$$\rho c \left[\frac{\partial T}{\partial t} + \overrightarrow{v} \cdot \nabla T \right] = \nabla \cdot (k \nabla T) + \sigma \left| \nabla \varphi \right|^2 . \tag{2.2}$$

Here, k is the thermal conductivity and the last term on the right hand side of (2.2) represents Ohmic heating. This model can be additionally simplified on assuming the following:

- (i) the thermal conductivity is constant independent on the temperature T;
- (ii) the food enters the heater with a temperature T_0 independent of its position along the channel;
- (iii) end effects for the problem can be neglected so that the potential is 0 at the start of the heater, $z = 0$, and $V(t)$ at the far (down-steam) end, $z = L$.

Here, z is the distance along the channel (i.e. in the direction of $\overrightarrow{v}$) which has parallel sides and owing assumptions $(i) - (ii)$ we then derive that the potential and temperature only vary with z and time t and satisfy the system

$$\frac{\partial}{\partial z} \left(\sigma \frac{\partial \varphi}{\partial z} \right) = 0 \, , \;\; 0 < z < L, \quad t > 0, \tag{2.3}$$

$$\rho c \left(\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial z^2} + \sigma \left(\frac{\partial \varphi}{\partial z} \right)^2 \, , \;\; 0 < z < L, \quad t > 0, \tag{2.4}$$

$$\varphi = 0 \, , \;\; T = T_0 \, , \;\; z = 0 \, ; \;\; \varphi = V \, , \;\; z = L. \tag{2.5}$$

Also, V is known if the potential difference across the device is specified (but has to be determined if the process is controlled in some other way). Equation (2.3) is integrated to give,

$$\sigma \frac{\partial \varphi}{\partial z} = J(t) \, , \;\; \text{so} \;\; V = J \int_0^L \frac{dz}{\sigma},$$

where J is the electric current density (along the channel) and finally (2.4) is transformed to

$$\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial z} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial z^2} + \frac{J^2}{\rho c \sigma} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial z^2} + \left(\frac{V^2}{\rho c} \right) \left(\frac{1}{\sigma} \right) \left(\int_0^L \frac{1}{\sigma} dz \right)^{-2} .$$

This is a variant of the non-local parabolic problem considered in [10, 29, 30]; heat transport is now happening both by convection and conduction.

It is convenient to scale the distance with the length of channel and the time to make the convective velocity 1. Additionally this scaling changes the temperature variable so that it becomes 0 at the inlet and its derivative becomes zero at the outlet, i.e. the right hand side of the device is thermally insulated. Then the dimensionless temperature u satisfies the following conduction-convection problem [22],

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left(\int_0^1 f(u)\,dx\right)^2}\ , \ 0 < x < 1\ , \ t > 0\ , \tag{2.6}$$

$$u(0,t) = u_x(1,t) = 0\ , \ t > 0\ , \tag{2.7}$$

$$u(x;0) = u_0(x)\ , \ 0 < x < 1\ , \tag{2.8}$$

with $f > 0$ being the dimensionless electrical reactivity $\left(\propto \frac{1}{\sigma}\right)$. The parameter $\lambda > 0$ can be identified with (the square of) applied potential difference; on occasions we shall absorb λ into f.

If the heater is part of a circuit, so that it is connected in series with a constant resistance, and a fixed EMF (Electromotive Force) is applied across the two, then the scaled non-local equation is replaced by

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left[a + b\int_0^1 f(u)\,dx\right]^2}\ , \ 0 < x < 1\ , \ t > 0\ , \tag{2.9}$$

where $a, b > 0$.

When the heating of the food is rapid, heat diffusion both in the direction of flow and normal to it can be neglected, i.e. $0 < k \ll 1$, as suggested in [36]. Following then the same steps as above, we end up with the model [31],

$$u_t + u_x = \frac{\lambda f(u)}{\left(\int_0^1 f(u)\,dx\right)^2}\ , \ 0 < x < 1\ , \ t > 0\ , \tag{2.10}$$

$$u(0,t) = 0\ , \ t > 0\ , \tag{2.11}$$

$$u(x;0) = u_0(x)\ , \ 0 < x < 1\ , \tag{2.12}$$

which is a non-local hyperbolic model.

If it is assumed that the density ρ and the velocity $\overrightarrow{v}$ of the food vary significantly with temperature T, then one has to take the change of mass of the food into account as well, so additionally to system (2.1)–(2.2) we have the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{v}) = 0, \tag{2.13}$$

expressing the conservation of mass.

Under assumptions $(i)-(iii)$ and for a rapid food heating, system (2.1), (2.2) and (2.13) is then reduced to

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial\varphi}{\partial x}\right)=0,\quad 0<x<L,\quad t>0, \tag{2.14}$$

$$\rho\,\frac{\partial T}{\partial t}+\rho\,v\frac{\partial T}{\partial x}=\sigma\left(\frac{\partial\varphi}{\partial x}\right)^2,\quad 0<x<L,\quad t>0, \tag{2.15}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0,\quad 0<x<L,\quad t>0. \tag{2.16}$$

We integrate (2.14) to derive

$$\rho\frac{\partial T}{\partial t}+\rho v\frac{\partial T}{\partial x}=\frac{I^2}{\sigma}=\frac{V^2}{\sigma}\left(\int_0^1\frac{dx}{\sigma}\right)^{-2}, \tag{2.17}$$

and then integrating (2.16) and substituting ρ back to (2.17) we deduce after proper scaling that the dimensionless temperature u satisfies the following hyperbolic non-local problem

$$\rho(u)u_t+\left(1-\int_0^x\rho'(u)u_t dy\right)u_x=\frac{\lambda f(u)}{\left(\int_0^1 f(u)dx\right)^2},\quad 0<x<1,\quad t>0, \tag{2.18}$$

$$u\,(0,t)=0\ ,\ t>0\ , \tag{2.19}$$

$$u\,(x;0)=u_0\,(x)\ \ ,\ 0<x<1\ , \tag{2.20}$$

where now $\lambda = V^2$, see also [24]. Obviously problem (2.18)–(2.20) reduces to (2.10)–(2.12) when ρ is constant.

Remarkably, non-local equations similar to (2.6) arise in various applications including the shear banding formation in metals being deformed under high strain rates, [1, 2], the investigation of the fully turbulent behavior of real flows, using invariant measures for the Euler equation, [3], to mention a few of them. Additionally, for the case of high strained metals and when there is lack of knowledge of certain physical parameters of the system, then we end up, [20], with the following stochastic system

$$\frac{\partial u}{\partial t}=\Delta u+\frac{\lambda\,e^u}{\left(\int_D e^u\,dx\right)^p}+\sigma(u)\,\partial_t W(x,t),\quad \text{in}\quad D_T:=D\times(0,T), \tag{2.21}$$

$$u(x,t)=0,\quad \text{o}\quad \partial D\times(0,T), \tag{2.22}$$

$$u(x,0)=\xi(x),\quad \text{in}\quad D, \tag{2.23}$$

for a bounded domain $D\in\mathbb{R}^N$, $N\geq 1$ and $0<p\leq 1$. Here $W(x,t)$ is a Wiener process and the multiplicative noise $\sigma(u)\ \partial_t W(x,t)$ encapsulates the present uncertainty in the system. In the current manuscript, only the deterministic version,

i.e. when $\sigma(u) \equiv 0$, of (2.21)–(2.23) is investigated, however the interested reader is referred to [20] for the inspection of the finite-time blow-up (occurrence of shear banding formation) of the stochastic problem (2.21)–(2.23).

2.1.2 *Local Existence and Monotonicity*

2.1.2.1 Parabolic Case

The well-possedeness of problem (2.6)–(2.8) can be obtained by using a Picard type argument. In particular, by $\{u^n\}_{n=1}^{\infty}$ satisfying

$$u_t^n + u_x^n = u_{xx}^n + \frac{\lambda u^{n-1}}{\left(\int_0^1 f(u^{n-1})dx\right)^2}\ ,\ 0 < x < 1\ ,\ t > 0\ ,$$
$$u^n\,(0,t) = u_x^n(1,t) = 0\ ,\ t > 0\ ,$$
$$u^n\,(x;0) = u_0\,(x)\ \ ,\ 0 < x < 1\ ,$$

we can easily prove the following

Theorem 2.1 *Fix $\lambda > 0$ and take f strictly positive satisfying a Lipschitz condition in the interval (α, β) where*

$$\alpha < \min\left\{0, \inf_{(0,1)} u_0(x)\right\} \quad and \quad \beta > \max\left\{0, \sup_{(0,1)} u_0(x)\right\}.$$

Assume also that $u_0 \in L^{\infty}(0,1)$ then there exists $T > 0$ such that problem (2.6)–(2.8) has a unique (classical) solution in $[0,1] \times [0,T]$.

For a proof of Theorem 2.1 see [18].

Remark 2.1.1 The (unique) solution provided by Theorem 2.1 continues to exist as long as it remains less than or equal to β. This argument implies that u ceases to exist only by blow-up; that is, there exists a sequence $(x_n, t_n) \to (x^*, t^*)$ with $t^* \le \infty$ such that $u(x_n, t_n) \to \infty$ as $n \to \infty$.

For the study of the long-time behavior of the solutions to problem (2.6)–(2.8) we will need the concept of lower and upper solutions.

Definition 2.1.2 A function $\overline{u}$ is an upper solution to problem (2.6)–(2.8) if it satisfies

$$\overline{u}_t + \overline{u}_x \geq \overline{u}_{xx} + \frac{\lambda f(\overline{u})}{\left(\int_0^1 f(\overline{u})\,dx\right)^2}, \quad 0 < x < 1, \quad t > 0,$$
$$\overline{u}(x,t) \geq u(x,t), \quad x = 0, 1, \quad t > 0,$$
$$\overline{u}(x,0) \geq u(x,0), \quad 0 < x < 1,$$

whereas a lower solution $\underline{u}$ to (2.6)–(2.8) satisfies the above inequalities but reversed, i.e.

$$\underline{u}_t + \underline{u}_x \leq \underline{u}_{xx} + \frac{\lambda f(\underline{u})}{\left(\int_0^1 f(\underline{u})\,dx\right)^2}, \quad 0 < x < 1, \quad t > 0,$$
$$\underline{u}(x,t) \leq u(x,t), \quad x = 0, 1, \quad t > 0,$$
$$\underline{u}(x,0) \leq u(x,0), \quad 0 < x < 1.$$

An importance observation is that if f is a decreasing function then we may use comparison methods, i.e. upper and lower solutions of problem (2.6)–(2.8) are ordered.

Indeed, if we define $v(x,t) := \overline{u}(x,t) - \underline{u}(x,t)$, by continuity there exists a time $T > 0$ such that $v \geq 0$ for $0 \leq t < T$ and

$$v_t + v_x \geq v_{xx} + I(s, \underline{u})\, v\,, \quad (x,t) \in Q_T := (0,1) \times (0,T),$$

$$v(x,t) \geq 0\,, \quad x = 0, 1\,, \quad 0 < t < T\,, \quad v(x,0) \geq 0, \quad 0 < x < 1,$$

where

$$I(s, \underline{u}) := \frac{\lambda f'(s)}{\left(\int_0^1 f(\underline{u})\,dx\right)^2},$$

is bounded for any $s \in (\underline{u}, \overline{u})$. The latter, by the maximum principle, implies that $v = \overline{u} - \underline{u} \geq 0$ for $0 \leq t \leq T$, see also [29].

This gives an alternative proof for the existence of a local-in-time solution u of problem (2.6)–(2.8) which lies in $(\underline{u}, \overline{u})$ under the monotonicity of f, [35, 39], and the monotonicity of f as well as the ordering of lower and upper solutions.

When f is increasing, however, the maximum principle is not applicable [29, 37]. Then upper and lower solutions are not necessarily ordered. Therefore, some new type of comparison functions should be defined, see also Chap. 1.

Definition 2.1.3 A pair of functions $v, z \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ is called a lower-upper solution pair of (2.6)–(2.8), if $v(x,t) \leq z(x,t)$ for $(x,t) \in Q_T$, $v(x,0) \leq u_0(x) \leq z(x,0)$ in $[0,1]$, $v(x,t) \leq 0 \leq z(x,t)$ for $x = 0, 1$, $0 < t < T$, and

$$v_t + v_x \le v_{xx} + \frac{\lambda f(v)}{\left(\int_0^1 f(z)\,dx\right)^2} \quad \text{in} \quad Q_T,$$

$$z_t + z_x \ge z_{xx} + \frac{\lambda f(z)}{\left(\int_0^1 f(v)\,dx\right)^2} \quad \text{in} \quad Q_T.$$

If the above inequalities are strict, then (v, z) is called a strict lower-upper solution pair.

Then we can prove local-in-time existence of the solution by similar arguments to Proposition 1.2.2.

2.1.2.2 Hyperbolic Case

Notably local-in-time existence for the hyperbolic problems (2.10)–(2.12) and (2.18)–(2.20) can be established through the theory of characteristic curves and applying a Picard type approach. In the following we focus only to the more general problem (2.18)–(2.20).

The characteristics of (2.18)–(2.20) are given as a solution of the following system of ordinary differential equations

$$\frac{dt}{d\tau} = \rho(u), \tag{2.24}$$

$$\frac{dx}{d\tau} = 1 - \int_0^x \rho'(u) u_t dy, \tag{2.25}$$

$$\frac{du}{d\tau} = \frac{\lambda f(u)}{(\int_0^1 f(u)\,dx)^2}. \tag{2.26}$$

Although discontinuities of u_0 or a mismatch between u_0 and the boundary condition give rise to irregular behavior of u, these are simply propagated along the characteristics and allow the existence of a (local) weak solution. So, in the following we will generally be thinking of u_0 being continuous (and normally, but not always, differentiable) with $u_0(0) = 0$.

Now if f is a Lipschitz continuous function and $\rho \in C^1(0, \infty)$, then the iteration argument implies the existence of a solution to (2.24)–(2.26); also nonexistence can only come about through blow-up with u becoming infinite after some finite time t^*, see [24]. Especially, (2.24) and (2.26), together with the iteration argument imply, since $\rho \in C^1(0, \infty)$, that u_t is bounded as far as u is bounded. Using the same arguments it is proved that u_x becomes unbounded only when u becomes unbounded.

Although (2.18)–(2.20) is a hyperbolic problem in the case where f is a decreasing function, more information can be gained by a comparison result. In fact, if f is a decreasing and Lipschitz continuous function, then $0 \leq f(\beta) - f(\alpha) \leq K(\alpha - \beta)$, where $\beta \leq \alpha \leq M$, $M > \sup_{(0,1)} u_0(x)$ for some positive constant $K \equiv K(M)$. Then a lower solution $\underline{u}$ and a solution u to problem (2.18)–(2.20) satisfy the inequality $\dfrac{dv}{d\tau} \leq \dfrac{\lambda f(0)}{f^2(M)}$ on a characteristic curve as long as they lie under M. So $\underline{u} \leq M$ and $u \leq M$ while

$$\tau \leq \frac{(M - \sup u_0) f^2(M)}{\lambda f(0)}.$$

Considering now $v_0 = \underline{u}$, it can be defined iteratively $\{v_n\}$ for $n \geq 1$ by

$$\frac{dv_n}{d\tau} + \frac{\lambda K}{f^2(M)} v_n = \frac{\lambda f(v_{n-1})}{\left(\int_0^1 f(v_{n-1}) dx\right)^2} + \frac{\lambda K}{f^2(M)} v_{n-1}, \tag{2.27}$$

with $v_n = u_0$ at $\tau = 0$ and $v_n = 0$ for $x = 0$.

Problem (2.27) has a unique solution since is linear and more precisely there holds $v_n \leq M$ for $\tau \leq T \equiv (M - \sup u_0) f^2(M)/\lambda(f(0) + KM) \leq T_1$; note also that

$$\frac{dv_1}{d\tau} + \frac{\lambda K}{f^2(M)} v_1 = \frac{\lambda f(v_0)}{\left(\int_0^1 f(v_0) dx\right)^2} + \frac{\lambda K}{f^2(M)} v_0 \geq \frac{dv_0}{d\tau} + \frac{\lambda K}{f^2(M)} v_0,$$

since $v_0 = \underline{u}$ is a lower solution to Eq. (2.26) and $v_1 \geq v_0$ for $\tau = 0$, $x = 0$; thus $v_1 \geq v_0$ for $0 \leq \tau \leq T_1$ $(0 \leq x \leq 1)$ and some $T_1 > 0$.

Moreover,

$$\begin{aligned}
\frac{dv_1}{d\tau} - \frac{\lambda f(v_1)}{\left(\int_0^1 f(v_1) dx\right)^2} &= \frac{\lambda f(v_1) \int_0^1 (f(v_1) + f(v_0)) dx \int_0^1 (f(v_1) - f(v_0)) dx}{\left(\int_0^1 f(v_1) dx\right)^2 \left(\int_0^1 f(v_0) dx\right)^2} \\
&\quad + \frac{\lambda (f(v_0) - f(v_1))}{\left(\int_0^1 f(v_0) dx\right)^2} + \frac{\lambda K}{f^2(M)} (v_0 - v_1) \\
&\leq \frac{\lambda f(v_1) \int_0^1 (f(v_1) + f(v_0)) dx \int_0^1 (f(v_1) - f(v_0)) dx}{\left(\int_0^1 f(v_1) dx\right)^2 \left(\int_0^1 f(v_0) dx\right)^2} \\
&\quad + \lambda K (v_0 - v_1) \left(\frac{1}{f^2(M)} - \frac{1}{\left(\int_0^1 f(v_0) dx\right)^2} \right) \leq 0,
\end{aligned}$$

provided that f is Lipschitz continuous and decreasing. It follows, inductively, that $\underline{u} = v_0 \le v_1 \le v_2 \le \ldots \le v_n \le \ldots \le M$ and so $v_n \to u \ge \underline{u}$ for some solution $u \le M$ and $0 \le \tau \le T$. The uniqueness of the solution for $\tau \in [0, T]$ is proved similarly. Supposing that there exist two solutions u_1, u_2 in $[0, T]$ then $0 \le u_1, u_2 \le M$ and using the Lipschitz continuity of f we get

$$\left|\frac{d}{d\tau}(u_1 - u_2)\right| \le \frac{\lambda f(u_1)\int_0^1 (f(u_1) + f(u_2))dx \int_0^1 |f(u_1) - f(u_2)|dx}{\left(\int_0^1 f(u_1)dx\right)^2 \left(\int_0^1 f(u_2)dx\right)^2} + \frac{\lambda|f(u_1) - f(u_2)|}{\left(\int_0^1 f(u_2)dx\right)^2} \le \Lambda|u_1 - u_2|, \tag{2.28}$$

where $\Lambda = (2\lambda f^2(0) + \lambda f^2(M))K/f^4(M)$. Since for $0 \le \tau \le T$ there holds $|u_1 - u_2| \le M$ due to (2.28) we get $|u_1 - u_2| \le \Lambda\, M\, T$ and inductively we obtain $|u_1 - u_2| \le \frac{M(\Lambda T)^n}{n!} \to 0$ as $n \to \infty$ resulting in $u_1 \equiv u_2$.

Using the same arguments but now starting at $\tau = T$ we deduce that $u \ge \underline{u}$ as long as they both exist. The proof that $u \le \overline{u}$, if $\overline{u}$ is an upper solution to (2.18)–(2.20), is similar.

2.1.3 *Stationary Problem*

The key point for the study of the long-time behavior of problems (2.6)–(2.8) and (2.18)–(2.20) is the study of the corresponding stationary problems. Henceforth, we assume that function f satisfies

$$f(s) > 0\,, \quad f'(s) < 0 \quad s \ge 0. \tag{2.29}$$

2.1.3.1 Parabolic Case

The corresponding steady-state problem to (2.6)–(2.8) is

$$w'' - w' + \mu f(w) = 0, \quad 0 < x < 1, \tag{2.30}$$

$$w(0) = w'(1) = 0, \tag{2.31}$$

where

$$\mu = \frac{\lambda}{\left(\int_0^1 f(w)\,dx\right)^2}, \tag{2.32}$$

is a positive local parameter (λ is the non-local one). Problem (2.30)–(2.31) has a unique solution for every $\mu > 0$, since f satisfies (2.29) (by using monotone type arguments), [5, 9, 34], and Lemma 2.1.5 holds, see below.

Here, due to the convection term, the steady problem is not symmetric, in contrast with the pure diffusion problem, [30]. In the sequel we shall investigate the spectrum of (2.30)–(2.31).

Integrating (2.30) over (0, 1) we obtain

$$w'(0) + M = \mu \int_0^1 f(w)dx,$$

and by virtue of (2.32) we finally derive

$$\lambda(M) = \frac{(w'(0) + M)^2}{\mu}, \tag{2.33}$$

where $M = ||w||_\infty = w(1)$.

Also multiplying (2.30) by w' and integrating we obtain

$$\frac{(w'(0))^2}{\mu} = 2\left[\int_0^M f(s)\,ds - \frac{1}{\mu}\int_0^1 (w'(x))^2\,dx\right] \le 2\int_0^M f(s)\,ds. \tag{2.34}$$

We now have:

Lemma 2.1.4 *If* $\int_0^\infty f(s)\,ds < \infty$ *then* $\frac{(w'(0))^2}{\mu} \to 2I_\infty$ *as* $\mu \to \infty$ *where* $I_\infty = \int_0^\infty f(s)\,ds$.

Proof We first consider the auxiliary problem:

$$z''(x) + \mu g(z(x)) = 0, \quad 0 < x < 1 - \delta, \tag{2.35}$$

$$z(x) = \sup_x w(x) = M, \quad z'(x) = 0 \quad \text{for} \quad 1 - \delta \le x \le 1 \tag{2.36}$$

$$z(0) = 0, \tag{2.37}$$

where $0 < g(s) < f(s)$, and z, z', are continuous at the point $x = 1 - \delta$.

Now, multiplying (2.35) by z' and integrating over the interval $(0, 1-\delta)$ we obtain

$$(z'(x))^2 = 2\mu \int_{z(x)}^M g(s)\,ds = 2\mu[G(z) - G(M)], \tag{2.38}$$

where $G(z) = \int_z^\infty g(s)\,ds$. Since $z'(x) > 0$ in $[0, 1 - \delta)$, then (2.38) entails

$$\int_0^M [G(z) - G(M)]^{-1/2}\,dz = (1 - \delta)\sqrt{2\mu}. \tag{2.39}$$

We next prove that the solution of problem (2.35)–(2.37) is a lower solution to problem (2.30)–(2.31). Indeed,

$$z''(x) - z'(x) + \mu f(z) = \mu f(z) > 0 \quad \text{for} \quad 1-\delta \le x \le 1.$$

Also taking into account (2.35)–(2.39),

$$\begin{aligned} z'' - z' + \mu f(z) &= z' + \mu(f(z) - g(z)) \\ &= -\sqrt{2\mu}\,[G(z) - G(M)]^{1/2} + \mu(f(z) - g(z)) \quad \text{for} \quad 0 < x < 1-\delta. \end{aligned} \tag{2.40}$$

Now choosing μ such that

$$\mu \ge \mu_0 = \sup_{x\in(0,M)} \frac{2[G(z) - G(M)]}{[f(z) - g(z)]^2}, \tag{2.41}$$

and $\delta < 1$, relations (2.40), (2.41) imply

$$z'' - z' + \mu f(z) > 0 \quad \text{for} \quad 0 < x < 1-\delta,$$

In addition $z(0) = z'(1) = w(0) = w'(1) = 0$, and thus z is a lower solution problem (2.30)–(2.31). Therefore

$$z(x) \le w(x) \quad \text{for} \quad 0 < x \le 1 \quad \text{and} \quad 0 < z'(0) \le w'(0). \tag{2.42}$$

Now we choose:

(a) g, such that $0 < g(s) < f(s)$ and $I_\infty - \epsilon \le G(0) = \int_0^\infty g(s)\,ds < I_\infty$,
(b) M such that $[G(0) - G(M)] > I_\infty - 2\epsilon$ for $\epsilon > 0$,
(c) μ to satisfy (2.41).

Note that since $G'(z) = -g(z) < 0$ and we have $G(0) \le I_\infty$ and then by virtue of (2.34), (2.38) and (2.42) we obtain

$$2I_\infty > \frac{(w'(0))^2}{\mu} \ge \frac{(z'(0))^2}{\mu} = 2[G(0) - G(M)] > 2(I_\infty - 2\epsilon), \quad \text{for any} \quad \epsilon > 0,$$

which implies the desired result. □

Also, by adapting some ideas from [9], we can prove following:

Lemma 2.1.5 *If w is the solution to (2.30)–(2.31) and $\int_0^\infty f(s)\,ds < \infty$ then $w(x;\mu) \to \infty$ as $\mu \to \infty$ for every x in* $(0, 1]$.

Proof We first prove that $\Phi(\mu) = \int_0^1 f(w(x;\mu))\,dx \to 0$ as $\mu \to \infty$. To this end we construct a lower solution of (2.30)–(2.31) of the form $z = \beta\phi_1$ for some $\beta > 0$ where ϕ_1 is the principal eigenfunction of

$$-\phi'' + \phi' = \lambda\phi, \quad 0 < x < 1, \tag{2.43}$$
$$\phi(0) = \phi'(1) = 0. \tag{2.44}$$

It is known that $\lambda_1 > 0$ and $\phi_1 > 0$, also we normalize ϕ_1 by taking $||\phi_1||_\infty = 1$. Now on choosing β to satisfy $\frac{\lambda_1\beta}{f(\beta)} \le \mu$, $\beta\phi_1$ becomes a lower solution of (2.30)–(2.31). Thus, it is sufficient to choose $\frac{\beta}{f(\beta)} = \frac{\mu}{\lambda_1}$.

This choice of β is unique for each $\mu > 0$. Indeed $\mathscr{L}(\beta) := \frac{\lambda_1\beta}{f(\beta)}$ is one to one since $\mathscr{L}'(\beta) > 0$ and maps $\mathbb{R}_+$ onto $\mathbb{R}_+$ since $\mathscr{L}((0,\infty)) = (0,\infty)$. Finally $\mathscr{L}$ is a diffeomorphism, hence to each μ corresponds a unique $\beta(\mu)\phi_1$ which is a lower solution to (2.30)–(2.31).

Therefore we obtain that

$$\Phi(\mu) = \int_0^1 f(w(x;\mu))\,dx \le \int_0^1 f(\beta(\mu)\phi_1(x))\,dx \to 0 \quad \text{as} \quad \mu \to \infty.$$

The last limit implies that $w(x;\mu) \to \infty$ as $\mu \to \infty$ for $0 < x \le 1$, otherwise we could find an x_0 so that $w(x;\mu) < \infty$ in $(0, x_0)$, but this would imply that $\lim_{\mu\to\infty}\Phi(\mu) > 0$, contradicting that $\Phi(\mu) \to 0$ as $\mu \to \infty$. □

An immediate consequence of Lemma 2.1.5 is that $M = w(1) \to \infty$ as $\mu \to \infty$. Furthermore, by using maximum principle to problem (2.30)–(2.31) we obtain the response local diagram, see Fig. 2.1d.

Now, multiplying (2.30) by $w' - w$, we obtain

$$\frac{(w'(0))^2}{\mu} = 2\left[\int_0^M f(s)\,ds - \int_0^1 f(w)w\,dx\right] + \frac{M^2}{\mu}. \tag{2.45}$$

In addition we have

$$\int_0^1 f(w)w\,dx = f(w(\xi;\mu))w(\xi;\mu), \quad \xi \in (0,1), \tag{2.46}$$

and hence we derive

$$\int_0^1 f(w)w\,dx \to 0 \quad \text{as} \quad \mu \to \infty, \tag{2.47}$$

by virtue of Lemma 2.1.5, taking also into account that f is decreasing.

Also by virtue of (2.45) we obtain, by taking the limit as $\mu \to \infty$,

$$\frac{M^2}{\mu} \to 0. \tag{2.48}$$

Henceforth, for convenience we normalize the integral

$$\int_0^\infty f(s)\,ds = I_\infty = 1. \tag{2.49}$$

Now we have the following:

Proposition 2.1.6 *If (2.49) holds then $\lambda(M) \to 2$ as $M \to \infty$ (or equivalently as $\mu \to \infty$).*

The proof is an immediate consequence of (2.33), (2.48) and Lemma 2.1.4.

We now consider the complementary case where

$$\int_0^\infty f(s)\,ds = \infty. \tag{2.50}$$

Proposition 2.1.7 *Let f satisfy (2.50) and w be the solution of (2.30)–(2.31). Then*

$$\frac{(w'(0))^2}{\mu} \to \infty \text{ as } \mu \to \infty \text{ and } \lambda(M) \to \infty \text{ as } M \to \infty.$$

Proof Let z satisfy

$$z''(x) + \mu g(z(x)) = 0, \quad 0 < x < 1 - \delta, \tag{2.51}$$

$$z(x) = M, \quad z'(x) = 0, \quad 1 - \delta \le x \le 1, \quad z(0) = 0. \tag{2.52}$$

It is easily proved that z is a lower solution of (2.30)–(2.31) provided that

(a) $0 < g(s) < f(s)$, $g'(s) < 0$ and $\int_0^\infty g(s)\,ds = \infty$, for instance g can be taken as $g(s) = \gamma f(s)$, $0 < \gamma < 1$,

(b) $\mu \ge \mu_0 = \sup_{z\in(0,M)}\{[2\int_z^M g(s)\,ds]/[f(z) - g(z)]^2\}$,

thus we obtain:

$$\frac{(w'(0))^2}{\mu} \ge \frac{(z'(0))^2}{\mu} = 2\int_0^M g(s)\,ds \to \infty,$$

as $\mu \to \infty$.

Hence

$$\lambda(M) = \frac{(w'(0) + M)^2}{\mu} \ge \frac{(w'(0))^2}{\mu} \to \infty \text{ as } M \to \infty.$$

□

From the above analysis we can obtain the main possible response diagrams, see Fig. 2.1. It is possible, see Fig. 2.1b, to have more than one turning points. This can occur even in the cases of Fig. 2.1a, c, see also [34, 42].

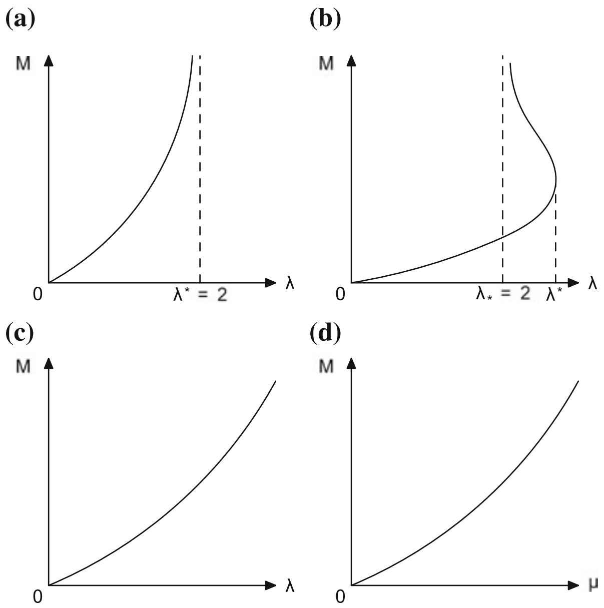

Fig. 2.1 λ, μ - diagrams represent the non-local, local response diagrams respectively of problem (2.30)–(2.31). (i) **(a)**, **(b)** for the case $\int_0^\infty f(s)\,ds = 1$ and (ii) **(c)** for the case $\int_0^\infty f(s)\,ds = \infty$ where $M = w(1)$

Remark 2.1 Solutions of the steady-state problem (2.30)–(2.31) for small values of the parameter λ can be constructed via monotone iteration techniques for both decreasing and increasing nonlinear functions f, see [1].

2.1.3.2 Hyperbolic Case

The steady-state problem to (2.18)–(2.20), which actually coincides with the one of (2.10)–(2.12), is

$$w' = \mu f(w), \quad 0 < x < 1, \quad w(0) = 0, \tag{2.53}$$

where again $\mu = \frac{\lambda}{\left(\int_0^1 f(w)dx\right)^2}$ is referred to as the local parameter while λ as the non-local one.

Equation (2.53) can be written

$$\frac{dw}{f(w)} = \mu\, dx, \quad 0 < x < 1, \tag{2.54}$$

from which by integration over (0, 1) we obtain

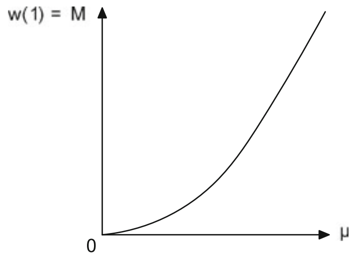

Fig. 2.2 The local response diagram to (2.53), where $M(\mu) = w(1; \mu) = w(1)$

$$\mu = \mu(M) = \int_0^M \frac{ds}{f(s)}, \quad M = ||w||_\infty = w(1). \tag{2.55}$$

The latter implies that $\mu'(M) = \frac{1}{f(M)} > 0$ leading to the response diagram appearing in Fig. 2.2. Also by integration of (2.53) over (0, 1) we get $\lambda = M^2/\mu$ and so $\lambda = \lambda(M) = M^2/\int_0^M \frac{ds}{f(s)}$. Since $\lim_{M\to\infty} \lambda(M) = 2\lim_{M\to\infty} Mf(M)$, we distinguish two cases:

(i) $\int_0^\infty f(s)ds < \infty$, then $Mf(M) \le 2\int_{M/2}^M f(s)\,ds \to 0$ as $M \to \infty$, and so there exists a λ^* such that for $0 < \lambda < \lambda^*$ problem (2.53) has at least two steady-state solutions while for $\lambda > \lambda^*$ there is no steady-state solution, see Fig. 2.3c.

(ii) $\int_0^\infty f(s)ds = \infty$, if $\lim_{M\to\infty} Mf(M)$ exists then two things might happen. Either $Mf(M) \to c$, $0 < c < \infty$ as $M \to \infty$ and so the spectrum to (2.53) is bounded, see Fig. 2.3b, or $Mf(M) \to \infty$ as $M \to \infty$ and (2.53) has at least one steady state for any $\lambda > 0$ ($\lambda^* = \infty$, see Fig. 2.3a).

Moreover, if $\mu(M) = \int_0^M \frac{ds}{f(s)} > M/2f(M)$ for $M > 0$, then

$$\lambda'(M) = \frac{M}{\mu(M)}\left[2\mu(M) - \frac{M}{f(M)}\right] > 0,$$

and thus there is a unique steady state to each $0 < \lambda < \lambda^*$. From the above analysis we get the possible non-local response diagrams of Fig. 2.3. Each diagram may contain more turning points than shown (so that for some λ there are more solutions).

2.1.4 Stability

2.1.4.1 Parabolic Case

We now study the stability of the steady solutions, for $0 < \lambda \le \lambda^* < \infty$ if $\displaystyle\int_0^\infty f(s)\,ds < \infty$ or for any $\lambda > 0$ if $\displaystyle\int_0^\infty f(s)\,ds = \infty$, by using comparison methods, see also [22].

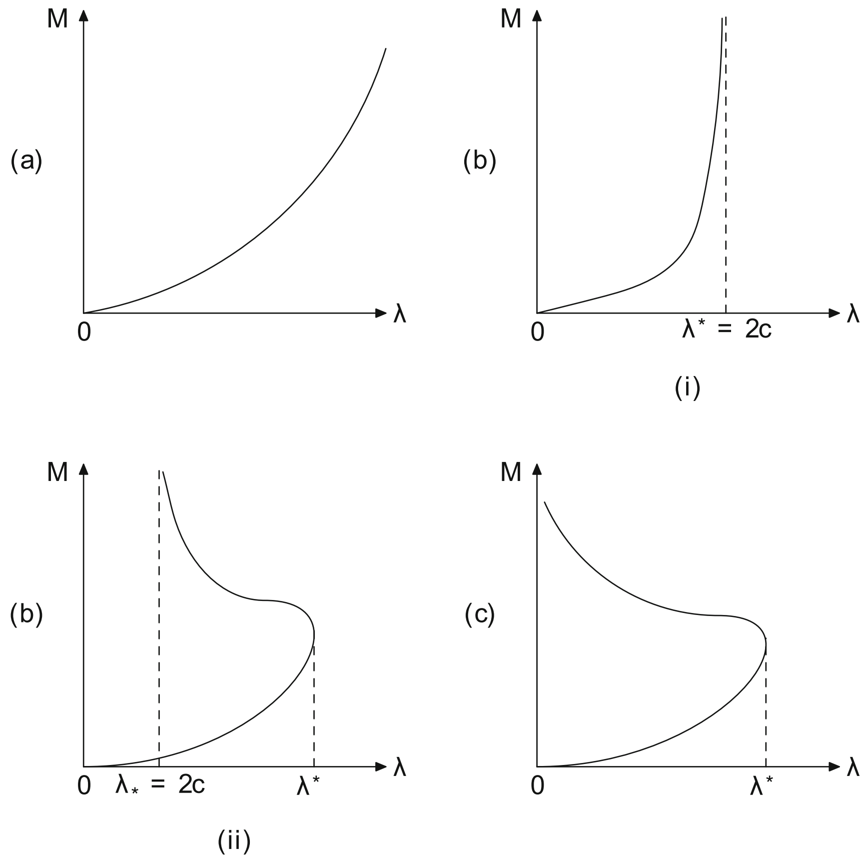

Fig. 2.3 Possible non-local response diagrams to (2.53). **(a)** $Mf(M) \to \infty$ as $M \to \infty$, **(b)** $Mf(M) \to c$, $0 < c < \infty$ as $M \to \infty$, **(c)** $Mf(M) \to 0$ as $M \to \infty$, where $M = w(1)$

In particular, we construct an upper solution $U(x,t) = w(x; \overline{\mu}(t))$ decreasing in time, and a lower solution $z(x,t) = w(x; \underline{\mu}(t))$ increasing in time, of problem (2.6)–(2.8). Both $\overline{\mu}$ and $\underline{\mu}$ satisfy the following initial value problem:

$$\dot{\nu} = h(\nu) \equiv \inf_{x \in (0,1)} \left\{ \frac{f(w)}{w_\nu} \right\} \frac{\lambda - \lambda(\nu)}{\left(\int_0^1 f(w)\, dx \right)^2}, \qquad \nu(0) = \nu_0\,, \tag{2.56}$$

where $w = w(x; \nu(t))$ is the solution of (2.30)–(2.31) with $\mu = \nu(t)$. Furthermore, $w = w(x; \mu)$ is the unique solution of problem (2.30)–(2.31), where $(\nu, M) = (\nu(t), M(t))$ is a point on the graph indicated in Fig. 2.1d. The w_ν–problem is derived by differentiating (2.30)–(2.31) with respect to μ and substituting ν for μ.

The existence of $w = w(x; \nu(t))$ is a consequence of the fact that to each $t > 0$ there corresponds μ so that $\mu = \nu(t)$ is continuous and maps $\mathbb{R}_+$ onto $[\nu_0, \infty)$. Moreover, the continuity of the function $h(\nu)$ and the form of diagrams in Fig. 2.1 imply the existence and uniqueness of the solution $\nu = \nu(t)$ of (2.56), see also [11].

The function $\nu = \nu(t)$ is global-in-time since ν is bounded: $\underline{\mu}(0) \le \nu(t) \le \overline{\mu}(0)$. We have also that ν is strictly monotone. More precisely, $\overline{\mu}(t)$ and $\underline{\mu}(t)$ are decreasing and increasing, respectively (see below). Moreover, it holds that $w_\nu > 0$ by the maximum principle, recalling $f'(s) < 0$. Also w_ν is finite; indeed for a fixed ν, any sufficiently large constant is an upper solution of the $w_\nu-$problem. Hence $\inf_{x\in(0,1)} \{f(w)/w_\nu\}$ is always positive since also $f(s)$ is bounded away from zero for $s \le \sup w < \infty$.

To construct upper and lower solutions, we choose ν_0 satisfying $w(x; \nu(0)) \ge u_0(x)$ and $w(x; \nu(0)) \le u_0(x)$, respectively. This is possible since $w_\nu > 0$. More precisely, this can be done by $u_0 \ge 0$ and $u_0 \in C^1([0, 1])$. Otherwise we choose $\nu(\epsilon)$ so that $w(\cdot; \nu(\epsilon)) \ge u(\cdot, \epsilon)$ for small $\epsilon > 0$.

If the response diagram is as in Fig. 2.1a, c, or b for $\lambda < \lambda_*$, a unique steady-state solution corresponds to each λ. Then we have a unique $M(\mu)$ and take either $\nu_0 > \mu$ or $\nu_0 < \mu$. In the complementary case, when two steady-states correspond to each λ denoted by $M_1 = M(\mu_1)$ and $M_2 = M(\mu_2)$, for example, we take $\mu(\lambda_*) = \mu_* < \nu_0 < \mu_1$, or $\mu_1 < \nu_0 < \mu_2$, or $\nu_0 > \mu_2$ for $\lambda_* < \lambda < \lambda^*$ as in Fig. 2.1b. The case with more than two turning points is similar, where each λ corresponds to more than two M's.

The above analysis implies that $\lambda - \lambda(\nu) < 0$, $\nu = \overline{\mu}(t)$ and $\lambda - \lambda(\nu) > 0$, $\nu = \underline{\mu}(t)$ for the upper and lower solutions, respectively. These inequalities extend to a proper region of ν, that is for the case of unique solution, $\lambda - \lambda(\nu) < 0$ and $\lambda - \lambda(\nu) > 0$ if $\nu > \mu$ and $\nu < \mu$, respectively, and for the case of two solutions, $\lambda - \lambda(\nu) < 0$ and $\lambda - \lambda(\nu) > 0$ if $\mu_1 < \nu < \mu_2$, $\mu_* < \nu < \mu_1$ and $\nu > \mu_2$, respectively.

These properties imply $U(x, t) = w(x; \overline{\mu}(t))$ and $z(x, t) = w(x; \underline{\mu}(t))$ are upper and lower solutions, respectively, and hence $\lambda - \lambda(\overline{\mu}) < 0 < \lambda^* - \lambda(\overline{\mu})$ and $\lambda - \lambda(\underline{\mu}) > 0 > \lambda_* - \lambda(\underline{\mu})$, respectively. Then it holds that $U_t = w_\nu \dot{\nu} < 0$ and $z_t = w_\nu \dot{\nu} > 0$, since $w_\nu > 0$, $\dot{\overline{\mu}}(t) < 0$, and $\dot{\underline{\mu}}(t) > 0$.

Returning to the case of a unique steady-state $w(x) \equiv w_1(x)$, the above construction implies that $z(x, t) \le u(x, t) \le U(x, t)$. Then $u(x, t)$ is a global-in-time solution and $z(x, t) \uparrow w(x)$ and $U(x, t) \downarrow w(x)$ as $t \to \infty$ uniformly in x, since $\overline{\mu}(t) \to \overline{l} = \mu+$, $\underline{\mu}(t) \to \underline{l} = \mu-$ as $t \to \infty$, which means that $\overline{\mu}(t) \to \mu$, $(\underline{\mu}(t) \to \mu)$ and $\overline{\mu}(t) > \mu$ $(\underline{\mu}(t) < \mu)$. The latter is true i.e. $\overline{l} = \underline{l} = \mu$, since assuming that $\overline{\mu}(t) \to \widehat{\mu}+ = \widehat{\mu} > \mu$, as $t \to \infty$ problem (2.56) would imply $\int_{\overline{\mu}(0)}^{\overline{\mu}(t)} \frac{ds}{h(s)} = t$, and by taking the limit as $t \to \infty$ we should have $\int_{\overline{\mu}(0)}^{\widehat{\mu}} \frac{ds}{h(s)} = \infty$. But this can occur if and only if $h(\widehat{\mu}) = 0$ or equivalently $\lambda(\widehat{\mu}) = \lambda$ which contradicts the uniqueness of the solution of the non-local steady problem, (otherwise we would have $\lambda(\widehat{\mu}) = \lambda(\mu) = \lambda$ with $\widehat{\mu} > \mu$, which is a contradiction, see Fig. 2.1c, d). The same argument applies to all other cases with more steady-states, giving always an extra steady solution. Similarly it is shown that $\underline{\mu}(t) \to \mu-$, as $t \to \infty$.

Consequently we deduce that w is a globally asymptotically stable solution. Here, if

$$\int_0^\infty f(s)ds = \infty,$$

then we can also prove that $u(x, t)$ is a global in time solution. Indeed we have

$$\dot{M}(t) \leq \frac{\lambda f(M)}{\left(\int_0^1 f(u)dx\right)^2} < \frac{\lambda}{f(M)} \quad \text{or} \quad \int_{M(0)}^{M(t)} f(s)\,ds < \lambda t,$$

which implies that $u(x, t)$ is global in time.

In case that more than one stationary solutions exist we can apply a similar analysis to show the stability alternates starting from a stable minimal stationary solution, proceeding to greater one which is unstable and so on, see [22].

Now we examine the long time behavior of u when f is an increasing function inspired by [14, 22]. For simplicity we consider Dirichlet boundary conditions,

$$u(0, t) = u(1, t) = 0\,,$$

and expand $u(x, t)$ as

$$u(x, t) = \sum_{n=1}^{\infty} E_n(t) y_n(x), \tag{2.57}$$

where $y_n(x) = e^{x/2}\sin(n\pi x), \quad n = 1, 2, \cdots$, are the eigenfunctions of the operator $-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}$, under Dirichlet boundary conditions.

Substituting (2.57) into Eq. (2.6) and taking into account the monotonicity of f, see also [22], we obtain the following estimate:

$$|E_n(t)| \leq |E_n(0)|e^{-\lambda_n t} + \frac{2\lambda}{n^2\pi^2 f(0)}[1 - e^{-\lambda_n t}],$$

hence

$$\limsup_{t\to\infty} |E_n(t)| \leq \frac{2\lambda}{n^2\pi^2 f(0)}\,,$$

where $\lambda_n = \frac{1}{4} + n^2\pi^2$.

By virtue of (2.57) we have

$$\limsup_{t\to\infty} u(x,t) \le e^{1/2} \limsup_{t\to\infty} \sum_{n=1}^{\infty} |E_n(t)| \le \frac{2\lambda e^{1/2}}{f(0)} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{\lambda e^{1/2}}{3f(0)},$$

since the series of $|E_n(t)|$ converges uniformly and hence in that case a global-in-time solution exists which is bounded for each time $t > 0$.

For the Neumann boundary condition

$$u_x(0,t) = u_x(1,t) = 0, \tag{2.58}$$

a similar analysis yields that the solution $u(x,t)$ of problem (2.6), (2.58) and (2.8) is global-in-time but not necessarily bounded, see [22].

2.1.4.2 Hyperbolic Case

Now we investigate the stability of steady states of (2.18)–(2.20), using the comparison result given in Sect. 2.1.2.

For this reason we consider comparison functions of the form $v(x,t)=w(x;\mu(t))$. Using $v_x = \mu f(v)$ and $\mu x = \int_0^w \frac{ds}{f(s)}$ obtained by (2.53) and (2.54), respectively, we get $v_t = \dot{\mu}(t) x f(v)$ with $v(0,t) = 0$. Thus by following the calculations in [22] we derive

$$\begin{aligned} \mathscr{H}(v) &:= \rho(v)v_t + \left(1 - \int_0^x \rho'(v)v_t dy\right) v_x - \frac{\lambda f(v)}{\left(\int_0^1 f(v)dx\right)^2} \\ &= f(v)\left[\dot{\mu}(t)x\rho(v) + \mu(t) - \dot{\mu}(t)\int_0^v y\rho'(s)ds - \lambda/\left(\int_0^1 f(v)dx\right)^2\right]. \end{aligned}$$

However, since $\int_0^1 f(v)dx = \frac{1}{\mu(t)}\int_0^1 v_x dx = \frac{M(t)}{\mu(t)}$ for $M(t) = \sup_x v(x,t) = v(1,t)$, choosing

$$\dot{\mu}(t) = \dot{\mu} = h(\mu(t)) \equiv \frac{1}{\rho(0)}\left(\frac{\lambda\mu^2}{M^2(\mu)} - \mu\right), \quad \text{for } t > 0, \tag{2.59}$$

we obtain

$$\mathscr{H}(v) = f(v)\left[\frac{1}{\rho(0)}\left(\frac{\lambda\mu^2}{M^2(\mu)} - \mu\right)\left(\rho(v)x - \int_0^v y\rho'(s)ds\right) + \mu - \frac{\lambda\mu^2}{M^2(\mu)}\right].$$

Let $\lambda > \dfrac{M^2(\mu)}{\mu}$. Using the fact that $\rho(s)$ is a positive decreasing function, we get $\mathscr{H}(v) \le 0$ for $0 < x < 1$. Thus in this case $v(x,t)$ is an increasing-in-time ($v_t = \dot{\mu}(t)xf(v) > 0$) lower solution to (2.18)–(2.20), provided that $v(x,0) = w(x;\mu(0)) \le u_0(x)$. Also for $\lambda < \dfrac{M^2(\mu)}{\mu}$ and $v(x,0) = w(x;\mu(0)) \ge u_0(x)$ we obtain that $v(x,t)$ is a decreasing-in-time upper solution to (2.18)–(2.20).

We start with the case that a unique steady state w exists. Then to each $0 < \lambda < \lambda^*$ ($\lambda^* = \infty$ when $\int_0^\infty f(s)ds = \infty$) there exists $\mu > 0$ such that $\lambda = \lambda(\mu) := \mu\left(\int_0^1 f(w)dx\right)^2$ and the function $\lambda(\mu)$ is increasing. For the case $u_0(x) \le w(x)$ we can choose $0 < \underline{\mu}(t) < \mu$, so $\lambda = \lambda(\mu) > \lambda(\underline{\mu}(t)) = \dfrac{M^2(t)}{\underline{\mu}(t)}$, satisfying the Eq. (2.59). Then $\underline{\mu}(t)$ satisfies the transcendental equation

$$\int_{\underline{\mu}(0)}^{\underline{\mu}(t)} \frac{ds}{h(s)} = t, \quad t > 0, \qquad \text{where} \quad h(s) = \frac{1}{\rho(0)}\left(\frac{\lambda s^2}{M^2(s)} - s\right). \tag{2.60}$$

Equation (2.60) has a unique solution in $[\underline{\mu}(0), \mu)$, for any $\underline{\mu}(0) \ge 0$. Hence in this case $\mathscr{G} : [\underline{\mu}(0), \mu) \to [0, \infty)$ with $\mathscr{G}(\xi) := \int_{\underline{\mu}(0)}^{\xi} \frac{ds}{h(s)}$ is a C^1–diffeomorphism, [22]. Thus (2.59) has a unique solution $\underline{\mu}(t)$ and since $w_{\underline{\mu}} = x\,f(w) \ge 0$ we can choose $\underline{\mu}(0) \ge 0$ such that $w(x;\underline{\mu}(0)) \le u_0(x)$. Hence $v(x,t) = w(x;\underline{\mu}(t))$ is an increasing-in-time lower solution to (2.18)–(2.20), so $v(x,t) \le u(x,t) \le w(x)$ for $x \in [0,1]$ and $t > 0$. Moreover $\underline{\mu}(t) \to \mu-$ as $t \to \infty$, because, otherwise there would be another steady state. Therefore, $v(\cdot,t) \to w(\cdot)$ as $t \to \infty$ uniformly in x resulting in $u(\cdot,t) \to w(\cdot)$ as $t \to \infty$ uniformly in x.

When $u_0(x) \ge w(x)$, it is possible to choose $\bar{\mu}(t) > \mu$ (so $\lambda < \lambda(\bar{\mu}(t))$) to satisfy (2.59) and construct a decreasing-in-time upper solution $z(x,t)$ to (2.18)–(2.20), provided that $z(x,0) = w(x;\bar{\mu}(0)) \ge u_0(x)$. The latter is achieved since $u_0(x)$, $u_0'(x)$ are bounded and $w_{\bar{\mu}} > 0$. Thus we obtain $w(x) \le u(x,t) \le z(x,t)$ and finally $u(\cdot,t) \to w(\cdot)$ as $t \to \infty$ uniformly in x, with $\bar{\mu}(t) \to \mu+$ as $t \to \infty$. Hence the unique steady state $w(x)$ is globally asymptotically stable and $u(x,t)$ is a global-in-time bounded solution.

We turn to the case where (2.53) has two steady states $w_1 = w(x;\mu_1)$ and $w_2 = w(x;\mu_2)$. Then for each $\lambda_* < \lambda < \lambda^*$ there exist μ_1 and μ_2 such that $\lambda = \lambda(\mu_1) = \lambda(\mu_2)$ and function $\lambda(\mu)$ is increasing for $0 < \mu < \mu^*$ and decreasing for $\mu > \mu^*$ with μ^* satisfying $\lambda'(\mu^*) = 0$. For $0 < u_0(x) < w_1(x)$, choosing $0 < \underline{\mu}(t) < \mu_1 < \mu^*$ such that (2.59), we get as above a lower solution $v(x,t) = w(x;\underline{\mu}(t))$ with $v(\cdot,t) \to w_1(\cdot)$ as $t \to \infty$ uniformly in x. Whereas for $w_1(x) < u_0(x) < w_2(x)$, on choosing $\mu_1 < \bar{\mu}(t) < \mu^*$, we construct an upper solution $z(x,t) = w(x;\bar{\mu}(t))$ such that $z(\cdot,t) \to w_1(\cdot)$ as $t \to \infty$ uniformly in x. Hence for $\lambda_* < \lambda < \lambda^*$ the minimal steady state w_1 is asymptotically stable with a region of attraction $[0, w_2]$, while for $0 < \lambda < \lambda_*$ w_1 is globally asymptotically stable. This implies that $u(x,t)$ is a global-in-time bounded solution.

If we consider $u_0(x) > w_2(x)$ and choose $\underline{\mu}(t) > \mu_2$ satisfying (2.59) then an unbounded lower solution $v(x,t) = w(x; \underline{\mu}(t))$ can be constructed. More precisely, $\underline{\mu}(t) \to \infty$ as $t \to T^* \leq \infty$. Otherwise there would be a third steady state which is a contradiction. Hence $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^* \leq T^* \leq \infty$, which means that $u(x,t)$ is unbounded. The latter implies that the maximal steady state w_2 is unstable.

Moreover, $w^*(x) = w(x; \lambda^*)$ is unstable. In fact, w^* is stable from below, $0 < u_0(x) < w^*(x)$, and unstable from above, $u_0(x) > w^*(x)$. If for each $\lambda_* < \lambda < \lambda^*$ more than two steady states exist, then the above arguments imply that the minimal steady state is stable, that the greater one is unstable, and so on.

We note that problem (2.18)–(2.20) has unbounded solutions for $\lambda > \lambda^*$. Indeed, in this case there holds $\lambda > \lambda(\mu) = \dfrac{M^2}{\mu}$ for $\mu > 0$. Hence we can construct a lower solution of the form $w(x; \mu(t))$, which actually becomes unbounded since $\mu(t) \to \infty$ as $t \to T^* \leq \infty$ and due to the fact that for $\lambda > \lambda^*$ there is no steady state. Consequently $u(x,t)$ becomes unbounded at some $t^* \leq T^* \leq \infty$, i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*-$.

2.1.5 *Finite-Time Blow-Up*

The phenomenon of finite-time blow-up. apart from its mathematical interest, is very significant for the process of sterilization of food. Indeed, in this context finite-time blow-up is closely associated with thermal runaway which finally leads to food burning and it should be avoided, since, otherwise the whole process of the food sterilization is inefficient. Therefore, it is important form the application point of view to investigate under which circumstances finite-time blow-up occurs so we can optimize the whole process.

In the following we provide a thorough study of the phenomenon of finite-time blow-up for the solutions of both (2.6)–(2.8) and (2.18)–(2.20). Here by finite-time blow-up we mean that there exists a point $(x^*, t^*) \in (0,1)$ and sequence $(x_n, t_n) \to (x^*, t^*)$ as $n \to +\infty$ such that $u(x_n, t_n) \to +\infty$ as $n \to +\infty$.

2.1.5.1 Parabolic Case

So far we have shown that when $\lambda \in (\lambda_*, \lambda^*]$ and $u_0(x) > w_2(x)$ or when $\lambda > \lambda^*$, the solution u of (2.6)–(2.8) is unbounded.

In the following, we actually prove that u blows up in finite time in these two cases. We first consider the case

$$\lambda > \lambda^* \quad \text{and} \quad \int_0^\infty f(s)\,ds < \infty, \tag{2.61}$$

and we construct a lower solution $z = z(x,t)$ to (2.6)–(2.8) of the form:

$$z_{xx} + \mu(t) f(z) = 0, \quad 0 < x < \delta(t), \quad t > 0, \tag{2.62}$$

$$z(0,t) = 0, \quad t > 0, \tag{2.63}$$

$$z(x,t) = M(t) = \sup_{x \in (0,\delta)} z(x,t), \quad z_x(x,t) = 0, \quad \delta(t) \leq x \leq 1, \quad t > 0. \tag{2.64}$$

Multiplying (2.62) by z_x and integrating we get

$$z_x(x,t) = \sqrt{2\mu}\, [F(z) - F(M)]^{1/2}, \tag{2.65}$$

where $F(\sigma) = \displaystyle\int_\sigma^\infty f(s)\,ds$, so $F'(\sigma) = -f(\sigma) < 0$.

The relation (2.65) implies

$$\int_0^M \frac{ds}{[F(s) - F(M)]^{1/2}} = \delta\sqrt{2\mu}. \tag{2.66}$$

Also

$$\begin{aligned} \int_0^1 f(z)\,dx &= -\int_0^\delta \frac{z_{xx}}{\mu}\,dx + (1-\delta) f(M) = \left(\frac{2}{\mu}\int_0^M f(s)\,ds\right)^{1/2} + (1-\delta) f(M) \\ &= \sqrt{\frac{2}{\mu}} + f(M) + o(1), \end{aligned} \tag{2.67}$$

as $\delta \to 0$ and $M \to +\infty$.

Let

$$\delta(M) = \frac{a}{2} f(M) \int_0^M [F(s) - F(M)]^{-1/2}\,ds, \tag{2.68}$$

where a is a suitable constant with $a > 1/\left[\left(\frac{\lambda}{2}\right)^{1/2} - 1\right]$. It is easily seen that such a choice of a entails $\Lambda = \frac{1}{2}\left[\frac{\lambda}{(1+a)^2} - \frac{2}{a^2}\right] > 0$. Then it holds that

$$\sqrt{\frac{2}{\mu}} = a\, f(M), \tag{2.69}$$

and thus, by using (2.29), (2.66) and (2.69), we obtain

$$\delta = \frac{1}{\sqrt{2\mu}} \int_0^M [F(s) - F(M)]^{-1/2}\,ds \leq a(M\, f(M))^{1/2},$$

which implies $\delta \to 0$ as $M \to \infty$ (or equivalently, as $\mu \to \infty$.)

Therefore we deduce that

$$\int_0^1 f(z)\,dx = (1+a)f(M) + o(1) \quad \text{as} \qquad M \to +\infty. \tag{2.70}$$

Now we can prove the following finite-time blow-up result.

Theorem 2.1.8 *Under the condition* (2.61) *the solution u of problem* (2.6)–(2.8) *blows up globally in finite time.*

Proof As the first step we show that z defined above is a blowing up lower solution of problem (2.6)–(2.8). Indeed, for $\delta \le x \le 1$ we have

$$\begin{aligned} \mathscr{F}(z) :&= z_t - z_{xx} + z_x - \frac{\lambda f(z)}{\left(\int_0^1 f(z)dx\right)^2} = \dot{M} - \frac{\lambda f(M)}{\left(\int_0^1 f(z)\,dx\right)^2} \\ &= \dot{M} - \frac{\lambda}{(1+a)^2 f(M)} + o(1) < \dot{M} - \frac{\Lambda}{f(M)} + o(1), \quad \text{as} \quad M \to +\infty. \end{aligned} \tag{2.71}$$

Hence $\mathscr{F}(z) \le 0$ provided that $0 < \dot{M} \le \Lambda/f(M)$ and for M sufficiently large. Notably it can be proven that $M(t)$ is differentiable almost every where following the approach in [15, 21].

Now we integrate relation (2.65) with respect to $x \in (0,\delta)$ and then differentiate with respect to t, to obtain

$$\begin{aligned} z_t &= \frac{x\dot{\mu}}{\sqrt{2\mu}}[F(z) - F(M)]^{1/2} + \frac{1}{2}\dot{M} f(M)[F(z) - F(M)]^{1/2} \int_0^z [F(s) - F(M)]^{-3/2} ds \\ &:= A + B. \end{aligned} \tag{2.72}$$

Owing to (2.66) and (2.69) we have

$$\begin{aligned} A &= -\frac{f'(M)\dot{M}}{f(M)}[F(z) - F(M)]^{1/2} \int_0^z [F(s) - F(M)]^{-1/2}\,ds \\ &\le -\frac{f'(M)\dot{M} f^{1/2}(z) M}{2 f^{3/2}(M)} \\ &\le \frac{\Lambda f(z)}{2 f^2(M)}, \end{aligned} \tag{2.73}$$

since $s(M^{1/2} - s) \le \frac{1}{4}M$, provided that $0 \le \dot{M} \le -\Lambda/Mf'(M)$.

Also we have

$$B \le \frac{1}{2}\dot{M} f(M) f^{1/2}(z)(M-z)^{1/2} f^{-3/2}(M) \int_0^z (M-s)^{-3/2} ds$$

$$\leq \frac{\dot{M} f^{1/2}(z)}{f^{1/2}(M)} \leq \frac{\Lambda f(z)}{2f^2(M)}, \tag{2.74}$$

as long as

$$0 \leq \dot{M} \leq \frac{\Lambda}{2f(M)}.$$

Finally, if $0 \leq \dot{M}(t) \leq \min\left\{\frac{\Lambda}{2f(M)}, \frac{-\Lambda}{M f'(M)}\right\}$ in $x \in (0, \delta]$, we obtain

$$\begin{aligned}\mathscr{F}(z) &\leq \frac{\Lambda f(z)}{f^2(M)} + \mu f(z) + \frac{2M^{1/2} f(z)}{a f^{3/2}(M)} - \frac{\lambda f(z)}{(1+a)^2 f^2(M)} + o(1)\\ &\leq \frac{f(z)}{f^2(M)}\left[\Lambda + \frac{2}{a^2} + \Lambda - \frac{\lambda}{(1+a)^2}\right] + o(1) = \frac{f(z)}{f^2(M)}[2\Lambda - 2\Lambda] + o(1) = o(1),\end{aligned} \tag{2.75}$$

as $M \to +\infty$, taking also into account that $\frac{2(f(M)M)^{1/2}}{a} \leq \Lambda$ for M sufficiently large.

Therefore, by choosing

$$\dot{M} = \min\left\{\frac{\Lambda}{2f(M)}, -\frac{\Lambda}{M f'(M)}\right\}, \tag{2.76}$$

there holds that $\mathscr{F}(z) \leq 0$ for $x \in (0, \delta) \cup (\delta, 1)$. Since z, z_x are continuous and $z(0, t) = z_x(1, t) = 0$, the function z is a lower solution of (2.6)–(2.8) for M large enough, after some time at which u is sufficiently large.

We shall show now that z blows up in finite time. Indeed relation (2.76) implies

$$\Lambda \frac{dt}{dM} = \max\{2f(M), -M f'(M)\} \leq 2f(M) - M f'(M) \quad \text{or}$$

$$\Lambda t \leq 3\int_0^M f(s)\, ds - M f(M) < 3\int_0^\infty f(s)\, ds < \infty,$$

since $M f(M) \to 0$ as $M \to \infty$. The latter implies that z blows up at $T^* = \frac{3}{\Lambda}\int_0^\infty f(s)\, ds < \infty$. Hence u must also blow up at some $t^* \leq T^* < \infty$.

Now, we claim that u blows up globally, which means that $u(x, t) \to \infty$ as $t \to t^*$ for all x in $(0, 1]$ and $u_x(0, t) \to \infty$ as $t \to t^*$. Indeed, noting that

$$M(t) = \sup_{[0,1]} u(\cdot, t) \quad \text{satisfies} \quad \dot{M} \leq \frac{\lambda f(0)}{\left(\int_0^1 f(u)\, dx\right)^2} = h(t),$$

we have $M(t) - M(0) \leq \int_0^t h(s)\,ds \to \infty$ as $t \to t^*-$, which implies $\int_0^1 f(u)\,dx \to 0$ as $t \to t^*-$. Thus for $\lambda > \lambda^*$ or for $\lambda_* < \lambda \leq \lambda^*$ and $u_0 > w_2$, u blows up globally and $u_x(0,t) \to \infty$ as $t \to t^*-$. □

An analogous result to Theorem 2.1.8 could be proved in the case

$$\lambda_* < \lambda \leq \lambda^* \quad \text{and} \quad \int_0^\infty f(s)\,ds < \infty,$$

by a similar construction of a blowing lower solution of problem (2.6)–(2.8). For more details see [22].

Remark 2.1.9 All the results obtained in Sects. 2.1.2–2.1.5 can be easily derived when the Dirichlet boundary condition $u(0,t) = 0$ is replaced by $u_x(0,t) - a\,u(0,t) = 0$, $t > 0$ for $a > 0$. Similar results can be derived if Eq. (2.6) is replaced by (2.9).

A nonlinearity is encountered quite often in application is the Heaviside function:

$$H(s) = \begin{cases} 0, & s < 0, \\ 1, & s \geq 0. \end{cases}$$

In fact, it is a good approximation for the food resistivity since in many occasions food materials change their resistivity during the sterilization process [40].

In that case problem (2.6)–(2.8) takes the form, [23],

$$u_t - u_{xx} + u_x = \frac{\lambda H(1-u)}{\left(\int_0^1 H(1-u)dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{2.77}$$

$$u(0,t) = u_x(1,t) = 0, \quad t > 0, \tag{2.78}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{2.79}$$

where now $u_0(x)$, $u_0'(x)$ are considered to be bounded with $u_0(x) \geq 0$ in $[0,1]$ (the last requirement is a consequence of the fact that for any $u_0(x)$ the solution u becomes non-negative throughout $0 < x < 1$ at some time t).

From the mathematical point of view, the Heaviside function is neither Lipschitz nor strictly positive however it is decreasing and so the techniques used in the previous sections can be modified to derive analogous results regarding the long time behavior of u. It should be pointed out that in this case the finite-time blow-up cannot occurs but instead finite-time quenching takes place when $\lambda > \lambda^*$ or when $0 < \lambda < \lambda^*$ and for large enough initial data. For a rigorous investigation of the long time behavior of the solution to (2.77)–(2.79) see [23].

2.1.5.2 Hyperbolic Case

It has been noted in Sect. 2.1.4 that the unbounded solutions to problem (2.18)–(2.20) exist either for $\lambda > \lambda^*$ or for $u_0(x)$ sufficiently large and $\lambda \le \lambda^*$. The exact behavior of such solutions to (2.18)–(2.20) depends upon the decreasing rate of $f(s)$. More precisely we have, [24],

Theorem 2.1.10 *If $\int_0^\infty f(s)ds < \infty$ and $\rho(s) \ge \gamma > 0$ for $s > 0$ then the unbounded solutions to (2.18)–(2.20) blow up globally in finite time, i.e. $u(x,t) \to \infty$ as $t \to t^* < \infty$ for any $x \in (0,1]$ and $u_x(0,t) \to \infty$ as $t \to t^*$.*

Proof As in Sect. 2.1.4 we can construct a lower solution of the form $v(x,t) = w(x;\nu(t))$ with $\nu(t)$ satisfying (2.59). Moreover, recalling that $M(\mu)$ is defined implicitly by

$$\mu(M) = \int_0^M \frac{ds}{f(s)},$$

we note that $M(0) = 0$, and thus Hardy's inequality [17] entails

$$\int_0^\nu \left(\frac{M(\sigma)}{\sigma}\right)^2 d\sigma \le 4\int_0^\nu (M'(\sigma))^2 d\sigma. \tag{2.80}$$

Also, by virtue of $M'(\nu) = f(M(\nu))$, then (2.80) implies

$$\int_0^\nu \left(\frac{M(\sigma)}{\sigma}\right)^2 d\sigma \le 4\int_0^\nu (M'(\sigma))^2 d\sigma = 4\int_0^{M(\nu)} f(s)ds < 4\int_0^\infty f(s)ds < \infty. \tag{2.81}$$

Since $\nu(t)$ satisfies (2.59), we obtain

$$t = \rho(0)\int_0^{\nu(t)} \frac{M^2(\sigma)}{\sigma^2\left(\lambda - \dfrac{M^2(\sigma)}{\sigma}\right)} d\sigma \quad \text{for any } t > 0. \tag{2.82}$$

Taking into account $\dfrac{M^2(\mu)}{\mu} = \dfrac{M^2(\mu)}{\int_0^M \frac{ds}{f(s)}} \to 0$ as $\mu \to \infty$ and $\int_0^\infty f(s) < \infty$, we deduce

$$\int_\beta^\infty \frac{M^2(\sigma)}{\sigma^2\left(\lambda - \dfrac{M^2(\sigma)}{\sigma}\right)} d\sigma = \frac{1}{\lambda}\int_\beta^\infty \left(\frac{M(\sigma)}{\sigma}\right)^2 d\sigma + o(1) \quad \text{as} \quad \beta \to +\infty. \tag{2.83}$$

Finally, combining (2.81), (2.82) and (2.83) we derive $\nu(t) \to \infty$ as $t \to T^*-$, where

$$T^* = \rho(0) \int_0^{\infty} \frac{M^2(\sigma)}{\sigma^2 \left(\lambda - \frac{M^2(\sigma)}{\sigma} \right)} d\sigma < \infty.$$

Hence $u(x,t)$ blows up in finite time, i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*- \le T^* < \infty$.

To prove global blow-up, we first note that $N(t) = max_{[0,1]} u(\cdot,t)$ satisfies

$$\frac{dN}{dt} = \frac{\lambda f(N)}{\rho(N) \left(\int_0^1 f(u)dx \right)^2} \le \frac{\lambda f(0)}{\gamma \left(\int_0^1 f(u)dx \right)^2} = h(t),$$

and since u blows up we take $N(t) - N(0) \le \int_0^t h(s)ds \to \infty$ as $t \to t^*$. The latter implies $h(t) \to \infty$ as $t \to t^*$ and so $\int_0^1 f(u)dx \to 0$ as $t \to t^*$, giving that $u(x,t) \to \infty$ as $t \to t^*$ for any $x \in (0,1]$ and $u_x(0,t) \ge w_x(0, v(t)) = v(t) f(0) \to \infty$ as $t \to t^*$. □

A complementary result to Theorem 2.1.10 is the following, [24],

Theorem 2.1.11 *If* $\int_0^\infty f(s)ds = \infty$ *and* $\rho(s) \ge \gamma > 0$ *for* $s > 0$ *then any unbounded solution* $u(x,t)$ *to* (2.18)–(2.20) *diverges or blows up in infinite time globally, i.e.* $u(x,t) \to \infty$ *as* $t \to \infty$ *for any* $x \in (0,1]$ *and* $u_x(0,t) \to \infty$ *as* $t \to \infty$.

Proof We consider the function $v = v(t) > 0$ such that

$$\frac{dv}{dt} = \frac{\lambda}{\gamma f(v)}. \tag{2.84}$$

Then there holds that

$$\begin{aligned} \mathcal{H}(z) &:= \rho(z)z_t + \left(1 - \int_0^x \rho'(z)z_t dy\right) z_x - \frac{\lambda f(z)}{\left(\int_0^1 f(z)dx \right)^2} \\ &= \rho(v)\frac{dv}{dt} - \frac{\lambda f(v)}{\left(\int_0^1 f(v)dx \right)^2} \ge \frac{\lambda}{f(v)} - \frac{\lambda}{f(v)} = 0. \end{aligned} \tag{2.85}$$

Choosing $v(0)$ such that $u_0(x) \le v(0)$, we see that $z(x,t) = v(t)$ is an upper solution to (2.18)–(2.20). Also (2.84) implies $\int_{v(0)}^{v(t)} f(s)ds = \frac{\lambda}{\gamma} t$, leading, due to the hypothesis $\int_0^\infty f(s)\,ds = \infty$, to $v(t) \to \infty$ as $t \to \infty$. Hence, $z(x,t)$ is a global-in-time unbounded upper solution to (2.18)–(2.20). This implies that $u(x,t)$ diverges or blows up in infinite time, i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to \infty$. Using similar arguments as in Theorem 2.1.10, it is proved that $\int_0^1 f(u)dx \to 0$ as $t \to \infty$. Thus $u(x,t) \to \infty$, for any $x \in (0,1]$ and $u_x(0,t) \to \infty$ as $t \to \infty$. This completes the proof. □

2.2 A Non-local Thermistor Problem

The second part of the current chapter is devoted to the study of the thermistor, a device for regulating electric current in a circuit.

The operation of the thermistor is described by the following system

$$u_t = \nabla \cdot (\kappa(u)\, \nabla u) + \rho(u)\, |\nabla \phi|^2, \quad x \in \Omega, \ \ t > 0, \tag{2.86}$$

$$\nabla \cdot (\rho(u)\, \nabla \phi) = 0, \quad x \in \Omega, \ \ t > 0, \tag{2.87}$$

together with some boundary conditions for $u(x, t)$ and $\phi(x, t)$ on $\partial\Omega$, see for example [3, 4, 13]. Here, Ω is assumed to be a smooth, bounded open set of $\mathbb{R}^N$, $N \geq 1$, and stands for the spatial domain occupied by the conductor (the body of the thermistor); the physical situation corresponds to $N = 3$. Moreover, $\phi(x, t)$ is the electrical potential, $u(x, t)$ the temperature inside the conductor, $\kappa(u) > 0$ the thermal conductivity, and $\rho(u) > 0$ stands for the electrical conductivity. The parabolic equation (2.86) describes the heat flow in the system, while the elliptic equation (2.87) describes the conservation of charge in the system, provided that its variation in space and time is not too rapid.

Using a similar approach to Sect. 2.1.1 we can reduce system (2.86)–(2.87) to the following non-local equation

$$u_t = \nabla \cdot (\kappa(u)\nabla(u)) + \frac{\lambda\, f(u)}{\left(\int_\Omega f(u)\, dx\right)^2}, \quad x \in \Omega, \quad t > 0, \tag{2.88}$$

which is also associated with boundary and initial conditions. The form of the boundary conditions depending on the flux conditions are applied at the edge of the thermistor device. Below we will mainly deal with zero-flux (Neumann), mixed (Robin), and Dirichlet type boundary conditions.

Local-in-time existence of problems for Eq. (2.88) can be obtained using similar ideas as in Sect. 2.1.2 and therefore we focus on the blowing-up behavior of the associated problems. Again blow-up behavior is closely linked with the occurrence of thermal runaway which might cause destruction of thermistor device. Therefore its thorough investigation is interesting from applications point of view as well.

2.2.1 Neumann Problem

We start our study from the simplest case, when $u(x, t)$ satisfies Neumann boundary conditions, i.e. the boundary of the thermistor is thermally insulated, and so the (dimensionless) temperature $u(x, t)$ satisfies the problem

$$u_t = \nabla \cdot (\kappa(u)\nabla(u)) + \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad x \in \Omega, \quad t > 0, \tag{2.89}$$

$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{2.90}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.91}$$

where $f(s) > 0, \quad f'(s) < 0$ and $\kappa(u) \geq c > 0$. Here $\partial/\partial\nu$ denotes the normal outward derivative to the boundary $\partial\Omega$. In the following we assume that $u_0(x) \geq 0$.

In this case the associated steady-state problem

$$\nabla \cdot (\kappa(w)\nabla(w)) + \frac{\lambda f(w)}{\left(\int_\Omega f(w)\,dx\right)^2} = 0, \quad x \in \Omega, \quad \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega, \tag{2.92}$$

does not permit any kind of solution for every $\lambda > 0$.

Otherwise, if we integrate the equation of problem (2.92) we get

$$0 = \frac{\lambda}{\int_\Omega f(w)\,dx},$$

which is a contradiction. The lack of stationary solutions is an indication that time-dependent solutions should be unbounded. From the physical point of view this is justified since the source term is positive, the system is provided with heat. On the other hand the boundary condition $\partial u/\partial\nu = 0$ prevents any heat from escaping. Thus, in such a situation, one expects the solution (concentration of heat) to become unbounded. This physical situation is described by the following.

Theorem 2.2.1 *There exists $t^* \leq \infty$ such that $||u(\cdot,t)||_\infty \to \infty$ as $t \to t^*$. If $\int_0^\infty f(s)\,ds < \infty$ then $u(x,t)$ blows up in finite time, i.e. $t^* < \infty$, whereas if $\int_0^\infty f(s)\,ds = \infty$ then blow-up in infinite-time occurs, i.e. $t^* = \infty$ and $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*$. Moreover in each case blow-up is global and flat (uniform) i.e.*

$$u(x,t) = ||u(\cdot,t)||_\infty(1 + o(1)) \quad \textit{as} \quad t \to t^* - \textit{ for a.e. } x \in \Omega.$$

Proof We first assume $\int_0^\infty f(s)\,ds < \infty$. Under this assumption and the positivity of u (which is a consequence of the maximum principle) the functional

$$Y(t) = \int_\Omega \int_{u(x,t)}^\infty f(\sigma)\,d\sigma\,dx,$$

is well-defined and nonnegative [3, 21].

Taking the derivative of $Y(t)$ with respect to t and using Eq. (2.89) we obtain

$$Y'(t) = -\int_\Omega f(u)\, u_t dx = -\int_\Omega f(u)\nabla\cdot(\kappa(u)\nabla(u))\ dx - \frac{\lambda\ \int_\Omega f^2(u)\ dx}{\left(\int_\Omega f(u)\ dx\right)^2}.$$

Using Jensen's inequality for $\phi(s) = s^2$ and integration by parts we have

$$Y'(t) \le -\int_\Omega f(u)\nabla\cdot(\kappa(u)\nabla(u))\ dx - \frac{\lambda}{|\Omega|} = \int_\Omega f'(u)\,\kappa(u)\,|\nabla u|^2\, dx - \frac{\lambda}{|\Omega|}.$$

Now using the monotonicity of $f(s)$ as well the positivity of $\kappa(u)$ then

$$Y'(t) \le -\frac{\lambda}{|\Omega|},$$

which finally yields

$$0 \le Y(t) \le Y(0) - \frac{\lambda}{|\Omega|}\, t.$$

The latter yields that $u(x,t)$ cannot exist beyond t^*, where

$$t^* \le T_u^* = \frac{|\Omega|\, Y(0)}{\lambda} < \infty. \tag{2.93}$$

Since the solution $u(x,t)$ of (2.89)–(2.91) ceases to exist only when it becomes unbounded, we deduce that $||u(\cdot,t)||_\infty \to \infty$ as $t \to t^*-$ (finite-time blow-up). An immediate result of relation (2.93) is that as the L^1-norm of the initial conditions increases then the bound T_u^* on the blow-up time, as it is expected, decreases. Moreover we can prove that $u(x,t)$ blows up globally, using similar ideas as in Theorem 2.1.8, and uniformly i.e. $u(x,t) \sim ||u(\cdot,t)||_\infty$ as $t \to t^*$ for a.e. $x \in \Omega$ (global and flat blow-up).

Applying a similar approach as in [15, 21], we obtain that $n(t) = \min_{x\in\bar{\Omega}} u(x,t)$ and $N(t) = \max_{x\in\overline{\Omega}} u(x,t)$ are differentiable almost for almost every t and thus it satisfies

$$\frac{dn}{dt} \ge \frac{\lambda\, f(n)}{\left(\int_\Omega f(u)\, dx\right)^2} \ge \frac{\lambda\, f(N)}{\left(\int_\Omega f(u)\, dx\right)^2} \ge \frac{dN}{dt}, \quad \text{a.e. in} \quad (0, t^*),$$

which implies $n(t) < N(t) \le n(t) + C,\ \ C = N(0) - n(0)$, for every $t \in (0, t^*)$. Hence the blow-up is uniform and it holds that

$$u(x,t) = ||u(\cdot,t)||_\infty(1 + o(1)) \quad \text{as} \quad t \to t^*- \ \ \text{for a.e.}\ \ x \in \Omega.$$

We now consider the case where $\int_0^\infty f(s)\, ds = \infty$. Again $N(t)$ satisfies

$$\frac{dN}{dt} \leq \frac{\lambda f(N)}{\left(\int_{\Omega} f(u)\, dx\right)^2} \leq \frac{\lambda}{f(N)\,|\Omega|^2} \quad \text{a.e in} \quad (0, t),$$

and integrating over $(0, t)$ yields

$$\int_{N(0)}^{N(t)} f(s)\, ds \leq \frac{\lambda}{|\Omega|^2}\, t,$$

for every $t > 0$. The latter implies that the solution cannot blow-up in finite time. By the inequality

$$\left(\int_{\Omega} u(x, t) dx\right)_t \geq \frac{\lambda}{|\Omega|\, f(0)} > 0,$$

we deduce $\lim_{t\to\infty} \int_{\Omega} u(x, t)\, dx = \infty$ and thus $N(t)$ diverges in infinite time. □

2.2.2 Robin Problem

We now investigate a problem with Robin-type boundary conditions, i.e. non-Newtonian cooling,

$$u_t = \nabla \cdot (\kappa(u)\nabla(u)) + \frac{\lambda f(u)}{\left(\int_{\Omega} f(u)\, dx\right)^2}, \quad x \in \Omega, \quad t > 0, \tag{2.94}$$

$$\frac{\partial u}{\partial \nu} + \beta\, u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{2.95}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{2.96}$$

where $\beta = \beta(x) > 0$. We still assume that $f(s) > 0$, $f'(s) < 0$ and $\int_0^{\infty} f(s)\, ds < \infty$; actually for simplicity we take

$$\int_0^{\infty} f(s)\, ds = 1. \tag{2.97}$$

Then the following blow-up result holds:

Theorem 2.2.2 *Assume $(f(s)\kappa(s))' < 0$ then if $u(x, t)$ is a solution of (2.94)–(2.96) then for λ sufficiently large there exists $t^* < \infty$ such that $||u(\cdot, t)||_{\infty} \to \infty$ as $t \to t^*-$, i.e. $u(x, t)$ blows up in finite time. Moreover blow-up is global and flat.*

Proof In this case the technique used for the Neumann problem fails because when integrating by parts the term containing $\partial u/\partial \nu$ does not vanish any more on the boundary. In order to overcome this difficulty we modify the form of the functional

$Y(t)$. This can be achieved by using an auxiliary function in the definition of $Y(t)$. More precisely we consider

$$Y(t) = \int_{\Omega} \Psi(x) \left(\int_{u(x,t)}^{\infty} f(\sigma)\, d\sigma \right) dx,$$

where $\Psi(x)$ is the Robin eigenfunction corresponding to the principal eigenvalue of the Laplacian, i.e. $\Psi(x)$ satisfies the problem

$$-\Delta\Psi = \mu_1 \Psi, \;\; x \in \Omega, \;\; \frac{\partial \Psi}{\partial \nu} + \beta \Psi = 0, \;\; x \in \partial\Omega. \tag{2.98}$$

It is known, see for example [2] Theorem 4.3, that μ_1 is positive and that $\Psi(x)$ does not change sign in $\overline{\Omega}$, so it can be chosen to be positive and for simplicity normalised so that

$$\int_{\Omega} \Psi(x)\, dx = 1. \tag{2.99}$$

Hence the functional $Y(t)$ is nonegative and well-defined due to (2.97).

Differentiating $Y(t)$ and using (2.94) we obtain

$$\begin{aligned} Y'(t) &\leq -\int_{\partial\Omega} \Psi\, f(u)\, \kappa(u)\, \frac{\partial u}{\partial \nu}\, ds + \int_{\Omega} \nabla\left(\Psi(x)\, f(u)\right)\, \kappa(u) \cdot \nabla(u)\, dx - \frac{\lambda\, m}{|\Omega|} \\ &= \beta \int_{\partial\Omega} \Psi\, f(u)\, \kappa(u)\, u\, ds + \int_{\Omega} \nabla\Psi(x) \cdot \nabla \left(\int_0^u f(\sigma)\kappa(\sigma) d\sigma \right) dx \\ &\quad + \int_{\Omega} \Psi(x) f'(u) \kappa(u) |\nabla u|^2 dx - \frac{\lambda\, m}{|\Omega|}, \end{aligned}$$

due to the boundary conditons. Then $f'(s) < 0$ implies

$$\begin{aligned} Y'(t) &\leq \beta \int_{\partial\Omega} \Psi\, f(u)\, \kappa(u)\, u\, ds - \beta \int_{\partial\Omega} \Psi \left(\int_0^u f(\sigma)\kappa(\sigma) d\sigma \right) ds \\ &\quad - \int_{\Omega} \Delta\Psi \left(\int_0^u f(\sigma)\kappa(\sigma) d\sigma \right) dx - \frac{\lambda\, m}{|\Omega|} \\ &= \mu_1 - \frac{\lambda\, m}{|\Omega|}, \end{aligned}$$

for $m = \min_{x \in \overline{\Omega}} \Psi(x) > 0$ by $(f(s)\kappa(s))' < 0$, (2.99), and (2.97).

The latter yields

$$0 < Y(t) \leq Y(0) - \left(\frac{\lambda\, m}{|\Omega|} - \mu_1 \right) t,$$

and finally, we conclude that the solution of (2.94)–(2.96) blows up globally in finite time by

$$T_u^* = \frac{Y(0)}{\frac{\lambda m}{|\Omega|} - \mu_1},$$

provided that $\lambda > \hat{\lambda} = \frac{\mu_1 |\Omega|}{m}$. It can also be proven that blow-up is global and uniform. □

The above method, apart from the existence of blow-up, provides us with an upper estimate of λ^*, the supremum of the spectrum of the corresponding steady-state problem. Indeed, we claim that $\lambda^* \leq \hat{\lambda}$, because otherwise, for initial conditions $u(x, 0) < w_m(x; \lambda)$ where $w_m(x; \lambda)$ is the minimal stationary solution corresponding to a $\hat{\lambda} < \lambda < \lambda^*$, we could prove, using similar ideas as in [30], that $u(x, t) \to w_m(x)$ as $t \to \infty$, which is a contradiction since blow-up occurs for every initial conditions if $\lambda > \hat{\lambda}$. This estimate does not seem to be optimal, at least for $f(s) = e^{-s}$. In fact, as Bebernes and Lacey conjectured in [5] that λ^* in this case will be proportional to $2\,|\partial\Omega|^2$. However, for the one-dimensional case the above estimate $\hat{\lambda}$ improves the ones in [30, 42].

As it has been pointed out in the second part of the current chapter, the occurrence of blow-up is linked with thermal runaway which might be responsible for the destruction of thermistor device. Therefore, it is important from applications point of view to estimate when finite blow-up takes place. Thus, in the following we provide some useful estimates of the blow-up time when $\lambda > \lambda^*$.

For simplicity we take a nonlinearity f satisfying

$$f(s),\ f''(s) > 0, \quad f'(s) < 0, \quad s \geq 0\,, \tag{2.100}$$

$$f(s) \leq \frac{c}{s^2}\,, \ \ c > 0 \ \text{ for } \ s \gg 1 \ \text{ and } \ \int_0^\infty f(s)\,ds < \infty, \tag{2.101}$$

and we also deal with the one-dimensional case, e.g. $\Omega = (-1, 1)$, for $\kappa(u) = 1$, i.e.,

$$u_t = u_{xx} + \frac{\lambda\, f(u)}{\left(\int_{-1}^{1} f(u)\,dx\right)^2}, \quad x \in (-1, 1), \quad t > 0, \tag{2.102}$$

$$u_x(x, t) \pm \beta\, u(x, t) = 0, \quad x = \pm 1, \quad t > 0, \tag{2.103}$$

$$u(x, 0) = u_0(x), \quad x \in (-1, 1). \tag{2.104}$$

In particular, in [26] is proven the following

Theorem 2.2.3 *Let f satisfy* (2.100) *and* (2.101)*. Assume, further, that the steady-state problem of* (2.102)–(2.104) *has at least a classical solution, say w^*, corresponding to λ^*. Then for any λ sufficiently close to λ^*, the blow-up time T_b of the solution of* (2.102)–(2.104) *can be estimated as follows*

$$T_l\,(\lambda - \lambda^*)^{-\frac{1}{2}} \leq T_b \leq T_u\,(\lambda - \lambda^*)^{-\frac{1}{2}}, \tag{2.105}$$

where T_l, T_u are positive constants.

The proof of Theorem 2.2.3 is based on several ideas introduced in [28] and is lengthy. Thus it is omitted here. In [26], some estimates of the blow-up via asymptotic and numerical methods are also provided for $f(s) = e^{-s}$ which actually confirm the results of Theorem 2.2.3.

2.2.3 Dirichlet Problem

We finally close our study with the Dirichlet case where for simplicity we also consider $\kappa(u) = 1$ and thus we focus on the following

$$u_t = \Delta u + \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad x \in \Omega, \quad t > 0, \tag{2.106}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{2.107}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.108}$$

In this case we expect for blow-up to occur if $f(s)$ is decreasing and satisfies (2.97). Indeed, under those conditions there exists a critical parameter $\lambda^* \geq 2\,|\partial\Omega|^2$, see Theorem 2.2 in [5], such that the steady-state problem

$$\Delta w + \frac{\lambda f(w)}{\left(\int_\Omega f(w)\,dx\right)^2} = 0, \quad x \in \Omega, \quad w(x) = 0, \quad x \in \partial\Omega, \tag{2.109}$$

has no solutions for $\lambda > \lambda^*$. Actually, in the one-dimensional case a thorough study of finite-time blow-up for $\lambda > \lambda^*$ is provided in [29, 30]. Here we provide an extension of those blow-up results in the higher dimensional case however we should restrict the class of the considered nonlinear functions f.

Notably the approach used in the case of Neumann and Robin boundary conditions fails here, for more details see [21]. Therefore, we will proceed for the proof of blow-up by constructing a proper blowing up lower solution. We actually have the following

Theorem 2.2.4 *For $\lambda > \lambda^*$ and sufficiently large initial conditions the solution of the problem (2.106)–(2.108) blows up globally and uniformly in finite time provided that $f(s)$ satisfies*

$$\int_0^\infty [2s\, f(s) - s^2 f'(s)]\, ds < \infty. \tag{2.110}$$

Proof For the construction of this lower solution we will need first to define a lower solution of the problem for the steady-state problem (2.109), which was first introduced in [5]. This lower solution has the form

$$v(x;\mu,\nu,\delta)=\begin{cases} V(\sqrt{\nu\,\delta}\,d(x,\partial\Omega);\mu), & d(x,\partial\Omega)<\dfrac{\mu}{\sqrt{\nu\,\delta}},\\ V(\mu;\mu)=M, & d(x,\partial\Omega)\geq\dfrac{\mu}{\sqrt{\nu\,\delta}}, \end{cases} \tag{2.111}$$

where V is the solution of the problem

$$V''+f(V)=0,\ \ 0<y<\mu,\ \ V(0;\mu)=0,\ \ V'(\mu;\mu)=0, \tag{2.112}$$

while $\delta=\lambda/(\int_\Omega f(w)\,dx)^2$ is the so called local parameter of (2.109) and ν is a constant to be determined later. Here $d(x,\partial\Omega)$ stands for the distance between $x\in\Omega$ and the boundary $\partial\Omega$. Also, d is smooth and more precisely $|\Delta d|<K$, for some K, in a neighborhood of the boundary if $\partial\Omega$ is smooth. In particular, such a neighborhood consists of all $x\in\Omega$ such that $d(x,\partial\Omega)\leq\frac{\mu}{\sqrt{\nu\,\delta}}<\rho$, where ρ is smaller than the infimum of the radius of the largest interior ball touching the boundary at some $z\in\partial\Omega$. Hence in the following δ is chosen large enough to ensure that $\frac{\mu}{\sqrt{\nu\,\delta}}<\rho$.

Obviously v satisfies the correct boundary condition and is C^1. Moreover there holds

$$\begin{aligned} \Delta v+\delta f(v) &= \delta f(V(\mu))>0 \quad \text{for} \quad d(x,\partial\Omega)\geq\frac{\mu}{\sqrt{\nu\,\delta}},\\ \Delta v+\delta f(v) &= \nu\,\delta\,|\nabla d|^2\,V''+\sqrt{\nu\,\delta}\,V'\,\Delta d+\delta\,f(V)\\ &= (1-\nu)\,\delta\,f(V)+\sqrt{\nu\,\delta}\,V'\,\Delta d \quad \text{where} \quad d(x,\partial\Omega)<\frac{\mu}{\sqrt{\nu\,\delta}}\\ &> (1-\nu)\,\delta\,f(V)-\sqrt{\delta}\,K\,V'\geq 0, \quad \text{for} \quad \nu<1, \end{aligned}$$

provided that $\nu\leq 1-\frac{K\,G(\mu)}{\sqrt{\delta}}$, where $G(\mu)=\sup_{y\in(0,\mu)}\frac{V'}{f(V)}$.

Therefore, by choosing $\nu=1-K\,G(\mu)/\sqrt{\delta}$, v is a lower solution of problem (2.109) for sufficiently large δ, i.e. for $\lambda<\lambda^*$ sufficiently close to λ^*.

Problem (2.112) also implies that V satisfies

$$V'^{\,2}=2\int_V^M f(s)\,ds=2\int_V^\infty f(s)\,ds-2\int_M^\infty f(s)\,ds=2(F(V)-F(M)),$$

where $F(\sigma)=\int_\sigma^\infty f(s)\,ds$ and thus the relation between μ and M is defined by

$$\mu=\mu(M)=\frac{1}{\sqrt{2}}\int_0^M[F(s)-F(M)]^{-1/2}\,ds. \tag{2.113}$$

We now consider as a candidate lower solution the function

$$z(x,t)=\begin{cases} V(\sqrt{\nu\,\delta}\,d(x,\partial\Omega);\mu), & d(x,\partial\Omega)<\dfrac{\mu}{\sqrt{\nu\,\delta}},\\ V(\mu;\mu)=M(t), & d(x,\partial\Omega)\geq\dfrac{\mu}{\sqrt{\nu\,\delta}}, \end{cases} \tag{2.114}$$

where the functions $\mu=\mu(t)$, $M=M(t)$, $\delta=\delta(t)$ and the constant ν will be determined below.

Indeed, due to (2.113) the relation between $M(t)$ and $\mu(t)$ is already determined, while we are free to chose the relation between $M(t)$ and $\delta(t)$. It is evident, from the definition of $z(x,t)$, that if the ratio $\mu/\sqrt{\nu\delta}$ decreases to 0 as t increases then the spatial independent behavior of $z(x,t)$ dominates the behavior near the boundary, i.e. the growth of $z(x,t)$ is uniform (flat) as t increases. In the following we will choose $M(t)$ and the dependence between M and δ such that, for large enough intitial conditions, $z(x,t)$ is a lower solution of problem (2.106)–(2.108) which blows up in finite time.

We choose

$$\delta=\delta(M)=\frac{M^2}{f(M)}, \tag{2.115}$$

while we impose $M(t)$ to satisfy

$$\dot{M}(t):=\frac{dM}{dt}=\frac{\lambda-\lambda^*}{\left(\int_\Omega f(z)\,dx\right)^2}\inf_{x\in\Omega}\left\{\frac{f(z)}{z_M}\right\}>0 \quad \text{for} \quad t>0, \tag{2.116}$$

and $M(0)$ is chosen large enough so that $\delta_0=\delta(M(0))=M^2(0)/f(M(0))$ is also sufficiently large (note that $\delta'(M)>0$).

Consequently, we have that

$$z_M=\frac{\partial z}{\partial M}=\frac{\partial V}{\partial y}\,y'(M)+\frac{\partial V}{\partial \mu}\,\mu'(M)>0,$$

where $y=y(M)=\sqrt{\nu\,\delta(M)}\,d(x,\partial\Omega)$. In fact, differentiating problem (2.112) with respect to y and using the monotonicity of $f(s)$ together with maximum principle arguments we get $\partial V/\partial y>0$. Besides, $y'(M)=\frac{1}{2}y(M)\,\delta'(M)/\delta(M)>0$ and $\mu'(M)$, $V_\mu(y;\mu)>0$.

Setting $Y=y/\mu$ then problem (2.112) can be written as

$$W''+\mu^2\,f(W)=0,\ \ 0<Y<1,\ \ W(0;\mu)=0,\ \ W'(1;\mu)=0,$$

where $W(Y)=V(y)$. Differentiating this problem with respect to μ we obtain

$$-W''_\mu-\mu^2\,f'(W)\,W_\mu=2\,\mu\,f(W)>0,\ \ 0<Y<1,\ \ W_\mu(0;\mu)=0,\ \ W'_\mu(1;\mu)=0,$$

which implies (due the maximum principle) that $W_\mu(Y;\mu)=V_\mu(y;\mu)>0$ for $0<y<\mu$ and $W_\mu(1;\mu)=M'(\mu)>0$, hence $\mu'(M)>0$.

Now if we choose δ_0 large enough then the fact that $\delta'(M) > 0$ and $M(t)$ satisfies (2.116) guarantee that $\delta(t)$ remains large enough for every $t > 0$.

Therefore

$$-\Delta z \leq \delta(t) f(z) \quad \text{for every} \quad t > 0, \tag{2.117}$$

provided that ν is chosen such that

$$\nu \leq \inf_{M > M(0)} \left(1 - \frac{K\, G(M)}{\sqrt{\delta(M)}}\right), \tag{2.118}$$

where $M(0)$ is also taken large enough. The latter choice is possible since

$$\lim_{M \to +\infty} \frac{G(M)}{\sqrt{\delta(M)}} = 0.$$

Therefore we derive

$$\begin{aligned}
z_t - \Delta z - \frac{\lambda f(z)}{\left(\int_\Omega f(z)\,dx\right)^2} &= z_M\, \dot{M}(t) - \Delta z - \frac{\lambda f(z)}{\left(\int_\Omega f(z)\,dx\right)^2} \\
&\leq z_M\, \dot{M}(t) + \delta(t)\, f(z) - \frac{\lambda f(z)}{\left(\int_\Omega f(z)\,dx\right)^2} \quad \text{(due to (2.11.7))} \\
&\leq z_M\, \dot{M}(t) + \frac{[\lambda^* - \lambda]\, f(z)}{\left(\int_\Omega f(z)\,dx\right)^2} + o(1) \\
&= z_M \left(\frac{\lambda - \lambda^*}{\left(\int_\Omega f(z)\,dx\right)^2} \inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} + \frac{[\lambda^* - \lambda]}{\left(\int_\Omega f(z)\,dx\right)^2} \frac{f(z)}{z_M} \right) + o(1) \\
&= \frac{z_M\, (\lambda - \lambda^*)}{\left(\int_\Omega f(z)\,dx\right)^2} \left[\inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} - \frac{f(z)}{z_M} \right] + o(1) \leq 0 \quad \text{as} \quad M \to +\infty,
\end{aligned}$$

since $\delta(t) \leq \lambda^* / \left(\int_\Omega f(z)\,dx\right)^2 + o(1)$ as $M \to +\infty$, and by choosing $M(0)$ sufficiently large.

Consequently, $z(x,t)$ is a lower solution of problem (2.106)–(2.108) provided that $M(t)$ satisfies (2.116) with sufficiently large initial conditions $M(0)$. Furthermore, we show that (2.116) guarantees the occurrence of finite-time blow-up for the lower solution $z(x,t)$.

Indeed, we first note that

$$\inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} = \min \left\{ f(M), \inf_{d(x,\partial\Omega) < \mu/\sqrt{\nu\,\delta}} \left\{ \frac{f(z)}{z_M} \right\} \right\},$$

where

$$\inf_{d(x,\partial\Omega) < \mu/\sqrt{\nu\,\delta}} \left\{ \frac{f(z)}{z_M} \right\} = \inf_{y \in (0,\mu)} \left\{ \frac{f(V(y;\mu))}{V_M(y;\mu)} \right\} = \frac{1}{\sup_{y \in (0,\mu)} \left\{ \frac{V_M(y;\mu)}{f(V(y;\mu))} \right\}}.$$

Since

$$V_M(y;\mu) = \frac{1}{2}\, y\, \frac{\delta'(M)}{\delta(M)}\, \frac{\partial V}{\partial y} + \frac{\partial V}{\partial \mu}\, \mu'(M),$$

then

$$\begin{aligned}\sup_{y\in(0,\mu)} \left\{ \frac{V_M(y;\mu)}{f(V(y;\mu))} \right\} &\le \frac{1}{2}\frac{\delta'(M)}{\delta(M)} \sup_{y\in(0,\mu)} \left\{ \frac{V_y(y;\mu)\, y}{f(V(y;\mu))} \right\} + \sup_{y\in(0,\mu)} \left\{ \frac{V_\mu(y;\mu)\,\mu'(M)}{f(V(y;\mu))} \right\} \\ &\le \frac{1}{2}\frac{\delta'(M)}{\delta(M)}\, G(\mu(M))\,\mu(M) + \sup_{y\in(0,\mu)} \left\{ \frac{V_\mu(y;\mu)}{f(V(y;\mu))} \right\} \mu'(M). \end{aligned} \tag{2.119}$$

Note that

$$\lim_{M\to+\infty} \frac{\mu(M)}{\sqrt{\delta(M)}} = 0, \tag{2.120}$$

and

$$\lim_{M\to+\infty} \frac{G(\mu(M))}{\sqrt{\delta(M)}} = 0. \tag{2.121}$$

For the former by virtue of (2.113) we have

$$\begin{aligned}\mu(M) &= \frac{1}{\sqrt{2}} \int_0^M [F(s) - F(M)]^{-1/2}\, ds \le \frac{1}{\sqrt{2}} \int_0^M [(M-s)\, f(M)]^{-1/2}\, ds \\ &= \sqrt{\frac{2\,M}{f(M)}},\end{aligned}$$

and thus

$$0 < \lim_{M\to+\infty} \frac{\mu(M)}{\sqrt{\delta(M)}} \le \lim_{M\to+\infty} \frac{\sqrt{\frac{2\,M}{f(M)}}}{\sqrt{\delta(M)}} = \lim_{M\to+\infty} \frac{\sqrt{\frac{2\,M}{f(M)}}}{\frac{M}{\sqrt{f(M)}}} = 0.$$

For the latter, taking into account the monotonicity of f we obtain that

$$f^2(M)\,(F(V) - F(M))/f^2(V) \le (M - V)\, f^2(M)/f(V) \le (M - V)\, f(M) \le M\, f(M),$$

which implies

$$G(\mu(M)) = \sup_{V\in(0,M)} \frac{\sqrt{2\,(F(V) - F(M))}}{f(V)} \le \sqrt{\frac{2\,M}{f(M)}},$$

and thus

$$0 < \lim_{M\to+\infty} \frac{G(\mu(M))}{\sqrt{\delta(M)}} \leq \lim_{M\to+\infty} \frac{\sqrt{\frac{2M}{f(M)}}}{\sqrt{\delta(M)}} = \lim_{M\to+\infty} \frac{\sqrt{\frac{2M}{f(M)}}}{\frac{M}{\sqrt{f(M)}}} = 0.$$

Furthermore,

$$\sup_{y\in(0,\mu)} \left\{ \frac{V_\mu(y;\mu)}{f(V(y;\mu))} \right\} \leq \frac{V_\mu(\mu;\mu)}{f(V(\mu;\mu))} = \frac{M'(\mu)}{f(M)}, \tag{2.122}$$

since

$$\frac{\partial^2 V(y;\mu)}{\partial\mu\,\partial y} = -\int_0^y f'(V(s;\mu))\, V_\mu(s;\mu)\, ds + \frac{\partial^2 V(0;\mu)}{\partial\mu\,\partial y} \geq 0.$$

Now by using (2.120)–(2.122) then relation (2.119) entails

$$\sup_{y\in(0,\mu)} \left\{ \frac{V_M(y)}{f(V(y))} \right\} < \frac{1}{2}\delta'(M) + \frac{1}{f(M)} = \frac{1}{2}\delta'(M)(1+o(1)) \quad \text{as} \quad M \to +\infty,$$

taking into account

$$\lim_{M\to\infty} \frac{\frac{1}{f(M)}}{\frac{\delta'(M)}{2}} = \lim_{M\to\infty} \frac{\frac{2}{\delta'(M)}}{f(M)} = \lim_{M\to\infty} \frac{2}{2M - \frac{M^2 f'(M)}{f(M)}} = 0,$$

and thus

$$\inf_{x\in\Omega} \left\{ \frac{f(z)}{z_M} \right\} \geq \min\left\{ f(M), \frac{2}{\delta'(M)} \right\} + o(M) = \frac{2}{\delta'(M)} + o(M) \quad \text{as} \quad M \to +\infty.$$

Consequently in view of relation (2.116) we obtain

$$\begin{aligned} \dot{M}(t) &\geq \frac{\lambda-\lambda^*}{\left(\int_\Omega f(z)\,dx\right)^2} \frac{2}{\delta'(M)} + o(M) = \frac{2(\lambda-\lambda^*)}{|\Omega|^2\, \delta'(M)\, f^2(M)} + o(M) \\ &= \frac{(\lambda-\lambda^*)}{|\Omega|^2\,[2M\,f(M) - M^2\, f'(M)]} + o(M), \quad \text{as} \quad M \to +\infty, \end{aligned}$$

since $\int_\Omega f(z)\,dx = f(M)\,|\Omega| + o(M)$ as $M \to +\infty$ due to (2.120) and the definition of $z(x,t)$. The latter implies that $M(t) \to \infty$ as $t \to t_1^*$, where

$$t_1^* \leq \frac{|\Omega|^2}{(\lambda-\lambda^*)} \int_{M(0)}^\infty [2\sigma\, f(\sigma) - \sigma^2\, f'(\sigma)]\, d\sigma + o(M) < \infty, \quad \text{as} \quad M \to +\infty, \tag{2.123}$$

recalling that f satisfies (2.110).

Consequently $z(x,t)$ is lower solution which blows up in finite time t_1^*, provided that $M(0)$ is chosen large enough. The latter implies that $u(x,t)$ blows up in finite time $t^* \leq t_1^*$, i.e. $||u(\cdot, t)||_\infty \to \infty$ as $t \to t^*$, for large enough initial conditions. It can be proven, using the same arguments as in the Neumann problem, that blow-up is global.

Finally, in this case we can also prove that blow-up is uniform (flat). Actually, if we consider the problem

$$v_t = \Delta v + h(t)\, f(N), \quad x \in \Omega, \quad t > 0,$$
$$v(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$
$$v(x,0) = 0, \quad x \in \Omega,$$

where $N = N(t) = \max_{x\in\Omega} u(x,t)$ and $h(t) = \lambda/(\int_\Omega f(u(x,t))\, dx)^2$ then $v(x,t)$ is a lower solution of problem (2.106)–(2.108) since $f(s)$ is decreasing.

Set $v(x,t) = \theta(x,t) + V(t)$ where $V(t)$ is the solution of the problem

$$\frac{dV(t)}{dt} = h(t)\, f(N), \quad V(0) = 0,$$

and $\theta(x,t)$ satisfies

$$\theta_t = \Delta\theta, \quad x \in \Omega, \quad t > 0,$$
$$\theta(x,t) = -V(t), \quad x \in \partial\Omega, \quad t > 0,$$
$$\theta(x,0) = 0, \quad x \in \Omega.$$

Thus θ has the integral representation

$$\theta(x,t) = \int_0^t V(\tau) \int_{\partial\Omega} \frac{\partial G(x,s,t-\tau)}{\partial\nu}\, ds d\tau,$$

where $G(x,y,t)$ is the Green's function for the heat equation in Ω with Dirichlet boundary conditions. Thus

$$v(x,t) = V(t) + \int_0^t V(\tau) \int_{\partial\Omega} \frac{\partial G(x,s,t-\tau)}{\partial\nu}\, ds d\tau, \tag{2.124}$$

and for any fixed $x \in \Omega$ the second term on the right-hand side of (2.124) is much smaller than the first as $t \to t^*$, due to the contribution of the Green's function term, and so $v(x,t) = V(t)(1 + o(1))$ as $t \to t^*$ for any $x \in \Omega$. Hence $N(t) \geq u(x,t) \geq V(t)(1 + o(1))$ as $t \to t^*$ for any $x \in \Omega$.

Now using the fact

$$\frac{dN(t)}{dt} \leq h(t)\, f(N) = \frac{dV(t)}{dt},$$

we obtain $N(t) \leq V(t) \leq N(t)(1+o(1))$ as $t \to t^*$ and thus it follows that $V(t) = N(t)(1+o(1))$ as $t \to t^*$ and consequently we deduce $u(x,t) = N(t)(1+o(1))$ as $t \to t^*-$ for any $x \in \Omega$ (uniform blow-up). □

Remark 2.2.5 Relation (2.110) is satisfied by $f(s) = e^{-s}$ as well as by $f(s) = 1/(1+s)^{1+k}$, for $k > 1$.

Remark 2.2.6 It should pointed out that the upper bound of blow-up time given by relation (2.123) is of the same form with the upper estimate obtained in Theorem 2.2.3, although the method used to derive (2.123) is independent of the spatial dimension.

In the case of a non-constant thermal conductivity $\kappa(u)$ some blow-up results not only for the Dirichlet case can be found in [19, 32].

The case $\lambda = \lambda^*$ seems to be critical since an infinite-time blow-up result holds for the one-dimensional case, e.g. $\Omega = (-1, 1)$. Indeed, we can construct an upper solution V to problem (2.106)–(2.108) which is global in time and unbounded.

Such an upper solution has the form

$$V(x,t) = w(y(x,t);\mu(t)), \quad \delta(t) \leq |x| \leq 1, \quad t > 0,$$
$$V(x,t) = M(t) = \max_{\delta(t)\leq|x|\leq 1} w(y(x,t);\mu(t)), \quad 0 \leq |x| < \delta(t), \quad t > 0,$$

where $0 \leq y(x,t) = \dfrac{|x| - \delta(t)}{1 - \delta(t)} \leq 1$, and $0 < \delta(t) < 1$ is a function can be chosen properly whereas $w(y;\mu(t))$ satisfies the following quasi steady-state problem

$$w_{yy} + \mu(t) f(w) = 0, \quad 0 < y < 1, \quad t > 0, \quad w'(0) = w(1) = 0,$$

with $\varepsilon(t) = 1 - \delta(t)$.

More precisely the following holds

Theorem 2.2.7 *Let $f(s)$ be a decreasing function satisfying relation (2.97) as well as the one of the following conditions*

$$\liminf_{s\to\infty} g(s) > c > 0$$

or

$$\liminf_{s\to\infty} g(s) = 0 \quad \text{and} \quad g(s) \geq \frac{\sqrt{2}\,(1+\alpha)}{\sqrt{\ln s}}, \ \alpha > 0 \ \ \text{for } s \gg 1,$$

where

$$g(s) = \frac{f(s)\sqrt{\mu(s)}}{\displaystyle\int_s^\infty f(\sigma)\,d\sigma}.$$

Then the solution $u^*(x,t)$ of problem (2.106)–(2.108) for $\lambda = \lambda^*$ is global-in-time. Moreover,

$$\lim_{t\to\infty} u^*(x,t) = \infty \quad \textit{for all} \ \in (-1,1),$$

and $u_x^*(\pm 1, t) \to \mp\infty$ as $t \to \infty$, i.e. u^* blows in infinite time (diverges) globally in x.

The proof of Theorem 2.2.7 is very lengthy and thus we omit it here, however it can be found in [27], where also a divergence result for the radial case is proven.

An analogous divergence result to Theorem 2.2.7 for the case of a non-constant thermal conductivity $\kappa(u)$ is given in [33].

Closing the current chapter we would like to specify that the following non-local parabolic equation

$$u_t = \nabla \cdot (\kappa(u)\nabla(u)) + \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^p}, \quad x \in \Omega, \quad t > 0, \tag{2.125}$$

where $p \neq 2$ can serve as a mathematical model for a variety of industrial processes. In particular, for $p \leq 1$ (2.125) can be used to describe the torsion test in metallurgy where finite-time blow-up is now associated with the phenomenon of shear banding formation, [6, 7].

For some finite-time blow-up results for Eq. (2.125) associated with different boundary conditions see [5, 7, 19, 25, 43] where a variety of methods are employed.

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