Chapter 2
Pareto Set Reduction Based on Elementary Information Quantum

The current chapter lays the foundation for the original axiomatic approach. First, we introduce the last (fourth) axiom on the invariance of preference relation. It is established that, within the accepted axiomatics, the DM's preference relation represents a cone relation with an acute convex cone. This feature allows employing the rich arsenal of convex analysis methods.

Next, we give the definition of an elementary information quantum about the unknown preference relation of the DM. The central result of the chapter is Theorem 2.5 that shows how the Pareto set can be reduced using an elementary information quantum.

In addition, different types of scales are discussed and the applicability of Theorem 2.5 to the multicriteria choice problems with criteria measured in arbitrary quantitative scales is justified.

2.1 Invariance Requirement for Preference Relation

2.1.1 Relations Invariant with Respect to Linear Positive Transformation

Recall the definition of an invariant relation given in Sect. 1.2. A binary relation $\Re$ defined on space R^m is called *invariant with respect to a linear positive transformation* if, for arbitrary vectors $y', y'' \in R^m$, any vector $c \in R^m$ and each positive number α, the relationship $y' \Re y''$ implies the relationship $(\alpha y' + c) \Re (\alpha y'' + c)$.

The inequality relations $>, \geqq, \geq$ defined on space R^m are the elementary examples of invariant binary relations. Obviously, a lexicographical relation (see Sect. 1.2) also belongs to the class of invariant binary relations.

In many application-oriented multicriteria choice problems, the preference relation $\succ$ can be considered invariant with respect to a linear positive

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transformation. Accordingly, let us supplement Axioms 1–3 by another one required for the development of a constructive mathematical theory.

Axiom 4 (preference relation invariance). *The preference relation $\succ$ is invariant with respect to a linear positive transformation.*

The invariance attributes of the relation $\succ$ are the properties of additivity and homogeneity. In other words, for any pair of vectors $y', y'' \in R^m$ such that $y' \succ y''$, the relationships $(y' + c) \succ (y'' + c)$ and $\alpha y' \succ \alpha y''$ hold for any vector $c \in R^m$ and any positive number α, respectively.

Lemma 2.1 *Owing to the transitivity and invariance of the preference relation $\succ$, the relationships $y \succ y'$ and $z \succ z'$ can be added termwise, i.e.,*

$$y \succ y', z \succ z' \Rightarrow y + z \succ y' + z'.$$

□ Add the vector z to both sides of the relationship $y \succ y'$. Using the additive property of the relation $\succ$, we obtain $y + z \succ y' + z$. The relationship $z \succ z'$ similarly implies $z + y' \succ z' + y'$. Now, taking advantage of the transitivity of the relation $\succ$, the relationships $y + z \succ y' + z$ and $z + y' \succ z' + y'$, we establish the desired result $y + z \succ y' + z'$. ■

2.1.2 Cone Relations

For further exposition, an important example of invariant binary relations is the class of cone relations. However, prior to the definition of a cone relation, we have to introduce some auxiliary notions of convex analysis.

A set A, $A \subset R^m$, is called *convex* if, together with any pair of points, it also contains the segment connecting them. In other words, a subset A of space R^m is convex if, for all pairs of points $y', y'' \in A$ and any number $\lambda \in [0, 1]$, we have the relationship $\lambda y' + (1 - \lambda) y'' \in A$. A set $K, K \subset R^m$, is called a *cone* if the inclusion $\alpha \cdot y \in K$ holds for each point $y \in K$ and any positive number α. A cone that is convex is called a *convex cone*. In other words, a convex set is a *convex cone* if, together with each point, it also contains the whole ray coming from the origin (generally, except the origin) to the given point. The origin (the *vertex* of a cone) may belong to this cone or not. As easily verified, the sum of any two (and more) elements of a convex cone belongs to this cone. A cone K is called *acute* if there exists no nonzero vector $y \in K$ satisfying $-y \in K$. A cone that is not acute contains at least one line passing through the origin (together with the origin or without it).

The set L of all solutions (vectors $x \in R^m$) of a homogeneous linear inequality $\langle c, x \rangle = c_1 x_1 + c_2 x_2 + \ldots + c_m x_m \geqq 0$, where c is a fixed nonzero vector from space R^m, represents some convex cone (known as the *closed half-space*).

□

Really, from $\langle c, x\rangle \geqq 0$ it follows that $\alpha\langle c, x\rangle = \langle c, \alpha x\rangle \geqq 0$ for any positive factor α. Hence, L is a cone. Make sure that this is a convex cone. To this end, take two arbitrary points x' and x'' of the cone L. They satisfy the inequalities $\langle c, x'\rangle \geqq 0$ and $\langle c, x''\rangle \geqq 0$. Multiply the first inequality by an arbitrary number $\lambda \in [0, 1]$ and the second one by $(1 - \lambda)$. The termwise addition of the resulting inequalities yields $\lambda\langle c, x'\rangle + (1 - \lambda)\langle c, x''\rangle = \langle c, \lambda x' + (1 - \lambda)x''\rangle \geqq 0$, which establishes the convexity of the cone L. ■

Note that the closed half-space is not an acute cone: together with the nonzero vector $\bar{x}$ satisfying the equality $\langle c, \bar{x}\rangle = 0$, it also contains the vector $-\bar{x}$, since multiplication by -1 does not violate the equality.

If a single linear homogeneous inequality is replaced by a finite system of such, then we get the system of linear homogeneous inequalities. This system also has a convex cone as the solutions set representing the intersection of a finite number of closed half-spaces (the so-called *polyhedral cone*). In the general case, this cone is not acute.

Consider a given collection of vectors $a^1, a^2, \ldots, a^p \in R^m$. It is easy to verify that the aggregate of all nonnegative linear combinations of these vectors (i.e., all vectors of the form $\lambda_1 a^1 + \lambda_2 a^2 + \ldots + \lambda_p a^p$ with nonnegative coefficients $\lambda_1, \lambda_2, \ldots, \lambda_p$) forms some convex *finitely generated* cone K in space R^m. In this case, we say that the collection of vectors $a^1, a^2, \ldots, a^p$ *generates* the convex cone K and write $K = \text{cone}\{a^1, a^2, \ldots, a^p\}$. The vertex belongs to this cone. According to duality theory in convex analysis (see [57], [62]), any finitely generated cone can be represented as the intersection of a finite number of closed half-spaces, i.e., being a polyhedral cone.

The vectors of a convex cone that are not representable as the linear combination of two other vectors of this cone with positive coefficients are called the *edges* or *generators* of the cone. As is well-known [57], any acute polyhedral cone not coinciding with the origin is generated by its edges.

If an acute polyhedral cone is the solution set for some system of linear homogeneous inequalities then the edges of this cone form a *fundamental system of solutions*. And arbitrary solution for this system of linear homogeneous inequalities can be represented as a linear combination of the fundamental system with non-negative coefficients. The fundamental system of solutions can be in principle obtained by exhaustion: just consider all possible subsystems of a definite number of the linear equations resulting from the original system of linear inequalities where inequality signs are replaced by equality ones.

The nonnegative orthant R^m_+ of space R^m, i.e.

$$R^m_+ = \{y \in R^m \mid y \geq 0_m\},$$

is a convex acute cone (without the vertex) generated by the unit vectors of this space. In the two-dimensional case ($m = 2$), this orthant has the form of the right angle coinciding with the first quarter (see Fig. 2.1). It is generated by the unit

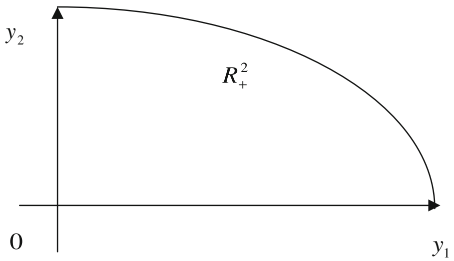

Fig. 2.1 The nonnegative orthant R^2_+

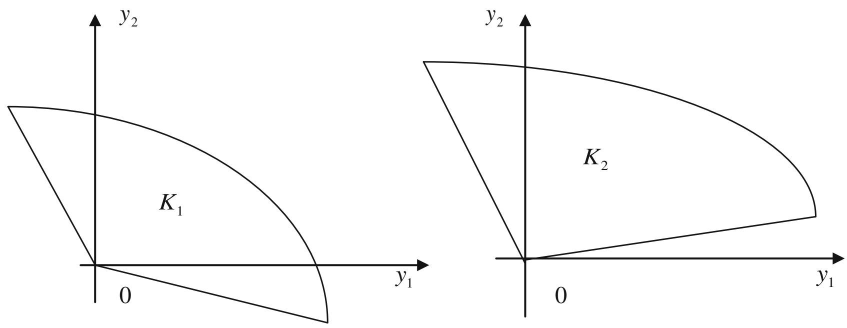

Fig. 2.2 Acute cones K_1 and K_2.

vectors $e^1 = (1,\ 0)$ and $e^2 = (0,\ 1)$, being the intersection of the right and upper closed half-planes (except the origin).

Other examples of acute plane cones, K_1 and K_2, can be observed in Fig. 2.2.

The upper half-plane represents a closed half-space, i.e., a convex cone that is not acute. The convex sets and cones are considered in more detail in [57, 62].

Definition 2.1 A binary relation $\Re$ defined on space R^m, i.e.$\Re \subset R^m \times R^m$, is called a *cone relation* if there exists a cone $K,\ K \subset R^m$, such that for any vectors $y'\ ,\ y'' \in R^m$ we have the equivalence

$$y' \Re y'' \Leftrightarrow y' - y'' \in K.$$

Often the right-hand side of the equivalence relationship is written in the form $y' \in y'' + K$ (see Fig. 2.3).

The inequality relations $>$ and $\geq$ considered on space R^m represent some cone relations with the cones $R^m_> = \{y \in R^m | y > 0_m\}$ and R^m_+, respectively.

It appears that any binary relation satisfying Axioms 2 and 4 is a cone relation. This follows from the result below.

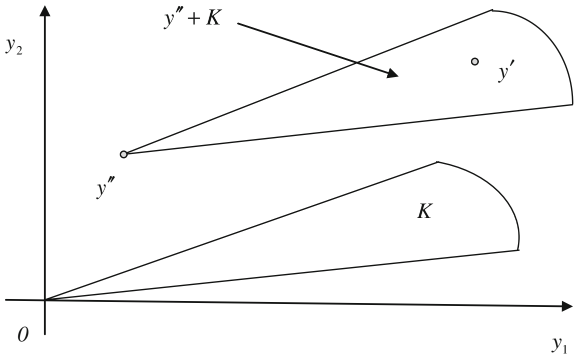

Fig. 2.3 Cone K and its translation $y'' + K$

Lemma 2.2 *Any binary relation $\Re$ defined on space R^m that has irreflexivity, transitivity and invariance with respect to a linear positive transformation is a cone relation with an acute convex cone not containing the origin. Conversely, each cone relation with the described cone represents a relation defined on R^m that has irreflexivity, transitivity and invariance with respect to a linear positive transformation.*

▢ Let $\Re$ be a binary relation defined on R^m that has irreflexivity, transitivity and invariance with respect to a linear positive transformation. Prove that $\Re$ represents a cone relation. To this end, introduce the set

$$K = \{y \in R^m |\ y \Re 0_m\}.$$

Owing to the homogeneity of the relation $\Re$, the set K is a cone. Moreover, for an arbitrary pair of vectors $y', y'' \in R^m$, by additivity we have

$$y' \Re y'' \Leftrightarrow (y' - y'') \Re 0_m \Leftrightarrow (y' - y'') \in K.$$

Therefore, the relation $\Re$ is actually a cone relation with the cone K. Now, it is necessary to verify that the cone K is convex, acute and does not contain the origin.

If $0_m \in K$, then $0_m \Re 0_m$ holds by the definition of the cone K. But this is inconsistent with the irreflexivity of the relation $\Re$. Hence, the cone K does not contain the origin.

To argue the convexity of the cone K, let us choose two arbitrary vectors $y', y'' \in K$ and a number $\alpha \in (0,\ 1)$(note that the values $\alpha = 1$ and $\alpha = 0$ can be omitted from further verification). Owing to the homogeneity of the relation $\Re$, the relationships $y' \Re 0_m$ and $y'' \Re 0_m$ imply $\alpha y' \Re 0_m$ and $(1 - \alpha)y'' \Re 0_m$, respectively. By additivity, the first relationship yields $(\alpha y' + (1 - \alpha)y'') \Re (1 - \alpha)y''$. Now, based on the transitivity of $\Re$, the second and the last relationships give $(\alpha y' + (1 - \alpha)y'') \Re 0_m$, or $(\alpha y' + (1 - \alpha)y'') \in K$, which establishes the convexity of the cone K.

To prove that the cone K is acute, conjecture the opposite. Let there exists a nonzero vector $y \in K$ satisfying the relationship $-y \in K$. For this vector, we have $y \Re 0_m$ and $-y \Re 0_m$. Hence, $(y - y) \Re (-y) \Re 0_m$ by the additive property of $\Re$. Owing to the latter's transitivity, this leads to the relationship $0_m \Re 0_m$ contradicting the irreflexivity of the relation $\Re$.

Now, prove the converse statement. Let $\Re$ be an arbitrary cone relation with an acute convex cone K not containing the origin.

Verify that this relation is irreflexive, transitive and invariant with respect to a linear positive transformation. First, this relation is actually irreflexive (otherwise, the cone K would contain the origin). Second, we verify its transitivity. To this end, select an arbitrary triplet of vectors $y', y'', y''' \in R^m$ satisfying the relationships $y' \Re y''$ and $y'' \Re y'''$. The last two relationships can be rewritten in the form $y' - y'' \in K$ and $y'' - y''' \in K$, whence it follows that there are two definite elements of the cone K. Since the sum of any two elements of a convex cone belongs to this cone, the last relationships yield $y' - y''' \in K$ or, equivalently, $y' \Re y'''$. This result testifies to the transitive property of the relation $\Re$.

And finally, the invariance of the relation $\Re$ follows from the relationships

$$y' \Re y'' \Leftrightarrow y' - y'' \in K \Leftrightarrow (y' + c) - (y'' + c) \in K \Leftrightarrow (y' + c) \Re (y'' + c),$$

$$y' \Re y'' \Leftrightarrow y' - y'' \in K \Leftrightarrow \alpha(y' - y'') \in K \Leftrightarrow \alpha y' - \alpha y'' \in K \Leftrightarrow \alpha y' \Re \alpha y'',$$

which hold for all vectors $c \in R^m$ and any positive number α. ■

Theorem 2.1 *Any binary relation* $\succ$ satisfying Axioms 2, 3 *and* 4 *is a cone relation with an acute convex cone containing the nonnegative orthant* R^m_+ *except the origin. Conversely, each cone relation with the described cone satisfies Axioms* 2, 3 *and* 4.

□ The binary relation $\succ$ satisfying Axioms 2–4 is irreflexive, transitive and invariant with respect to a linear positive transformation.

Necessity. Based on Lemma 2.2, it remains to show that the cone K of the binary relation $\succ$ includes the nonnegative orthant. By Lemma 1.3 from Sect. 1, the Pareto axiom (in terms of vectors) holds, i.e.,

$$y' \geq y'' \Rightarrow y' \succ y''.$$

Rewrite this axiom as the implication

$$y' - y'' \in R^m_+ \Rightarrow y' - y'' \in K.$$

The difference $y' - y''$ can be any vector of the nonnegative orthant R^m_+, and so the above implication means the inclusion $R^m_+ \subset K$.

Sufficiency. If a cone relation is generated by an acute convex cone (without the origin), then by Lemma 2.2 the corresponding cone relation is irreflexive, transitive and invariant with respect to a linear positive transformation (Axioms 2 and 4 are satisfied). On the other hand, this cone contains the nonnegative orthant R^m_+, and therefore the corresponding cone relation also satisfies the Pareto axiom. Obviously, the Pareto axiom implies Axiom 3, and the cone relation under consideration satisfies Axioms 2–4. ■

According to Theorem 2.1, the binary relations satisfying Axioms 2–4 (which are assumed true in the sequel) admit a simple geometrical interpretation. Namely, they represent cone relations with acute convex cones except the origin, and also these cones include the nonnegative orthant R^m_+.

Theorem 2.1 makes it possible to involve convex analysis results for Pareto set reduction.

2.2 Definition of Elementary Information Quantum

2.2.1 Original Multicriteria Choice Problem

The subsequent analysis is dedicated to the multicriteria choice problem that includes

- the set of feasible alternatives X,
- the vector criterion $f = (f_1, f_2, \ldots, f_m)$,
- the preference relation $\succ_X$.

Note that many aspects of this problem become simpler if stated and solved in terms of vectors. As mentioned in previous chapter, all results obtained in terms of alternatives can be easily reformulated in terms of vectors and vice versa. Therefore, further exposition will repeatedly address the *multicriteria choice problem in terms of vectors* that includes

- the set of feasible vectors Y, $Y \subset R^m$,
- the preference relation $\succ$ defined on space R^m.

Recall that the set of feasible vectors is defined by the equality

$$Y = f(X) = \{y \in R^m | y = f(x) \text{for some } x \in X\},$$

while the preference relation $\succ$ represents the extension to the whole space R^m of the preference relation $\succ_Y$ naturally connected to the preference relation $\succ_X$ defined on the set of feasible alternatives X.

Throughout the book below, we assume that Axioms 1–4 hold. Within these conditions (see Lemma 1.3), the Pareto axiom is true, which states that (in terms of vectors) any pair of vectors $y', y'' \in R^m$ such that $y' \geq y''$[1] satisfy the relationship $y' \succ y''$, i.e.,

$$y' \geq y'' \Rightarrow y' \succ y''. \tag{2.1}$$

Under the above assumptions, the DM can compare any two vectors $y',\ y''$ from the criterion space R^m using the irreflexive and transitive relation $\succ$. And one and only one of the following cases is realized then:

- $y' \succ y''$, i.e., y' is preferable to y'';
- $y'' \succ y'$, i.e., y'' is preferable to y';
- neither the relationship $y' \succ y''$ nor the relationship $y'' \succ y'$ holds.

2.2.2 *Elementary Information Quantum: Motivation*

Introduce the criteria index set

$$I = \{1, 2, \ldots, m\},$$

and consider the simplest choice problem with two vectors $y', y'' \in R^m$ and the minimum number of different components.

If the vectors y' and y'' have only one different component, e.g., $y'_i \neq y''_i$ and $y'_s = y''_s$ for all $s \in I \backslash \{i\}$, then the relationship $y' \geq y''$ or $y'' \geq y'$ holds. Hence, by Axiom 3, we have $y' \succ y''$ or $y'' \succ y'$, respectively. Therefore, in the elementary case considered, the choice from the two vectors is determined by Axiom 3.

Now, suppose that the vectors y' and y'' have two different components, i.e.,

$$y'_i \neq y''_i, y'_j \neq y''_j; \quad y'_s = y''_s \quad \text{for all}\, s \in I \backslash \{i, j\},$$

and the equalities $y'_i = y'_j,\ y''_i = y''_j$ do not hold simultaneously. Then one and only one of the following four cases is realized:

$$\begin{array}{llll} (1) \quad y'_i > y''_i, \quad y'_j > y''_j; & (2) \quad y''_i > y'_i, \quad y''_j > y'_j; \\ (3) \quad y'_i > y''_i, \quad y''_j > y'_j; & (4) \quad y''_i > y'_i, \quad y'_j > y''_j. \end{array}$$

Assume that the DM chooses one of these two vectors, i.e., either the relationship $y' \succ y''$ or the relationship $y'' \succ y'$ takes place. Without loss of generality owing to clear symmetry, we believe that the first relationship $y' \succ y''$ is true. And the following question arises immediately. How can the DM's choice be explained?

[1]Recall that the inequality $y' \geq y''$ means $y' \geqq y''$ and $y' \neq y''$.

If the first case from the four ones above is realized, the relationship $y' \succ y''$ results from the Pareto axiom. The second case is impossible: otherwise, by the Pareto axiom, we have the relationship $y'' \succ y'$ that is inconsistent with the relationship $y' \succ y''$ due to the asymmetry of $\succ$.

Now, examine the last two cases. As they are symmetric, it suffices to consider one of them, e.g., the third case. The inequality $y'_i > y''_i$ means that, in terms of criterion i, the vector y' is preferable to y'' for the DM. On the other hand, in terms of criterion j, the vector y'' is preferable to the vector y' since $y''_j > y'_j$. In the final analysis, we have two mutually contradicting conditions and the question is: why does the DM choose the vector y' between the vectors y' and y'' under the existing contradictions? What is the reason of such choice?

Apparently, the most rational explanation for this fact consists in the following. In the contradictory case, the DM is willing to compromise, losing in terms of criterion j for gaining in terms of more important criterion i.

2.2.3 Definition of Elementary Information Quantum

The above arguments applying to the simplest choice problem from an arbitrary pair of vectors motivate the following definition.

Definition 2.2 Let $i, j \in I, i \neq j$. We say that there is *an elementary information quantum about the DM's preference relation with given positive parameters* w_i^*, w_j^* if, for all vectors $y', y'' \in R^m$ such that

$$y'_i - y''_i = w_i^*, y''_j - y'_j = w_j^*, y'_s = y''_s \quad \text{for all } s \in I \backslash \{i, j\}, \qquad (2.2)$$

the relationship $y' \succ y''$ holds. Also in this case we say that *criterion* f_i *is more important than criterion* f_j *with parameters* w_i^*, w_j^*.

Remark 2.1 This definition is invariant with respect to the multiplication of the parameters by arbitrary positive number. More specifically, due to the homogeneity of the relation $\succ$, the specification of an elementary information quantum with parameters w_i^*, w_j^* actually generates a similar quantum with the parameters $\alpha \cdot w_i^*$, $\alpha \cdot w_j^*$ for any positive α.

Given an elementary information quantum, the DM that chooses from a pair of vectors (2.2) is willing to sacrifice the quantity w_j^* in terms of criterion f_j for gaining the quantity w_i^* in terms of criterion f_i (the values of all other criteria are fixed).

And the correlation between the quantities w_i^* and w_j^* gives a quantitative estimation for the degree of compromise. For instance, it is possible to consider the ratio w_j^*/w_i^* taking any positive values. However, a more convenient approach is to operate a normalized value from 0 to 1. Apply the transformation $y = x/(1 + x)$ to this ratio, arriving at the following notion.

Definition 2.3 Let $i,j \in I, i \neq j$, and there is an elementary information quantum with positive parameters w_i^* and w_j^*. In this case, the number

$$\theta_{ij} = \frac{w_j^*}{w_i^* + w_j^*} = \frac{1}{w_i^*/w_j^* + 1} \in (0,1)$$

will be called the DM's *coefficient* (or *degree*) *of compromise* for this pair of criteria.

This coefficient shows the share of loss in terms of criterion j the DM accepts against the sum of loss and gain in terms of criterion i. If the coefficient θ_{ij} is close to 1, then the DM incurs a sufficiently large loss in terms of criterion j for obtaining a relatively small gain in terms of criterion i. In this situation, the criterion i has a high importance relatively to the criterion j. Whenever this coefficient is close to 0, the DM is willing to lose in terms of criterion j only for gaining much in terms of the more important criterion. In other words, the degree of importance of criterion i against criterion j is relatively small; and this state of things corresponds to the small degree of compromise. If $\theta_{ij} = 1/2$, then the DM agrees with a definite gain in terms of a more importance criterion at the expense of loss in terms of a less important criterion provided that the loss coincides with the gain.

In addition, take notice that the value of θ_{ij} quantitatively depends on the type of scale used for criteria measurement. For details, we refer to Sect. 2.4.

2.2.4 Properties of Elementary Information Quantum

Let us explore the properties of an elementary information quantum.

Theorem 2.2 *If criterion f_i is more important than criterion f_j with given positive parameters w_i^*, w_j^* then criterion f_i is more important than criterion f_j with any pair of positive parameters w_i', w_j' satisfying the inequality $(w_i', -w_j') \geq (w_i^*, -w_j^*)$. In other words, if the DM's degree of compromise is $\theta_{ij} \in (0, 1)$, then this DM possesses any degree of compromise $\theta_{ij}' < \theta_{ij}$.*

▢ Choose arbitrarily two positive numbers $w_i', w_j', (w_i', -w_j') \geq (w_i^*, -w_j^*)$, and two vectors $y', y'' \in R^m$ such that

$$y_i' - y_i'' = w_i', y_j'' - y_j' = w_j', y_s' = y_s'' \quad \text{for all } s \in I \backslash \{i,j\}.$$

Prove that $y' \succ y''$.

Consider a vector $z \in R^m$ of the form

$$z_i = y_i'' + w_i^* = y_i' - w_i' + w_i^*,\ z_j = y_j'' - w_j^* = y_j' + w_j' - w_j^*,\ z_s' = y_s' \quad \text{for all } s \in I \backslash \{i,j\}.$$

Since $(w'_i, -w'_j) \geq (w^*_i, -w^*_j)$, we have $y' \geq z$. Hence, by the Pareto axiom, $y' \succ z$.

Recall that criterion f_i is more important than criterion f_j with the parameters w^*_i, w^*_j. Using this we get the relationship $z \succ y''$. Together with $y' \succ z$, it leads to the desired result $y' \succ y''$ owing to the transitive property of the relation $\succ$.

Now, prove the second part of the theorem. Let $\theta'_{ij} < \theta_{ij}$. By virtue of Remark 2.1, we may introduce the parameters

$$w'_i = 1 - \theta'_{ij}, w'_j = \theta'_{ij}; \quad w^*_i = 1 - \theta_{ij}, w^*_j = \theta_{ij}.$$

Obviously, for these parameters we have

$$\frac{w'_j}{w'_i + w'_j} = \theta'_{ij}, \quad \frac{w^*_j}{w^*_i + w^*_j} = \theta_{ij}$$

and, in addition, $(w'_i, -w'_j) > (w^*_i, -w^*_j)$.

In this case, using the first part of the theorem (see above), we establish the existence of an elementary information quantum with the parameters w'_i, w'_j, ergo with the degree of compromise θ'_{ij}. ■

The content of Theorem 2.2 well fits the intuitive idea of compromise. In particular, if the DM is willing to lose w^*_j in terms of criterion f_j for gaining w^*_i in terms of criterion f_i, then the DM obviously agrees with a smaller loss w'_j $(w'_j < w^*_j)$ as well as with a greater gain w'_i $(w'_i > w^*_i)$.

Based on the definition of an elementary information quantum and Theorem 2.2, let us analyze the possible cases for an arbitrary pair of different criteria f_i, f_j.

In fact, one and only one of the three cases are possible as follows:

1. At least one positive number from the interval $(0, 1)$ represents the degree of compromise for criteria f_i and f_j, and at least one number does not;
2. None of the positive numbers from the interval $(0, 1)$ is the degree of compromise for criteria f_i, f_j. In this case, we shall say that *criterion f_i is not more important than criterion f_j*;
3. Any positive number from the interval $(0, 1)$ is the degree of compromise for criteria f_i, f_j. In this case, we shall say that *criterion f_i is incomparably more important than criterion f_j*.

Let us investigate the first case. If at least one number $\theta_{ij} \in (0, 1)$ is the degree of compromise, then by Theorem 2.2 any smaller number within this interval is also the degree of compromise for the pair of criteria under consideration. Construct two disjoint sets A and B using the following procedure. Add on to the former set all numbers from the interval $(0, 1)$ that are the degrees of compromise for this pair of criteria; naturally, $A \neq \emptyset$. The latter set B comprises all numbers from the interval that are not the degrees of compromise; by the data, $B \neq \emptyset$. Clearly, the described procedure yields $A \cup B = (0,\, 1)$, and the inequality $a < b$ holds for all $a \in A,\ b \in B$. This means that the sets A and B define a section of the interval $(0,\, 1)$. By the Dedekind principle, there exists a unique number $\bar{\theta}_{ij} \in (0,\, 1)$ implementing this section, further called the *limit degree of compromise*.

Note that the number $\bar{\theta}_{ij}$ may be the degree of compromise or not. In other words, either $\bar{\theta}_{ij} \in A$ or $\bar{\theta}_{ij} \notin A$ holds.

2.2.5 Connection to Lexicographic Relation

The preference relation $\succ$ satisfying Axioms 2–4 and the lexicographic[2] relation have a certain connection revealed by the next statement in terms of an ordered collection of incomparably more important criteria.

Theorem 2.3 *The irreflexive, transitive and invariant relation $\succ$ defined on space R^m is a lexicographic relation if and only if criterion f_1 is incomparably more important than criterion f_2, criterion f_2 is incomparably more important than criterion f_3,..., criterion f_{m-1} is incomparably more important than criterion f_m.*

▢ Necessity. Let the relation $\succ$ be lexicographic. In this case, for arbitrary vectors $y',\, y'' \in R^m$, we have the logical propositions

$$
\begin{array}{ll}
(1) & y'_1 > y''_1 \quad \Rightarrow \quad y' \succ y'', \\
(2) & y'_1 = y''_1, y'_2 > y''_2 \quad \Rightarrow \quad y' \succ y'', \\
(3) & y'_1 = y''_1, y'_2 = y''_2, y'_3 > y''_3 \quad \Rightarrow \quad y' \succ y'' \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m) & y'_i = y''_i, i = 1, 2, \ldots, m-1; \quad y'_m > y''_m \quad \Rightarrow \quad y' \succ y''.
\end{array}
$$

The first proposition implies the relationship $y' \succ y''$ for two arbitrary vectors $y',\, y'' \in R^m$ satisfying $y'_1 > y''_1, y'_2 < y''_2, y'_3 = y''_3, \ldots, y'_m = y''_m$. This means that criterion f_1 is incomparably more important than criterion f_2.

Similarly, using the second proposition, we conclude that criterion f_2 is incomparably more important than criterion f_3, and so on; in the final analysis, the incomparably higher importance of criterion f_{m-1} against criterion f_m follows from the $(m-1)$-th proposition.

Sufficiency.[3] For each $i = 1, 2, \ldots, m-1$, let criterion f_i be incomparably more important than criterion f_{i+1}. Prove that the relation $\succ$ is lexicographic.

[2]The definition of a lexicographic relation can be found in Sect. 1.2.

[3]The proof is suggested by O.V. Baskov.

Consider two arbitrary vectors y', $y'' \in R^m$. If they coincide, then none of them is lexicographically greater than the other, which agrees with the definition of a lexicographic relation.

Let $y' \neq y''$. Denote by i the minimum index such that $y'_i \neq y''_i$. Without loss of generality, assume that $y'_1 < y''_1$. It is required to show that $y'' \succ y'$. The proof has the form of an algorithm with the following steps.

Step 1. Compare the numbers y'_m and y''_m. If $y'_m = y''_m$, proceed to Step 2 by setting $z^1 = y'$. If $y'_m < y''_m$, introduce the vector $z^1 = (y'_1, \ldots, y'_{m-1}, y''_m)$, which satisfies the relationship $z^1 \succ y'$ due to the compatibility axiom. Then pass to Step 2.
If $y'_m > y''_m$, fix an arbitrary $\alpha > y'_{m-1}$ and introduce the vector $z^1 = (y'_1, \ldots, y'_{m-2}, \alpha, y''_m)$. Since the criterion f_{m-1} is incomparably more important than the criterion f_m, we obtain $z^1 \succ y'$. Next, move to Step 2.

Step 2. By analogy, continue the comparison of z^1_{m-1} and y''_{m-1}. And so on.

Step k+1. At this step, we have $z^k_j = y''_j$, $j = i+2, \ldots, m$. Compare z^k_{i+1} and y''_{i+1}. If $z^k_{i+1} = y''_{i+1}$, then the compatibility axiom dictates that $y'' \succ z^k$. In the case $z^k_{i+1} < y''_{i+1}$, we get $y'' \geq z^k$. Owing to the Pareto axiom, hence it appears that $y'' \succ z^k$. If $z^k_{i+1} > y''_{i+1}$, then $y'_i = z^k_i < y''_i$ and the incomparably higher importance of the criterion f_i against the criterion f_{i+1} again give $y'' \succ z^k$.

As a result, we arrive at the chain of relationships $y'' \succ z^k \succ \ldots z^1 \succ y'$ where (at some but not all positions) the preference symbol $\succ$ can be replaced by the equality sign. In combination with the transitivity of the preference relation, this leads to the desired relationship $y'' \succ y'$. ■

2.3 Pareto Set Reduction Using Elementary Information Quantum

2.3.1 Simplification of Basic Definition

Definition 2.2 reveals the whole essence of an elementary information quantum about the DM's preference relation. This definition involves two numerical parameters used to measure the degree of compromise.

To verify that criterion f_i is more important than criterion f_j, by Definition 2.2 we have to compare infinitely many pairs of vectors y', $y'' \in R^m$ such that

$$y'_i - y''_i = w^*_i > 0, y''_j - y'_j = w^*_j > 0, y'_s = y''_s \quad \text{for all } s \in I \backslash \{i,j\}. \tag{2.3}$$

And if for any pair above, the first vector y' every time appears preferable to the second one y'', then by Definition 2.2 there is a given elementary information quantum with the corresponding parameters.

It is absolutely clear that such verification appears non-implementable in practice due to infinitely many pairs of vectors for comparison. Actually, this verification is not required, as the preference relation possesses invariance. The whole procedure can be reduced to comparing merely a pair of vectors y', y'' satisfying (2.3). The following result gives the details.

Theorem 2.4 *In Definition 2.2, the vectors y', y'' can be assumed fixed. Particularly,*

$$y'_i = w^*_i, y'_j = -w^*_j \text{ and } y'_s = 0 \quad \text{for all } s \in I \backslash \{i,j\}, y'' = 0_m, \tag{2.4}$$

or

$$y'_i = 1 - \theta_{ij},\ y'_j = -\theta_{ij} \text{ and } y'_s = 0 \quad \text{for all } s \in I \backslash \{i,j\}, y'' = 0_m, \tag{2.4'}$$

where θ_{ij} is the degree of compromise.

▢ Consider two arbitrary vectors y' and y'' satisfying (2.3). Obviously,

$$y'_i > y''_i \Leftrightarrow y'_i - y''_i > 0,$$
$$y''_j > y'_j \Leftrightarrow y''_j - y'_j > 0.$$

Denote $\bar{y}_i = y'_i - y''_i = w^*_i$, $\bar{y}_j = y'_j - y''_j = -w^*_j$, where $\bar{y}_s = 0$ for all $s \in I \backslash \{i,j\}$. By the additivity of the preference relation $\succ$, we have

$$y' \succ y'' \Leftrightarrow (y' - y'') \succ 0_m \Leftrightarrow \bar{y} \succ 0_m,$$

where the vector $\bar{y}$ has only two nonzero components, namely, components i and j being $\bar{y}_i$ and $\bar{y}_j$, respectively. This means that Definition 2.2 in the general form is equivalent to itself in the "simplified" form with the fixed vectors $y' = \bar{y}$ and $y'' = 0_m$.

Hence, in Definition 2.2 the vectors y', y'' can be assumed fixed.

Now, we prove the remainder of Theorem 2.4. The relationship $\bar{y} \succ 0_m$ for the above vector $\bar{y}$ is equivalent to the relationship $\alpha \bar{y} \succ 0_m$ with any positive number α by the homogeneity of the preference relation $\succ$. Choosing $\alpha = -\theta_{ij}/\bar{y}_j$ and setting $\hat{y} = \alpha \bar{y}$ yield

$$\hat{y}_i = \alpha\bar{y}_i = -\frac{\theta_{ij}\bar{y}_i}{\bar{y}_j} = \frac{\theta_{ij}w_i^*}{w_j^*} = \frac{w_i^*}{w_i^* + w_j^*} = 1 - \theta_{ij},$$
$$\hat{y}_j = \alpha\bar{y}_j = -\frac{\theta_{ij}\bar{y}_j}{\bar{y}_j} = -\theta_{ij},$$
$$\hat{y}_s = \alpha\bar{y}_s = \alpha 0 = 0 \quad \text{for all } s \in I\backslash\{i,j\}.$$

Therefore, the relationship $\bar{y} \succ 0_m$ is equivalent to the relationship $\hat{y} \succ 0_m$, where the vector $\hat{y}$ has the same components

$$\hat{y}_i = 1 - \theta_{ij}, \hat{y}_j = -\theta_{ij}; \hat{y}_s = 0 \quad \text{for all } s \in I\backslash\{i,j\},$$

as the vector y' from (2.4′). ■

According to the aforesaid, the preference relation $\succ$ is supposed invariant with respect to a linear positive transformation. Using Theorem 2.4, we introduce a new (simplified) definition of an elementary information quantum.

Definition 2.4. Let $i,j \in I$, $i \neq j$. We say that there is a *given elementary information quantum with positive parameters* w_i^*, w_j^* (*with the degree of compromise* $\theta_{ij} \in (0,1)$) if the relationship $y' \succ 0_m$ holds for the vector $y' \in R^m$ of form (2.4) (form (2.4′), respectively).

To verify that criterion f_i is more important than criterion f_j with the degree of compromise $\theta_{ij} \in (0,1)$, by Definition 2.4 it suffices to check that the vector y' of form (2.4) is preferable to the zero vector, i.e. $y' \succ 0_m$. For instance, if the vector $(0.7, -0.3, 0)$ appears preferable to $(0,\ 0,\ 0)$ for the DM, then the first criterion is more important for the DM than the second criterion with the degree of compromise $\theta_{12} = 0.3$.

2.3.2 *Pareto Set Reduction Based on Elementary Information Quantum*

The next result shows how the available information about the preference relation in the form of an elementary quantum can be used for reducing the search space of selectable vectors.

Theorem 2.5 (in terms of vectors). *Assume that there exists an elementary information quantum with positive parameters* w_i^* *and* w_j^* (*with the degree of compromise* $\theta_{ij} \in (0,1)$). *Then for any set of selectable vectors* $C(Y)$ *we have*

$$C(Y) \subset \hat{P}(Y) \subset P(Y), \tag{2.5}$$

where $\hat{P}(Y)$ *is the set of feasible vectors corresponding to the set of Pareto optimal alternatives in the multicriteria problem with the initial set of feasible* *alternatives* X *and the "new" vector criterion* $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_m)$ (*i.e.*, $\hat{P}(Y) = f(P_{\hat{f}}(X))$) *with the components calculated by*

$$\hat{f}_j = w_j^* f_i + w_i^* f_j, \hat{f}_s = f_s \quad \text{for all } s \in I \backslash \{j\}, \tag{2.6}$$

or

$$\hat{f}_j = \theta_{ij} f_i + (1 - \theta_{ij}) f_j, \hat{f}_s = f_s \quad \text{for all } s \in I \backslash \{j\}. \tag{2.6'}$$

▢ The proof consists of four parts.

I. Denote by K the acute convex cone (without the origin) of the cone preference relation $\succ$. By the hypothesis of Theorem 2.5 and Definition 2.4, the vector y' described by equalities (2.4) satisfies the relationship $y' \succ 0_m$. The latter is equivalent to the inclusion $y' \in K$.

Consider the collection of unit vectors $e^1, e^2, \ldots, e^m$ of space R^m; here component s of the vector e^s is 1 and the other components are 0. Let M be the convex cone (without the origin) generated by the collection of linear independent[4] vectors

$$e^1, \ldots, e^{i-1}, y', e^{i+1}, \ldots, e^m. \tag{2.7}$$

The cone M coincides with the set of all vectors representable as the linear combinations

$$\lambda_1 e^1 + \ldots + \lambda_{i-1} e^{i-1} + \lambda_i y' + \lambda_{i+1} e^{i+1} + \ldots + \lambda_m e^m$$

of the vectors from collection (2.7) with the nonnegative coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ that are not zero simultaneously.

Check that the cone M is acute. If not, there exists a nonzero vector $y \in M$ such that $-y \in M$. According to the aforesaid,

$$y = \lambda_1 e^1 + \ldots + \lambda_{i-1} e^{i-1} + \lambda_i y' + \lambda_{i+1} e^{i+1} + \ldots + \lambda_m e^m,$$
$$-y = \lambda'_1 e^1 + \ldots + \lambda'_{i-1} e^{i-1} + \lambda'_i y' + \lambda'_{i+1} e^{i+1} + \ldots + \lambda'_m e^m,$$

where all coefficients of the linear combinations are nonnegative and each of the collections $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\lambda'_1, \lambda'_2, \ldots, \lambda'_m$ is not zero simultaneously. The sum of

[4]Indeed, vectors (2.7) form a linear independent system, since the matrix composed of them has rank m.

two elements of a cone belongs to this cone; hence, by summing up the last two equalities, we obtain

$$0_m = (\lambda_1 + \lambda_1')e^1 + \ldots + (\lambda_{i-1} + \lambda_{i-1}'')e^{i-1} + (\lambda_i + \lambda_i')y' + (\lambda_{i+1} + \lambda_{i+1}')e^{i+1} + \ldots + (\lambda_m + \lambda_m')e^m,$$

where at least one coefficient of the linear combination in parentheses is nonzero. However, owing to the linear independence of vectors (2.7), the last equality implies that all coefficients of the linear combination are zero. This contradiction to the initial hypothesis testifies that the cone M is acute.

II. Now, demonstrate that the cone M coincides with the set of all nonzero solutions to the following system of linear homogeneous inequalities:

$$\begin{aligned} y_s &\geqq 0 \qquad \text{for all } s \in I \backslash \{j\}, \\ w_j^* y_i + w_i^* y_j &\geqq 0. \end{aligned} \tag{2.8}$$

To this end, find the fundamental system of solutions for the system of inequalities (2.8) and make sure that it coincides with collection (2.7).

For obtaining the fundamental system of solutions for the system of inequalities (2.8), consider the corresponding collection of linear equations

$$\begin{aligned} & y_s = 0 \quad \text{for all } s \in I \backslash \{j\}, \\ & w_j^* y_i + w_i^* y_j = 0, \end{aligned} \tag{2.9}$$

which can be rewritten as[5]

$$\begin{aligned} & \langle e^s, y \rangle = 0 \quad \text{for all } s \in I \backslash \{j\}, \\ & \langle \tilde{y}, y \rangle = 0, \end{aligned} \tag{2.10}$$

where $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m)$ and

$$\tilde{y}_i = w_j^*, \tilde{y}_j = w_i^*, \tilde{y}_s = 0 \quad \text{for all } s \in I \backslash \{i, j\}.$$

The number of equations in (2.10) is m. An arbitrary collection of $m-1$ vectors obtained from $e^1, \ldots, e^{j-1}, \tilde{y}, e^{j+1}, \ldots, e^m$ by removing a single vector appears linearly independent. And so, to find the fundamental system of solutions for the system of inequalities (2.8), it suffices to go over the nonzero solutions to each subsystem constructed from $m-1$ equalities of the original system (2.10). Among them, one should choose the vectors satisfying the system of inequalities (2.8).

[5]Recall that, for m-dimensional vectors a and b, the notation $\langle a, b \rangle$ gives their *scalar product*: $\langle a, b \rangle = \sum_{i=1}^{m} a_i b_i$.

We will sequentially eliminate one equation from system (2.10), seeking for the nonzero solutions to the resulting "truncated" system. With the last equation eliminated from (2.10), e.g., the vector e^j is a nonzero solution to the "truncated" system. After elimination of the equation $\langle e^s, y\rangle = 0$ (where $s \neq i$), the vector e^s can be chosen as a nonzero solution to the "truncated" system. As easily verified, the "truncated" system without the equation $\langle e^i, y\rangle = 0$ has the nonzero solution y'. This procedure yields the fundamental system of solutions $e^1, \ldots, e^{i-1}, y', e^{i+1}, \ldots, e^m$ to the system of inequalities (2.8). The fundamental system coincides with the vector collection (2.7) generating the cone M of the cone preference relation $\succ$. Therefore, the cone M represents the set of nonzero solutions to the system of linear inequalities (2.8).

III. As mentioned in the beginning of the proof, the inclusion $y' \in K$ takes place. By Theorem 2.1, we have the relationship $R^m_+ \subset K$. The cone R^m_+ is generated by the collection of unit vectors $e^1, e^2, \ldots, e^m$. Since K represents a convex cone, together with vectors (2.7) it surely contains all nonzero linear combinations of vectors (2.7) with nonnegative coefficients, i.e., $M \subset K$. Finally, we get the inclusions

$$R^m_+ \subset M \subset K,$$

yielding

$$\text{Ndom } Y \subset \hat{P}(Y) \subset P(Y), \tag{2.11}$$

where

$$P(\hat{Y}) = \{y^* \in Y |\text{ there exists no } y \in Y \text{ such that } y - y^* \in M\}$$

is the set of nondominated elements of the set Y with respect to the cone relation with the cone M.

IV. Choose arbitrarily two elements $x, x^* \in X, y = f(x), y^* = f(x^*)$ such that $f(x) \neq f(x^*)$. As shown in part II, the cone M coincides with the set of nonzero solutions to the system of linear inequalities (2.8), and hence the inclusion $f(x) - f(x^*) \in M$ takes place if and only if the vector $y = f(x) - f(x^*)$ is a nonzero solution to (2.8), i.e.,

$$\begin{pmatrix} f_1(x) - f_1(x^*) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_{j-1}(x) - f_{j-1}(x^*) \\ w_j^*(f_i(x) - f_i(x^*)) + w_i^*(f_j(x) - f_j(x^*)) \\ f_{j+1}(x) - f_{j+1}(x^*) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_m(x) - f_m(x^*) \end{pmatrix} \geq 0_m.$$

The last inequality can be rewritten in the compact form $\hat{f}(x) - \hat{f}(x^*) \in R^m_+$, or $\hat{f}(x) \geq \hat{f}(x^*)$ where $\hat{f}$ is defined by (2.6). And therefore the relationship $y - y^* \in M$ for the vectors $y = f(x), y^* = f(x^*)$ is equivalent to the inequality $\hat{f}(x) \geq \hat{f}(x^*)$. Subsequently, $\hat{P}(Y) = f(P_{\hat{f}}(X))$.

By the hypothesis of the current theorem and Lemma 1.2, we have the inclusion $C(Y) \subset \text{Ndom}\ Y$ for arbitrary set $C(Y)$. And so, inclusions (2.11) lead to inclusions (2.5), which were to be established.

The vector criterion (2.6′) is obtained from (2.6) by dividing component j of the latter by the positive number $w_i^* + w_j^*$. Such an operation clearly do not modify the Pareto set $\hat{P}(Y)$. ■

According to the Edgeworth-Pareto principle, all selectable vectors must belong to the Pareto set. If the multicriteria choice problem includes additional information about the DM's willingness to compromise while comparing the values of two certain criteria, then Theorem 2.5 serves for Pareto set reduction based on this information without losing any selectable vectors. In other words, some vectors can be eliminated from the Pareto set, since they would not be selected for sure.

For justice' sake, we have to note the following. In definite cases (especially if the degree of compromise is close to 0, viz. the criteria f_j and $\hat{f}_j$ almost coincide), the reduction may fail due to the identical Pareto sets in terms of the "old" and "new" vector criteria, i.e. $\hat{P}(Y) = P(Y)$. One can say that in such cases the available information about the preference relation is not rich in content.

Theorem 2.4 acquires the following form in terms of alternatives.

Theorem 2.6 (in terms of alternatives). *Assume that criterion f_i is more important than criterion f_j with given positive parameters w_i^*, w_j^* (with the degree of compromise $\theta_{ij} \in (0,\ 1)$). Then for any set of selectable alternatives $C(X)$ we have*

$$C(X) \subset P_{\hat{f}}(X) \subset P_f(X), \tag{2.12}$$

where $P_{\hat{f}}(X)$ is the set of Pareto optimal alternatives in the multicriteria problem with the set of feasible alternatives X and the "new" vector criterion $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_m)$ with the components calculated by formulas (2.6) *or* (2.6′).

Figure 2.4 illustrates the inclusions (2.12).

Commenting on Theorem 2.5, first of all we emphasize its universalism. Namely, there exist no requirements to the set of feasible alternatives X and the

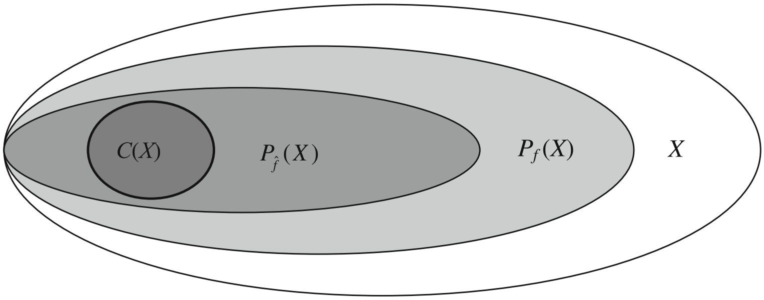

Fig. 2.4 Nested sets

vector criterion f. This theorem is hence applicable to any multicriteria choice problem satisfying Axioms 1–4. And the set of feasible alternatives (and vectors) may be finite or infinite, while the functions $f_1, f_2, \ldots, f_m$ may belong to an arbitrary class (being nonlinear, nonconvex, nonconcave or discontinuous). The only constraint in the conditions of Theorem 2.5 concerns the DM's behavior: during the choice process, the DM must act "reasonably" in the sense that its preference relation necessarily meets Axioms 1–4. Second, the "new" criterion $\hat{f}$ is recalculated using the "old" one f by a very simple formula, see (2.6). According to it, the "new" vector criterion is obtained from the "old" counterpart by replacing the less important criterion f_j for the positive linear combination of the criteria f_i and f_j with the parameters w_i^*, w_j^*. The other "old" criteria remain the same. As easily seen, this "recalculation" of criterion j does not affect many fruitful optimization-oriented properties of the criteria f_i and f_j. For instance, if these criteria are continuous, concave, convex or linear, the new criterion $\hat{f}_j$ inherits the same properties.

The simplest recalculation formula appears in the case of linear criteria. We state the corresponding result below.

Corollary 2.1 *In addition to the hypothesis of Theorem 2.5, let $X \subset R^n$ and let the criteria f_i and f_j be linear, i.e.,*

$$f_k(x) = \langle c^k, x \rangle = \sum_{l=1}^{n} c_l^k x_l, \quad k = i, j,$$

where $c^k = (c_1^k, c_2^k, \ldots, c_n^k)$. Then the new criterion j has the form $\hat{f}_j(x) = \langle \hat{c}, x \rangle$ with $\hat{c} = w_j^* c^i + w_i^* c^j$, or

$$\hat{c} = \theta_{ij} c^i + (1 - \theta_{ij}) c^j. \tag{2.13}$$

This result immediately follows from formula (2.6) and the linear property of the scalar product of vectors.

Equality (2.13) admits a clear interpretation if the set of feasible alternatives is a subspace in the two-dimensional vector space, i.e., $X \subset R^2$ (see Fig. 2.5).

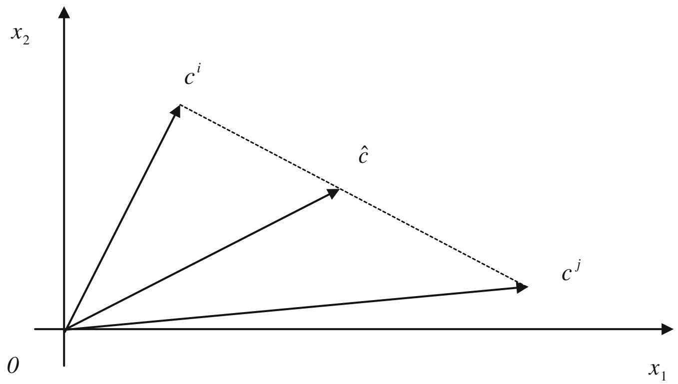

Fig. 2.5 Vectors c^i, c^j and $\hat{c}$

The closer is the degree of compromise θ_{ij} to 0, the closer is the end of the vector $\hat{c}$ to that of the vector c^j. As we increase θ_{ij} within the interval (0, 1), the vector "attracts" the vector $\hat{c}$ associated with the new criterion j. In the case $\theta_{ij} = 0.5$, the end of the vector $\hat{c}$ is in the middle of the line segment connecting the ends of the two vectors c^i and c^j. If the degree of compromise is close to 1, then the vector $\hat{c}$ slightly differs from c^i, which means that the vector criterion $\hat{f}$ includes two almost identical criteria. And the impact of the less important criterion f_j associated with the vector c^j on the solution of the multicriteria choice problem becomes negligible.

2.3.3 Geometrical Aspects

As a rule, the preference relation $\succ$ guiding the DM choice process is not completely defined (i.e., fragmentary) in the multicriteria choice problems. Throughout this book, we assume that it merely satisfies Axioms 1–4. Under these conditions, by Theorem 2.1 the preference relation $\succ$ is a cone relation with an (unknown) acute convex cone K except the origin. Furthermore, the cone K contains the nonnegative orthant, i.e., $R^m_+ \subset K$. This gives the inclusion Ndom $Y \subset P(Y)$, in combination with $C(Y) \subset$ Ndom Y yielding

$$C(Y) \subset P(Y). \tag{2.14}$$

The last inclusion expresses the Edgeworth-Pareto principle, which states that the choice should be performed within the Pareto set. As mentioned in Sect. 1.4, this principle is applicable to any multicriteria choice problem satisfying Axioms 1–3. Here is an alternative formulation of the principle: *the Pareto set represents an upper estimate for the set of selectable vectors.*

Now, suppose that (besides Axioms 1–4 satisfied by the multicriteria choice problem) we have additional information that criterion f_i is more important than criterion f_j with the degree of compromise $\theta_{ij} \in (0, 1)$. In geometrical terms, the existence of such information means the specification of a vector $y' \in R^m$ of form (2.4) with the inclusion $y' \in K$. Consequently, the cone K contains not only the nonnegative orthant, but also the vector y' beyond this orthant.

Consider the cone M coinciding with the set of all nonzero nonnegative linear combinations of the vectors $e^1, \ldots, e^{i-1}, y', e^{i+1}, \ldots, e^m$, see the proof of Theorem 2.5. In the course of this proof, we have also established the inclusions $R^m_+ \subset M \subset K$, where $M \neq R^m_+$. These inclusions imply

$$C(Y) \subset \text{Ndom}\ Y \subset \text{Ndom}_M\ Y \subset P(Y),$$

where

$$\text{Ndom}\ Y = \{y^* \in Y |\ \text{there exists no}\ y \in Y\ \text{such that}\ y - y^* \in K\},$$

$$\text{Ndom}_M Y = \{y^* \in Y |\ \text{there exists no}\ y \in Y\ \text{such that}\ y - y^* \in M\},$$

$$P(Y) = \{y^* \in Y |\ \text{there exists no}\ y \in Y\ \text{such that}\ y - y^* \in R^m_+\}.$$

Hence, the upper estimate (2.14) for the unknown set of selectable vectors is refined to

$$C(Y) \subset \text{Ndom}_M\ Y.$$

Note that, the wider is the cone M in comparison with the nonnegative orthant R^m_+, the narrower is the set $\text{Ndom}_M Y$ in comparison with $P(Y)$.

Thus, using an elementary information quantum, one can extract in the unknown cone K a cone M wider than R^m_+ (see Fig. 2.6), thereby constructing a more precise upper estimate for the set of selectable vectors as against the estimate yielded the Edgeworth-Pareto principle.

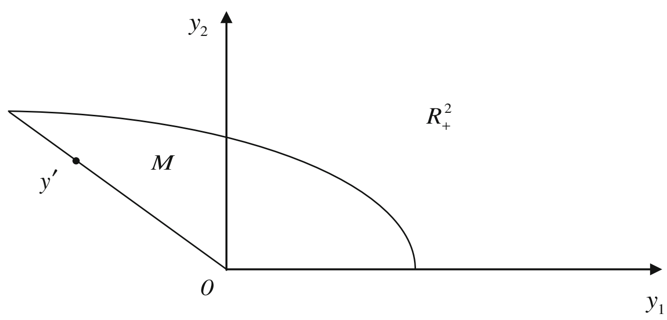

Fig. 2.6 Cones M and R^2_+.

Example 2.1 Let $m = 2$ and $Y = \{y^1, y^2, y^3\}$, where

$$y^1 = (4,1), y^2 = (3,2), y^3 = (1,3).$$

Here all the three feasible vectors are Pareto optimal. In other words, the Edgeworth-Pareto principle does not assist in reducing the search space of selectable vectors.

Imagine that the first criterion is more important than the second one with the degree of compromise 0.5. Geometrically, this means that $y' = (0.5, -0.5) \in K$.

Figure 2.7 shows the three feasible vectors and the cone M translated into the points corresponding to the second and third feasible vectors.

Clearly, neither the second nor third vector can be selected, as both have dominating vectors:

$$y^2 \in y^3 + M, y^1 \in y^2 + M.$$

And the only selectable vector is hence the first one y^1. In other words, if the set of selectable vectors is non-empty in this problem, then it consists of the first vector only.

The same conclusion can be drawn using Theorem 2.5. Really, by formula (2.6) the new criterion 2 acquires the form $0.5y_1 + 0.5y_2$ and, as easily found,

$$\hat{f}(X) = \{(4, 2.5), (3, 2.5), (1, 2)\}.$$

In this set, the first vector is the only Pareto optimal one; it corresponds to the vector y^1. Therefore, this (and only this vector) can appear selectable from Y if the selectable vectors exist.

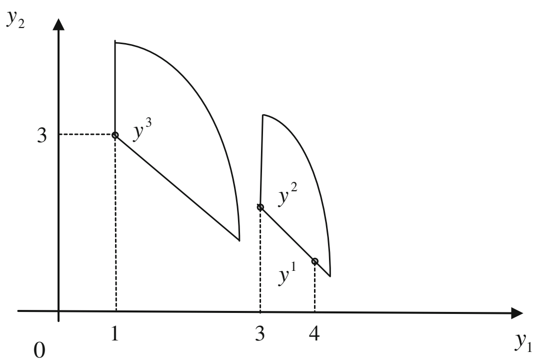

Fig. 2.7 The dominated vectors y^2 and y^3

2.4 Scales of Criteria and Invariance of Measurements

2.4.1 Quantitative and Qualitative Scales

As mentioned earlier, all criteria $f_1, f_2, \ldots, f_m$ in the multicriteria choice problem statement have numerical values. Therefore, the inclusion $y_i = f_i(x) \in R$ holds for any $x \in X$ and each $i = 1, 2, \ldots, m$. This information about the criteria is enough for the rigorous mathematical formulation of the multicriteria choice problem.

However, in real applications the numerical values of the criteria are the measurement results in a certain scale. For instance, if a criterion reflects the value, cost or profit of a project, these quantities can be expressed in RUB, USD, EURO or other monetary units. The lengths of different objects are measured in meters, inches, foots, yards, and so on. Hours, seconds, years, millions of years, etc. are used for time intervals. Consequently, in specific applications the values of criteria are associated with a certain scale, being expressed in definite units of measure.

There exist various measurement scales. Whenever it is required to count the number of objects, people, items, etc., one adopts the so-called *absolute scale*. This scale has a fixed reference point (0) and a fixed spacing (1). Two individuals performing independent measurements of same quantities in the absolute scale (two measurers) must obtain the identical results. In addition, note that this scale has a unique unit of measure for all measurers.

Different units of measure are used for measuring the physical characteristic of mass. As is well-known, the mass of an object can be expressed in kilograms, pounds, tones, poods, etc. Here only the reference point (0) is fixed for all measurers, which corresponds to the absence of mass; and the scale spacing may vary for measurers. Thereby, for a same object, the measurements results y_i' and y_i'' obtained by two measurers in different units of measure differ by some fixed positive factor α_i, i.e., $y_i' = \alpha_i y_i''$. In this case, the measurement results are defined within the transformation $\phi_i(y_i) = \alpha_i y_i$, $\alpha_i > 0$. Such a scale is called the *ratio scale*, which can be explained as follows. Regardless of the unit of measure, the measurements in this scale yield the same ratios for different measurers. Really, assume that, for two objects, measurers 1 and 2 obtain the values y_i', y_i'' and $\tilde{y}_i', \tilde{y}_i''$, respectively. Since $\tilde{y}_i' = \alpha_i y_i'$ and $\tilde{y}_i'' = \alpha_i y_i''$ for some $\alpha_i > 0$, we have the equalities

$$\frac{\tilde{y}_i'}{\tilde{y}_i''} = \frac{\alpha_i y_i'}{\alpha_i y_i''} = \frac{y_i'}{y_i''},$$

which mean that the ratios of the measurements are preserved for two different measurers in the ratio scale. And so, if a measurer concludes that, e.g., the mass of an object is twice as much as that of the other, then another measurer (operating different units of measure) must come to the same conclusion. This testifies that, while comparing the measurement results in the ratio scale, the statement "object 1 is α_i times greater (smaller) than object 2" actually makes sense.

Obviously, such quantities as profit, costs, etc. expressed in currency units should be measured in the ratio scale, too.

Another measurement scale has a given spacing and an unfixed reference point (for different measurers). A possible example is the chronology scale–passing from one chronology to another requires an appropriate variation in the reference point. More precisely, the *difference scale* is a scale in which the measurement results are defined within the transformation $\phi_i(y_i) = y_i + c_i$ with a fixed constant c_i. The measurements in this scale preserve the differences between two different measurements performed by distinct measurers. In other words, for the measurements in the difference scale, a sensible statement has the form "object 1 is greater (smaller) than object 2 by the constant c_i." For example, the reign of Tsar Nicholas II in Russia calculated according to the Gregorian and Julian calendars is the same (as well as in any other calendar).

The interval scale is a scale in which the measurement results are defined within (are invariant with respect to) the linear positive transformation $\phi_i(y_i) = \alpha_i y_i + c_i$, where $\alpha_i > 0$ and c_i represent fixed constants. A typical example of such a scale is a temperature scale. As is well-known, the Celsius scale and the Fahrenheit scale serve for temperature measurements. Transition from one scale to the other employs the formula $\tilde{y}_i = \alpha_i y_i + c_i$.

Each measurer choosing the interval scale can have a specific reference point and a specific spacing. And the measurements performed in the interval scale by different measurers satisfy the ratio of the differences:

$$\frac{\tilde{y}_i - \tilde{y}_i'}{\tilde{y}_i'' - \tilde{y}_i'''} = \frac{\alpha_i y_i + c_i - (\alpha_i y_i' + c_i)}{\alpha_i y_i'' + c_i - (\alpha_i y_i''' + c_i)} = \frac{y_i - y_i'}{y_i'' - y_i'''}.$$

The above-mentioned scales (absolute scale, ratio scale, difference scale and interval scale) belong to *quantitative scales*. Naturally, the measurement results that are invariant with respect to the linear positive transformation of the general form $\tilde{y}_i = \alpha_i y_i + c_i$ inherit this property with respect to the transformations $\tilde{y}_i = a_i y_i$ or $\tilde{y}_i = y_i + c_i$. It explains why the interval scale is most "general" among the quantitative scales. In this context, all assertions established for the measurements in the interval scale remain in force for the measurements in the ratio scale and in the difference scale (and, of course, in the absolute scale).

Besides quantitative scales there exist *qualitative scales*. A typical representative of this class is the *ordinal scale* in which the measurement results are defined within a transformation $\phi_i(y_i)$ where ϕ_i denotes an arbitrary strictly increasing function. As examples, we refer to Mohs' scale for the hardness of minerals, the ordering scale for different works based on their importance, as well as various rating scales. The ordinal scales have no fixed reference point, possibly involving different spacing. Figuratively speaking, different measurers may even employ variable spacing between the marks. The statements "object 1 is α_i times greater (smaller) than object 2" and "object 1 is greater (smaller) than object 2 by the constant c_i" appear meaningless for the measurement results in the ordinal scale. Only the "greater-smaller" relation makes sense here.

All assertions established for the measurements in a qualitative scale remain in force for the measurements in a quantitative scale, but the converse fails. Thus, the quantitative scales are "richer" than the qualitative ones, as yielding more substantial assertions (though, for a narrower class of problems).

2.4.2 Pareto Set Invariance with Respect to Strictly Increasing Transformation of Criteria

Recall the definition of the set of Pareto optimal vectors:

$$P(Y) = \{y^* \in Y | \text{ there exists no } y \in Y \text{ such that } y \geq y^*\}.$$

The inequality $y \geq y^*$ in the definition of the Pareto set means the component-wise inequalities $y_i \geqq y_i^*$ for all $i = 1, 2, \ldots, m$, with at least one of them being strict.

Let ϕ_i be a strictly increasing numerical function of single variable defined on the whole real axis, i.e.,

$$y_i > y_i' \Leftrightarrow \phi_i(y_i) > \phi_i(y_i')$$

for all $y_i, y_i' \in R$. Obviously, the equality $y_i = y_i'$ holding for a strictly increasing function ϕ_i is equivalent to the equality $\phi_i(y_i) = \phi_i(y_i')$. Next, by the definition of this function, the inequality $y_i > y_i'$ takes place if and only if $\phi_i(y_i) > \phi_i(y_i')$ is the case.

Hence, the definition of the Pareto set does not change essentially if a strictly increasing transformation is applied to the values of criteria. In other words, the Pareto set has invariance with respect to the above transformation, and *the notion of the Pareto set can be used whenever the criteria are measured at least in the ordinal scale* (all the more, in any quantitative scale).

2.4.3 Invariance of Theorem 2.5 with Respect to Linear Positive Transformation

Theorem 2.5 shows how an elementary information quantum can be used for Pareto set reduction. As stated in the previous section, this reduction proceeds from the inclusions

$$C(X) \subset P_{\hat{f}}(X) \subset P_f(X), \tag{2.12}$$

where $P_{\hat{f}}(X)$ is the set of Pareto optimal alternatives in the multicriteria choice problem with the initial set of feasible alternatives X and the “new” vector criterion $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_m)$ calculated by the formulas

$$\hat{f}_j = w_j^* f_i + w_i^* f_j, \hat{f}_s = f_s \quad \text{for all } s \in I \backslash \{j\}. \tag{2.6}$$

Since the quantitative approach considered in the book presupposes measuring the criteria values in quantitative scales, the invariance of inclusions (2.12) with a linear positive transformation of the criteria is of certain practical interest. Note that, without such invariance, the suggested approach would be inapplicable to the real multicriteria problems with quantitative criteria.

Theorem 2.7 *Inclusions* (2.5) *and* (2.12) *are invariant with respect to a linear positive transformation of the criteria.*

□ First of all, observe that the inclusion $C(X) \subset \text{Ndom}\, X$ holds for any set of selectable alternatives under the hypothesis of Theorem 2.5. Moreover, the definition of the set of selectable alternatives $C(X)$ makes no mention of the criteria. Hence, this definition does not depend on the choice of the criteria scales, being invariant with respect to any transformation of the criteria.

In subsection 2.4.2, we have established the Pareto set invariance with respect to a strictly increasing transformation. A linear positive transformation is a special case of a strictly increasing transformation. Therefore, the Pareto set $P_f(X)$ from (2.12) inherits invariance with respect to a linear positive transformation of the criteria. To prove the invariance of the set $P_{\hat{f}}(X)$, it suffices to verify the invariance of the strict inequality $\hat{f}_j = w_j^* y_i + w_i^* y_j > w_j^* \bar{y}_i + w_i^* \bar{y}_j = \bar{f}_j$ incorporating the new criterion f_j, since for an arbitrary criterion f_i, $i \neq j$, the invariance of the corresponding inequalities is checked in the same elementary way as in subsection 2.4.2.

At the beginning, recall that

$$w_i^* = y_i' - y_i'', \quad w_j^* = y_j'' - y_j',$$

where $y_k' = f_k(x')$, $y_k'' = f_k(x'')$ $(k = i, j)$ and w_i^*, w_j^* are fixed positive numbers.

Replace y_k with $\tilde{y}_k = \alpha_k y_k + c_k$ $(\alpha_k > 0)$, $k = i, j$, in formula (2.6) defining the new criterion $\hat{f}_j$. This replacement yields the transformed criterion

$$\tilde{\hat{f}}_j = (\alpha_j y_j'' + c_j - \alpha_j y_j' - c_j) \cdot (\alpha_i y_i + c_i) + (\alpha_i y_i' + c_i - \alpha_i y_i'' - c_i)(\alpha_j y_j + c_j).$$

And trivial simplifications lead to

$$\tilde{\hat{f}}_j = \alpha_i \alpha_j w_j^* y_i + \alpha_i \alpha_j w_i^* y_j + C, \tag{2.15}$$

where the constant

$$C = \alpha_j w_j^* c_i + \alpha_i w_i^* c_j$$

does not depend on y_i, y_j.

Now, assume that the inequality

$$\hat{f}_j = w_j^* y_i + w_i^* y_j > w_j^* \bar{y}_i + w_i^* \bar{y}_j = \bar{f}_j \tag{2.16}$$

holds for arbitrary fixed numbers $y_i, y_j, \bar{y}_i, \bar{y}_j$. Having (2.15) in mind, multiply by the positive number $\alpha_i \alpha_j$ and add the constant C to both sides of inequality (2.16) to get

$$\tilde{\hat{f}}_j > \tilde{\bar{f}}_j = \alpha_i \alpha_j w_j^* \bar{y}_i + \alpha_i \alpha_j w_i^* \bar{y}_j + C. \tag{2.17}$$

Subsequently, inequality (2.16) implies inequality (2.17). In a similar way, inequality (2.16) can be obtained from inequality (2.17). This means the equivalence of the two inequalities. ■

Note that the degree of compromise θ_{ij} is not invariant with respect to a linear positive transformation of the criteria. Furthermore, as easily verified, the degree of compromise is not invariant with respect to the transformations $\tilde{y}_k = a_k y_k$ and $\tilde{y}_k = y_k + c_k$, $k = i, j$, which indicates of the following. *For different measurers (different DMs), the degrees of compromise may differ even if the DMs are considered in the same choice problem, have identical preferences and perform measurements in a scale of the same type*. This fact contains no contradiction, as the DMs may adopt different units of measure for the same criteria.

Really, imagine two DMs with identical preferences, who measure the values of the first criterion in USD and, RUB respectively. Suppose that the values of the second criterion are measured by them in the absolute scale (e.g., in pcs). For the DM 1 operating USD and willing to compromise 10 pcs for the gain of \$1000, the degree of compromise for the first criterion in comparison with the second one makes up

$$\theta'_{12} = \frac{10}{1000 + 10} \approx 0.01.$$

The other DM 2 operating RUB and acting in the same way must be willing to compromise 10 pcs for the gain of 60,000 RUB, since at the moment of decision-making \$1 = 60 RUB. Therefore, for DM 2 the degree of compromise constitutes

$$\theta''_{12} = \frac{10}{60000 + 10} \approx 0.00016,$$

which is considerably smaller than for DM 1. But this result is correct, since the latter operates the much more "expensive" currency than the former.

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