Tight Welfare Guarantees for Pure Nash Equilibria of the Uniform Price Auction

Georgios Birmpas[1], Evangelos Markakis[1], Orestis Telelis[2(✉)], and Artem Tsikiridis[1]

[1] Department of Informatics, Athens University of Economics and Business, Athens, Greece
{gebirbas,markakis,tsikiridis15}@aueb.gr
[2] Department of Digital Systems, University of Piraeus, Piraeus, Greece
telelis@gmail.com

Abstract. We revisit the inefficiency of the uniform price auction, one of the standard multi-unit auction formats, for allocating multiple units of a single good. In the uniform price auction, each bidder submits a sequence of non-increasing marginal bids, one for each additional unit. The per unit price is then set to be the highest losing bid. We focus on the pure Nash equilibria of such auctions, for bidders with submodular valuation functions. Our result is a tight upper and lower bound on the inefficiency of equilibria, showing that the Price of Anarchy is bounded by 2.188. This resolves one of the open questions posed in previous works on multi-unit auctions.

1 Introduction

The Uniform Price Auction is one of the standard multi-unit auction formats, for allocating multiple units of a single good, at a uniform price per unit. Multi-unit auctions have been in use for a long time, with important applications, such as the auctions offerred by the U.S. and U.K. Treasuries for selling bonds to investors. They are also being deployed in various platforms, including several online brokers [9,10]. In the literature, multi-unit auctions have been a subject of study ever since the seminal work of Vickrey [13], and some formats were conceived even earlier, by Friedman [6]. The quantification of their inefficiency at equilibrium has been the subject of recent works [4,7,8,12], via derivation of upper and lower bounds on the Price of Anarchy, in the full and incomplete information models. The outcomes of these works are quite encouraging, as they establish that the inefficiency is bounded by a small constant.

In this work we derive tight bounds for the Price of Anarchy of pure Nash equilibria of the uniform price auction. For such an auction on k units, each bidder is required to submit a sequence of non-increasing bids, one for each additional unit. Among all submitted bids the k highest win the auction and

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each bidder receives as many units of the good as the number of his winning bids. The highest losing bid is then chosen as the uniform price that each bidder pays, per unit won. The simplicity of this auction format is counterbalanced however by the fact that it does not support truthful bidding in dominant strategies, thus encouraging strategic behavior. The underlying strategic game induced by the auction is known to possess pure Nash equilibria, and a polynomial time algorithm for computing such an equilibrium is developed in [8]. However, [8] derives a tight bound for the Price of Anarchy of only a strict subset of such equilibria (in undominated strategies). For the full set of pure Nash equilibria, the results of [7] show that the Price of Anarchy is between $2 - \frac{1}{k}$ and 3.146. The source of inefficiency in the uniform price auction is partly due to a *demand reduction* effect discussed in [1], where bidders may have incentives to understate their demand, so as to receive less units at a lower price per unit, and partly due to the use of dominated strategies. These effects motivate the further study and quantification of the inefficiency for this auction format.

Contribution. We focus on the pure Nash equilibria of the uniform price auction, for bidders with submodular valuation functions. Our results are tight upper and lower bounds on the inefficiency of pure equilibria for submodular bidders, showing that the Price of Anarchy is bounded by 2.188. This resolves one of the questions left open from [7,8]. The proof of the upper bound is based on carefully analyzing the performance of equilibria with respect to bidders who receive less units than in the optimal assignment. Our lower bound is obtained by an explicit construction, which matches our upper bound as the number of units becomes large enough.

2 Model and Definitions

We consider a multi-unit auction, involving the allocation of k units of a single item, to a set $\mathcal{N}$ of n bidders, $\mathcal{N} = \{1, \ldots, n\}$. Each bidder $i \in \mathcal{N}$ has a private valuation function $v_i : \{0, 1, \ldots, k\} \mapsto \mathbb{R}^+$, defined over the quantity of units that he receives, with $v_i(0) = 0$. In this work, we assume that each function v_i is a non-decreasing submodular function.

Definition 1. *A valuation function* $f : \{0, 1, \ldots, k\} \mapsto \mathbb{R}^+$ *is called* **submodular**, *if for every* $x < y$, $f(x) - f(x-1) \geqslant f(y) - f(y-1)$.

A valuation function can also be specified through a sequence of *marginal values*, corresponding to the value that each additional unit yields for the bidder. For the j-th additional unit, the bidder obtains marginal value $v_i(j) - v_i(j-1)$, which we denote by m_{ij}. Then, the function v_i can be determined by the vector $\mathbf{m}_i = (m_{i1}, \ldots, m_{ik})$. For submodular functions, $m_{i1} \geqslant \ldots \geqslant m_{ik}$, by definition. We will often use the representation of v_i by $\mathbf{m}_i$ in the sequel. We will also use the following well known properties of submodular functions:

Proposition 1. *Given $x, y \in \{0, 1, \ldots, k\}$ with $x \leqslant y$, any non-decreasing submodular function f, with $f(0) = 0$, satisfies $yf(x) \geqslant xf(y)$. Moreover, when $x < y$, for any $j = 1, \ldots, y - x$ the function f satisfies: $(f(x + j) - f(x))/j \geqslant (f(y) - f(x))/(y - x)$.*

The standard Uniform Price Auction requires that bidders submit their non-increasing marginal value for each additional unit; every bidder i is asked to declare his valuation curve, as a *bid vector* $\mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{ik})$, satisfying $b_{i1} \geqslant b_{i2} \geqslant \ldots \geqslant b_{ik}$. Thus, b_{ij} is the *declared* marginal bid of i, for obtaining the j-th unit of the item. Note that each b_{ij} may differ from the bidder's actual marginal value, m_{ij}. Given a *bidding profile* $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, the auction allocates the k units to the k highest marginal bids. We denote this allocation by $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}), \ldots, x_n(\mathbf{b}))$, where $x_i(\mathbf{b})$ is the number of units allocated to bidder i. Each bidder pays a uniform price $p(\mathbf{b})$ *per received unit*, which equals the highest rejected marginal bid, i.e., the $(k+1)$-th highest marginal bid. The total payment of bidder i is then $x_i(\mathbf{b}) \cdot p(\mathbf{b})$, and his utility for the allocation is: $u_i(\mathbf{b}) = v_i(x_i(\mathbf{b})) - x_i(\mathbf{b}) \cdot p(\mathbf{b})$.

The (utilitarian) *Social Welfare* achieved under a bidding profile $\mathbf{b}$ is defined as the sum of utilities of all interacting parties, inclusively of the auctioneer's revenue. This sum equals the sum of the bidders' values for their allocations:

$$SW(\mathbf{b}) = \sum_{i=1}^{n} v_i(x_i(\mathbf{b}))$$

Our goal is to derive upper and lower bounds on the Price of Anarchy (PoA) of pure Nash equilibria of the Uniform Price Auction. This is the worst-case ratio of the optimal welfare, over the welfare achieved at a pure Nash equilibrium. If $\mathbf{x}^*$ denotes an optimal allocation, then

$$PoA = \sup_{\mathbf{b}} \frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})}$$

where the supremum is taken over pure equilibrium profiles.

Finally, following previous works on equilibrium analysis of auctions, e.g., [2, 3, 8], we focus on *non-overbidding* profiles $\mathbf{b}$, wherein no bidder ever outbids his value, for any number of units. That is, for any $\ell \leqslant k$, we assume $\sum_{j=1}^{\ell} b_{ij} \leqslant v_i(\ell)$. Note that, this *does not* necessarily imply $b_{ij} \leqslant m_{ij}$, except for when $j = 1$: i.e., $b_{i1} \leqslant m_{i1} = v_i(1)$. In our analysis, we refer to non-overbidding vectors, $\mathbf{b}_i$, and profiles, $\mathbf{b}$, as *feasible*.

3 Inefficiency Upper Bound

In this section we develop tight welfare guarantees for feasible (non-overbidding) pure Nash equilibrium profiles of the uniform price auction, when the bidders have submodular valuation functions. By the results of [7], it is already known that for submodular valuations on k units, $2 - \frac{1}{k} \leqslant PoA \leqslant 3.146$. We show that:

Theorem 1. *The Price of Anarchy of non-overbidding pure Nash equilibria of the Uniform Price Auction with submodular bidders is at most:*

$$\frac{2 + \mathcal{W}_0(-e^{-2})}{1 + \mathcal{W}_0(-e^{-2})} \approx 2.188$$

where $\mathcal{W}_0$ is the first branch of the Lambert W function.

The Lambert W function is the multi-valued inverse function of $f(W) = We^W$. For more on the properties of this function, see [5].

Before we continue, we introduce first some notation to be used throughout the section. Let $\mathbf{b}$ denote a feasible bidding profile. We denote the k winning marginal bids under $\mathbf{b}$ by $\beta_j(\mathbf{b})$, $j = 1, \ldots, k$, so that $\beta_j(\mathbf{b})$ is the j-th lowest winning bid under $\mathbf{b}$, thus, $\beta_1(\mathbf{b}) \leqslant \beta_2(\mathbf{b}) \leqslant \ldots \leqslant \beta_k(\mathbf{b})$. We will often apply this notation to profiles of the form $\mathbf{b}_{-i}$, for some bidder $i \in \mathcal{N}$. For a profile of valuation functions $(v_1, v_2, \ldots, v_n)$, we denote the *socially optimal* – i.e., welfare maximizing – allocation by $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$. If there are multiple such allocations, we fix one for the remainder of the analysis. We define a partition of the set of bidders, $\mathcal{N}$, with reference to $\mathbf{x}^*$ and any arbitrary allocation $\mathbf{x}$, into two subsets, $\mathcal{O}$ and $\mathcal{U}$, as follows:

$$\mathcal{N} = \mathcal{O} \cup \mathcal{U}, \quad \mathcal{O} = \{i \in \mathcal{N} :\, x_i \geqslant x_i^*\}, \quad \mathcal{U} = \{i \in \mathcal{N} :\, x_i < x_i^*\}.$$

The set $\mathcal{O}$ contains the *"overwinners"*, i.e., bidders that receive in $\mathbf{x}$ at least as many units as in $\mathbf{x}^*$. The set $\mathcal{U}$ contains respectively the *"underwinners"*. In our analysis, the allocations we refer to are determined by some profile $\mathbf{b}$, i.e., $\mathbf{x} \equiv \mathbf{x}(\mathbf{b})$. Consequently, the sets $\mathcal{O}$ and $\mathcal{U}$ will depend on $\mathbf{b}$; for simplicity, we omit this dependence from our notation. The following lemma states that, under a feasible bidding profile $\mathbf{b}$, every bidder $i \in \mathcal{O}$ retains value at least equal to a convex combination of her socially optimal value, $v_i(x_i^*)$, and of the sum of her winning bids.

Lemma 1. *Let $\mathbf{b}$ be a feasible bidding profile, and let $\mathcal{O}$ be the set of overwinners with respect to the allocation $\mathbf{x}(\mathbf{b})$. For every $\lambda \in [0, 1]$, and every bidder $i \in \mathcal{O}$:*

$$v_i(x_i(\mathbf{b})) \geqslant \lambda \cdot v_i(x_i^*) + (1 - \lambda) \cdot \sum_{j=1}^{x_i(\mathbf{b})} b_{ij} \tag{1}$$

Proof. Indeed, for every $i \in \mathcal{O}$: $v_i(x_i(\mathbf{b})) = \lambda v_i(x_i(\mathbf{b})) + (1 - \lambda) v_i(x_i(\mathbf{b}))$, which is at least equal to $\lambda v_i(x_i^*) + (1-\lambda) v_i(x_i(\mathbf{b}))$, by definition of $\mathcal{O}$. Then, (1) follows by our no-overbidding assumption on $\mathbf{b}$. □

By definition, each overwinner is capable of *"covering"* her socially optimal value. Conversely, the underwinners are the cause of social inefficiency. We will bound the total inefficiency by transforming the leftover fractions of winning bids of bidders in $\mathcal{O}$, i.e., the term $(1-\lambda) \cdot \sum_{j=1}^{x_i(\mathbf{b})} b_{ij}$ for each bidder $i \in \mathcal{O}$ in (1), into fractions of the value attained by bidders in $\mathcal{U}$ in the optimal allocation. In this manner, we will quantify the value that the underwinners are missing (due to their strategic bidding), and determine the worst-case scenario that can arise at a pure Nash equilibrium. The following claim can be inferred from [8], and will be used to facilitate this transformation. We present the proof for completeness.

Claim 1. *Let* $\mathbf{b}$ *be any bidding profile. Then it holds that:*

$$\sum_{i\in\mathcal{U}}\sum_{j=1}^{x_i^*-x_i(\mathbf{b})}\beta_j(\mathbf{b}) \leqslant \sum_{i\in\mathcal{O}}\sum_{j=x_i^*+1}^{x_i(\mathbf{b})} b_{ij}. \tag{2}$$

Proof. For every unit missed under $\mathbf{b}$ by any bidder $i \in \mathcal{U}$ (with respect to the units won by i in the optimal allocation), there must exist some bidder $\ell \in \mathcal{O}$ that obtains this unit. If i missed $x_i^* - x_i(\mathbf{b}) > 0$ units under $\mathbf{b}$, there are at least as many bids issued by bidders in $\mathcal{O}$ who obtained collectively these units. The sum of these bids cannot be less than the sum $\sum_{j=1}^{x_i^*-x_i(\mathbf{b})}\beta_j(\mathbf{b})$ of the $x_i^* - x_i(\mathbf{b})$ lowest winning bids in $\mathbf{b}$. Summing over $i \in \mathcal{U}$ yields the desired inequality. □

Next, we develop a characterization of upper bounds on the Price of Anarchy. To this end, let us first define the following set, $\Lambda(\mathbf{b})$, for any bidding profile $\mathbf{b}$.

$$\Lambda(\mathbf{b}) = \left\{\lambda \in [0,1] :\, v_i(x_i(\mathbf{b})) + (1-\lambda)\sum_{j=1}^{x_i^*-x_i(\mathbf{b})}\beta_j(\mathbf{b}) \geqslant \lambda v_i(x_i^*),\ \forall i\ \in \mathcal{U}\right\} \tag{3}$$

Notice that, for every $\mathbf{b}$, $\Lambda(\mathbf{b}) \neq \emptyset$, because $\lambda = 0 \in \Lambda(\mathbf{b})$. The following lemma helps us understand how one can obtain upper bounds on the Price of Anarchy.

Lemma 2. *If there exists* $\lambda \in [0,1]$ *such that* $\lambda \in \Lambda(\mathbf{b})$*, for every feasible pure Nash equilibrium profile* $\mathbf{b}$ *of the Uniform Price Auction, then the Price of Anarchy of feasible pure Nash equilibria is at most* λ^{-1}*.*

Proof. Fix a feasible pure Nash equilibrium profile $\mathbf{b}$ and consider any $\lambda \in \Lambda(\mathbf{b})$. Then, we can apply consecutively the partition $\mathcal{N} = \mathcal{O} \cup \mathcal{U}$ with respect to $\mathbf{b}$, Lemma 1, Claim 1 and, finally, the definition of $\Lambda(\mathbf{b})$, to obtain:

$$\begin{aligned}
SW(\mathbf{b}) &= \sum_{i\in\mathcal{O}} v_i(x_i(\mathbf{b})) + \sum_{i\in\mathcal{U}} v_i(x_i(\mathbf{b})) \\
&\geqslant \lambda\sum_{i\in\mathcal{O}} v_i(x_i^*) + (1-\lambda)\sum_{i\in\mathcal{O}}\sum_{j=1}^{x_i(\mathbf{b})} b_{ij} + \sum_{i\in\mathcal{U}} v_i(x_i(\mathbf{b})) \\
&\geqslant \lambda\sum_{i\in\mathcal{O}} v_i(x_i^*) + (1-\lambda)\sum_{i\in\mathcal{O}}\sum_{j=x_i^*+1}^{x_i(\mathbf{b})} b_{ij} + \sum_{i\in\mathcal{U}} v_i(x_i(\mathbf{b})) \\
&\geqslant \lambda\sum_{i\in\mathcal{O}} v_i(x_i^*) + \sum_{i\in\mathcal{U}}\Big((1-\lambda)\sum_{j=1}^{x_i^*-x_i(\mathbf{b})}\beta_j(\mathbf{b}) + v_i(x_i(\mathbf{b}))\Big) \\
&\geqslant \lambda\sum_{i\in\mathcal{O}} v_i(x_i^*) + \sum_{i\in\mathcal{U}} \lambda\cdot v_i(x_i^*) = \lambda\cdot SW(\mathbf{x}^*)
\end{aligned}$$

□

Using $\lambda = 0$ with Lemma 2, yields the trivial upper bound of ∞. To obtain better upper bounds, Lemma 2 shows that we need to understand better the sets $\Lambda(\mathbf{b})$, and whether underwinners can extract at equilibrium a good fraction of their value under the optimal assignment. By the definition of these sets, the next step towards this is to derive lower bounds on every $\beta_\ell(\mathbf{b})$ for each underwinner $i \in \mathcal{U}$, and every value $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$. The lower bound that we will use is formally expressed below.

Lemma 3. *Let* $\mathbf{b}$ *be a pure Nash equilibrium of the Uniform Price Auction and* $\mathbf{x}^*$ *be a socially optimal allocation. For every underwinning bidder* $i \in \mathcal{U}$ *under* $\mathbf{b}$ *and for every* $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$*:*

$$\beta_\ell(\mathbf{b}) \geqslant \frac{1}{x_i(\mathbf{b}) + \ell} \cdot \Big(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b}))\Big) \tag{4}$$

We defer the proof of this statement, in order to explain first how – along with Lemma 2 – it leads to the proof of Theorem 1.

Proof (***of Theorem*** 1). Using Lemma 2, we identify values of λ that belong to every $\Lambda(\mathbf{b})$. Fix any feasible pure Nash equilibrium profile $\mathbf{b}$ and, for every bidder $i \in \mathcal{U}$, let $q_i(\mathbf{b}) = x_i^* - x_i(\mathbf{b})$. To simplify the notation, we use hereafter x_i for $x_i(\mathbf{b})$, p for $p(\mathbf{b})$, q_i for $q_i(\mathbf{b})$, and β_j for $\beta_j(\mathbf{b})$, (always with respect to the Nash equilibrium $\mathbf{b}$).

Consider an arbitrary $\lambda \in [0, 1]$ and, keeping everything else fixed, define $h(\lambda) = v_i(x_i) + (1 - \lambda) \cdot \sum_{j=1}^{q_i} \beta_j$. We can now have the following implications.

$$\begin{aligned} h(\lambda) &= v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \beta_j \\ &\geqslant v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \frac{1}{j + x_i} \cdot \Big(v_i(x_i + j) - v_i(x_i)\Big) \qquad (5) \\ &= v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \left(\frac{j}{j + x_i} \cdot \frac{v_i(x_i + j) - v_i(x_i)}{j}\right) \\ &\geqslant v_i(x_i) + (1-\lambda) \cdot \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \sum_{j=1}^{q_i} \frac{j}{j + x_i}. \qquad (6) \end{aligned}$$

In the derivation above, inequality (5) follows by applying (4) from Lemma 3, for every β_j, $j = 1, \ldots, q_i$. Inequality (6) follows by application of the second statement of Proposition 1, which yields $\frac{v_i(x_i+j) - v_i(x_i)}{j} \geqslant \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i}$, for any $j = 1, \ldots, q_i$.

Suppose now that under the equilibrium $\mathbf{b}$, there exists $i \in \mathcal{U}$ such that $x_i = 0$. In order for some λ to belong to $\Lambda(\mathbf{b})$, we would need to have $h(\lambda) \geqslant \lambda v_i(x_i^*)$. Using (6), for the underwinners with $x_i = 0$, and substituting $v_i(x_i) = 0$, we obtain: $h(\lambda) \geqslant (1-\lambda)v_i(x_i^*)$. If we now impose that $(1-\lambda)v_i(x_i^*) \geqslant \lambda v_i(x_i^*)$, we obtain $\lambda \leqslant 1/2$. Thus, any value of λ in $[0, 1/2]$ satisfies the constraint in the definition of $\Lambda(\mathbf{b})$ for bidders in $\mathcal{U}$ with $x_i = 0$. It remains to consider the more interesting case, which is for bidders in $\mathcal{U}$ with $x_i > 0$. We continue from (6) to bound $h(\lambda)$ as follows:

$$
\begin{aligned}
h(\lambda) &\geqslant \lambda v_i(x_i) + (1-\lambda)\cdot\left(v_i(x_i) + \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i}\cdot\sum_{j=1}^{q_i}\frac{j}{j+x_i}\right)\\
&\geqslant \lambda\cdot v_i(x_i) + (1-\lambda)\cdot\left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right)\cdot\left(1+\frac{x_i}{x_i^*-x_i}\cdot\left(1-\sum_{j=1}^{q_i}\frac{1}{j+x_i}\right)\right)\\
&\geqslant \lambda\cdot v_i(x_i) + (1-\lambda)\cdot\left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right)\cdot\left(1+\frac{x_i}{x_i^*-x_i}\cdot\left(1-\int_{x_i}^{x_i^*}\frac{1}{y}dy\right)\right)\\
&\geqslant \lambda\cdot v_i(x_i) + (1-\lambda)\cdot\left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right)\cdot\left(1+\frac{x_i}{x_i^*-x_i}\cdot\left(1+\ln\frac{x_i}{x_i^*}\right)\right)\\
&= \lambda\cdot v_i(x_i) + (1-\lambda)\cdot\left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right)\cdot\left(1+\frac{\frac{x_i}{x_i^*}}{1-\frac{x_i}{x_i^*}}\cdot\left(1+\ln\frac{x_i}{x_i^*}\right)\right)
\end{aligned}
$$

The second inequality follows from the fact that $v_i(x_i(\mathbf{b})) \geqslant \frac{x_i}{x_i^*-x_i}\cdot\sum_{j=x_i+1}^{x_i^*} m_{ij}$, which is an implication of the first statement of Proposition 1. We have bounded the sum of harmonic terms by using $\sum_{k=m}^{n} f(k) \leqslant \int_{m-1}^{n} f(x)dx$, which holds for any monotonically decreasing positive function.

To continue, we minimize the function $f(y) = 1 + \frac{y}{1-y}\cdot(1+\ln y)$ over $(0, 1)$, since x_i/x_i^* belongs to this interval.

Fact 1. *The minimum of the function* $f(y) = 1 + \frac{y}{1-y}\cdot(1+\ln y)$ *over* $(0, 1)$, *is achieved at* $y = -\mathcal{W}_0(-e^{-2})$, *where* $\mathcal{W}_0$ *is the first branch of the Lambert W function.*

By substituting, we obtain a new lower bound on $h(\lambda)$ as follows:

$$
h(\lambda) \geqslant \lambda\cdot v_i(x_i(\mathbf{b})) + (1-\lambda)\cdot\left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right)\cdot\left(1+\mathcal{W}_0(-e^{-2})\right)
$$

If we now set the right hand side of the above to be greater than or equal to $\lambda v_i(x_i^*)$, we can check which values of λ can belong to $\Lambda(\mathbf{b})$. In particular, we notice that by using $\lambda^* = (1+\mathcal{W}_0(-e^{-2}))/(2+\mathcal{W}_0(-e^{-2})) \approx 0.457$, we have that $h(\lambda^*) \geqslant \lambda^* v_i(x_i^*)$ for every bidder $i \in \mathcal{U}$ with $x_i > 0$. Since for bidders with $x_i = 0$, we found earlier that $\lambda \leqslant 1/2$ suffices, and since $\lambda^* < 1/2$, we conclude that $\lambda^* \in \Lambda(\mathbf{b})$. Hence, the theorem follows by Lemma 2. □

To complete our analysis, we provide the proof of Lemma 3.

Proof (***of Lemma*** 3). Let $\mathbf{b}$ denote a feasible pure Nash equilibrium profile and $p(\mathbf{b})$ be the uniform price under $\mathbf{b}$. Fix an underwinning bidder $i \in \mathcal{U}$. We explore whether i is able to deviate from $\mathbf{b}$ feasibly and unilaterally to obtain ℓ additional units for $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$. Consider the following deviation $\mathbf{b}_i'$, for bidder i, and for any such ℓ.

$$\mathbf{b}_i' = \Big(\underbrace{b_{i1}, \cdots, b_{ir}}_{r \text{ bids}}, \underbrace{\beta_\ell(\mathbf{b}_{-i}) + \epsilon, \beta_\ell(\mathbf{b}_{-i}) + \epsilon, \ldots, \beta_\ell(\mathbf{b}_{-i}) + \epsilon}_{x_i(\mathbf{b}) + \ell - r \text{ bids}}, 0, 0, \ldots, 0\Big),$$

where $0 \leqslant r \leqslant x_i(\mathbf{b})$ is the index of the last bid in $\mathbf{b}_i$, up to position $x_i(\mathbf{b})$, that is strictly greater than $\beta_\ell(\mathbf{b}_{-i}) + \epsilon$, and $\epsilon > 0$ is any *sufficiently small* positive constant. The last index of $\mathbf{b}_i'$ with a value of $\beta_\ell(\mathbf{b}_{-i}) + \epsilon$ is the $(x_i + \ell)$-th bid. All subsequent bids are set to 0. Observe that such a bidding vector (should it be feasible) would grant bidder i exactly $x_i(\mathbf{b}) + \ell$ units in total in the profile $(\mathbf{b}_i', \mathbf{b}_{-i})$: the first r bids of $\mathbf{b}_i'$ were already winning bids in $\mathbf{b}$ and each of the next $x_i(\mathbf{b}) + \ell - r$ bids exceed the ℓ-th lowest winning bid of the other bidders, $\beta_\ell(\mathbf{b}_{-i})$. Moreover, the price at $(\mathbf{b}_i', \mathbf{b}_{-i})$ would be $\beta_\ell(\mathbf{b}_{-i})$; this is now the highest losing bid (issued by some other bidder in the auction).

Note that $\mathbf{b}_i'$ may not always be a feasible deviation, since it may not obey the no-overbidding assumption. We continue by examining two cases separately.

Case 1: The bidding vector $\mathbf{b}_i'$ is a feasible deviation. Then bidder i obtains ℓ additional units by deviating. But since $\mathbf{b}$ is a pure Nash equilibrium, the utility of the bidder at $(\mathbf{b}_i', \mathbf{b}_{-i})$ cannot be higher than the utility obtained by the bidder at $\mathbf{b}$, i.e.:

$$v_i(x_i(\mathbf{b}) + \ell) - (x_i(\mathbf{b}) + \ell) \cdot \beta_\ell(\mathbf{b}_{-i}) \leqslant v_i(x_i(\mathbf{b})) - x_i(\mathbf{b}) \cdot p(\mathbf{b})$$

Thus, for a bidder that may feasibly perform such a deviation, a lower bound for β_ℓ is, for $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$:

$$\beta_\ell(\mathbf{b}_{-i}) \geqslant \frac{1}{\ell + x_i(\mathbf{b})} \cdot \Big(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b})) + x_i(\mathbf{b}) \cdot p(\mathbf{b})\Big)$$

By dropping the non-negative term $x_i(\mathbf{b}) \cdot p(\mathbf{b})$ and since $\beta_\ell(\mathbf{b}) \geqslant \beta_\ell(\mathbf{b}_{-i})$ for every $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$, we obtain (4).

Before continuing to examine the second case, we identify first a useful inequality pertaining to the feasibility of $\mathbf{b}_i'$, given the initial feasible profile $\mathbf{b}$.

Claim 2. *For $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$, the vector $\mathbf{b}_i'$ is feasible if and only if*

$$v_i(x_i(\mathbf{b}) + \ell) \geqslant \sum_{j=1}^{x_i(\mathbf{b})+\ell} b_{ij}'$$

Proof. If $\mathbf{b}_i'$ is a feasible deviation, the inequality holds, by definition of no-overbidding. For the reverse direction, we will show that if $\mathbf{b}_i'$ is not feasible, i.e., it violates the no-overbidding assumption, then $v_i(x_i(\mathbf{b}) + \ell) < \sum_{j=1}^{x_i(\mathbf{b})+\ell} b_{ij}'$. When $\mathbf{b}_i'$ is not feasible, we know there exists an index $t \leqslant x_i(\mathbf{b}) + \ell$, such that $v_i(t) < \sum_{j=1}^{t} b_{ij}'$. Note also that $t > r$, because $b_{ij}' = b_{ij}$ for $j \leqslant r$ and $\mathbf{b}$ is a feasible bidding vector. Assume that $t < x_i(\mathbf{b}) + \ell$ since, otherwise, we are done. We can decompose the sum of bids in our inequality as:

$$v_i(t) < \sum_{j=1}^{t} b_{ij}' = \sum_{j=1}^{r} b_{ij}' + \sum_{j=r+1}^{t} b_{ij}' = \sum_{j=1}^{r} b_{ij} + (t-r)(\beta_\ell(\mathbf{b}_{-i}) + \epsilon)$$

By rearranging the terms we obtain:

$$\begin{aligned}(t-r)(\beta_\ell(\mathbf{b}_{-i}) + \epsilon) > v_i(t) - \sum_{j=1}^{r} b_{ij} &= v_i(t) - v_i(r) + v_i(r) - \sum_{j=1}^{r} b_{ij} \\ &\geqslant v_i(t) - v_i(r) = \sum_{j=r+1}^{t} m_i(j)\end{aligned}$$

This means that there exists an index $s \in \{r+1, \ldots, t\}$ such that $m_{is} < \beta_\ell(\mathbf{b}_{-i}) + \epsilon$. Then, by definition of $\mathbf{b}_i'$ and by the non-increasing marginal values of the submodular valuation function, we derive that $m_{ij} < b_{ij}'$, for $j = s+1, \ldots, x_i(\mathbf{b}) + \ell$. Hence, since the no-overbidding assumption was violated at index t, it will continue to be violated if we include all the non-zero bids of $\mathbf{b}_i'$, up until the index $x_i(\mathbf{b}) + \ell$. Thus, $v_i(x_i(\mathbf{b}) + \ell) < \sum_{j=1}^{x_i(\mathbf{b})+\ell} b_{ij}'$, as claimed. □

Case 2: Suppose $\mathbf{b}_i'$ is not feasible, i.e., it involves overbidding. Then we can still infer a lower bound on $\beta_\ell(\mathbf{b})$, by exploiting Claim 2, as follows:

$$\begin{aligned}v_i(x_i(\mathbf{b}) + \ell) < \sum_{j=1}^{x_i(\mathbf{b})+\ell} b_{ij}' &= \sum_{j=1}^{r} b_{ij} + (x_i(\mathbf{b}) + \ell - r) \cdot \Big(\beta_\ell(\mathbf{b}_{-i}) + \epsilon\Big) \\ &\leqslant \sum_{j=1}^{x_i(\mathbf{b})} b_{ij} + (x_i(\mathbf{b}) + \ell) \cdot \Big(\beta_\ell(\mathbf{b}_{-i}) + \epsilon\Big) \\ &\leqslant v_i(x_i(\mathbf{b})) + (x_i(\mathbf{b}) + \ell) \cdot \Big(\beta_\ell(\mathbf{b}_{-i}) + \epsilon\Big)\end{aligned}$$

where the last inequality holds because $\mathbf{b}$ is feasible. By rearranging, we obtain:

$$\beta_\ell(\mathbf{b}_{-i}) > \frac{1}{\ell + x_i(\mathbf{b})} \cdot \Big(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b}))\Big) - \epsilon.$$

Observe that the above *strict* inequality holds for any sufficiently small constant $\epsilon > 0$. Since also $\beta_\ell(\mathbf{b}_{-i}) \leqslant \beta_\ell(\mathbf{b})$, inequality (4) follows. □

4 A Matching Lower Bound

We now present a lower bound, establishing that our upper bound is tight[1].

Theorem 2. *For any $k \geqslant 8$, the Price of Anarchy of the Uniform Price Auction for pure Nash equilibria and submodular bidders is at least*

$$1 + \frac{(1-\frac{1}{k})(1+\mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1} + 1 + (-(1-\frac{1}{k})\mathcal{W}_0(-e^{-2}) - \frac{1}{k})\ln(-\mathcal{W}_0(-e^{-2}) + \frac{1}{k})}$$

and approaches $(2+\mathcal{W}_0(-e^{-2}))/(1+\mathcal{W}_0(-e^{-2})) \approx 2.188$ *as k grows.*

Proof. We construct an instance of the auction with two bidders and $k \geqslant 8$ units. Let $x \in \{1, 2, \dots, k-2\}$ be a parameter that we will set later on. The valuation function of bidder 1 assigns value only for the first unit and equals

$$m_{11} = \frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}$$

For the remaining units, we have $m_{1j} = 0$, for any $j \geqslant 2$.

The valuation function of bidder 2 is given by the following marginal values:

$$m_{2j} = \begin{cases} 1, & j = 1, \dots, k-1 \\ 0, & j = k \end{cases}$$

Hence, the optimal allocation is for bidder 1 to obtain 1 unit and for bidder 2 to obtain $k-1$ units. Consider a bidding profile $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$ defined as follows:

$$b_{1j} = \left\{\begin{array}{ll} 1 - \frac{x}{k-1}, & j = 1 \\ 1 - \frac{x}{k-j+1}, & j = 2, \dots, k-x \\ 0, & \text{otherwise} \end{array}\right\} \quad \text{and} \quad b_{2j} = \left\{\begin{array}{ll} \epsilon, & j = 1, \dots, x \\ 0, & j > x \end{array}\right\}.$$

Here, $\epsilon > 0$ is any small positive quantity no larger than 1.

We will see that this construction yields better lower bounds than the previously known bound of $2 - \frac{1}{k}$, when $k \geqslant 11$. For example, for $k = 11$ and $x = 2$ we obtain the following instance:

$$\mathbf{m}_1 = (5.942, 0, 0, \dots, 0, 0, 0), \ \mathbf{b}_1 = \left(\frac{8}{10}, \frac{8}{10}, \frac{7}{9}, \frac{6}{8}, \frac{5}{7}, \dots, \frac{1}{3}, 0, 0\right)$$
$$\mathbf{m}_2 = (1, 1, \dots, 1, 1, 0), \ \mathbf{b}_2 = (\epsilon, \epsilon, 0, 0, \dots, 0, 0)$$

It is easy to verify that this instance already yields a lower bound of 2.007, on the Price of Anarchy. Coming back now to the analysis for general k and x, we will first ensure that both bidding vectors $\mathbf{b}_1$, $\mathbf{b}_2$ adhere to no-overbidding. For the vector $\mathbf{b}_1$, it suffices to note that

$$\sum_{j=1}^{k-x} b_{1j} = \frac{k-1-x}{k-1} + \sum_{j=2}^{k-x} \frac{k-j+1-x}{k-j+1} = \frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i} = m_{11}$$

[1] We note that Theorem 2 holds for any deterministic tie-breaking rule.

where the last equality holds by changing indices and setting $i = k - j + 1 - x$. Therefore, we have that $\sum_{j=1}^{k-x} b_{1j} = v_1(k-x)$. And this directly implies that for any $\ell < k - x$, we have $\sum_{j=1}^{\ell} b_{1j} < v_1(\ell)$. It is also straightforward that for $\ell > k - x$, the no-overbidding assumption cannot be violated. Similarly, for the vector $\mathbf{b}_2$, it is easy to check that it complies to no-overbidding.

Under $\mathbf{b}$, bidder 1 obtains $k - x$ units and bidder 2 obtains x units. Notice that in this profile the uniform price is 0, as there is no contest for any unit; bidder 1 bids for exactly $k - x$ units, while bidder 2 bids for x units. All other bids are 0. We also note that $\mathbf{b}_1$ is a weakly dominated strategy.

We now argue that $\mathbf{b}$ is a pure Nash equilibrium. Bidder 1 clearly has no incentive to deviate. She is interested only in the first unit, and there is no incentive to win more units, since that would increase the price. She is also not interested in deviating to receive less units than in the current profile. Such a deviation, would either grant her zero units, and thus no utility, or would grant her at least one unit with the same utility as in $\mathbf{b}$.

Let us examine the case of bidder 2. Since bidder 2 is not interested in the last unit, we can consider only deviation vectors $\mathbf{b}'_2$ with $b'_{2k} = 0$. Note that under $\mathbf{b}$, $u_2(\mathbf{b}) = x$. Hence, bidder 2 does not have an incentive to obtain less than x units, since the price will then still remain 0, and she will only have a lower utility. It therefore suffices to consider what happens when she tries to obtain ℓ additional units, where $\ell = 1, \ldots, k - x - 1$. To do so, bidder 2 must outbid some of the winning bids of $\mathbf{b}_1$. In particular, to obtain ℓ additional units at the minimum possible price, she must outbid the bid b_{1t} of bidder 1, where t is the index $t = k - x - (\ell - 1)$. If she issues a bid $\mathbf{b}'_2$, where the first $x + \ell$ coordinates outbid b_{1t} and the remaining bids are 0, then she will obtain exactly $x + \ell$ units, and the new price (i.e., the new highest losing bid) will be precisely b_{1t}. However, any such attempt will grant bidder 2 utility equal to $u_2(\mathbf{b})$, since

$$\begin{aligned} u_2(\mathbf{b}_1, \mathbf{b}'_2) &= v(x+\ell) - (x+\ell) \cdot b_{1t} \\ &= x + \ell - (x+\ell) \cdot (1 - \frac{x}{x+\ell}) = x = u_2(\mathbf{b}). \end{aligned}$$

We conclude that the profile $\mathbf{b}$ is a pure Nash Equilibrium. The ratio of the optimal Social Welfare to the one in $\mathbf{b}$ is at least:

$$\begin{aligned} &\frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})} = \frac{v_1(1) + v_2(k-1)}{v_1(k-x) + v_2(x)} \\ &= 1 + \frac{k-1-x}{\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i} + x} = 1 + \frac{k-1-x}{\frac{k-1-x}{k-1} + k - 1 - x \sum_{i=x+1}^{k-1} \frac{1}{i}} \\ &\geqslant 1 + \frac{k-1-x}{\frac{k-1-x}{k-1} + k - 1 - x \int_{x+1}^{k} \frac{1}{y} dy} \geqslant 1 + \frac{k-1-x}{\frac{k-1-x}{k-1} + k - 1 - x \ln \frac{k}{x+1}} \end{aligned} \tag{7}$$

At this point we set $x = \lfloor -(k-1)\mathcal{W}_0(-e^{-2}) \rfloor$, where $\mathcal{W}_0$ is the first branch of the Lambert W function. To continue from (7), we will need to ensure that $-\mathcal{W}_0(-e^{-2})(k-1)-1>0$, which holds for $k \geqslant 8$. Substitution of x in (7) and standard algebraic manipulations yield a lower bound for (7) equal to:

$$f(k) = 1 + \frac{(1-\frac{1}{k})(1+\mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1}+1+(-(1-\frac{1}{k})\mathcal{W}_0(-e^{-2})-\frac{1}{k})\ln(-\mathcal{W}_0(-e^{-2})+\frac{1}{k})}$$

So far, we have shown that $SW(\mathbf{x}^*)/SW(\mathbf{b}) \geqslant f(k)$. The theorem follows by observing that, as k goes to ∞:

$$\lim_{k\to\infty} f(k) = 1 + \frac{1+\mathcal{W}_0(-e^{-2})}{1-\mathcal{W}_0(-e^{-2})\cdot\ln(-\mathcal{W}_0(-e^{-2}))} = \frac{2+\mathcal{W}_0(-e^{-2})}{1+\mathcal{W}_0(-e^{-2})},$$

where we use that $\ln(-\mathcal{W}_0(y)) = -\mathcal{W}_0(y)+\ln(-y)$, for $y \in [-e^{-1}, 0)$. This stems from the definition of the Lambert W function, particularly of $\mathcal{W}_0$ [5]. □

5 Conclusions

We presented a tight bound for the Price of Anarchy of pure Nash equilibria and for bidders with submodular valuation functions. There are still several intriguing open questions for future research in multi-unit auctions. First, it is not clear to us if our proof can be recast into the smoothness framework of [11,12]. Second, when going beyond submodular valuations to the superclass of subadditive functions, the bounds are not tight. It still remains elusive to produce lower bounds tailored for subadditive functions, and the best known upper bound is 4, due to [7]. Finally, a major open problem is to tighten the known gaps for the set of Bayes-Nash equilibria.

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