Chapter 2
Confined Electromagnetic Waves—Cavities

In this chapter I will briefly focus on the description of the electromagnetic field inside a cavity. Starting from Maxwell's equations, one can derive a formalism treating such field modes inside a resonator as harmonic oscillators. These field modes can then be quantized and treated as quantum harmonic oscillators. In the latter, I will use this formalism to describe electrical circuits and microwave cavities. This concepts are a broad field for itself and are a key in quantum optics and cavity quantum electro dynamics (cQED), and a comprehensive introduction can be found in the following textbooks [1–8] on which the following sections are based on.

2.1 Electromagnetic Radiation

2.1.1 Mode Expansion in Free Space

The starting point for describing the electromagnetic field in free space is the set of Maxwell equations

$$\vec{\nabla}\vec{E} = 0, \vec{\nabla}\vec{B} = 0, \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \text{ and } \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} \tag{2.1}$$

assuming no free charges and currents. Derived from Eq. 2.1, wave equations for the electric $\vec{E}$ and magnetic $\vec{B}$ field components read

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial^2}\vec{E} = 0 \text{ and } \Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial^2}\vec{B} = 0\ . \tag{2.2}$$

For which a solution can be found using a wave expansion for electro magnetic field components

S. Putz, *Circuit Cavity QED with Macroscopic Solid-State Spin Ensembles*,
Springer Theses, DOI 10.1007/978-3-319-66447-7_2

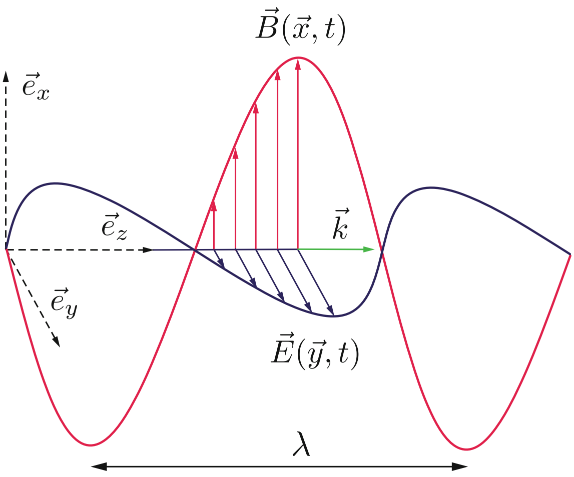

Fig. 2.1 Visualization of a transverse electromagnetic (TEM) wave

$$\vec{E}_y = \mathrm{i}\frac{\omega}{c\sqrt{V}}\sum_k \vec{e}_y A(t)\mathrm{e}^{\mathrm{i}\vec{k}\vec{z}} + c.c.\text{and}$$
$$\vec{B}_x = \mathrm{i}\frac{1}{\sqrt{V}}\sum_k \vec{k}\times\vec{e}_y A(t)\mathrm{e}^{\mathrm{i}\vec{k}\vec{z}} + c.c. \tag{2.3}$$

with $A(t) = A_0\mathrm{e}^{-\mathrm{i}\omega t}$. It follows that the amplitudes of the electric $\vec{E}$ and magnetic field $\vec{B}$ are in phase but perpendicular to each other. The polarization of the magnetic field $\vec{e}_x$ follows immediately by choosing the polarization direction of the electric field in the $\vec{e}_y$ direction and letting the wave propagate in the $\vec{e}_z$ direction, denoted by the wave vector $\vec{k} = \vec{e}_x \times \vec{e}_y$. Such purely transversal polarized electromagnetic waves are known as transverse electromagnetic (TEM) waves (Fig. 2.1). In free space there is an infinite number of possible wave modes with different wavelengths λ and frequencies following the dispersion relation $\omega = kc$ and therefore one has to sum over all possible $k = |\vec{k}|$ values in Eq. 2.3.

2.1.2 Modes Inside a Cavity

In the previous section the electromagnetic field in free space was described by a mode expansion. Thus the energy of the electromagnetic field inside a volume V follows by summation or integration over a continuum of all possible field modes. The number of modes in a volume V can be restricted by defining boundary conditions for the electromagnetic field. Here boundary condition simply mean that the electromagnetic field has to obey constraints on interfaces like walls. This is realized by assuming e.g. two reflecting surfaces spatially separated by a distance l. On the mirrors incident $\vec{E}_i$ and reflected electric field $\vec{E}_r$ experience a π phase shift, since $\vec{k} = \vec{E}_i \times \vec{B}_i =$

$-\vec{E}_r \times \vec{B}_r$ must be fulfilled. Reflections and phase shifts on both surfaces create interference and standing waves between the mirrors are formed. For such a standing wave, maxima and minima of the electric and magnetic field amplitudes do have a $\pi/2$ phase shift. Therefore after performing a wave expansion for the field inside the cavity the electromagnetic field components read

$$E_y(z,t) = \frac{1}{\sqrt{V}} \sum_n \frac{\omega_c}{c} \sin(k_n z) A_0 (\mathrm{e}^{\mathrm{i}\omega_c t} + \mathrm{e}^{-\mathrm{i}\omega_c t})$$
$$B_x(z,t) = \frac{\mathrm{i}}{\sqrt{V}} \sum_n \frac{\omega_c}{c} \cos(k_n z) A_0 (\mathrm{e}^{\mathrm{i}\omega_c t} - \mathrm{e}^{-\mathrm{i}\omega_c t}) \qquad (2.4)$$

with $k_n = \pi n/l$. Therefore the condition for creating a standing wave between the two mirrors (Fig. 2.2) (i.e. resonance) is given by the distance l which must be an integer multiple n of half of the wavelength i.e. $\lambda/2$ and $k_n = \frac{2\pi n}{\lambda}$. This means that after one round trip, the phase shift is equal to the initial phase of the wave. Hence a resonance condition follows and gives a fundamental resonance of the cavity field mode at $\omega_c = \frac{2\pi}{\lambda} c$.

The derived expressions for the electric and magnetic field inside the resonator, stated in Eq. 2.4, would include all higher harmonics of the cavity. However, in the latter only the fundamental resonance $\omega_c = \pi/l$ is of interest and all higher harmonics are ignored. This is usally assumed in cavity QED and known as a single mode cavity. The total energy in the cavity and single field mode ($n = 1$) is then given by integrating over the entire cavity volume V

$$\mathcal{E} = \frac{1}{2} \int_V (\epsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2) dV \rightarrow \frac{1}{2} \frac{\omega^2}{c^2} A_0^2 \qquad (2.5)$$

with vacuum permittivity ϵ_0 permeability μ_0. The total energy is thus proportional to the squared amplitude of $A(t)$ introduced in the previous section. The total energy $\mathcal{E}$ is constant but oscillating between electric and magnetic field components with the cavity eigenfrequency ω_c. This fact allows to treat the electric and magnetic field

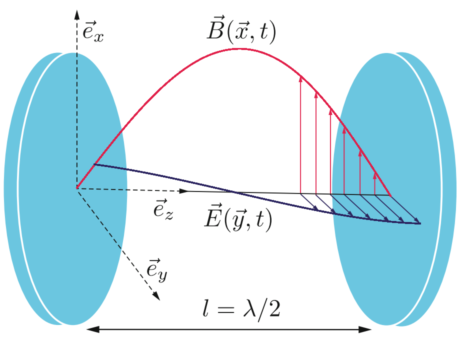

Fig. 2.2 Visualization of a the electromagnetic field between two mirrors creating a standing wave in a $\lambda/2$ cavity. The stored energy is oscillating between magnetic and electric field amplitudes with the cavity frequency ω_c

amplitudes inside the cavity as real and complex parts or field quadratures of the electromagnetic field as $A = E_x + \mathrm{i}B_y$. As we already know E_x and B_y experience a $\pi/2$ phase shift inside a cavity and span what we will later call phase space.

2.2 Single Cavity Modes—Harmonic Oscillators

2.2.1 Canonical Variables of the Electromagnetic Field

In the previous section (Eq. 2.5) the total energy of the electromagnetic field inside a cavity was derived. Here we show how a single field mode can be treated as a single harmonic oscillator. If the resonator is restricted to the fundamental resonance (i.e. single mode), the energy density is given by

$$\mathcal{H} = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) \tag{2.6}$$

with vacuum permittivity ϵ_0 permeability μ_0 and is a Hamiltonian function $\mathcal{H}$. As already discussed E and B are the field quadratures and are $\pi/2$ phase shifted, which is basically a set of canonical variables. The energy of the electromagnetic wave inside a cavity can be interpreted to be equivalent to the sum of the kinetic and potential energy of the system and allows to map this problem to the harmonic oscillator model from classical mechanics. In such, the total system energy can be written as a Hamiltonian reading

$$\mathcal{H} = \frac{1}{2}(\frac{p^2}{2m} + kx^2) \tag{2.7}$$

with momentum p, position x and force constant k. Both x and p can be renormalized such that

$$\mathcal{H} = \frac{\omega}{2}(X^2 + P^2)\ . \tag{2.8}$$

A linear equation of motion for $A = X + \mathrm{i}P$ follows

$$\omega^2 A + \ddot{A} = 0 \tag{2.9}$$

for which a solution is given by $A(t) = A_0 \mathrm{e}^{-i\omega t}$. Hence position $X(t) = \mathrm{Re}(A(t))$ and momentum $P(t) = \mathrm{Im}(A(t))$ are complex and imaginary parts of $A = X + \mathrm{i}P$. If compared to Eq. 2.5 this suggests that the energy of the electromagnetic field inside the cavity can be rewritten as

$$\mathcal{E} = \frac{\omega}{2}\{|A_0 \cos(\omega t)|^2 + |A_0 \sin(\omega t)|^2\} \tag{2.10}$$

with a field amplitude $|A| = \sqrt{\mathrm{Re}(A)^2 + \mathrm{Im}(A)^2}$ and phase $\phi = \tan \frac{\mathrm{Re}(A)}{\mathrm{Im}(A)}$ for the cavity field. Note that in comparison to the previous section $A(t)$ is renormalized and directly related to X and P by $\omega P = \dot{X}$. Form Hamilton's equations $\frac{\partial \mathcal{H}}{\partial X} = -\dot{P}$ and $\frac{\partial \mathcal{H}}{\partial P} = \dot{X}$ one further can conclude that X and P are canonical variables.

2.2.2 *Drive and Dissipation of a Classical Oscillator*

As shown above the electromagnetic field inside a cavity can be described by a harmonic oscillator. However, before quantizing the resonator field mode it is instructive to look on a classical damped and driven harmonic oscillator in order to give an intuitive picture first (Fig. 2.3).

The equation of motion for a mass m on a spring k_i with an internal damping constant γ_i reads

$$\ddot{x} + 2(\frac{\gamma_i + \gamma_c}{m})\dot{x} + (\frac{k_i + k_c}{m})x = \frac{k_c}{m} D\mathrm{e}^{\mathrm{i}\omega t} \tag{2.11}$$

with position x off the mass. A driving force is applied by a damped γ_c spring k_c which is periodically deflected with an amplitude D and frequency ω. By defining a frequency $\sqrt{\frac{k_c+k_i}{m}} = \omega_0$ the equation of motion can be transformed to

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \omega^2 D\mathrm{e}^{\mathrm{i}\omega t} \tag{2.12}$$

with $\gamma = (\gamma_i + \gamma_c)/m$. On should note that any coupling of a harmonic oscillator to an additional spring also changes its true eigenfrequency $\sqrt{\frac{k_i}{m}}$. With the Ansatz $x = A_0\mathrm{e}^{\mathrm{i}\omega t}$ a solution (i.e. $\dot{A} = 0$) for the stationery amplitude

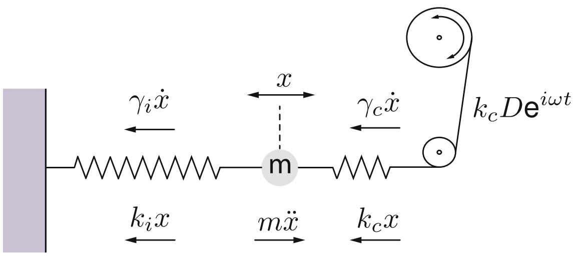

Fig. 2.3 Schematic drawing of a classical harmonic oscillator. A mass m on a spring k_i determines the natural eigenfrequency of the damped oscillator. Coupled to a second damped spring k_c this can be interpreted as a one sided optical cavity

$$|A_0|^2 = \frac{\omega^4 D^2}{(\omega_0^2 - \omega^2)^2 + (\omega 2\gamma)^2} \tag{2.13}$$

is found. The amplitude on resonance (i.e. $\omega_0^2 - \omega^2 = 0$) can be derived as

$$|A_{\mathrm{Res}}| = \frac{\omega}{2\gamma} D = QD \tag{2.14}$$

with the introduced quantity $Q = \omega/2\gamma$. The resonator quality factor Q states that the resonator amplitude is equal to the driving amplitude enhanced by a factor of Q. One should note that in the latter the quality factor Q is defined in the standard way by the ratio of stored to lost energy rather than terms of resonator amplitudes.

For cavities with high Q factors Eq. 2.13 can be well approximated by

$$|A|^2 = \frac{\eta^2}{(\omega_0 - \omega)^2 + \kappa^2} \tag{2.15}$$

if the damping constant γ becomes small. This Lorentzian function with a full-width at half-maximum (FWHM) line width 2κ and drive amplitude $\eta = \sqrt{\omega\kappa}D$ will be used in the latter to describe the cavity response. In Fig. 2.4 the difference between Eqs. 2.13 and 2.15 is plotted and also shows how the line shape of a harmonic oscillator approaches a Lorentzian function for small values of γ. From the Ansatz $x = Ae^{\mathrm{i}\omega t}$ it follows that the amplitude A oscillates sinusoidaly with frequency ω, thus a constant energy transfer between kinetic and potential stored energy. After the periodic driving force is switched off ($D = 0$) the amplitude $|A(t)|$ will decay with the rate κ while the intensity $|A(t)|^2$ decays twice as fast with 2κ. One should note that the squared amplitude of the harmonic oscillator has to be compared to the Lorentz function in order to estimate κ.

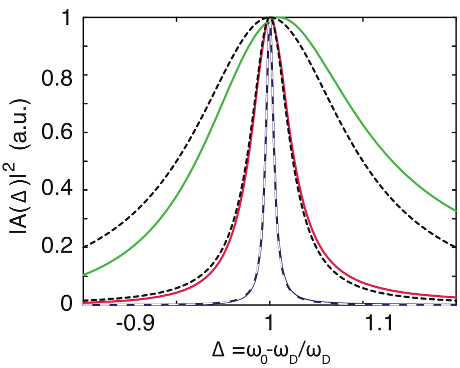

Fig. 2.4 Difference between Lorentzian oscillator model and low Q harmonic oscillators. As the damping constant γ gets smaller (*yellow* $\gamma = \omega_0/10$, *red* $\gamma = \omega_0/20$ and *blue* $\gamma = \omega_0/100$) the line is well approximated by a Lorentzian function with a line width of 2κ

2.2.3 *Quantum Harmonic Oscillator*

A single cavity mode can be treated as a harmonic oscillator and quantized in full quantum mechanical fashion. The starting point is the canonical transformation of the cavity field $A = X + \mathrm{i}P$ to the quantum mechanical creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ reading

$$\hat{a} = \frac{\hat{X} + \mathrm{i}\hat{P}}{\sqrt{2\hbar}} \text{ and } \hat{a}^\dagger = \frac{\hat{X} - \mathrm{i}\hat{P}}{\sqrt{2\hbar}} . \tag{2.16}$$

With the commutation relation for position and momentum operator $[\hat{P}, \hat{X}] = -\mathrm{i}\hbar$ the quantized expression for the Hamiltonian for a single mode cavity follows as

$$\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{2.17}$$

where the operator hats in $\hat{a}(\hat{a}^\dagger) \to a(a^\dagger)$ were dropped as in the rest of this thesis.

The mean energy stored in the system is equal to the number of stored excitations and an operator for the photon number then reads $N = a^\dagger a$. This operator uses a new eigenbasis representation for the energy in the cavity called Fock space or number states $\rangle n$. Creation and annihilation operators $a^\dagger$ and a raise and lower the number of excitations, since $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. This also means that although the cavity is empty $|0\rangle$ there will be a zero point energy $\hbar\omega/2$ due to vacuum fluctuations. Since the Hamiltonian is an expression combining kinetic and potential energy we can calculate its eigenstates or wavefunction by solving the time independent Schrödinger equation

$$\mathcal{H}\rangle\Psi = \mathcal{E}\rangle\Psi \tag{2.18}$$

where $\mathcal{E}$ is an energy eigenvalues found by Eq. 2.17. One finds a set of solutions for $|\Psi(x)\rangle$ using the Hermite polynomials [9] and can calculate the probability $|\Psi(x)|^2$ of finding the oscillator in the potential given by $\frac{\omega^2}{2}X^2$. As is shown in Fig. 2.5 by the plotted squared wave function of the quantum harmonic oscillator, the energy variance in the system becomes more and more classical in a sense when the number of excitations grows.

The quantized harmonic oscillator model and its ladder operators $a^\dagger$ and a, can be used to quantize the electromagnetic field inside the cavity. The field quadratures read

$$\begin{aligned} X &= \mathrm{Re}(A) = \sqrt{\frac{\hbar}{2}}(a + a^\dagger) \to E_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(a + a^\dagger)\sin(kz) \\ P &= \mathrm{Im}(A) = -\mathrm{i}\sqrt{\frac{\hbar}{2}}(a - a^\dagger) \to B_y = -\mathrm{i}\sqrt{\frac{\hbar\omega\mu_0}{2V}}(a - a^\dagger)\cos(kz) \end{aligned} \tag{2.19}$$

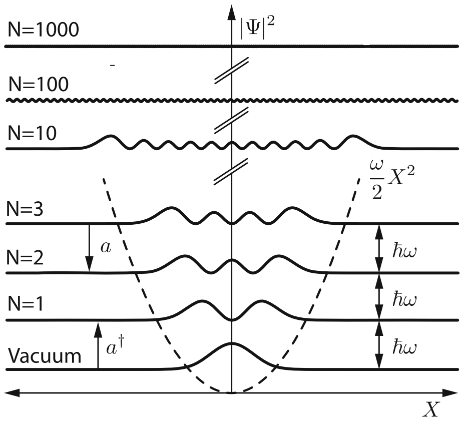

Fig. 2.5 The probability distribution of the quantum harmonic oscillator $|\Psi(x)|^2$ of finding the oscillator somewhere in the potential given by $\frac{\omega^2}{2}X^2$. The wave function and hence the probability of the oscillator becomes more and more de-localized for growing number of excitations $\langle N\rangle$

in a unit volume of size V with the vacuum permittivity ϵ_0 and permeability μ_0. The electromagnetic field is therefore quantized in the same way as the quantum harmonic oscillator. From Eq. 2.19 one finds that the expectation values for the electromagnetic field components read

$$\langle n|E|n\rangle = 0 = \langle n|B|n\rangle \tag{2.20}$$

and become zero. For the described harmonic oscillator this is equivalent to having zero amplitude, which is a possible but rather special case. Therefore, a formalism to introduce states with well defined field amplitudes are coherent states

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{2.21}$$

which are eigenstates of the annihilation operator a. These states $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\phi}$, with amplitude $|\alpha|$ and phase ϕ, can be expanded in the Fock or number basis states

$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.22}$$

since coherent states $\alpha|\alpha\rangle$ are eigenstates of a. The probability of detecting n photons in a coherent state $|\langle n|\alpha\rangle|^2$ obeys a Poisonian distribution and becomes Gaussian for large $|\alpha\rangle$. The expectation values of the field quadratures therefore read

$$\begin{aligned}\langle X\rangle &= \sqrt{\frac{\hbar}{2}}(\alpha+\alpha^*) = \mathrm{Re}(\alpha)\\ \langle P\rangle &= -\mathrm{i}\sqrt{\frac{\hbar}{2}}(\alpha-\alpha^*) = \mathrm{Im}(\alpha)\end{aligned} \tag{2.23}$$

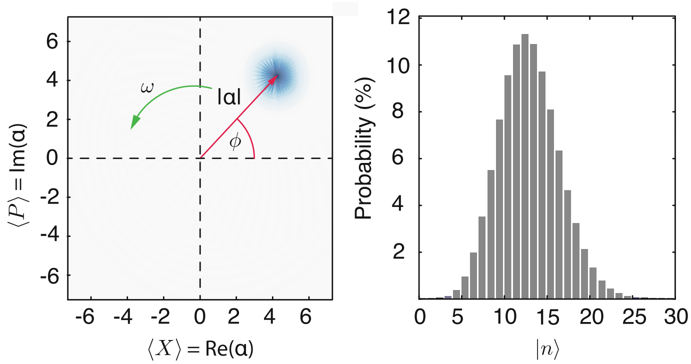

Fig. 2.6 *Left* Visualization of a coherent cavity state with amplitude $|\alpha| = \sqrt{\langle X\rangle^2 + \langle P\rangle^2}$ rotating with ω in phase space. *Right* Since coherent states are eigenstates of the annihilation operator a they can be expanded in the Fock basis $|n\rangle$. For a state $|\alpha\rangle$ we find a photon distribution $\langle N\rangle = |\alpha|^2$ with variance $\Delta N = |\alpha|$

and one finds the average energy of a coherent cavity state to be

$$\langle \mathcal{E}\rangle = \hbar\omega|\alpha|^2 = \hbar\omega\langle N\rangle \tag{2.24}$$

with variance $\Delta\mathcal{E} = \hbar\omega|\alpha|$. For large amplitudes the relative variance $\Delta\mathcal{E}/\langle\mathcal{E}\rangle$ goes to zero and the state becomes more and more well defined or "classical" as is shown in Fig. 2.6.

2.3 Electrical Oscillators—"From Classical to Quantum"

2.3.1 Quantization of a LC Oscillator

In this section an electric LC circuit is discussed as a harmonic oscillator which can be fully quantized. The previously introduced formalism is used to describe such a circuit which can be considered as the counterpart to an optical cavity. The starting point as is shown in Fig. 2.7 is a circuit consisting of an inductance L and capacitor C in a parallel configuration. The inductance is a measure of how much magnetic flux Φ is created in a circuit carrying a given current I, hence the relation $L = \Phi/I$. Additionally it follows from Faraday's law that by an an alternating current a voltage $V_L = \frac{d\Phi}{dt} = L\frac{dI}{dt}$ is induced in the circuit. Contrary, the capacitance C of two parallel plates is given by the ratio of the charging[1] $\pm Q$ and the voltage V across both plates, given by $V_C = Q/C$. For a closed circuit the potential across the circuit has to be

[1] *Nota bene:* in this section the charge is abbreviated by capital Q and should not be confused with the cavity quality factor Q through out the rest of this thesis.

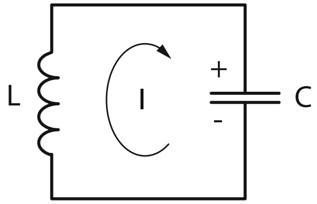

Fig. 2.7 Visualization of a LC resonant circuit. The energy is oscillating back and forth between the capacitor C and inductor L. The charged capacitor C stores the kinetic energy, while the inductor L stores the kinetic energy of the harmonic LC oscillator model

zero ($V_C + V_L = 0$) and at any node of the circuit the sum of in and out flowing currents hast to be zero ($I_C = I_L$) according to Kirchhoff's circuit laws. From this the following equation of motion for an electrical LC circuit

$$Q + \frac{1}{LC}\ddot{Q} = 0 \tag{2.25}$$

can be derived with a resonance frequency given by $\omega = \frac{1}{\sqrt{LC}}$ and characteristic impedance $Z = \sqrt{\frac{L}{C}}$. The energy stored in the inductance L and capacitance C is given by $\mathcal{E}_L = \frac{1}{2}LI^2 = \frac{\Phi^2}{2L}$ and $\mathcal{E}_C = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$, respectively.

With the total energy density in the circuit one can make a transition to Hamilton's mechanic

$$\mathcal{H} = \frac{1}{2}(\frac{Q^2}{C} + L\dot{Q}^2) = \frac{1}{2}(\frac{Q^2}{C} + \frac{\Phi^2}{L}) \tag{2.26}$$

with $\partial\mathcal{H}/\partial\Phi = \dot{Q} = \Phi/L$ and $\partial\mathcal{H}/\partial Q = -\dot{\Phi} = Q/C$. The magnetic flux Φ is considered as the generalized momentum with kinetic energy $\mathcal{E}_{\text{kin}} = \frac{\Phi^2}{2L}$ stored in the inductor. Whereas the electric charge Q is the generalized coordinate connected to the potential energy stored in the capacitor $\mathcal{E}_{\text{pot}} = \frac{Q^2}{2C}$. Therefore the electric charge and magnetic flux are the canonical variables of the harmonic oscillator. This circuit can be quantized in a quantum mechanical way by using the commutator relation $[\Phi, Q] = -\mathrm{i}\hbar$. Annihilation a and creation $a^\dagger$ operators can be expressed in terms of Q and Φ as

$$Q = \sqrt{\frac{\hbar}{2Z}}(a + a^\dagger) \text{ and } \Phi = -\mathrm{i}\sqrt{\frac{\hbar Z}{2}}(a - a^\dagger) \tag{2.27}$$

and the Hamiltonian for a loss-less quantized LC circuit reads

$$\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2})\ . \tag{2.28}$$

In analogy to the quantization of the electromagnetic field inside a cavity, voltage and current operator reading

$$V = \sqrt{\frac{\hbar\omega}{2C}}(a + a^\dagger) \text{ and } I = -\mathrm{i}\sqrt{\frac{\hbar\omega}{2L}}(a - a^\dagger) \tag{2.29}$$

can be found as field quadratures. The zero point energy in the cavity, ($a^\dagger a = 0$), caused by vacuum fluctuations is then given by

$$\mathcal{E}_{\text{vac}} = \mathcal{E}_C + \mathcal{E}_L = \frac{1}{2}(CV^2 + LI^2) = \frac{\hbar\omega}{2}\,. \tag{2.30}$$

2.3.2 Drive and Dissipation in a Classical *LC* Oscillator

After deriving an equation of motion and the quantization of a simple electrical LC resonator dissipation and coupling to external ports has to be added to such a circuit. In Fig. 2.8 a circuit diagram of a lumped element resonator in parallel LC configuration is shown. The circuit includes an internal resistance R accounting for dissipation and is coupled by two capacitors C_k to a feedline with $Z_0 = 50\ \Omega$ at port one and two. The input impedance of the uncoupled classical electrical resonator is given by [8]

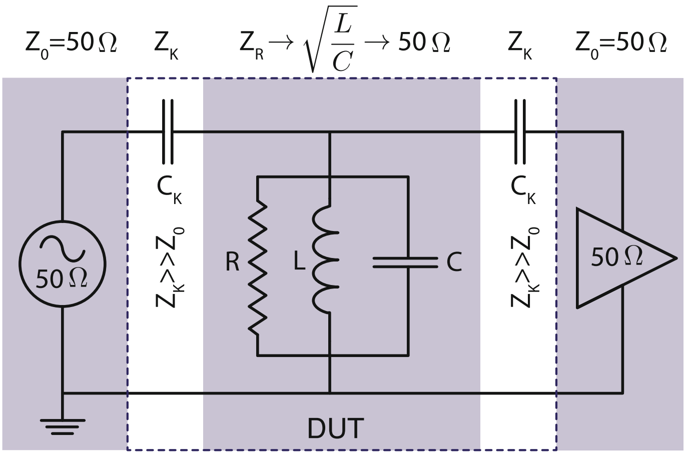

Fig. 2.8 Lumped element equivalent circuit diagram of a RLC resonant circuit connected to a 50 Ω load via two coupling capacitors C_k. The total quality-factor Q_{tot} has contributions from the internal quality factor Q_I determined by R and a coupling quality-factor Q_k which is $\propto C_k^2$

$$Z_{in} = \left(\frac{1}{R} + \frac{1}{\mathrm{j}\omega L} + j\omega C\right)^{-1} \tag{2.31}$$

with a characteristic impedance of the circuit $Z_R = \sqrt{\frac{L}{C}}$. The energy dissipated by the resonator is given by $P_d = \frac{1}{2}|V|^2/R$ and the maximum energy stored is given by $\mathcal{E} = \frac{1}{2}V^2 C$ since on resonance electric and magnetic energy are equal. For such a parallel RLC resonator one can then derive an quantity for the internal quality factor which reads

$$Q_I = \omega RC = \frac{\omega}{\kappa_I} \tag{2.32}$$

and is the relation of maximum stored to dissipated energy per cycle. The quality factor depends on the internal resistance which determines the total dissipation. By defining $R = Z_0/\alpha$ with an attenuation α the Q factor would go to infinity for a perfect loss less LC resonator.

If a feedline with characteristic impedance $Z_0 = 50\ \Omega$ and a $50\ \Omega$ load R_L is coupled to the cavity by two capacitors C_k with matched impedance given by $Z_R = \sqrt{\frac{L}{C}} = 50\ \Omega$ the total Q factor of the resonator is modified. The small input coupling capacitors have a huge impedance compared to the loaded feedline, which is a discontinuity creating a standing wave between both C_k's. The coupling strength to the feedline can be controlled by C_k and the coupling quality factor follows as $Q_k \sim C/C_K^2 R_L \omega$. Therefore the total quality factor of the coupled cavity reads

$$\frac{1}{Q_{tot}} = \frac{1}{Q_I} + \frac{1}{Q_k} \tag{2.33}$$

and we can state a total energy dissipation rate κ as

$$\frac{1}{\kappa} = R_{tot} C_{tot} \tag{2.34}$$

with total resistance R_{tot} and capacitance C_{tot}. One should also note that the coupling capacitors also change the resonance frequency of the circuit since ω is given by $1/\sqrt{LC_{tot}}$. The ratio of coupling to internal quality factor

$$g_c = \frac{Q_I}{Q_k} = \frac{\kappa_k}{\kappa_I} \tag{2.35}$$

is an important figure of merit and one can identify three different cases for g_c:

- $g_c < 1$ the resonator is under-coupled, and Q_{tot} is dominated by Q_I and κ_I
- $g_c > 1$ the resonator is over-coupled, and Q_{tot} is dominated by Q_k and κ_k
- $g_c = 1$ the resonator is critically coupled, meaning $Q_I = Q_k$ and $\kappa_I = \kappa_k$

As already discussed the resonator is connected to a matched feedline as is shown in Fig. 2.8. The feedline further is connected to a 50 Ω load on both sides, a voltage source at the left port (number one) and to a detector on the right port (number two). Through the source an alternating voltage can be applied and is transmitted through the cavity and detected at port two. The impedance matrix Z_{ij} of the circuit relates voltages V_i and currents I_j on ports i and j by

$$Z_{ij} = \frac{V_i}{I_j} \tag{2.36}$$

with $I_j = 0$ for $j = i$. A more convenient formalism is to use scattering matrices relating input V^- and output V^- voltages at every port by the relation $V^- = SV^+$ (see Fig. 2.9). The generalized scattering parameters for the voltages at port one and two read

$$V_i = \frac{1}{Z_0}(V_i^- + V_i^+) \tag{2.37}$$

which are generally used if the characteristic impedance changes over a multiport network. The relation of voltages at both ports then follow as

$$\begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix} \tag{2.38}$$

where the scattering matrix describes the response of a resonator or more generally speaking a device under test (DUT).

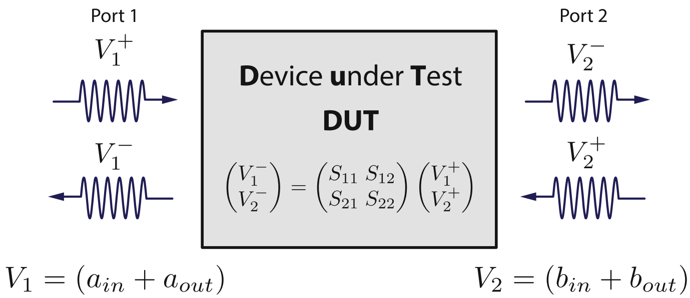

Fig. 2.9 Visualization of a device under test (DUT). A voltage at port 1 $V_1 = V_1^+ + V_1^-$ is related by a scattering matrix S to a $V_2 = V_2^+ + V_2^-$ at port 2

2.3.3 Quantum Mechanical Description of a Real LC Cavity

The resonator and its input and output signals are related by the scattering matrix of the cavity. A different treatment of this problem is known as input output formalism in which the Hamiltonian in Eq. 2.28 is used in a Master equation approach. In the Heisenberg picture an operator equation is given therefore by

$$\dot{a}(t) = -\frac{\mathrm{i}}{\hbar}[a(t), \mathcal{H}] - (2\kappa_k + \kappa_I)a(t) + \sqrt{2\kappa_k}(a_{in}(t) + b_{in}(t)) \tag{2.39}$$

for the cavity field operator a. The rate κ_I introduces damping by an internal reservoir. The external modes $a_{in}(t) \propto V_1^+$ and $b_{in}(t) \propto V_2^+$ are coupled to the resonator mode by $\sqrt{\kappa_k}$ according to a first Markov approximation. This linear operator equation can thus be written as

$$\dot{a}(t, \omega_p) = -\mathrm{i}(\omega - \omega_p)a(t) - (2\kappa_k + \kappa_I)a(t) + \sqrt{2\kappa_k}(a_{in}(t) + b_{in}(t)) \tag{2.40}$$

for a symmetrically coupled two port cavity (i.e. $\kappa_k \to 2\kappa_k$) with an external driving voltage with frequency ω_p ($V = V_0(\exp(\mathrm{i}\omega_p) + \exp(-\mathrm{i}\omega_p))$). The frequency dependent scattering matrix is obtained by solving the cavity operator equation for the stationary state (i.e. $\dot{a}(t, \omega_p) = 0$), with the boundary conditions at each port reading

$$a_{in} + a_{out} = \sqrt{\kappa_k}\, a \text{ and } b_{in} + b_{out} = \sqrt{\kappa_k}\, a\,. \tag{2.41}$$

After introducing the total dissipation rate as $\kappa = \kappa_I + \kappa_k$ one finds then

$$b_{out} = \frac{\kappa_k(a_{in} + b_{in}) - b_{in}(\kappa - \mathrm{i}(\omega - \omega_p))}{\kappa - \mathrm{i}(\omega - \omega_p)}\,. \tag{2.42}$$

Therefore the resulting scattering matrix elements can be calculated as $S_{21} = \frac{b_{out}}{a_{in}}$ and $S_{11} = \frac{a_{out}}{a_{in}}$. Equations for the transmitted and reflected power then read

$$\begin{aligned}|T|^2 = |S_{21}|^2 &= \frac{\kappa_k^2}{\kappa^2 + (\omega - \omega_p)^2} \text{ and} \\ |R|^2 = |S_{11}|^2 &= 1 - \frac{\kappa_k^2}{\kappa^2 + (\omega - \omega_p)^2}\end{aligned} \tag{2.43}$$

since the average power delivered e.g. to port one is given by $P_1 = \frac{1}{2}(|a_{in}|^2 + |a_{out}|^2)$. Again three different coupling scenarios for the ratio of $g_c = \frac{\kappa_k}{\kappa_I}$ are found as introduced in the previous section and are shown in Fig. 2.10. In the discussed experiments later on over-coupled cavities are used and the dissipation rate of the resonator is approximated by a single dissipation constant $\kappa \sim 2\kappa_k$.

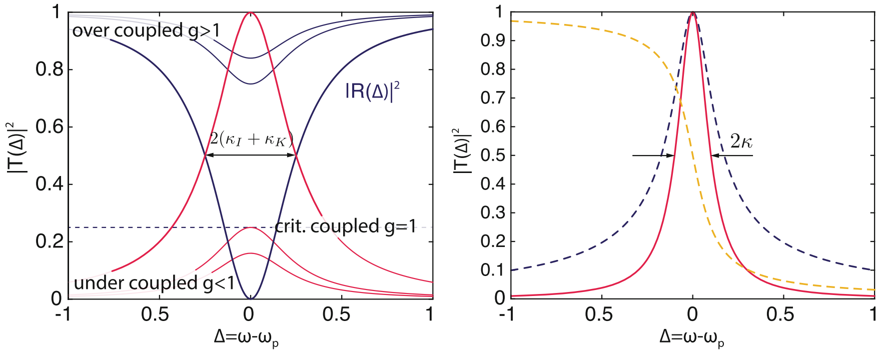

Fig. 2.10 *Left* Transmission $|T|^2 = |S_{21}|^2$ and reflection $|R|^2 = |S_{11}|^2$ of a two port *RLC* resonator. If $\kappa_I > \kappa_k$ the resonator is under coupled, critically coupled for $\kappa_I = \kappa_k$ and over coupled for $\kappa_I < \kappa_k$. *Right* The Lorentz oscillator model where only one dissipation rate κ determines the line width. *Red solid line* corresponds to the transmitted squared cavity amplitude, *blue dotted line* cavity amplitude and *yellow dotted line* to the phase of the resonator

A Hamiltonian for the coupled cavity (i.e. LC resonator) that includes a driving term then reads

$$\mathcal{H}_c = \hbar\omega_c a^\dagger a + \mathrm{i}\hbar\eta(a\mathrm{e}^{\mathrm{i}\omega_p t} - a^\dagger \mathrm{e}^{-\mathrm{i}\omega_p t}) \tag{2.44}$$

with a cavity eigenfrequency ω_c, driving frequency ω_p and transmitted drive η. In the rotating frame one can replace ω_c with $\Delta = \omega_c - \omega_p$ and work in a frame rotating with ω_c. Dissipation can be introduced as a complex frequency contribution $-\mathrm{i}\kappa$ to ω_c and from a Heisenberg equation

$$\dot{a} = -\frac{i}{\hbar}[a, \mathcal{H}_c] \tag{2.45}$$

the linear cavity operator equations can be derived reading $\dot{a} = (\kappa - \mathrm{i}\Delta)a + \eta$. The steady state solution of a can be found by setting $\dot{a} = 0$ and reads

$$a = \frac{\eta}{\kappa - \mathrm{i}\Delta} \; . \tag{2.46}$$

which is a complex function. The expectation value of a can be identified as the cavity field amplitude $|A|$ since $\langle a^\dagger a\rangle = |A|^2$ and an expressions for squared cavity amplitude and phase read

$$|A|^2 = \frac{\eta^2}{\Delta^2 + \kappa^2} \rightarrow |T|^2 = \frac{\kappa^2}{\Delta^2 + \kappa^2} \text{ and } \phi = \tan^{-1}(\frac{\Delta}{\kappa}) \tag{2.47}$$

with $|A|^2$ having the form of a Lorentzian function of width of 2κ (Fig. 2.11). If a coherent drive field is injected into the cavity a coherent cavity state $|\alpha\rangle$ is created

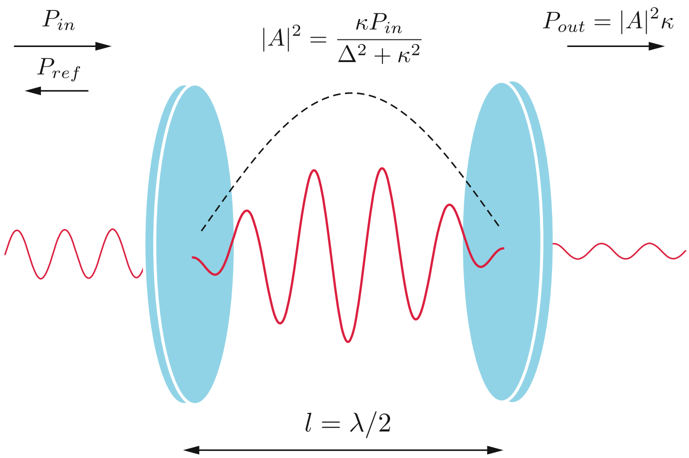

Fig. 2.11 Visualization of a the electromagnetic field between two mirrors forming a cavity. Incident P_{in} and output power P_{out} are related by the cavity and it's response can be approximate near resonance by a Lorentzian function

with a mean photon number $|A|^2 \rightarrow |\alpha|^2 = \langle N \rangle$. The mean photon number for a given input power P_{in} [W] is then obtained by replacing η^2 with $\tilde{\eta}^2 = \kappa P_{\mathrm{in}}$ (J/s^2), from which the mean photon number in the cavity follows as $|\alpha|^2 = \tilde{\eta}^2/\hbar\omega_c\kappa^2 = P_{\mathrm{in}}/\hbar\omega_c\kappa$ with $\Delta = 0$. Therefore the driving power needed to have on average one photon in the cavity corresponds to $\eta^2 = \kappa^2$, since $\langle N \rangle = \eta^2/\kappa^2 = 1$.

The cavity amplitude sets into a steady state under the action of a continuous drive. After the drive is switched off again the intra-cavity field $|A|^2$ decays, which is known as cavity ring down. The time dependent cavity amplitude after a coherent drive with power P_{in} is switched off is given by

$$\dot{\alpha(t)} = -\kappa|\alpha(0)| \tag{2.48}$$

with initial amplitude $\alpha(0)$ and the solution $\alpha(t) = |\alpha(0)|\mathrm{e}^{-(\kappa+\mathrm{i}\omega)t}$. The energy and mean photon number decays exponentially as

$$\langle N(t) \rangle = |\alpha(0)|^2\mathrm{e}^{-2\kappa t} \tag{2.49}$$

with a time constant $1/2\kappa$. The cavity Hamiltonian in Eq. 2.44 is solved by a Master equation approach using a quantum optics toolbox,[2] with a collapse operator $\sqrt{2\kappa}a$ and under resonant driving $\omega_p = \omega_c$. In Fig. 2.12 the time dependent mean cavity

[2]I will use http://qutip.org and can highly recommend this free python based Quantum optics tool box.

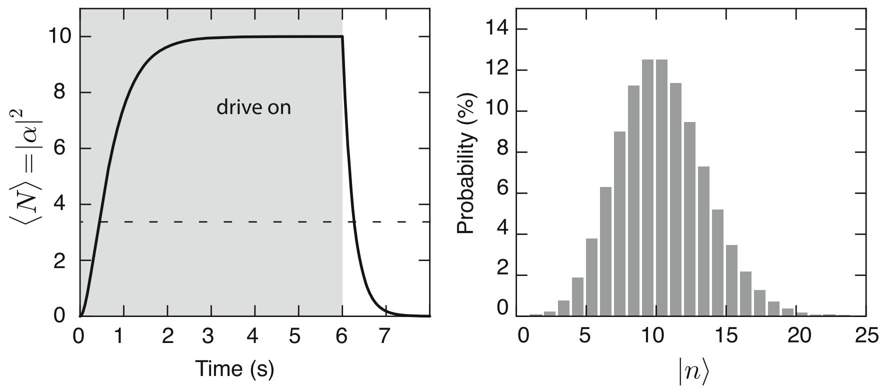

Fig. 2.12 *Left* time dependent mean cavity photon number $\langle N \rangle$ under the action of a rectangular coherent drive pulse. *Right* the corresponding photon distribution in Fock space for the cavity steady state under action of a coherent drive, derived from a full quantum mechanical calculation using http://qutip.org

photon number is plotted for $\kappa/2\pi = 2$ Hz and a drive $\eta^2 = 40$ photons/s^2. After the drive is switched on the cavity field will build up and reaches its steady state value of $\langle N \rangle = \eta^2/\kappa^2 = 10$ photons and exponentially decays with 2κ after switch off.

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Circuit Cavity QED with Macroscopic Solid-State Spin Ensembles
Putz, S.
2017, XVIII, 124 p. 75 illus., 65 illus. in color., Hardcover
ISBN: 978-3-319-66446-0