Chapter 2
Models of Composite Materials and Mathematical Methods of Their Investigation

2.1 General Relations of the Linear Theory of Elasticity

The linear theory of elasticity yields the following relations between the displacements u_i, deformations ε_{ij} and stress σ_{ij} in a continuous matter:

(i) Cauchy equations:

$$\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}, \quad \varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}. \tag{2.1}$$

(ii) Equations of deformation compatibility:

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0,\\
&\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} - \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0,\\
&\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} - \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0,\\
&2\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} + \frac{\partial^2 \varepsilon_{23}}{\partial x_1^2} - \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_3} - \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_2} = 0,\\
&2\frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} + \frac{\partial^2 \varepsilon_{13}}{\partial x_2^2} - \frac{\partial^2 \varepsilon_{12}}{\partial x_2 \partial x_3} - \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_2} = 0,\\
&2\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} + \frac{\partial^2 \varepsilon_{12}}{\partial x_3^2} - \frac{\partial^2 \varepsilon_{13}}{\partial x_2 \partial x_3} - \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_3} = 0.
\end{aligned} \tag{2.2}$$

In what follows, we consider solutions to the elastic problems regarding displacements, where the relations (2.2) are satisfied as identities.

I.V. Andrianov et al., *Asymptotical Mechanics of Composites*,
Advanced Structured Materials 77, https://doi.org/10.1007/978-3-319-65786-8_2

(iii) Equilibrium equations:

$$\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+f_1=0,\\
&\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+f_2=0,\\
&\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+f_3=0,
\end{aligned} \tag{2.3}$$

(iv) Hooke's law for anisotropic body:

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{bmatrix}=\begin{bmatrix}J_{11}&J_{12}&J_{13}&J_{14}&J_{15}&J_{16}\\ \cdot&J_{22}&J_{23}&J_{24}&J_{25}&J_{26}\\ \cdot&\cdot&J_{33}&J_{34}&J_{35}&J_{36}\\ \cdot&\cdot&\cdot&J_{44}&J_{45}&J_{46}\\ \cdot&\cdot&\cdot&\cdot&J_{55}&J_{56}\\ \cdot&\cdot&\cdot&\cdot&\cdot&J_{66}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}. \tag{2.4}$$

In the so far reported relations f_i stand for components of the volume force, $[J_{mn}]$ is the symmetric matrix of the elasticity coefficients and $J_{mn}=J_{nm}$, $\sigma_{ij}=\sigma_{ji}$, $\varepsilon_{ij}=\varepsilon_{ji}$; $i,j=1,2,3$; $m,n=1,\dots,6$.

In the case of the orthotropic materials having symmetric properties with respect to three perpendicular planes, the coefficients of the matrix $[J_{mn}]$ are equal to:

$$\begin{aligned}
&J_{11}=1/E_1,\quad J_{22}=1/E_2,\quad J_{33}=1/E_3,\\
&J_{12}=-\mu_{21}/E_2,\quad J_{13}=-\mu_{31}/E_3,\quad J_{23}=-\mu_{32}/E_3,\\
&J_{44}=1/G_{23},\quad J_{55}=1/G_{13},\quad J_{66}=1/G_{12},\\
&J_{14}=J_{15}=J_{16}=J_{24}=J_{25}=J_{26}=J_{34}=J_{35}=J_{36}=J_{45}=J_{46}=J_{56}=0,
\end{aligned} \tag{2.5}$$

where E_i—Young moduli; G_{ij}—shear moduli; μ_{ij}—Poisson's coefficients and $\mu_{ij}E_j=\mu_{ji}E_i$.

One-directional fibre composites (Fig. 3.1) exhibit a property of the transversal orthogonality. Let axis x_3 coincides with the fibres direction. Hook's law (2.4) and (2.5) can be recast to the following form

$$\begin{aligned}
&\varepsilon_{11}=\frac{\sigma_{11}}{E_T}-\mu_T\frac{\sigma_{22}}{E_T}-\mu_L\frac{\sigma_{33}}{E_L},\\
&\varepsilon_{22}=-\mu_T\frac{\sigma_{11}}{E_T}+\frac{\sigma_{22}}{E_T}-\mu_L\frac{\sigma_{33}}{E_L},\\
&\varepsilon_{33}=-\mu_L\frac{\sigma_{11}}{E_L}-\mu_L\frac{\sigma_{22}}{E_L}+\frac{\sigma_{33}}{E_L},\\
&\varepsilon_{23}=\frac{\sigma_{23}}{G_L},\quad \varepsilon_{13}=\frac{\sigma_{13}}{G_L},\quad \varepsilon_{12}=\frac{\sigma_{12}}{G_T}.
\end{aligned} \tag{2.6}$$

Properties of such composites can be described by sixth independent elastic constant: $E_L=E_3$, $E_T=E_1=E_2$, $G_L=G_{13}=G_{23}$, $G_T=G_{12}$, $\mu_L=\mu_{31}=\mu_{32}$, $\mu_T=\mu_{12}=\mu_{21}$, where $\mu_{13}=\mu_{23}=\mu_LE_T/E_L$.

Hexagonal lattice composed of cylindrical fibres (Fig. 3.1b) exhibits a transversal isotropy. In this case, relations (2.6) are supplemented by the dependence $G_T = E_T/[2(1+\mu_T)]$, whereas a number of independent elastic constants achieve five.

In grain composites (Fig. 4.1), three planes of symmetry can be mutually changed. The Hook's law in this case takes the following form

$$\begin{aligned}
\varepsilon_{11} &= \frac{\sigma_{11}}{E} - \mu\frac{\sigma_{22}}{E} - \mu\frac{\sigma_{33}}{E},\\
\varepsilon_{22} &= -\mu\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E} - \mu\frac{\sigma_{33}}{E},\\
\varepsilon_{33} &= -\mu\frac{\sigma_{11}}{E} - \mu\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E},\\
\varepsilon_{23} &= \frac{\sigma_{23}}{G}, \quad \varepsilon_{13} = \frac{\sigma_{13}}{G}, \quad \varepsilon_{12} = \frac{\sigma_{12}}{G},
\end{aligned} \tag{2.7}$$

where $E = E_1 = E_2 = E_3$, $G = G_{12} = G_{13} = G_{23}$, $\mu = \mu_{ij}$. A number of independent elastic constants are now reduced to three.

Finally, in the case of an isotropic matter, in addition to Eq. (2.7), the relation $G = E/[2(1+\mu)]$ is added. In this case, we have only two elastic constants.

In practical problems in order to describe properties of the materials often, the so-called volume modulus K exhibiting relations between the volume deformation and the sum of normal stress is introduced. In the case of a plane transversal deformation of the fibre composites ($\varepsilon_{33} = 0$) we have $\sigma_{11} + \sigma_{22} = 2K_T(\varepsilon_{11} + \varepsilon_{22})$, where $K_T = \left[\frac{2(1-\mu_T)}{E_T} - \frac{4\mu_L^2}{E_L}\right]^{-1}$. In the case of deformation of the grain composites the following relation holds: $\sigma_{11}+\sigma_{22}+\sigma_{33} = 3K(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$, where $K = \frac{E}{3(1-2\mu)}$.

2.2 General Relations of the Linear Theory of Viscoelasticity

In the case of many composite materials with components situated on a polymer basis, the deformation processes depends on velocity and time. Properties of those composites can be described in frame of the theory of viscoelasticity. The linear isothermic relations between stresses $\sigma(t)$ and deformations $\varepsilon(t)$ take the following form [1]:

$$\begin{aligned}
\sigma(t) &= \int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau,\\
\varepsilon(t) &= \int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau,
\end{aligned} \tag{2.8}$$

where $E(t)$—relaxation function, $J(t)$—creep function.

The choice of the function $E(t)$, $J(t)$ and determination of their parameters is mainly based on the experimental investigations. In practical problems aimed on modelling of the relaxation processes Duffing's power kernel, Rzhanitsyn's exponential power kernel, Rabotnov's fractional exponential kernel, and product of the exponential kernel and Rabotnov's kernel [2] are employed. It should be noted that while describing the volume and shear deformations, the various rheological models are applied. It is often assumed that the volume deformations are elastic, whereas shear deformations are viscoelastic.

Hashin [3, 4] has shown that elastic moduli and viscoelastic moduli (and creep compliances) of heterogeneous materials of identical phase geometry are related by the analogy (correspondence principle). That is, a solution for an elastic effective property can be interpreted as a viscoelastic solution in the Laplace transform domain by replacing the elastic properties through the transform parameter multiplied Laplace transforms of the corresponding viscoelastic properties. The final step of the solution of the viscoelastic problem involves the inverse Laplace transform (Sect. 2.5).

Let us apply Laplace transform (2.21) to relations (2.8). Using the convolution theorem (2.28), one obtains

$$\bar{\sigma}(s) = s\bar{E}(s)\bar{\varepsilon}(s), \quad \bar{\varepsilon}(s) = s\bar{J}(s)\bar{\sigma}(s), \tag{2.9}$$

where

$$s^2\bar{E}(s)\bar{J}(s) = 1. \tag{2.10}$$

Relations (2.9) are mathematically equivalent to the relations given by theory of elasticity. Therefore, solutions to static viscoelastic problems can be obtained on a basis of the known solutions for the elastic matter assuming that instead of the stiffness moduli and the flexibility coefficients the quantities $s\bar{E}(s)$ and $s\bar{J}(s)$ are taken, whereas the elastic stress and deformations ε will be exchanged by the counterpart transforms $\bar{\sigma}(s)$ and $\bar{\varepsilon}(s)$.

The given principle can be directly employed for determination of the effective functions of relaxation $E_0(t)$ and creep $J_0(t)$ of the composite materials. Solutions for the effective elastic characteristics can be interpreted as solutions for the transforms $s\bar{E}_0(s)$, $s\bar{J}_0(s)$, where instead of the elastic properties of the components one may substitute their viscoelastic counterparts $s\bar{E}^{(a)}(s)$, $s\bar{J}^{(a)}(s)$. The final solutions are found through the inversed transformation $\bar{E}_0(s)$, $\bar{J}_0(s)$, which can be realized via numerous methods and algorithms [5–8].

In dynamic problems the viscoelastic properties of materials can be described using a concept of the complex moduli. Consider a regime of stationary vibrations, where stress and deformations are harmonic functions of the form

$$\sigma(t) = \sigma_0 \exp(i\omega t), \quad \varepsilon(t) = \varepsilon_0 \exp(i\omega t), \tag{2.11}$$

where σ_0, ε_0 are amplitudes, and ω is a frequency.

Substituting (2.11) into (2.8), and carrying out the Laplace transformation, we get

$$\sigma_0 = E^*(\omega)\varepsilon_0, \quad \varepsilon_0 = J^*(\omega)\sigma_0,$$
$$\sigma(t) = E^*(\omega)\varepsilon(t), \quad \varepsilon(t) = J^*(\omega)\sigma(t), \tag{2.12}$$

where $E^*(\omega)$ stands for the complex stiffness modulus, $J^*(\omega)$ is the complex coefficient of flexibility $E^*(\omega) = 1/J^*(\omega)$, and

$$E^*(\omega) = i\omega\bar{E}(s), \quad J^*(\omega) = i\omega\bar{J}(s). \tag{2.13}$$

Relations (2.10) and (2.13) imply that for a given problem $s = i\omega$, and hence $E^*(\omega) = i\omega\bar{E}(i\omega)$, $J^*(\omega) = i\omega\bar{J}(i\omega)$.

Physical meaning of relations (2.12) is as follows: for a given constant amplitude of deformation $\varepsilon_0 =$ const, the amplitude of stress generated in a viscoelastic material is a function of a frequency, i.e. $\sigma_0 = \sigma_0(\omega)$, and vice versa.

Observe that formulas (2.12) are mathematically equivalent to the Hooke's law (2.4). Consequently, the effective complex moduli of viscoelastic composites can be defined with a help of solutions for the effective elastic characteristics, if instead of the real elastic properties of the components we take their complex analogue.

Based on relations (2.12) we obtain

$$\sigma(t) = \varepsilon_0 \exp\{i[\omega t + \psi(\omega)]\}, \tag{2.14}$$

where $\exp[i\psi(\omega)] = E^*(\omega)$. Comparing (2.14) and (2.11), one may interpret the parameter $\psi(\omega)$ as the phase angle of a deformation delay with respect to the stress. In practical problems the following quantity is introduced:

$$E^*(\omega) = E_R^*(\omega) + iE_I^*(\omega), \quad \tan[\psi(\omega)] = E_I^*(\omega)/E_R^*(\omega), \tag{2.15}$$

where $E_R^*(\omega) = \omega\int_0^\infty E(t)\sin(\omega t)dt$, $E_I^*(\omega) = \omega\int_0^\infty E(t)\cos(\omega t)dt$. Tangents of losses belongs to the important characteristics of the viscoelastic matter, since it governs intensity of a relaxation process. More larger $\tan[\psi(\omega)]$, more fast dynamical effects are damped.

2.3 General Relations of the Nonlinear Theory of Elasticity

A geometric nonlinearity is defined by deriving relations between deformations ε_{ij} and displacements u_i. A geometrical nonlinearity appears due to nonlinear relations between the elastic strains and the gradients of displacements and is described by the Cauchy-Green strain tensor. In the Lagrangian formulation with respect to the reference configuration, it reads [9]:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i}\right), \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad i, j, k = 1, \ldots, 3. \tag{2.16}$$

Here and further the summation is carried out for the repeated indices.

The first Piola-Kirchhoff stress tensor is defined via density W of the potential energy of deformation as follows

$$\sigma_{ij} = \frac{\partial W}{\partial\left(\partial u_i/\partial x_j\right)}. \tag{2.17}$$

Equilibrium equations take the following form

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{2.18}$$

Physical nonlinearity displays a deviation of the stress–strain relations from the classical Hooke's law. It can be modelled representing the energy of deformation as a series expansion in powers of invariants of the strain tensor and it takes into account the higher-order terms. Nowadays, the latter series is known as the Murnaghan elastic potential [10]. It should be noted that the origin of this approach can be found in the work of Landau and Rumer [11], whereas the five time-points model of elasticity has been proposed by Voigt in 1893 [12].

The Murnaghan [10] elastic potential reads

$$W = \frac{\lambda}{2}\varepsilon_{ii}^2 + \mu\varepsilon_{ij}^2 + \frac{A}{3}\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ik} + B\varepsilon_{ii}\varepsilon_{ij}\varepsilon_{ji} + \frac{C}{3}\varepsilon_{ii}^3 + O\left(\varepsilon_{ij}^4\right). \tag{2.19}$$

Here, two first terms correspond to the linear elastic model, whereas λ are μ are the elasticity moduli of the second order (the Lamé elastic constants). Next terms present influence of the physical nonlinearity, and A, B and C denote elasticity moduli of the third order (these are so called the Landau elastic constants). It should be emphasized that nowadays there are known the values of the Landau coefficients for many materials [9, 13–17], and a few examples are given in Table 2.1.

It should be mentioned that the Murnaghan model is suitable only for the case of small deformations. Observe that the nonlinearity increases with the increase in the

Table 2.1 Elastic properties of some materials and [17], GPa

Material	λ	μ	A	B	C
Helca steel 37	111	82.1	−720	−280	−180
Aluminium D16T	57	27.6	−260	−180	−110
Plexiglas	3.9	1.9	−14.4	−7.2	−4
Polystyrene	1.7	0.95	−10	−8	−11
Granite	22	23.6	−14070	−20230	−1150
Limestone	22.7	20.6	−9730	−6435	−1870
Sandstone	1.9	6.3	−17530	−5670	−2230

amplitude of deformations. Series (2.19) is applicable for practical computations, if the ratio of the three last terms to the two first terms is of the order 10^{-1} or less. In the case of majority of the design materials including metals and polymers, the elasticity moduli of the third order are negative and they are of an order larger than the counterpart moduli of the second order with respect to their absolute values. Therefore, the areas of applicability of the series (2.19) are bounded by the magnitude of the maximum possible deformations $\varepsilon \leq 10^{-2}$ (in rigid bodies, the elastic deformations do not reach 10^{-3}).

The Murnaghan model can be unsuitable for the materials and elastomers, allowing for large elastic deformations (up to 10^{0}), as well as for which are characterized by large values of the elastic moduli of the third order (Table 2.1). In the latter case, other formulas for the internal energy W are available [18].

2.4 Elementary Provisions of the Percolation Theory

If the distribution of inclusions in the composite materials is completely random, then with an increase in their volume fraction $c^{(2)}$, the chains of the contacting inclusions (clusters) are created in the material (Fig. 2.1).

The critical value $c^{(2)} = c_p$, for which the cluster of an infinite length is formed, is called the percolation threshold. The properties of such composite materials cannot be

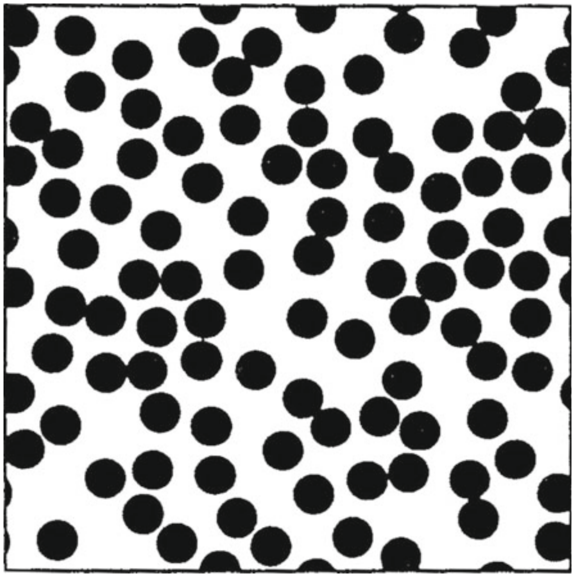

Fig. 2.1 Distribution of identical cylindrical inclusions at $c^{(2)} \approx 0.35$ [19]

described within the framework of regular or quasi-regular models, and it is necessary to use the theory of percolation. This theory was intensively developed in the recent decades [20, 21]. Originally, it was related to the problem of determination of critical concentration of the existing system of channels filled with liquid, beginning from which they merge into the infinite connected set of channels, so that liquid can flow between two parallel planes located on any large distance from each other. Similarly, the random system of electrical conductors, heat-conducting elements, etc., can be examined instead of the system of channels. The objectives of the theory of percolation consist in description of the correlations between the appropriate physical and geometrical characteristics of the objects under study.

In to construct a common percolation model, consider a regular lattice, for example, a square lattice, and make it a random network by randomly establishing sites (vertices) or bonds (edges) with a statistically independent probability. At the percolation threshold, a long-range connectivity appears first. According to the Shklovskii–De Gennes classical model [20, 21], the skeleton of an infinite cluster can be represented in the form of a network with a characteristic size called a radius of the correlation R expressed as follows:

$$R \sim \frac{l}{\left|c^{(2)} - c_p\right|^{\nu}} \quad \text{for} \quad c^{(2)} \to c_p, \tag{2.20}$$

where l is the distance between the centres of contacting inclusions, c_p is the critical volume fraction of the conductive phase, ν is the critical exponent of percolation.

If $c^{(2)} < c_p$, the magnitude of R defines the maximum size of the finite clusters.

Effective characteristics k_0 of a composite near the percolation threshold ($c^{(2)} \to c_p$) are defined by the asymptotic relations like $k_0 \sim \left|c^{(2)} - c_p\right|^t$, where t is the critical index of the corresponding physical property.

Indices t and ν are connected by the following relations: $t = \nu$ for 2D case and $t = 2\nu$ for 3D case [20, 21].

Different models of percolation media and corresponding methods of calculation of the critical indices are reviewed in [20, 21]. It is worth to note that until now there is a certain discrepancy between the results of different authors, especially in the 3D case.

According to the conventional view, the critical indices depend only on the dimensionality of space: $\nu \approx 1.33$ for 2D case and $\nu \approx 0.85$ for 3D case [20, 21].

Percolation threshold depends on the shape of inclusions. In what follows, we consider the circular inclusions in 2D case and spherical inclusions in 3D case, and we will assume accordingly, $c_p = 0.5$ in 2D case and $c_p = 0.16$ in 3D case [22].

The results of the theory of percolation are especially important in the vicinity of percolation threshold. Far from the threshold, the known effective media approximation methods can be used, for example, the Maxwell's formula. It is important to derive the universal formulas, which describe the transport properties of a composite material for the entire range of volume fractions of inclusions. Application of the theory of analytical functions leads to a need to solve numerically the systems of linear algebraic equations [23]. Bruggeman's formula makes it possible to qualitatively describe the percolation threshold [22, 24, 25]. This is related to the fact that Bruggeman's formula represents the two-point Taylor's formula, in which limiting cases are the composites with the small volume fraction of inclusions and the small volume fraction of the matrix material, respectively. However, the accuracy of Bruggeman's formula is low. In the paper [24], a modification of Maxwell's formula is proposed based on the single-point Padé approximant, which provides a qualitative explanation of the existence of the percolation threshold.

2.5 Integral Transforms

Integral transforms allow to transit from a given space to adjoint one, and they belong to efficient tools for solving numerous problems of the theory of composite materials. In particular, they allow to reduce the problems governed by PDEs to the problems of ODE and the problems governed by ODEs to a set of algebraic equations [26, 27].

The Laplace transform of a function f(t), defined for all real numbers $t \geq 0$, is the function $\bar{f}(s)$, which has the following form

$$\bar{f}(s) = \int_0^\infty f(t)\exp(-st)dt. \tag{2.21}$$

The inverse Laplace transform is given by the following complex integral, which is known by various names (the Bromwich integral, the Fourier–Mellin integral and Mellin's inverse formula):

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT}\bar{f}(s)\exp(st)ds, \tag{2.22}$$

where α is a real number so that the contour path of integration is in the region of convergence of $\bar{f}(s)$.

The latter procedure can be treated as a solution to the Fredholm integral equation of the first kind, so this is an ill-posed problem. Using FEM is difficult, because it may lead to unstable numerical computations. That is why analytical solutions are useful not only in the engineering practice, but also in evaluating test results calculated by BE or FE methods. Generally speaking, one needs a regularization procedure for solving this problem. Rational approximations have regularization properties [28–30] and can be successfully used for inverting Laplace transform [31].

It is recommended to check a possibility of finding a required formula in known handbooks, see, e.g. [32]. It is also possible to apply theory of the function of a complex variable, see, e.g. [33]. For numerical inversion, the Gaver algorithm can be used [34] as follows:

$$f(t)=\frac{n\ln 2}{t}\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{f}\left[\frac{(n+k)\ln 2}{t}\right], \tag{2.23}$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, and the parameter n defines accuracy of the introduced approximation $n=1,2,3,\ldots$.

Let us consider the inverse Laplace transform with a help of the two-point Padé approximants (Sect. 2.13). Assume that we deal with the function:

$$f(t)=\left(1+t^2\right)^{-0.5}. \tag{2.24}$$

Asymptotics of original are as follows:

$$f(t)\cong\begin{cases}1-0.5t^2+\ldots \text{ at } t\to 0,\\ t^{-1}+\ldots \text{ at } t\to\infty.\end{cases}$$

The two-point Padé approximants (TPPA) in this case can be written in the following way.

Numerical results are shown in Fig. 2.2. Approximated solution (2.25) (upper curve)

$$f(t)=\frac{1+0.5t}{1+0.5t+0.5t^2} \tag{2.25}$$

well coincides with the original (2.24) (down curve) for arbitrary values of the argument.

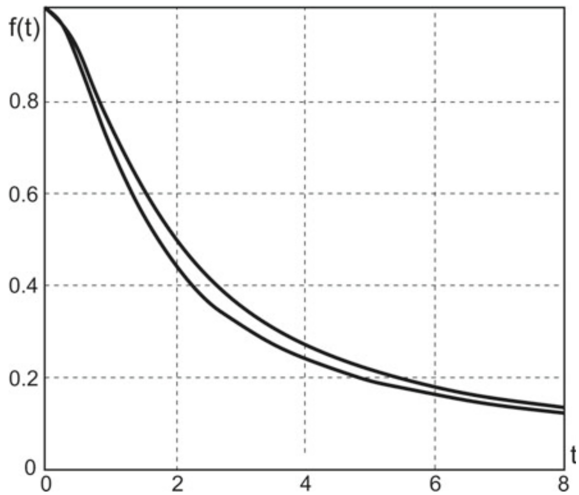

Fig. 2.2 Exact and approximate results of the inverse Laplace transform

In order to define TPPA, the asymptotics for the limiting cases $t = 0$ and $t \to \infty$ can be employed. The initial value theorem and the final value theorem give the following relation between $T(y,t)$ and $\overline{T}(y,s)$ [26]:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\bar{f}(s), \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s). \tag{2.26}$$

Method of asymptotically equivalent functions (Sect. 2.14) also can be used.

In the theory of viscoelasticity, the convolution theorem [26] is very useful. It can be formulated as follows. For convolution of functions $f_1(\tau)$ and $f_2(\tau)$, i.e.

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

one has

$$\int_0^\infty [f_1(t) * f_2(t)] \exp(-st) dt = \bar{f}_1(s) \bar{f}_2(s). \tag{2.27}$$

Exponential Fourier transform has the form

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x) \exp(-isx) dx, \tag{2.28}$$

whereas an inverse exponential Fourier transform is given by the equation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f}(s) \exp(isx) ds. \tag{2.29}$$

Cosine and sine Fourier transforms can be written as follows:

$$\overline{f}(s) = \int_0^\infty f(x) \cos(sx) dx, \quad f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}(s) \cos(sx) ds, \tag{2.30}$$

$$\overline{f}(s) = \int_0^\infty f(x) \sin(sx) dx, \quad f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}(s) \sin(sx) ds. \tag{2.31}$$

In the case of finite domains, the finite integral transforms can be employed [27]

$$\bar{f}(s) = \int_0^H f(x) \cos \frac{\pi s x}{H} dx, \tag{2.32}$$

$$f(x) = \frac{2}{\pi} \sum_{s=1}^{\infty} \bar{f}(s) \cos \frac{s\pi x}{H}, \tag{2.33}$$

$$\bar{f}(s) = \int_{0}^{H} f(x) \sin \frac{\pi s x}{H} dx, \tag{2.34}$$

$$f(x) = \frac{2}{\pi} \sum_{s=1}^{\infty} \bar{f}(s) \sin \frac{s\pi x}{H}. \tag{2.35}$$

2.6 The Method of Multiple Scales

"The method of multiple scales is so popular that it is being rediscovered just about every 6 months" [35]. Its idea is simple and clear. Let us briefly explain its algorithm on example of the strongly damped oscillator, i.e. the so-called system with 1/2 DoF (degree-of-freedom)

$$\varepsilon \ddot{x} + \dot{x} + x = 0, \tag{2.36}$$

$$x(0) = a, \quad \dot{x}(0) = 0. \tag{2.37}$$

Instead of one independent variable, t, we introduce two of them, i.e.: the slow coordinate ($t_1 = t$) and fast coordinate $\tau = \varepsilon^{-1} t$. The first one serves for description of the fundamental state/configuration, whereas the second one plays a role of a boundary layer. The full derivative can be computed in the following way

$$\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon^{-1} \frac{\partial}{\partial \tau}. \tag{2.38}$$

Note that the variable x_0 depends now on two parameters: t_1 and τ. Substituting (2.38) into (2.36) and into boundary conditions (2.37) yields

$$\frac{\partial^2 x}{\partial \tau^2} + \frac{\partial x}{\partial \tau} + \varepsilon \left(\frac{\partial}{\partial t_1} + 1 + 2 \frac{\partial^2}{\partial^2 \partial t_1} \right) x + \varepsilon^2 \frac{\partial^2 x}{\partial t_1^2} = 0, \tag{2.39}$$

$$x(0, 0) = a, \qquad \left. \left(\frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial t_1} \right) \right|_{\tau=0, t_1=0} = 0. \tag{2.40}$$

Observe that instead of singularly perturbated ODE (ordinary differential equation), we obtain a PDE (partial differential equation), but without a small parameter standing by a higher-order derivative. This form of regularization is known as a passing to the space of higher dimension [36]. Now, we seek a solution in the form of the following asymptotic series

$$x = x_0(t_1, \tau) + \varepsilon x_1(t_1, \tau) + \dots. \tag{2.41}$$

Substituting formula (2.41) into Eq. (2.39) and into initial condition (2.40), we obtain

$$\frac{\partial^2 x_0}{\partial \tau^2} + \frac{\partial x_0}{\partial \tau} = 0, \tag{2.42}$$

$$\frac{\partial^2 x_1}{\partial \tau^2} + \frac{\partial x_1}{\partial \tau} = -\frac{\partial x_0}{\partial t_1} - x_0 - 2\frac{\partial^2 x_0}{\partial \tau \partial t_1}, \tag{2.43}$$

$$\dots,$$

for $t_1 = 0$, $\tau = 0$

$$x_0 = a, \tag{2.44}$$

$$\frac{\partial x_0}{\partial \tau} = 0, \tag{2.45}$$

$$x_1 = 0, \tag{2.46}$$

$$\frac{\partial x_1}{\partial \tau} = -\frac{\partial x_0}{\partial t_1}, \tag{2.47}$$

$$\dots$$

General solution of equation (2.42) has the following form

$$x_0 = C(t_1) + C_1(t_1)e^{-\tau}. \tag{2.48}$$

Initial conditions (2.44) and (2.45) yield

$$C_1(0) = 0, \qquad C(0) = a. \tag{2.49}$$

The function $C(t_1)$ is not defined yet. However, it can be defined through the Eq. (2.43), i.e. we have

$$\frac{\partial^2 x_1}{\partial \tau^2} + \frac{\partial x_1}{\partial \tau} + 2\frac{\partial^2 x_0}{\partial \tau \partial t_1} = -\frac{\partial C}{\partial t_1} - C. \tag{2.50}$$

If the right-hand side of Eq. (2.50) is not equal to zero, then the solution x_1 possesses a secular term. It is removed, when the following relation holds

$$\frac{\partial C}{\partial t_1} + C = 0.$$

This and condition (2.49) yield $C = ae^{-t_1}$. General solution of Eq. (2.43) can be presented in the following form

$$x_1 = C^1(t_1) + C_1^1(t_1)e^{-\tau}.$$

The initial conditions (2.46) and (2.47) give

$$C^1(0) = -C_1^1(0), \qquad C_1^1(0) = a.$$

The function $C^1(t_1)$ is defined from the condition of a lack of secular term in the second-order equations.

A particular advantage of the two-scale method relies on its ability to solve the problems involving terms of strong difference in their changes. As an illustrative example, we consider the following singular third-order ODE

$$\varepsilon^2 y''' + y' + y = 0 \tag{2.51}$$

with the following initial conditions

$$y(0) = a, \quad y'(0) = 0, \quad y''(0) = 0, \quad a = const. \tag{2.52}$$

We take the slow and fast coordinate in the form $x_1 = x$ and $\xi = \varepsilon^{-1}x$, respectively, and we seek the solution in the following form

$$y = y_0(x_1, \xi) + \varepsilon y_1(x_1, \xi) + \ldots.$$

Therefore, the following recurrent set of the successive equations and boundary conditions is obtained

$$\frac{\partial^3 y_0}{\partial \xi^3} + \frac{\partial y_0}{\partial \xi} = 0, \tag{2.53}$$

$$\frac{\partial^3 y_1}{\partial \xi^3} + \frac{\partial y_1}{\partial \xi} = -\frac{\partial y_0}{\partial x_1} - y_0 - 3\frac{\partial^3 y_0}{\partial \xi^2 \partial x_1}, \tag{2.54}$$

$$\ldots$$

$$y_0(0,0) = a, \quad \frac{\partial y_0(0,0)}{\partial \xi} = 0, \quad \frac{\partial^2 y_0(0,0)}{\partial \xi^2} = 0, \tag{2.55}$$

$$y_1(0,0) = 0, \quad \frac{\partial y_1(0,0)}{\partial \xi} = \frac{\partial y_0(0,0)}{\partial x_1}, \quad \frac{\partial^2 y_1(0,0)}{\partial \xi^2} = 0, \tag{2.56}$$

$$\ldots$$

First equation of (2.53) supplemented by initial conditions (2.55) allows to find

$$y_0 = C(x_1), \qquad C(0) = a,$$

whereas the condition of a lack of secular terms in solution of equation (2.54) yields

$$C(x_1) = ae^{-x_1}.$$

Finally, fast part of solution to equation (2.54), taking into account the initial conditions (2.56), takes the following form

$$y_1 = -a \sin \xi.$$

Therefore, the employed method of two scales allowed to separate solution with essentially different changes, assuming a lack of localization phenomenon. The method of multiple scales extends and generalizes the two-scale method. Besides, in some problems the fast coordinate can be assumed as a function depending both on the slow coordinates and on the small parameter (for instance, $\tau = \varphi(t, \varepsilon)/\varepsilon$). Numerous examples of such functions (so called regularization functions), that include irregular dependences on ε are reported in [36].

2.7 Differential Equations with Periodically Discontinuous Coefficients

In what follows, we consider application of the method of homogenization in order to solve ODEs with periodically discontinuous coefficients. As a model example, let us consider deformation of a membrane reinforced by fibres.

Equation of equilibrium in intervals $kl < y < (k+1)l$ is as follows:

$$\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} = Q(x_1, y_1). \tag{2.57}$$

Conditions of sewing together of the neighbourhood membrane parts, called conjugate or jump conditions, have the following form

$$\begin{gathered} \lim_{y_1 \to kl+0} u \equiv u^+ = \lim_{y_1 \to kl-0} u \equiv u^-; \quad k = 0, \pm 1, \pm 2, \dots, \\ \left(\frac{\partial u}{\partial y_1}\right)^+ - \left(\frac{\partial u}{\partial y_1}\right)^- = p\frac{\partial^2 u_1}{\partial x_1^2}, \end{gathered} \tag{2.58}$$

where the parameter p characterizes the relative stiffness of the fibre. The boundary conditions for $x = 0, H$ are as follows

$$u = 0. \tag{2.59}$$

Let the external load is periodically distributed with respect to y_1, and its period L is essentially larger than the distance between the successive fibres. Therefore, it is useful to employ the average description taking into account the quantity $\varepsilon = l/L$ as a small/perturbation parameter. Instead of the variable y_1, we introduce the fast $(\eta = y_1/l)$ and slow $(y = y_1/L)$ variables, and hence

$$\frac{\partial}{\partial y_1} = \frac{1}{L}\left(\frac{\partial}{\partial y} + \varepsilon^{-1}\frac{\partial}{\partial \eta}\right). \tag{2.60}$$

The function u can be approximated by the following series

$$u = u_0(x, y) + \varepsilon^{\alpha}[u_{01}(x,\ y) + u_1(x,\ y, \eta)] + \varepsilon^{\alpha_1}[u_{02}(x,\ y) + u_2(x,\ y, \eta)] + \ldots, \tag{2.61}$$

where $0 < \alpha < \alpha_1 < \ldots$, $x = x_1/L$.

Substituting Ansatz (2.61) into Eq. (2.57) and into contact condition (2.58), and taking into account expression for the derivative (2.60), we get

$$\nabla^2 u_0 + \varepsilon^{\alpha-2}\frac{\partial^2 u_1}{\partial \eta^2} + 2\varepsilon^{\alpha-1}\frac{\partial^2 u_1}{\partial y \partial \eta} + \varepsilon^{\alpha_1-2}\frac{\partial^2 u_2}{\partial \eta^2}$$

$$+ 2\varepsilon^{\alpha_1-1}\frac{\partial^2 u_2}{\partial y \partial \mu} + O(\varepsilon^{\alpha}) = q(x, y), \tag{2.62}$$

$$[u_0 + \varepsilon^{\alpha}(u_{01} + u_1) + \ldots]^+ = [u_0 + \varepsilon^{\alpha}(u_{01} + u_1) + \ldots]^-, \tag{2.63}$$

$$\varepsilon^{\alpha-1}\left[\left(\frac{\partial u_1}{\partial \eta}\right)^+ - \left(\frac{\partial u_1}{\partial \eta}\right)^-\right] + O(\varepsilon^{\alpha}) = p_1\left[\frac{\partial^2 u_0}{\partial x^2} + O(\varepsilon^{\alpha})\right],$$

where $q = L^2 Q$, $p_1 = p/L$, $\nabla^2 u_0 = \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}$, $(\ldots)^{\pm} = \lim_{\eta \to k \pm 0} u$.

It should be noted that majority of the works aimed on homogenization of the periodic systems and in particular those based on purely mathematical approaches employ an implicit assumption that the analysed system parameters are of a unit order. However, the character of construction of the asymptotics essentially depends on the order of the relative fibre stiffness p_1 in comparison with the parameter ε. Let us introduce the parameter β, characterizing this order ($p_1 \sim \varepsilon^{\beta}$), and let us analyse a possible structure of the limiting systems versus the parameters α, β.

Owing to Eq. (2.62), the different limiting systems appear for $0 < \alpha < 2$, $\alpha = 2$ and $\alpha > 2$, namely

$$\text{for} \quad 0 < \alpha < 2, \quad \frac{\partial^2 u_1}{\partial \eta^2} = 0, \tag{2.64}$$

$$\text{for} \quad \alpha = 2, \quad \nabla^2 u_0 + \frac{\partial^2 u_1}{\partial \eta^2} = q, \tag{2.65}$$

$$\text{for} \quad \alpha > 2, \quad \nabla^2 u_0 = q. \tag{2.66}$$

The limiting relations, yielded by Eq. (2.63) for $\varepsilon \to 0$, have the following form

$$\text{for} \quad \beta < \alpha - 1, \quad \frac{\partial^2 u_0}{\partial x^2} = 0, \tag{2.67}$$

$$\text{for} \quad \beta = \alpha - 1, \quad \left(\frac{\partial u_1}{\partial \eta}\right)^+ - \left(\frac{\partial u_1}{\partial \eta}\right)^- = p_1 \varepsilon^{1-\alpha} \frac{\partial^2 u_0}{\partial x^2}, \tag{2.68}$$

$$\text{for} \quad \beta > \alpha - 1, \quad \left(\frac{\partial u_1}{\partial \eta}\right)^+ = \left(\frac{\partial u_1}{\partial \eta}\right)^-. \tag{2.69}$$

The plane-quarter of the parameters $\beta > 0$ and $\alpha > 0$ is splitted into nine zones (Fig. 2.3).

Let us analyse those zones in more detail. We begin with $\beta < \alpha - 1$, which means that the fibres are stiff. Equation (2.67) yields $u_0 = 0$, and hence the homogenization approach is not applicable here. The corresponding limiting equation for zones 1–3 takes the following form

$$\frac{\partial^2 u_1}{\partial \eta^2} = q. \tag{2.70}$$

The case $\beta > \alpha - 1$ is associated with zones 4–6. Physically, this case corresponds to weak fibres having a marginal influence, and the limiting equation has the form of (2.66).

Zones 7 and 8 correspond to the cases without a physical interpretation. The most important case corresponds to $\alpha = 2$, $\beta = 1$ (zone 9) being associated with the averaged stiffness with respect to the fibre. Now, the limiting system is governed by Eqs. (2.65) and (2.68), whereas the condition of transition takes the following form

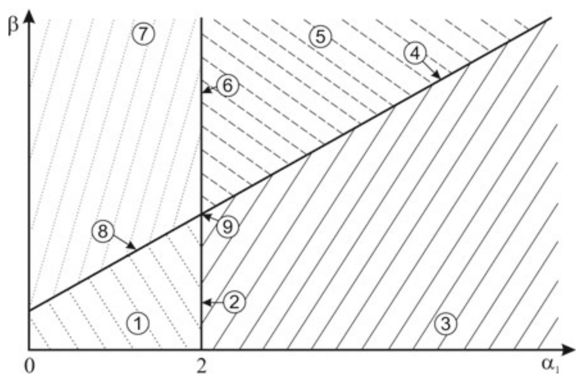

Fig. 2.3 Splitting of the α, β, plane into zones of different asymptotics

$$u_1^+ = u_1^-, \tag{2.71}$$

$$\left(\frac{\partial u_1}{\partial \eta}\right)^+ - \left(\frac{\partial u_1}{\partial \eta}\right)^- = p_2 \frac{\partial^2 u_0}{\partial x^2}, \tag{2.72}$$

where $p_2 = p/l$.

Equation (2.64) yields

$$u_1 = 0.5(q - \nabla^2 u_0)\eta^2 + C(x, y)\eta + C_1(x, y).$$

The constant $C_1(x, y)$ should be matched with the term u_{01}, being defined by averaged equations of the successive approximations. Condition (2.71) yields

$$C(x, y) = -0.5(q - \nabla^2 u_0)L. \tag{2.73}$$

Yet we need to satisfy condition (2.72), but we have not additional required constants. However, observe that condition (2.72) generates at once the sought homogenized equation. Indeed, substituting into (2.72) the found u_1, one gets

$$\nabla^2 u_0 + p_2 \frac{\partial^2 u_0}{\partial x^2} = q. \tag{2.74}$$

Equation (2.74) should be integrated for the boundary conditions

$$u_0 \quad \text{for} \quad x = 0, H/L.$$

Physically, a transition to Eq. (2.74) corresponds to … of the fibres stiffness, i.e. we transit to the structurally-orthotropic theory. The function u_1 can be finally recast to the following form

$$u_1 = 0.5 p_2 \frac{\partial^2 u_0}{\partial x^2} \eta(\eta - 1).$$

In general, boundary conditions on the edges of the studied zones are not satisfied. The boundary error changes fast with respect to η and implies occurrence of the boundary layer u_b. We can construct u_b introducing the variable $\xi = x_1/l$ and by employing the following series

$$u_b = \varepsilon^{\gamma_1} u_{b1}(x, y, \xi, \eta) + \varepsilon^{\gamma_2} u_{b2}(x, y, \xi, \eta) + \ldots,$$

where $0 < \gamma_1 < \gamma_2 < \ldots$.

The required equations u_b follow

$$\frac{\partial^2 u_{b1}}{\partial \xi^2} + \frac{\partial^2 u_{b1}}{\partial \eta^2} = 0,$$

$$u_{b1}|_{\eta=k} = 0, \quad k = 0, \pm 1, \ldots.$$

The boundary conditions (we consider only one edge, since the same algorithm can be used for the second edge) for $x = \xi = 0$ take the form

$$u_{b1} = -u_1.$$

In order to construct the boundary layer, one may apply the Kantorovich method. Namely, substituting u_{b1} in the form satisfying the boundary condition for $\eta = 0, l$, we get

$$u_{b1} = \Phi(\xi)\eta(\eta - l),$$

and hence one may employ the standard procedure of the Kantorovich method [37].

Let us more precisely define the used notion of the fast- and slow-changing load. The function $f(\varepsilon, \theta)$ is called oscillating function with velocity ε^{-1} on the interval 2π, if [28]

$$0 < C_1 \le \int_0^{2\pi} |f(\varepsilon, \theta)|^2 d\theta \le C_2 < \infty, \quad |\int_0^{\alpha} f(\varepsilon, \theta) d\theta| \le C\varepsilon, 0 \le \alpha \le 2\pi,$$

where C, C_1, C_2 are certain constants.

2.8 Homogenization Approach for Differential Equation with Rapidly Changing Coefficients

We begin with a brief introduction of the homogenization method using an example of 1D problem governed by the following equation [39]

$$\frac{d}{dx}\left[a\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right] = q(x), \tag{2.75}$$

$$u = 0 \quad \text{for} \quad x = 0, L, \tag{2.76}$$

where (x/ε) stands for a periodic function with respect to x and period ε.

The right-hand side of Eq. (2.75) changes slowly, but the coefficient (x/ε) exhibits high changes. Therefore, one may apply the method of two scales by introducing the fast $\eta = \varepsilon$ and slow $y = x$ variables. The associated derivative takes the following form

$$\frac{d}{dx} = \frac{\partial}{\partial y} + \varepsilon^{-1}\frac{\partial}{\partial \eta}, \tag{2.77}$$

and hence instead of the input ODE, we obtain a PDE. Its solution can be found in the form of the following series

$$u = u_0(\eta,\ y) + \varepsilon u_1(\eta,\ y) + \ldots, \tag{2.78}$$

where $u_0, u_1, \ldots$ are periodic functions with respect to η and with the period of unity.

Substituting relations (2.77) and (2.78) into the input Eqs. (2.75) and boundary condition (2.76) and comparing the terms standing by the same powers of ε, the following recurrent system of equations is obtained

$$\frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_0}{\partial\eta}\right] = 0, \tag{2.79}$$

$$\frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_0}{\partial y}\right] + a(\eta)\frac{\partial^2 u_0}{\partial y\partial\eta} + \frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_1}{\partial\eta}\right] = 0, \tag{2.80}$$

$$\frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_2}{\partial\eta}\right] + a(\eta)\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_1}{\partial y}\right] + a(\eta)\frac{\partial^2 u_1}{\partial y\partial\eta} = q(y), \tag{2.81}$$

$$\cdots$$

$$u_j = 0 \quad \text{for} \quad y = 0, L; \quad \eta = 0, L/\varepsilon; \quad j = 0, 1, 2, \ldots. \tag{2.82}$$

Equation (2.79), owing to periodicity of the function u_0 with respect to η, yields $u_0 = u_0(y)$, i.e. u_0 plays a role of an averaged value of the function not dependent on the fast variable. In many physical problems, the existence of the averaged part is motivated already by the nature of the solution, and, hence, one may assume that the first term of series (2.78) does not depend on the fast variable. Equation (2.80) takes the following form

$$\frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_1}{\partial\eta}\right] = -\frac{\partial a(\eta)}{\partial\eta}\frac{du_0}{dy}. \tag{2.83}$$

This equation is studied on interval ($0 \le \eta \le 1$), and hence it is referred as "the problem on cell" or the local problem. It is clear that a solution with regard to one cell is essentially simpler than that aimed on solve the problem of the whole space. In our case, we have

$$\frac{\partial u_1}{\partial\eta} = -\frac{\partial u_0}{\partial y} + \frac{C(y)}{a}. \tag{2.84}$$

The periodicity conditions with respect to the first improvement of the homogenized solution $u_1|_0^1 = 0$ allow to estimate the constant

$$C(y) = \hat{a}\frac{du_0}{dy}, \quad \hat{a} = \left[\int_0^1 a^{-1} d\eta\right]^{-1}.$$

Removing the function $\partial u_1/\partial\eta$ from Eq. (2.81) yields

$$\frac{\partial}{\partial\eta}\left(a\frac{\partial u_2}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(a\frac{\partial u_1}{\partial y}\right)+\hat{a}\frac{d^2u_0}{dy^2}=q(y). \tag{2.85}$$

Now, in order to separate from (2.85) the slow components, one may apply the homogenization procedure by action of the averaged operator $\int_0^1(\ldots)d\eta$ on each of the term of the equation. Two first terms, due to the periodicity condition and averaging procedure, are equal to zero. Finally, Eq. (2.85) yields

$$\hat{a}\frac{d^2u_0}{dy^2}=q(y). \tag{2.86}$$

Equation (2.86) is supplemented by the following boundary condition

$$u_0=0 \quad \text{for} \quad y=0,L. \tag{2.87}$$

The function u_1 follows from (2.84), and it takes the following form

$$u_1=\frac{du_0}{dy}\left(\hat{a}\int\limits_0^1 a^{-1}d\eta-\eta\right), \quad 0\le\eta\le 1.$$

Furthermore, the function u_1 is periodically extended with respect to coordinate η with the unit period. Observe that the found value of u_1 generally does not satisfy boundary conditions (2.76), and the associated errors are of the order of ε. In order to compensate their influence, we go back to original variables and solve the following problem

$$\frac{d}{dx}\left[a\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right]=0,$$

$$u|_{x=0}=A=u_1|_{y=\eta=0}, \quad u|_{x=l}=B=u_1|_{y=L,\eta=L/\varepsilon}.$$

The method of homogenization is employed once more, and the first step of approximations yields

$$\hat{a}\frac{d^2u_{01}}{dy^2}=0, \quad u_{01}|_{y=0}=A, \quad u_{01}|_{y=L}=B.$$

The latter form of the problem motivates us to seek the solution in the form of the following series

$$u = u_0(y) + \varepsilon[u_{01}(y) + \varepsilon u_{02}(y) + \varepsilon^2 u_{03}(y) + \ldots] \\ +\varepsilon[u_1(\eta,\ y) + \varepsilon u_2(\eta,\ y) + \varepsilon^2 u_3(\eta,\ y) + \ldots], \tag{2.88}$$

where $u_i(\eta, y)$ are functions with a zero-averaged value with respect to the period.

In what follows, we consider one more model example associated with the non-linear ODE of the form

$$\frac{d}{dx}\left[a\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right] + b\left(\frac{x}{\varepsilon}\right)u^3 = q(x), \tag{2.89}$$

$$u = 0 \quad \text{for} \quad x = 0, L. \tag{2.90}$$

Introducing the fast and slow variables η and y and presently the function u in the form of (2.78), the following recurrent relations are obtained

$$\frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_1}{\partial\eta}\right] + \frac{da(\eta)}{d\eta}\frac{du_0}{dy} = 0, \tag{2.91}$$

$$\frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_2}{\partial\eta}\right] + \frac{\partial}{\partial\eta}\left[a(\eta)\frac{\partial u_1}{\partial y}\right] + a(\eta)\frac{\partial^2 u_1}{\partial y\partial\eta} + a(\eta)\frac{d^2 u_0}{dy^2} + b(\eta)u_0^3 = q(y), \tag{2.92}$$

$$\ldots$$

$$u_0 = 0 \quad \text{for} \quad y = 0, l, \\ u_1 = 0 \quad \text{for} \quad y = 0, l, \quad \eta = 0, L/\varepsilon, \tag{2.93}$$

$$\ldots$$

Equation (2.91) coincides with Eq. (2.80), and hence the local problem is unchanged with respect to higher derivative in spite of occurrence of new terms. Now, using solution (2.84), the following averaged equation is derived

$$\hat{a}\frac{d^2 u_0}{dy^2} + \hat{b}u^3 = q(y), \quad \hat{b} = \int_0^1 b(\eta)d\eta. \tag{2.94}$$

Boundary conditions for Eq. (2.94) take the form of (2.93). Observe the following interesting feature, i.e. though $u = u_0 + O(\varepsilon)$, but $\frac{du}{dx} = \frac{du_0}{dy} + \frac{\partial u_1}{\partial\eta} + O(\varepsilon)$. In other words, though a solution u_0 to the homogenized equation approximates the function u with the accuracy up to the terms of order ε, but in relation with respect to a derivative the term with u_1 should be taken, since they power increase strongly with differentiation. This damps a possibility of the direct analytical computation.

In what follows, we briefly discuss the physical meaning of the coefficients of the homogenized Eq. (2.94). First, the averaged stiffness b is equal and the counterpart flexibility is $1/a$. The averaging with respect to stiffness is often called the

Voigt averaging, whereas the averaging with respect to flexibility is called the Reuss averaging. Those estimations present the averaged arithmetic and averaged harmonic characteristics of the matrix and inclusions for the composites. It is known for a wide class of the problems that the real values of the coefficients of the averaged equations (2.94), $\tilde{a}_{ij}$, are localized in between the coefficients of the averaged Voigt ($\bar{a}_{ij}$) and Reuss ($\hat{a}_{ij}$) estimations

$$\hat{a}_{ij} \le \tilde{a}_{ij} \le \bar{a}_{ij} \tag{2.95}$$

Estimation (2.95) is also referred to as to the Voigt–Reuss boundaries or Hill boundaries, though it has been first obtained by Wiener [40]. For example, in Fig. 2.4 the results of computation of the averaged conductivity d of the composite material composed of the matrix and squared inclusions are shown. The input problems are described by the Laplace equation validated in the periodic non-homogenous matter. The cell of periodicity presents a square of the side 1, whereas the inclusions are located symmetrically with respect to this square centre, and has a side of length 1/3, whereas a ratio of the matrix and inclusion is denoted by d_0. The dotted curve corresponds to the Voigt estimation, whereas the dashed curve is associated with the Reuss estimation. The solid curve presents the results of homogenization obtained numerically while solving the problems on a cell [41].

Figure 2.4 allows to choose the most suitable approximation (2.95) to be practically applied. Consider now the problems of eigenvalues associated with the following ODE

$$\frac{d}{dx}\left[a\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right] + \lambda u = 0,$$

$$u = 0 \quad \text{for} \quad x = 0, L. \tag{2.96}$$

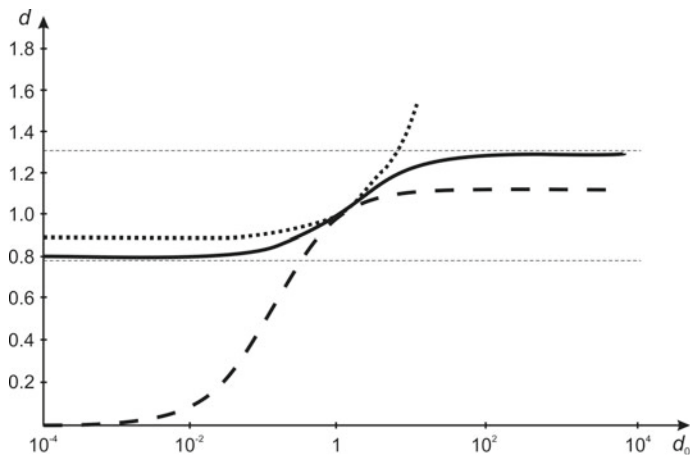

Fig. 2.4 Comparison of the results of homogenization of the Laplace equation with periodically non-homogeneous matter with Voigt and Reuss estimations

We present the being sought eigenform in the form (2.88), whereas the eigenvalue λ can be presented in the form of the following series

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots . \tag{2.97}$$

Substituting the series (2.88) and (2.97) into the input boundary problem (2.96) and taking into account formulas for the derivative (2.77), the following system of the recurrent equations is obtained

$$\frac{\partial a}{\partial \eta}\frac{du_0}{dy} + \frac{\partial}{\partial \eta}\left[a\frac{\partial u_1}{\partial \eta}\right] = 0, \tag{2.98}$$

$$\frac{\partial}{\partial \eta}\left(a\frac{\partial u_2}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(a\frac{\partial u_1}{\partial y}\right) + a\frac{\partial^2 u_1}{\partial y \partial \eta} + \frac{\partial a}{\partial \eta}\frac{du_{01}}{dy} + a\frac{d^2 u_0}{dy^2} + \lambda_0 u_0 = 0, \tag{2.99}$$

$$\frac{\partial}{\partial \eta}\left(a\frac{\partial u_3}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(a\frac{\partial u_2}{\partial \eta}\right) + \frac{\partial a}{\partial \eta}\frac{du_{02}}{dy}$$

$$+ a\frac{\partial^2 u_2}{\partial y \partial \eta} + a\frac{d^2 u_{01}}{dy^2} + \lambda_1 u_0 + \lambda_0(u_{01} + u_1) = 0, \tag{2.100}$$

$$\ldots$$

$$u_0 = 0 \quad \text{for} \quad y = 0, L, \tag{2.101}$$

$$u_1 + u_{01} = 0 \quad \text{for} \quad \eta = 0, L/\varepsilon, \tag{2.102}$$

$$\ldots$$

Equations (2.98) define the value of $\partial u_1/\partial \eta$, and after its substitution to Eq. (2.99) and the boundary conditions (2.101) and after carrying out the averaging, the following boundary problem is obtained to be solved with respect to u_0, λ_0:

$$\hat{a}\frac{d^2 u_0}{dy^2} + \lambda_0 u_0 = 0, \quad u_0 = 0 \quad \text{for} \quad y = 0, L.$$

Now, Eq. (2.99) yields

$$\frac{\partial u_2}{\partial \eta} = -\frac{\partial u_1}{\partial y} - \frac{du_{01}}{dy} + \frac{C_1(y)}{a}.$$

The periodicity coefficients for the functions u_2 with respect to the variable η are defined via the following equation

$$C_1 = \hat{a}\frac{du_{01}}{dy} + \hat{a}\frac{\partial \hat{u}_1}{\partial y},$$

where

$$\hat{u}_1 = \int_0^1 u_1 d\eta.$$

Substituting the solved values u_1, u_2 into Eq. (2.100) and carrying out the averaging procedure, we have

$$\hat{a}\frac{d^2 u_{01}}{dy^2} + \lambda_0 u_{01} + \hat{a}\frac{\partial^2 \hat{u}_1}{\partial y^2} + \lambda_0 \hat{u}_1 + \lambda_1 u_0 = 0. \quad (2.103)$$

Boundary conditions for Eq. (2.103) are yielded by conditions (2.102), and they have the following form

$$u_{01} = -\widehat{u}_1 \quad \text{for} \quad y = 0, L. \quad (2.104)$$

The improvement term to the frequency λ_1 is defined in a usual way of the theory of perturbations [35, 42], and then a slow correction term to the averaged solution u_{01} is found from a solution to the boundary problems (2.103) and (2.104).

The so far presented approach allows to find a solution with respect to an arbitrary approximation regarding ε. One of the additional benefits is connected with the general aspect of the approach. Indeed, if a solution to the local problem is found, then without any difficulties a solution to the input problem can be found, as well as the associated eigenvalues. If we add to the equations the nonlinear terms in a way not to change the higher derivatives, then a construction of further steps is still simple. The local problem remains exactly the same as in the linear case, and also the higher-order approximations will be linear. The whole nonlinearity is localized in the averaged boundary problems with smooth coefficients, which can be solved either numerically or via employment of the variational methods.

2.9 Homogenization of Periodically Perforated Media. Schwarz Alternating Method

We consider the Poisson equation

$$\nabla^2 u = f(x, y) \quad (2.105)$$

in the multiconnected domain Ω (Fig. 2.5). The small parameter ε characterizes the characteristic dimension of the repeated part cell with respect to the membrane dimension. On the boundaries of holes, there are given Neumann boundary conditions

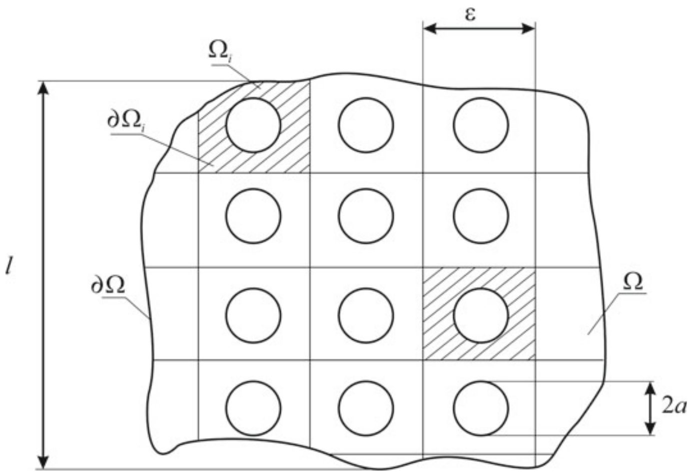

Fig. 2.5 Performed matter

$$\frac{\partial u}{\partial \mathbf{n}_i} = 0 \quad \text{on} \quad \partial \Omega_i, \tag{2.106}$$

where $\mathbf{n}_i$ corresponds to a normal to the contour of the ith hole. Edges of the membrane are rigidly clamped, i.e.

$$u = 0 \quad \text{on} \quad \partial \Omega. \tag{2.107}$$

Let us introduce fast variables $\xi = x/\varepsilon$, $\eta = y/\varepsilon$. A solution is searched in the form of the following series

$$u = u_0(x, y) + \varepsilon u_1(x, y, \xi, \eta) + \varepsilon^2 u_2(x, y, \xi, \eta) + \ldots, \tag{2.108}$$

where u_j $(j = 1, 2, \ldots)$ are periodic functions with respect to both ξ, η with the period 1.

The partial derivatives take now the following form

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y} + \varepsilon^{-1}\frac{\partial}{\partial \eta}. \tag{2.109}$$

The shape of periodically repeated cell with a hole presented in fast coordinates has the form shown in Fig. 2.6.

Substituting (2.108) into the boundary value problem (2.105)–(2.107) and taking into account relations (2.109) yield, after splitting with respect to ε, the following recurrent set of the boundary value problem

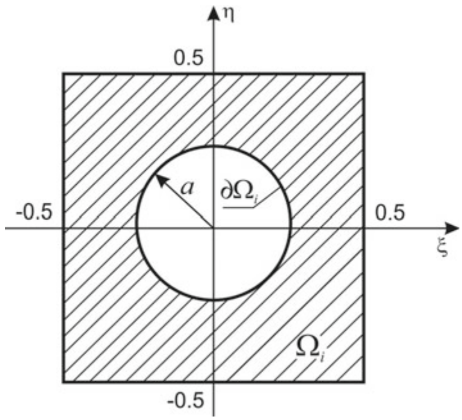

Fig. 2.6 Periodically repeated cell

$$\frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \eta^2} = 0 \quad \text{in} \quad \Omega_i, \tag{2.110}$$

$$\frac{\partial u_1}{\partial \mathbf{k}} + \frac{\partial u_0}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Omega_i, \tag{2.111}$$

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\left(\frac{\partial^2 u_1}{\partial x \partial \xi} + \frac{\partial^2 u_1}{\partial y \partial \eta}\right) + \frac{\partial^2 u_2}{\partial \xi^2} + \frac{\partial^2 u_2}{\partial \eta^2} = f \quad \text{in} \quad \Omega_i, \tag{2.112}$$

$$\frac{\partial u_2}{\partial \mathbf{k}} + \frac{\partial u_1}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Omega_i, \tag{2.113}$$

$$\dots$$

$$u_i = 0, \quad i = 0, 1, 2, \dots \quad \text{on} \quad \Omega_i. \tag{2.114}$$

Here, **k** denotes the external normal to the hole contour presented in fast coordinates. We define the averaging operator in the following form

$$\tilde{\Phi}(x, y) = \iint\limits_{\Omega_i} \Phi(x, y, \xi, \eta) d\xi d\eta. \tag{2.115}$$

Equation (2.112), after action of the averaging operator (2.115), yields

$$\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)\left(1 - \pi a^2\right) + \iint\limits_{\Omega_i} \left(\frac{\partial^2 u_1}{\partial x \partial \xi} + \frac{\partial^2 u_1}{\partial y \partial \eta}\right) d\xi d\eta = \left(1 - \pi a^2\right) f. \tag{2.116}$$

The associated boundary condition takes the form

$$u_0 = 0 \quad \text{Ha} \quad \partial\Omega. \tag{2.117}$$

Now, we need to solve the problem on the cell (2.110) and (2.111) with a condition of the periodic extension, i.e. conditions of equality of the function u_1 and their partial derivatives of first order with respect to the corresponding variables on the opposite cell sides.

Note that a reduction of the periodic problems to boundary value problem has been presented in [39]. For our studied case, the periodic problem can be splitted into two problems. In both cases, the displacements on two opposite sides of the external cell border are equal to zero, and the same holds for the derivative on two remaining sides.

In order to solve local problem, we will use the Schwarz alternating method [37]. This method, named after Hermann Schwarz, is an iterative method to find the solution of a partial differential equations on a domain which is the union of two overlapping subdomains, by solving the equation on each of the two subdomains in turn, taking always the latest values of the approximate solution as the boundary conditions.

Let a diameter of the circle hole 2 is small in comparison with the cell dimension. Then, in the first iteration of the Schwarz alternating method, one may solve the problem regarding holes in an infinite plane, ignoring the external cell border

$$\frac{\partial^2 u_1^{(1)}}{\partial \xi^2} + \frac{\partial^2 u_1^{(1)}}{\partial \eta^2} = 0, \tag{2.118}$$

$$\frac{\partial u_1^{(1)}}{\partial \mathbf{k}} + \frac{\partial u_0}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega_i, \tag{2.119}$$

$$u_1^{(1)} \to 0 \quad \text{for} \quad \xi^2 + \eta^2 \to \infty. \tag{2.120}$$

In polar coordinates, the boundary problem (2.118)–(2.120) can be written as follows

$$\frac{\partial^2 u_1^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u_1^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1^{(1)}}{\partial \theta^2} = 0, \tag{2.121}$$

$$\left.\frac{\partial u_1^{(1)}}{\partial r}\right|_{r=a} = -\frac{\partial u_0}{\partial x}\cos\theta - \frac{\partial u_0}{\partial y}\sin\theta, \tag{2.122}$$

$$u_1^{(1)} \to 0 \quad \text{for} \quad r \to \infty. \tag{2.123}$$

The solution to the boundary value problem (2.121)–(2.123) is as follows

$$u_1^{(1)} = \frac{a^2}{r}\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right). \tag{2.124}$$

Observe that function $u_1^{(1)}$ does not satisfy the periodicity conditions. For discrepancy compensation, we use the second iteration of the Schwarz alternating method. In this case, we ignore the hole and consider the square $|\xi| \leq 0.5$; $|\eta| \leq 0.5$. Boundary conditions in this case are defined in a way to compensate discrepancy in the periodicity conditions. In result, we obtain for the function $u_1 (u_1 \approx u_1^{(1)} + u_1^{(2)})$ the following boundary value problem

$$\Delta u_1^{(2)} = 0 \quad \text{in} \quad \Omega_i^*,$$

$$u_1^{(2)} (0.5,\ \eta) - u_1^{(2)} (-0.5,\ \eta) = u_1^{(1)} (-0.5,\ \eta) - u_1^{(1)} (0.5,\ \eta)\,,$$

$$u_1^{(2)} (\xi,\ 0.5) - u_1^{(2)} (\xi,\ -0.5) = u_1^{(1)} (\xi,\ -0.5) - u_1^{(1)} (\xi,\ 0.5)\,,$$

$$u_{1\xi}^{(2)} (0.5,\ \eta) - u_{1\xi}^{(2)} (-0.5,\ \eta) = u_{1\xi}^{(1)} (-0.5,\ \eta) - u_{1\xi}^{(1)} (0.5,\ \eta)\,,$$

$$u_{1\eta}^{(2)} (\xi,\ 0.5) - u_{1\eta}^{(2)} (\xi,\ -0.5) = u_{1\eta}^{(1)} (\xi,\ -0.5) - u_{1\eta}^{(1)} (\xi,\ 0.5)\,.$$

Now, let us as present $u_1^{(2)}$ in the form

$$u_1^{(2)} = u_1^{(12)} + u_1^{(22)}, \tag{2.125}$$

where function $u_1^{(12)}$ satisfies the homogeneous boundary condition with respect to ξ and non-homogeneous boundary conditions with respect to η; the function $u_1^{(22)}$ is obtained from the function $u_1^{(12)}$ by a permutation of the variables $(\xi \leftrightarrow \eta;\ x \leftrightarrow y)$.

Then, in order to obtain $u_1^{(12)}$, the following boundary value problem appears

$$\Delta u_1^{(12)} = 0 \quad \text{in} \quad \Omega_i^*, \tag{2.126}$$

$$u_1^{(12)}(0.5, \eta) = u_1^{(12)}(-0.5, \eta), \qquad u_{1\xi}^{(12)}(0.5, \eta) = u_{1\xi}^{(12)}(-0.5, \eta), \tag{2.127}$$

$$\begin{aligned} u_1^{(12)} (\xi, 0.5) - u_1^{(12)} (\xi, -0.5) &= u_1^{(1)} (\xi, -0.5) - u_1^{(1)} (\xi, 0.5)\,, \\ u_{1\eta}^{(12)} (\xi, 0.5) - u_{1\eta}^{(12)} (\xi, -0.5) &= u_{1\eta}^{(1)} (\xi, -0.5) - u_{1\eta}^{(1)} (\xi, 0.5)\,. \end{aligned} \tag{2.128}$$

A general solution to Eq. (2.126) takes the following form

$$\begin{aligned} u_1^{(12)} = A_0 + B_0\eta + \sum_{n=1}^{\infty} [(A_n \cosh(2\pi n\eta) + B_n \sinh(2\pi n\eta)) \cos(2\pi n\xi) \\ + C_n \cosh(2\pi n\eta) + D_n \sinh(2\pi n\eta)) \sin(2\pi n\xi)], \end{aligned} \tag{2.129}$$

where A_n, B_n, C_n, D_n are arbitrary constants.

Let us recast the boundary conditions (2.128) to the following form

$$u_1^{(12)}(\xi, 0.5) - u_1^{(12)}(\xi, -0.5) = -\frac{\partial u_0}{\partial y} a^2 (\xi^2 + 0.25)^{-1}, \tag{2.130}$$

$$u_{1\eta}^{(12)}(\xi, 0.5) - u_{1\eta}^{(12)}(\xi, -0.5) = 2\frac{\partial u_0}{\partial x} a^2 \xi (\xi^2 + 0.25)^{-2}. \tag{2.131}$$

Developing the right-hand side of relations (2.130) and (2.131) into Fourier series and substituting (2.129) into (2.130) and (2.131), we obtain

$$A_n = D_n = 0 \quad n = 0, 1, \ldots, \quad B_0 = -\frac{\partial u_0}{\partial y} \pi a^2 = \frac{\partial u_0}{\partial y} B_0^*,$$

$$B_n = -\frac{\partial u_0}{\partial y} \frac{2a^2}{sh\pi n} [e^{-\pi n} \operatorname{Im} E_1(\pi n(i-1)) - e^{\pi n} \operatorname{Im} E_1(\pi n(i+1))] = \frac{\partial u_0}{\partial y} B_n^*,$$

$$C_n = B_n, \quad \text{where} \quad \frac{\partial u_0}{\partial y} \Rightarrow \frac{\partial u_0}{\partial x}, \quad n = 0, 1, 2, \ldots .$$

Here, $E_1(\ldots)$ is the integral exponential function [43]; $i = \sqrt{-1}$. Therefore, we get

$$\bar{u}_1^{(2)} = \frac{\partial u_0}{\partial y} B_0^* \eta + \sum_{n=1}^{\infty} B_n^* \left(\frac{\partial u_0}{\partial y} \sinh(2\pi n\eta) \cos(2\pi n\xi) + \frac{\partial u_0}{\partial x} \cosh(2\pi n\eta) \sin(2\pi n\xi) \right).$$

We find the function $u_1^{(22)}$ in the analogous way. Substituting relation $u_1 = u_1^{(1)} + u_1^{(2)}$ into Eq. (2.116) yields the following averaged equation

$$q \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \right) = f, \tag{2.132}$$

where

$$q = 1 - \pi a^2 + \frac{8\pi^2 a^4}{1 - \pi a^2} \sum_{n=1}^{\infty} \frac{n}{sh\pi n} \left(e^{-\pi n} \operatorname{Im} E_1(\pi n(i-1)) - e^{\pi n} \operatorname{Im} E_1(\pi n(i+1)) \right). \tag{2.133}$$

The obtained series (2.133) is absolutely convergent with fast decreasing parts: $|a_{n+1}/a_n| \to \exp(-\pi)$.

The averaged boundary condition for Eq. (2.132) takes the form (2.117). Now, we shortly stop on the paradox pointed out by [44]. The authors considered two cases. In the first step, the averaging procedure is carried out for the matter with holes. In the second step, the matter with a few inclusions has been averaged, and then the obtained averaged relations regarding the inclusion characteristics have been equalled to zero. However, the obtained corresponding limiting systems have not coincided. A reason is that the right-hand side of the input Poisson equations for inclusions has been averaged with respect to the whole cell area, whereas in the case of the matter with holes, only for the cell area without the hole. In our problem, the occurred paradox is solved in a relatively simple way. Namely, for the holes the right-hand side can be averaged with respect to the cell area without a hole, whereas the coefficients regarding the left-hand parts can be obtained by employment a limiting transition from the problem dealing with inclusions.

2.10 Boundary Perturbation Method

In this section, we consider the free undamped vibrations of a membrane, which form differs slightly from the circle [45–47]. The input equation has the following form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \lambda^2 u = 0, \tag{2.134}$$

which holds inside of the space bounded by the line $r = 1 + \varepsilon\varphi(\theta)$, $\varepsilon \ll 1$. We take the following boundary conditions

$$u(1 + \varepsilon\varphi(\theta), \theta) = 0, \tag{2.135}$$

$$|u(0, 0)| < \infty. \tag{2.136}$$

A solution to the boundary value problem (2.134)–(2.136) is searched in the form of the following series

$$u(r, \theta) = u_0(r, \theta) + \varepsilon u_1(r, \theta) + \varepsilon^2 u_2(r, \theta) + \ldots. \tag{2.137}$$

The boundary condition (2.135) holds for $r = 1 + \varepsilon\varphi(\theta)$, therefore while substituting series (2.137) into boundary condition (2.135), the small parameter ε appears not only in the series coefficients but also in the argument of function $u(r, \theta)$. Consequently, we must shift the boundary condition from line $r = 1 + \varepsilon\varphi(\theta)$ onto a circle $r = 1$ with a help of the Taylor series

$$u(1 + \varepsilon\phi(\theta), \theta) = u(1, \theta) + \frac{\partial u(1, \theta)}{\partial r}\varepsilon\phi(\theta) + \frac{1}{2}\frac{\partial^2 u(1, \theta)}{\partial r^2}\varepsilon^2\phi^2(\theta) + \ldots = 0. \tag{2.138}$$

After splitting with regard to ε, the following recurrent set of boundary conditions appear as follows:

$$\frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_0}{\partial \theta^2} - \lambda_0^2 u_0 = 0, \tag{2.139}$$

$$u_0(1, \theta) = 0, \tag{2.140}$$

$$|\, u_0(0, \theta)\, | < \infty, \tag{2.141}$$

$$\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} - \lambda_0^2 u_1 - \lambda_1^2 u_0 = 0, \tag{2.142}$$

$$u_1(1,\theta) - \frac{\partial u_0(1,\theta)}{\partial r}\varphi(\theta) = 0, \tag{2.143}$$

$$|\, u_1(0,\theta)\,| < \infty, \tag{2.144}$$

$$\dots$$

Solution to equation (2.139) satisfying condition (2.141) takes the following form

$$u_0(r,\theta) = C I_n(\lambda_0 r)\cos n\theta,$$

whereas the non-dimensional frequencies λ_0 are defined by the following transcendental equation

$$I_n(\lambda_0) = 0,$$

where I_n stands for the Bessel function [43]. In what follows, we begin with the boundary problem of the first approximation (2.142)–(2.144). In order to make the boundary condition (2.142) homogeneous, let

$$u_1 = u_{11} - \frac{\partial u_0(1,\theta)}{\partial r}\varphi(\theta).$$

Now, the function $u_{11}(r,\theta)$ is involved into the following boundary value problem

$$\frac{\partial^2 u_{11}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{11}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_{11}}{\partial \theta^2} - \lambda_0^2 u_{11} - \lambda_1^2 u_0 = \psi(\theta), \tag{2.145}$$

$$u_{11}(1,\theta) = 0, \tag{2.146}$$

$$|\, u_{11}(0,\theta)\,| < \infty, \tag{2.147}$$

where

$$\psi(\theta) = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left[\frac{\partial u_0(1,\theta)}{\partial r}\varphi(\theta)\right] - \lambda_0^2 \frac{\partial u_0(r,\theta)}{\partial r}\varphi(\theta).$$

Multiplying Eq. (2.145) by the function $u_0(r,\theta)$ and integrating with respect to the membrane, we have

$$\lambda_1^2 = \frac{\int_0^1 \int_0^{2\pi} \psi(\theta) u_0(r,\theta) d\theta dr}{\int_0^1 \int_0^{2\pi} u_0(r,\theta) d\theta dr}.$$

This process can be further extended.

2.11 The Papkovich–Fadle Approach

In many cases, a ratio of the geometric dimensions of the problem can be taken as a small/perturbation parameter. The mentioned simplification is widely applied in the hydrodynamics and in the theory of composites, and it is referred as the lubrication theory [48]. We use terms lubrication theory or densely packed model approach, though Christensen [1] also suggests a concentrated suspension model or lubrication approximation. From the mathematical standpoint, we apply thin-layer approach [49]. It means simplified models by assuming that length scales in (say) x direction are much smaller than those normal to it. The formal procedure used is to rescale the x variable with a small parameter expressing the ratio of the relative length scales.

We consider a bending of the clamped plate, where its one side length (L_1) is essentially less than the other side length (L). The governing equations and boundary conditions have the following form

$$D\nabla^4 w = Q(x, y), \tag{2.148}$$

$$\begin{aligned} &\text{for} \quad x = 0, L \quad w = w_x = 0, \\ &\text{for} \quad x = 0, L_1 \quad w = w_y = 0. \end{aligned} \tag{2.149}$$

After changing the variables:

$$\xi = x/L, \qquad \eta = y/L_1,$$

equation (2.148) is recast to the following form

$$\varepsilon^4 w_{\xi\xi\xi\xi} + 2\varepsilon^2 w_{\xi\xi\eta\eta} + w_{\eta\eta\eta\eta} = q, \tag{2.150}$$

where $q = QL_1^4/D$ and $\varepsilon = L_1/L \ll 1$. The boundary conditions (2.149) in new coordinates take the following form

$$\text{for} \quad \xi = 0, 1 \quad w = w_\xi = 0, \tag{2.151}$$

$$\text{for} \quad \eta = 0, 1 \quad w = w_\eta = 0. \tag{2.152}$$

Let us present a solution to the boundary value problem (2.151) and (2.152) in the following form

$$w = w_0 + \varepsilon^2 w_1 + \varepsilon^4 w_2 + \ldots, \tag{2.153}$$

and in result, we obtain a recurrent sequence of the following boundary value problem

$$w_{0\eta\eta\eta\eta} = q, \tag{2.154}$$

$$w_{1\eta\eta\eta\eta} = -2w_{0\eta\eta\xi\xi}, \tag{2.155}$$

$$w_{i\eta\eta\eta\eta} = -2w_{i-1\eta\eta\xi\xi} - w_{i-2\eta\eta\eta\eta}, \quad j = 0, 1, \ldots, \tag{2.156}$$

where $\eta = 0, 1 \quad w_j = w_{j\eta} = 0, \quad j = 0, 1, \ldots$.

The constructed solution does not satisfy the boundary conditions for $\xi = 0$ and $\xi = 1$. The occurred errors introduced to solution w_n play a role of the boundary layer. It can be constructed in the following way. In the equation

$$\nabla^4 w = 0$$

we change the variables $\varphi = x/L_1$ and $\eta = y/L_1$, and we present w_n in the following form

$$w_b = \varepsilon w_{b0} + \varepsilon^2 w_{b1} + \varepsilon^3 w_{b2} + \ldots.$$

Equations modelling the boundary layer have the following form

$$\left(\frac{\partial^4}{\partial\phi^4} + 2\frac{\partial^4}{\partial\phi^2\partial\eta^2} + \frac{\partial^4}{\partial\eta^4}\right) w_{bi} = 0, \quad i = 0, 1, 2, \ldots. \tag{2.157}$$

For $\varphi = 0, \varepsilon^{-1}$ we have:

$$w_{b0} = 0, \quad w_{b0\varphi} = -w_{0\xi}\big|_{\xi=0},$$

$$w_{b1} = -w_0, \qquad w_{b1\phi} = -w_{1\xi}\big|_{\xi=0},$$

$$w_{bi} = -w_{i-1}, \qquad w_{bi\varphi} = -w_{i\xi}\big|_{\xi=0},$$

$$\text{for} \quad \eta = 0, 1 \qquad w_{bj} = w_{bj\eta} = 0, \quad j = 0, 1, 2, \ldots. \tag{2.158}$$

Since a distance between the plate sides $\varphi = 0$ and $\varphi = \varepsilon^{-1}$ is large, one may construct the corresponding solutions separately in the vicinity of each side, i.e. take into account bending.

Boundary value problems (2.157) and (2.158) are essentially two-dimensional. In order to solve the problem, one may apply the Papkovich–Fadle approach [50, 51]. Equivalently, it can be also applied the series with respect to eigenfunctions. Owing to the Papkovich–Fadle approach, a solution to equations (2.157) for the boundary conditions (2.158) can be presented in the following form (we consider the edge $\varphi = 0$, whereas for the edge $\varphi = \varepsilon^{-1}$ the solution can be obtained similarly)

$$w_{nj} = \sum_{k=1}^{\infty} \left[Y_k(\eta_1) a_k e^{-is_k\phi} + \bar{Y}_k(\eta_1)\bar{a}_k e^{is_k\varphi} \right], \tag{2.159}$$

where $\eta_1 = \eta - 0.5$, $\overline{(\ldots)}$ denotes the complex conjugated values $(\ldots)$, $i = \sqrt{-1}$.

The eigenfunctions $Y_k(\eta)$ are as follows:

(i) For even components with respect to the line $\eta_1 = 0$, we have

$$Y_k = \frac{\cosh(s_k\eta_1)}{\cosh(0.5s_k)} - \frac{2\eta_1\sinh(s_k\eta_1)}{\sinh(0.5s_k)}, \tag{2.160}$$

where the constants s_k are roots of the transcendental equation

$$\sinh(s_k) = -s_k; \tag{2.161}$$

(ii) For odd components for the line $\eta_1 = 0$, we obtain

$$Y_k = \frac{\sinh(s_k\eta_1)}{\sinh(0.5s_k)} - \frac{2\eta_1\cosh(s_k\eta_1)}{\cosh(0.5s_k)}, \tag{2.162}$$

and the constants s_k are the roots of the following transcendental equation

$$\sinh(s_k) = s_k. \tag{2.163}$$

We stop now on solution to the transcendental equation (2.161) (for Eq. (2.163) the analogous results are obtained). Besides the trivial zero solution, it has only the complex roots $s_k = \alpha_k + i\beta_k$, where α_k, β_k satisfy the system of the following transcendental equations

$$\sinh\alpha_k\cos\beta_k = -\alpha_k; \tag{2.164}$$

$$\cosh\alpha_k\sin\beta_k = \beta_k. \tag{2.165}$$

For small values of α_k and β_k, the system (2.164) and (2.165) is solved numerically, whereas for large values α_k, β_k Eqs. (2.164) and (2.165) yield the asymptotics

$$\cos\beta_k \cong 0, \quad \beta_k \cong \frac{\pi}{2}(4k+1) \quad \text{and} \quad \exp\alpha_k \cong \beta_k, \quad \alpha_k \cong \ln\beta_k.$$

In order to define constants a_k, $\bar{a}_k$ in solution (2.159), the boundary conditions (2.154) should be projected onto functions Y_k, $\bar{Y}_k$. In other words, after presentation of the solution in the form (2.159), the conditions (2.151) should be multiplied by Y_k, $\bar{Y}_k$ ($k = 1, 2, \ldots$), respectively, and then integrated w η from zero to one. Finally, we obtain the infinite systems of coupled algebraic equations. The exact solution is possible only in a few cases. As a rule, the reduction method for solving systems of linear algebraic equations should be employed.

2.12 The Padé Approximants

> One can calculate only a few terms of a perturbation expansion, usually not more than two or three, and almost never more than seven. The resulting series is often slowly convergent, or even divergent. Yet those few terms contain a remarkable amount of information, which the investigator should do his best to extract [52].

The Padé approximants (PA) one of the most popular and useful transformation of the power series into fractional-rational function [53–57]. We give a definition following [53].

Let us consider the power series

$$f(\varepsilon) = \sum_{i=1}^{\infty} c_i \varepsilon^i, \tag{2.166}$$

represented by the function $f(z)$. The PA is the rational function

$$f_{[n/m]}(\varepsilon) = \frac{a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n}{1 + b_1\varepsilon + \cdots + b_m\varepsilon^m}, \tag{2.167}$$

and its coefficients are defined as follows

$$\begin{aligned} &\left(1 + b_1\varepsilon + \cdots + b_m\varepsilon^m\right)\left(c_0 + c_1\varepsilon + c_2\varepsilon^2 + \ldots\right) \\ &\quad = a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n + O\left(\varepsilon^{n+m+1}\right). \end{aligned} \tag{2.168}$$

Comparing the coefficients of the same powers of ε, we obtain the following system of linear algebraic equations

$$\begin{aligned} &b_m c_{n-m+1} + b_{m-1} c_{n-m+2} + \cdots + c_{n+1} = 0, \\ &b_m c_{n-m+2} + b_{m-1} c_{n-m+3} + \cdots + c_{n+2} = 0, \\ &\ldots \\ &b_m c_n + b_{m-1} c_{n+1} + \cdots + c_{n+m} = 0, \end{aligned} \tag{2.169}$$

where $c_j = 0$ for $j < 0$. The latter equations allow to find the coefficients b_i. The coefficients a_i are found from relations (2.168), and they are compared with the coefficients standing by the same powers of ε to yield:

$$\begin{aligned} &a_0 = c_0, \\ &a_1 = c_1 + b_1 c_0, \\ &\ldots \\ &a_n = c_n + \sum_{i=1}^{p} b_i c_{n-i}, \end{aligned} \tag{2.170}$$

Table 2.2 Padé table

m	n			
	0	1	2	...
0	$f_{[0/0]}(\varepsilon)$	$f_{[1/0]}(\varepsilon)$	$f_{[2/0]}(\varepsilon)$	...
1	$f_{[0/1]}(\varepsilon)$	$f_{[1/1]}(\varepsilon)$	$f_{[2/1]}(\varepsilon)$	...
2	$f_{[0/2]}(\varepsilon)$	$f_{[1/2]}(\varepsilon)$	$f_{[2/2]}(\varepsilon)$	...
...	...	...	...	...

where $p = \min(n, m)$. Equations (2.169) and (2.170) are called Padé equations. In the case, when system (2.169) is solved, the Padé equations yield coefficients of the PA nominator and denominator. The function $f_{[n/m]}(\varepsilon)$ for different values of n and m defines the choices, which usually are presented in the form of the Padé table (Table 2.2).

Terms of the first row of the Padé table correspond to finite sums of the McLaurin series. When the powers of the polynomials are equal ($n = m$), the diagonal of PA is obtained, which belongs to the mostly distributed on practice. It should be emphasized that in the Padé table, a few indices n, m are omitted for which the PA does not exist.

In what follows, we briefly present a few important properties of PA [54, 55]:

1. If PA for given m and n exists, then it is unique.
2. If the PA is convergent to a given function, then the roots of its denominator tend to the poles of the function. It allows to find an efficient large number of the series terms to define the poles, and hence to continue the analytical investigation.
3. PA allows for the meromorphic continuation of the given function by the power series.
4. PA computed with respect of the inversed function is equal to PA of the original function. This property is called duality ... and it can be formulated in the more formal form as follows

$$q(\varepsilon) = f^{-1}(\varepsilon) \quad \text{and} \quad f(0) \neq 0, \quad \text{then} \quad q_{[n/m]}(\varepsilon) = f^{-1}_{[n/m]}(\varepsilon) \tag{2.171}$$

   under condition that at least one of the mentioned approximations exists.
5. Diagonal PAs are invariant with respect to linear-fractional transformation of the argument. Let the function is given by the series (2.166). We consider the linear-fractional transformation including the coordinates origin $W = \frac{a\varepsilon}{1+b\varepsilon}$ and the function $q(W) = f(\varepsilon)$. Therefore, we have

$$q_{[n/n]}(W) = f_{[n/n]}(\varepsilon).$$

   under condition that one of the approximations exists.
6. The diagonal PAs are invariant with regard to linear-fractional transformation of the functions. Let we have given function (2.166), and let

$$q(\varepsilon) = \frac{a + bf(\varepsilon)}{c + df(\varepsilon)}.$$

If $c + df(0) \neq 0$, then

$$q_{[n/n]}(\varepsilon) = \frac{a + bf_{[n/n]}(\varepsilon)}{c + df_{[n/n]}(\varepsilon)}$$

under condition that $f_{[n/n]}(\varepsilon)$ exists. Owing to this property, the infinite values of PA can be considered as the finite.

7. PA allows to get the upper and lower estimation of the function. For the diagonal PA, the following estimation holds

$$f_{[n/n-1]}(\varepsilon) \leq f_{[n/n]}(\varepsilon) \leq f_{[n/n+1]}(\varepsilon). \quad (2.172)$$

As a rule, this estimation is validated and holds for the function, i.e. $f_{[n/n]}(\varepsilon)$ in the relation (2.172) should be substituted by $f(\varepsilon)$.

8. Diagonal PAs and PAs close to them often exhibit the property of autocorrection [28–30]. In order to define the coefficients of nominator and denominator, we need to solve a system of linear algebraic equations. However, the so far described procedure is not correct, and hence the PA coefficients are defined with large errors. Since the errors are self-corrected, the accuracy of PA is relatively high. Here, a radial difference between PA and Taylor series, and its computation the error only increases with a number of employed terms. The autocorrection property is numerically verified for the series of special functions.

Now, let us consider the following problems. The given mathematical results regarding convergence of PA allow to increase reliability of the solution with respect to practical problems. Gonchar's theorem [56] yields the following statement. If no one from the diagonal PA $f_{[n/n]}(\varepsilon)$ has poles in the circle of the radius R, then the sequence $f_{[n/n]}(\varepsilon)$ is uniformly convergent in this circle to the original $f(\varepsilon)$. Furthermore, the lack of poles of the sequence $f_{[n/n]}(\varepsilon)$ in the circle of the radius R implies a convergence of the input Taylor series in the mentioned circle. Since the diagonal elements of PA are invariant with respect to the linear-fractional transformation $\varepsilon \to \varepsilon/(a\varepsilon + b)$, the theorem is valid for an arbitrary opened circle including the point of series development, and for an arbitrary space linking those circles. However, the essential drawback of this approach relies on requirement of checking all diagonals of the PA. Observe that in a circle of radius R, there are no poles in any subsequence of the sequence of diagonals of PA, and then its uniform convergence to the original holomorphic in the given circle function is guaranteed only for $r < r_0$, where $0.583R < r_0 < 0.584R$ [57]. The question arises: How we may employ those results? Assume that there is an interval of the series related to theory of perturbation and we want to estimate its convergence radius R. Consider the interval $[0, \varepsilon_0]$, on which the series interval and the last diagonal differ no more than 5% (the required technical accuracy). If no one of the previous PA diagonals has poles in the circle of radius ε_0, then with a high reliability one may assume that $R \geq \varepsilon_0$.

The procedure of construction of the PA is less tedious than the construction of the higher approximations of the theory of perturbations. The PA can be employed not only regarding the power series, but also on the series being orthogonal with respect to polynomials. In other words, the PAs are locally the best rational approximations for a given power series and they can be constructed directly from the series coefficients. They allow to realize the effective analytical extension of this series outside of the circle of convergence, and their holes in some sense localize the singular points, i.e. poles and their multiplicity, hence allowing for the extended functions in the corresponding area of convergence and on its border. This feature stands in contrary to the rational approximations with fixed poles, including the polynomial approximations, where all poles are located in one point located in infinity. The so far mentioned property allows to solve the problem of effective analytical extension of the power series, which lies in the fundamental problems occurred in practice.

This is why the theory of PA can be treated as the independent chapter of theory of approximations. The Padé approximations have found a wide spectrum of direct application to theory of rational approximations as well as to theory of perturbations. In what follows, we exhibit briefly the fundamental benefits of the PA in comparison with the Taylor series.

1. A speed of convergence of the rational approximations essentially exceeds the convergence velocity of the polynomial approximations. For instance, the function z in its circle of convergence is approximated by the rational polynomials $P_n(\varepsilon)/Q_n(\varepsilon)$ in 4^n of four times better than employ an algebraic polynomials of order $2n$. This statement is more evidently validated in the case of the functions with the bounded smoothness. Since the function $|\varepsilon|$ cannot be approximated on interval $[-1, 1]$ by the algebraic polynomials in order to achieve the order of approximation better than $1/n$, when n stands for the polynomial order, the AP yield the convergence velocity of $\sim \exp(-\sqrt{2n})$.
2. Typically the domain of convergence of the rational approximations is more wider in comparison with the power series. In the case of *arctan*, the Taylor polynomials are convergent only for $|\varepsilon| \leq 1$, whereas the PA everywhere in $C \setminus ((-i\infty, -i] \cup [i, i\infty))$.
3. PA allows to estimate a position of the function singularities.

The so far described PA is known as the one-point PA. The multipoint PA for the function $f(\varepsilon)$ is called a rational function $f_{[n/m]}(\varepsilon)$ of the form (2.167) having its values for $n + m + 1$ arguments coinciding with the values of $f(\varepsilon)$ in the mentioned points. This property allows for getting a system of linear algebraic equations yielding the searched coefficient of a nominator and denominator polynomials of the constructed PA.

In order to find PA, it is necessary to compute in one point the associated derivatives of the being studied function of a high order.

If a function is given in a tabular form, this method cannot be practically applied, and in particular when the given values are obtained with a relatively large error. Therefore, the functions given in tabular forms are suitable to be studied for the multipoint PA [58]. The rational interpolation can be effectively employed for approximations of the functions in those cases, where the ranks of an approximation are less than a number of points, where the function is defined. For instance through the method of least squares, the multipoint PA not only guarantee obtaining of the interpolation formulas, but they are also allow to extrapolate the values of functions defined on a limited interval beyond this interval. Furthermore, PA of a second order gives a possibility to estimate a position of the real poles of the being approximated functions lying beyond the interpolation interval.

2.13 Two-Point Padé Approximants

Analysis of the given numerous examples allows to formulate the following rule: if for $\varepsilon \to 0$, one may construct the physically validated asymptotics, then there exist non-trivial asymptotics for $\varepsilon \to \infty$. The most difficult case from a point of view of the asymptotic approach is the case $\varepsilon \sim 1$. Though in this condition usually numerical approaches can be employed, however, if we need to estimate a solution versus the parameter ε, then one needs to construct different solutions in different spaces. However, a construction of only one solution fitting the whole areas of solution existence does not belong to easy tasks. Let us illustrate the mentioned problem in more detail. It is known the function behaviour in zones I and III (Fig. 2.7), but we need to construct a solution in zone II. For this purpose, we apply the two-point PA (TPPA) [54].

Let us suppose

$$F(\varepsilon) = \sum_{i=0}^{\infty} c_i \varepsilon^i \quad \text{for} \quad \varepsilon \to 0, \tag{2.173}$$

$$F(\varepsilon) = \sum_{i=0}^{\infty} d_i \varepsilon^{-i} \quad \text{for} \quad \varepsilon \to \infty. \tag{2.174}$$

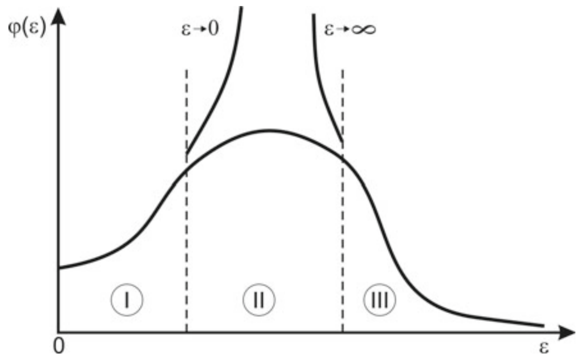

Fig. 2.7 Matching of asymptotic solutions

The TPPA is the fractional-rational function of the form (2.167), k coefficients of which are defined by condition (2.168), and the remaining coefficients are defined in an analogous way for ε^{-1}.

As an illustrative example, we consider the problem of finding a solution to the classical van der Pol equation

$$\ddot{x} + \varepsilon \dot{x}(x^2 - 1) + x = 0.$$

Asymptotic estimation of the oscillations period for both small and large values of ε is as follows [47]:

$$T = 2\pi \left(1 + \frac{\varepsilon^2}{16} - \frac{5\varepsilon^4}{3072}\right) \quad \text{for} \quad \varepsilon \to 0, \tag{2.175}$$

$$T = \varepsilon(3 - 2\ln 2) \quad \text{for} \quad \varepsilon \to \infty. \tag{2.176}$$

Let us construct the TPPA using four conditions for $\varepsilon \to 0$ and two conditions for $\varepsilon \to \infty$

$$T(\varepsilon) = \frac{a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3}{1 + b_1\varepsilon + b_2\varepsilon^2}, \tag{2.177}$$

where

$$a_0 = 2\pi, \quad a_1 = \frac{\pi^2\,(3 - 2\ln 2)}{4\,(3 - 2\ln 2)^2 - \pi^2},$$

$$a_2 = \frac{\pi\,(3 - 2\ln 2)^2}{2\,\left(4\,(3 - 2\ln 2)^2 - \pi^2\right)}, \quad a_3 = \frac{\pi^2\,(3 - 2\ln 2)}{16\,\left(4\,(3 - 2\ln 2)^2 - \pi^2\right)},$$

$$b_1 = \frac{\pi\,(3 - 2\ln 2)}{2\,\left(4\,(3 - 2\ln 2)^2 - \pi^2\right)}, \quad b_2 = \frac{\pi^2}{16\,\left(4\,(3 - 2\ln 2)^2 - \pi^2\right)}.$$

Table 2.3 reports the result of comparison of numerical estimation of the period given in [59] versus the computational results obtained using the formula (2.177).

2.14 Method of Asymptotically Equivalent Functions

Unfortunately, the case when both limiting asymptotics possess a power series development and hence they are suitably fitted by TPPA, is rather rarely met in practice. This is why another method of constructing of the uniformly suitable solution in the whole interval of the asymptotic parameter should be employed. We mention here one of the approaches based on the method of asymptotically equivalent functions, proposed by Slepyan and Yakovlev, while finding the inverses of integral transformations (see [60]).

Table 2.3 Comparison of numerical and TPPA results

ε	T	T Padé
1	6.66	6.61
2	7.63	7.37
3	8.86	8.40
4	10.20	9.55
5	11.61	10.81
6	13.06	12.15
7	14.54	13.54
8	16.04	14.96
9	17.55	16.42
10	19.08	17.89
20	34.68	33.30
30	50.54	49.13
40	66.50	65.10
50	82.51	81.14
60	98.54	97.20
70	114.60	113.29
80	130.67	129.40
90	146.75	145.49
100	162.84	161.61

Let Laplace transform of the function $f(t)$ (see formula (2.21)) is given. In order to get estimation of its original, we need to know the behaviour of the transform in points $s = 0$ and $s = \infty$. We need also to establish a character of location of its singular points lying on the exact border of the regular behaviour or in its vicinity.

Achieving that the transform $F(s)$ is substituted by the function $F_0(s)$, where the latter one guarantees an exact transition back to the original. This function should satisfy the following conditions.

1. The function $F_0(s)$ and $F(s)$ are asymptotically equivalent for $s \to \infty$ and $s \to 0$, i.e. we have
$$F_0(s) \sim F(s) \quad \text{for} \quad s \to 0 \quad \text{and} \quad s \to \infty.$$
2. Singular point of the functions $F_0(s)$ and $F(s)$ are located on the exact border of the space of regularity, and they coincide.

The free parameters of the function $F_0(s)$ are chosen in a way to satisfy the conditions of the approximate function $F(s)$ in the sense of the minimal relative error for all real values of $s \geq 0$:

$$\max \left| \frac{F_0(s, \alpha_1, \alpha_2, \ldots, \alpha_k)}{F(s)} - 1 \right| = \min . \tag{2.178}$$

Condition (2.178) is achieved via variation of the free parameters α_i. In many cases, a satisfaction of the formula

$$\int_0^\infty F_0(s)ds = \int_0^\infty F(s)ds$$

or $F'_0(s) \sim F'(s)$ for $s \to 0$ yields an efficient accuracy of estimating the condition (2.178).

Let us give an example of construction of the asymptotically equivalent function. We are aimed on finding the original of the transform given by the modified Bessel function of the following form [43]:

$$K_0(s) = -\ln(s/2)I_0(s) + \sum_{k=0}^{\infty} \frac{s^{2k}}{2^{2k}(k!)^2}\psi(k+1), \tag{2.179}$$

where $\psi(z)$ is the Euler's psi function [43].

When we have purely marginally values of the argument $s (s = iy, 0 < |y| < \infty)$, the function $K_0(s)$ does not have singular points. Consequently, one may limit the consideration to study its behaviour only for $s \to 0$ and $s \to \infty$. Let us write the associated asymptotic relations [43]:

$$\begin{aligned} K_0(s) &= -[\ln \frac{s}{2} + \gamma] + O(s), s \to 0, \\ K_0(s) &= \sqrt{\frac{\pi}{2s}} e^{-s} \left[1 + O\left(\frac{1}{s}\right)\right], s \to \infty, \end{aligned} \tag{2.180}$$

where γ is the Euler's constant ($\gamma = 1.781\ldots$) (let us mention a type in the first formula of (2.3) in [60]).

The being analysed transform possesses a bifurcation point of the logarithmic type, a bifurcation point of the algebraic type and purely singular point. All of the mentioned singular points should be preserved in a structure of the zero-order approximations.

The most simple form of the solution is obtained by matching two of the asymptotic formulas (2.180) in a way not to violate their meaning via their interaction and to contain the free parameters

$$F_0(s) = e^{-s}\left[\ln \frac{s+\alpha}{s} + \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{s+\beta}}\right]. \tag{2.181}$$

In result, the following zero-order approximation is obtained, where α and β are free parameters. It is not difficult to see that formula (2.181) has a proper asymptotics for $s \to \infty$. The free parameters are defined through the condition of equality of the asymptotics $K_0(s)$ and $F_0(s)$ for $s \to 0$, and conservation of equality of the integrals

$$\int_0^\infty F_0(s)ds = \int_0^\infty K_0(s)ds.$$

The carried out computations yield a system of the following transcendental equations

$$\ln\alpha + \sqrt{\frac{\pi}{2\beta}} = \ln 2 - \gamma,$$

$$\ln\alpha - e^{\alpha} Ei(-\alpha) + \gamma + \frac{\pi}{\sqrt{2}} e^{\beta}[1 - erf(\sqrt{\beta})] = \frac{\pi}{2},$$

where $Ei(\ldots)$ is the sine integral [43] and $erf(\ldots)$ is the error function [43] (the types in those formulas in reference [60] are reported).

Solving the problem numerically, we get $\alpha = 0.3192$, $\beta = 0.9927$.

Original of zero-order approximation is as follows:

$$f_0(t) = \left\{\frac{1 - \exp[-\alpha(t-1)]}{t-1} + \frac{\exp[-\beta(t-1)]}{\sqrt{2(t-1)}}\right\} H(t-1). \tag{2.182}$$

Observe that the exact value of the original is

$$f(t) = \frac{1}{\sqrt{t^2 - 1}} H(t-1). \tag{2.183}$$

Comparison of exact (2.183) (solid curve) and approximate (2.182) (dashed curve) solution are shown in Fig. 2.8. One may observe that the efficient result is achieved already in the first approximation.

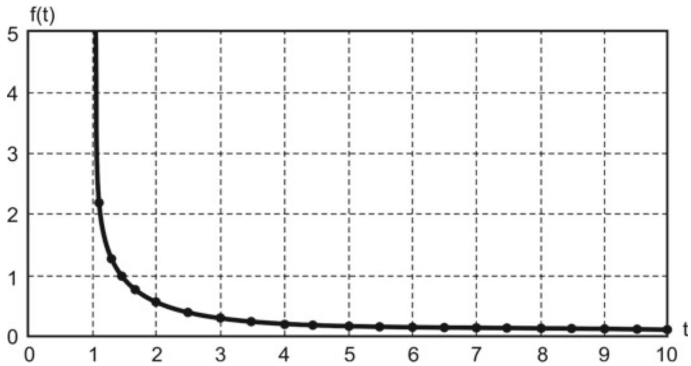

Fig. 2.8 Comparison of exact inverse Laplace transform versus the inverse obtained via the method of equivalent functions

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Asymptotical Mechanics of Composites
Modelling Composites without FEM
Andrianov, I.V.; Awrejcewicz, J.; Danishevskyy, V.V.
2018, XI, 329 p. 139 illus., 7 illus. in color., Hardcover
ISBN: 978-3-319-65785-1