Chapter 2
Electromagnetic Fields and Particle Motion

The design of accelerator magnets requires an understanding of the generation of magnetic fields together with the motion of particles within these fields. The basic equations are presented here to clarify the utilised symbols. Maxwell's equations and the magneto quasistatic approximation (MQS) is presented in [1, 2]. The equations of particle motion is based on analytic mechanics.

2.1 Maxwell's Equations

The field descriptions presented in this treatise are based on Maxwell's equations. These are given in the differential form by

$$\nabla \cdot \mathcal{B} = 0 \tag{2.1}$$

$$\nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} \tag{2.2}$$

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} \tag{2.3}$$

$$\nabla \cdot \mathcal{D} = \rho_e, \tag{2.4}$$

with $\mathcal{B}$ the magnetic induction, $\mathcal{H}$ the magnetising field, t the time, $\mathcal{J}$ the electric current density, $\mathcal{D}$ the electric displacement field, $\mathcal{E}$ the electric field and ρ_e the electric charge density. Calligraphic letters denote vector fields. Any stationary material is modelled by

$$\mathcal{B} = \mu_0 \left(\mathcal{H} + \mathcal{M}(\mathcal{H})\right) \tag{2.5}$$

$$\mathcal{D} = \epsilon_0 \mathcal{E} + \mathcal{P}_e(\mathcal{E}), \tag{2.6}$$

with $\mathcal{M}$ the magnetic polarisation and $\mathcal{P}_e$ the electric polarisation.

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2.1.1 Magnetic Quasistatic Approximation

Here only magnetic fields are treated; electric fields only need to be considered as so far as they are having an impact on the magnetic fields. Given that field changes are slow the time dependence of the term $\frac{\partial \mathcal{D}}{\partial t}$ can be considered not significant within the domain of interest, so the above equations reduce to

$$\nabla \cdot \mathcal{B} = 0 \tag{2.7}$$

and the magnetising $\mathcal{H}$-field is only induced by the current

$$\nabla \times \mathcal{H} = \mathcal{J} + \underbrace{\frac{\partial \mathcal{D}}{\partial t}}_{\approx 0} \approx \mathcal{J}\,. \tag{2.8}$$

This equation implies

$$\nabla \cdot \mathcal{J} = 0\,, \tag{2.9}$$

as no electric currents nor electric displacement currents are considered within the magneto quasi-static approximation (MQS).

If the current density vanishes, $\mathcal{J} = 0$, then from (2.8) follows (e.g. [1])

$$\nabla \times \mathcal{H} = 0, \tag{2.10}$$

which implies that a potential can be defined by

$$\mathcal{H} = -\nabla \Phi'. \tag{2.11}$$

In the absence of material $\mathcal{B} = \mu_0 \mathcal{H}$. Inserting this equation in (2.1) leads to

$$\nabla \cdot \mu_0 \mathcal{H} = 0, \tag{2.12}$$

and the Laplace equation for the scalar magnetic potential Φ',

$$\Delta \Phi' = 0 \tag{2.13}$$

is obtained. As this treatise focuses on fields in air gaps, it will relate any further potential to the magnetic induction $\mathcal{B} = \mu_0 \mathcal{H} = -\nabla \Phi$.

2.1.2 Magnetic Field in Linear Material

The magnetic material Eq. (2.5) shows two components. The first one describes the relation of the magnetising field to the magnetic field in vacuum. Material, which is isotropic and magnetically linear, can be modelled by

$$\mathcal{M} = \mu_r \mathcal{H} \tag{2.14}$$

or more commonly by

$$\mathcal{B} = \mu_0 \mu_r \mathcal{H}\,, \tag{2.15}$$

with μ_r the scalar magnetisation. For theoretical considerations one writes

$$\mathcal{B} = \mu \mathcal{H}. \tag{2.16}$$

This μ is used to describe "linear material". Any material reacting to the magnetising field in an isotropic manner and independent of the strength of the magnetising field fulfils this property. Vacuum and air are two common practical materials of this category. For vacuum $\mu_r = 1$ while for air $\mu_r \approx 1 + 0.4 \times 10^{-6}$. The fields are treated here with an accuracy of not more than 1 ppm, thus the difference between air and vacuum can be neglected. The characteristics of linear material allow a further simplification of Maxwell's equations.

2.1.2.1 Poisson and Laplace Equation for the Vector Potential

Given that magnetic fields are solenoidal (see (2.1)) and using the assumption that only fields in linear material need to be considered here, one can rewrite (2.1) by

$$\nabla \cdot \mathcal{B} = \nabla \cdot \mu_0 \mathcal{H}. \tag{2.17}$$

A vector potential $\mathcal{A}$ can now be introduced by

$$\nabla \times \mathcal{A} = \mu_0 \mathcal{H}\,. \tag{2.18}$$

From (2.8), one can deduce that

$$\nabla \times (\nabla \times \mathcal{A}) = \mu_0 \mathcal{J}\,. \tag{2.19}$$

The above operator is recalculated to

$$\nabla \times (\nabla \times \mathcal{A}) = \nabla(\nabla \cdot \mathcal{A}) - \nabla^2 \mathcal{A} \tag{2.20}$$

following vector analysis rules. However, the vector potential is not unique. It may be defined as purely solenoidal: $\nabla \cdot \mathcal{A} = 0$ (Coulomb gauge) which in turn yields $\nabla(\nabla \cdot \mathcal{A}) = 0$. Thus one obtains an equation resembling the Poisson equation for the vector potential

$$\Delta \mathcal{A} = -\mu_0 \mathcal{J}. \tag{2.21}$$

Many problems are constant with respect to one of the coordinates; then only a xy- plane needs to be considered with currents flowing only perpendicular to this plane. If the third coordinate is a Cartesian one, say z, the above equation simplifies to

$$\Delta A_z = -\mu_0 J_z \,. \tag{2.22}$$

In the free aperture of the magnet no currents or charges are found. The field is described by the homogeneous equation corresponding to the above equations. But in this case it is more advantageous to use a scalar magnetic potential Φ, which is a solution of the potential equation

$$\Delta \Phi = 0 \,. \tag{2.23}$$

The derivation of this equation from Maxwell's equation was given in Sect. 2.1.1.

2.1.2.2 Solutions of the Laplace Equation

The solution of the Laplace Equation (2.13) depends, as for any partial differential equation, on the boundary conditions next to the domain. Furthermore, the set of functions to be used depends on the coordinate system (see e.g. [3]). Here they are given for a circular cylinder. Solutions for other coordinate systems are given in Chap. 4. The Laplace Equation in cylindrical circular coordinates is given by

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \vartheta^2} = 0 \,, \tag{2.24}$$

assuming that the problem is independent of z, and thus, $\frac{\partial^2 \Phi}{\partial z^2} = 0$. This equation can be solved by separation using $\Phi = R(r)\Theta(\vartheta)$ (see e.g. [3]), forming the differential equations

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{a_2}{r^2} = 0, \quad \text{and} \tag{2.25}$$

$$\frac{\mathrm{d}^2 \Theta}{\mathrm{d}\vartheta^2} + a_2 \Theta = 0. \tag{2.26}$$

For $a_2 = 0$ the solutions are

$$R = A_0 + B_0 \ln(r) \tag{2.27}$$
$$\Theta = C_0 + D_0\,\vartheta\,. \tag{2.28}$$

For $a_2 = n^2$ the general solution is given by

$$R = A'_n r^n + B'_n r^{-n} \tag{2.29}$$
$$\Theta = F'_n \sin(n\vartheta) + G'_n \cos(n\vartheta)\,. \tag{2.30}$$

Fields considered within this treatise are always within a reference volume; in this section it is a circular cylinder. A regular solution within this boundary is demanded, thus only the following functions are considered:

$$\sin(n\vartheta),\ \cos(n\vartheta),\ r^n\,. \tag{2.31}$$

Hence the potential Φ is given by

$$\Phi(r,\vartheta) = -\sum_{n=1}^{\infty} r^n \left(F_n \sin(n\vartheta) + G_n \cos(n\vartheta)\right)\,. \tag{2.32}$$

The associated field components are given by

$$B_r(r,\vartheta) = -\frac{\partial \Phi}{\partial r} = \sum_{n=1}^{\infty} n\, r^{n-1} \left(F_n \sin(n\vartheta) + G_n \cos(n\vartheta)\right) \quad \text{and} \tag{2.33}$$
$$B_\vartheta(r,\vartheta) = -\frac{1}{r}\frac{\partial \Phi}{\partial \vartheta} = \sum_{n=1}^{\infty} n\, r^{n-1} \left(F_n \cos(n\vartheta) - G_n \sin(n\vartheta)\right)\,. \tag{2.34}$$

Now these field components, given in the cylindrical circular coordinate system shall be converted into a Cartesian coordinate system represented by complex coordinates. The Cartesian components are given by

$$B_y = B_r \sin\vartheta + B_\vartheta \cos\vartheta \qquad \text{and} \qquad B_x = B_r \cos\vartheta - B_\vartheta \sin\vartheta\,. \tag{2.35}$$

The geometric extent of magnetic fields, found in accelerators, is typically much longer in one coordinate than in the other two. The 3D field can be treated by beam optics as an infinite thin object (equivalent to a thin lens in geometric optics). Complex representation simplifies the calculation in 2D space. In 3D Cartesian coordinates Maxwell's equations of the magnetic field in linear material are described by

$$\nabla \cdot \mathcal{B} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) = 0 \tag{2.36}$$

and without currents within the reference volume by

$$\nabla \times \mathcal{B} = \begin{pmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ -\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{pmatrix} = 0\,. \tag{2.37}$$

As the problem is to be independent of the longitudinal coordinate z one can set $\frac{\partial B_y}{\partial z} = 0$, $\frac{\partial B_x}{\partial z} = 0$. Furthermore the field component B_z shall be invariant with respect to z, thus $\frac{\partial B_z}{\partial x} = 0$ and $\frac{\partial B_z}{\partial y} = 0$. This simplifies the above equations to

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} \quad \text{and} \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}. \tag{2.38}$$

These are exactly the Cauchy-Riemann equations for $B_y + \mathrm{i}B_x$ with $x + \mathrm{i}y$. Using complex notation (2.35) can be expressed by [4]

$$B_y + \mathrm{i}B_x = (B_\vartheta + \mathrm{i}B_r)e^{-\mathrm{i}\vartheta}\,. \tag{2.39}$$

$B_\vartheta + \mathrm{i}B_r$ are thus (using (2.33) and (2.34))

$$B_\vartheta + \mathrm{i}B_r = \sum_{n=1}^{\infty} nr^{n-1}\,(F_n + \mathrm{i}G_n)\,e^{\mathrm{i}n\vartheta}\,. \tag{2.40}$$

Furthermore it is good practice to use dimensionless coefficients. Therefore the radius is scaled by

$$\rho = \frac{r}{R_{\mathrm{Ref}}}\,, \tag{2.41}$$

with R_{Ref} the reference radius. Using the relation between the polar and Cartesian field (2.39) and the expansion (2.40) the complex field $\mathbf{B}(\mathbf{z})$ is given by

$$\mathbf{B}(\mathbf{z}) = B_y(x + \mathrm{i}y) + \mathrm{i}B_x(x + \mathrm{i}y) = \sum_{n=1}^{\infty} \mathbf{C_n} \left(\frac{\mathbf{z}}{R_{\mathrm{Ref}}}\right)^{n-1}\,, \tag{2.42}$$

with $\mathbf{C_n} = B_n + A_n$, $B_n = n\,F_n\,R_{\mathrm{Ref}}^{n-1}$, and $A_n = n\,G_n\,R_{\mathrm{Ref}}^{n-1}$. The B_n are called the normal components and the A_n the skew components. $\mathbf{C_n}$, B_n and A_n are called "absolute harmonics". Their dimension is T at R_{Ref} (Tesla at reference radius). For accelerators the so called normalised harmonics $\mathbf{c_n} = b_n + \mathrm{i}a_n$ are introduced by

$$\mathbf{c_n} = \frac{\mathbf{C_n}}{\mathbf{C_m}}10^4\,, \tag{2.43}$$

with $\mathbf{C_m}$ the main harmonic of the magnet. The main harmonic is $\mathbf{C_1}$ for a dipole, $\mathbf{C_2}$ for a quadrupole, and so on. These coefficients are dimensionless. The scaling factor 10^4 is used so that the size of $\mathbf{c_n}$ is in the order of 1 for a typical accelerator magnet. In the accelerator community one then typically speaks of a "harmonic of the size of 1 unit".

2.1.2.3 Treating Fringe Fields as Averaged 2D Fields

Also fringe fields can be treated as 2D fields, if the 3D field is integrated over the longitudinal coordinate z [5]. An averaged potential $\overline{\Phi}$ is introduced, which is obtained integrating Laplace's equation in z over a distance z_L. Then for this case the equation can be reformulated to

$$\frac{\partial^2 \overline{\Phi}}{\partial x^2} + \frac{\partial^2 \overline{\Phi}}{\partial y^2} + \frac{1}{z_L} \frac{\partial \Phi}{\partial z}\bigg|_{z_0}^{z_0+z_L} = 0\,, \tag{2.44}$$

with z_0 the integration limit. The field $\mathcal{B}$ is obtained by $-\nabla\Phi$. So $\frac{\partial \Phi}{\partial z}$ corresponds to B_z. Then the last term vanishes if B_z fulfils

$$B_z(z_0) = 0 \quad \text{and} \quad B_z(z_0 + L) = 0\,. \tag{2.45}$$

So end fields can be treated as averaged 2D fields if these conditions are fulfilled.

2.2 Particle Motion in Magnetic Fields

Particles guided in accelerators, typically electrons, protons or ions, are charged. Due to their speed, $\vec{v}$, often between 10% to closely to 100% of the speed of light c, the Lorentz Force

$$\vec{F}_{\mathrm{L}} = q\,\vec{v} \times \mathcal{B} \tag{2.46}$$

is much larger for technically achievable magnetic fields (typically between 0.3 to 2 T for conventional magnets) than the force on the charge q in a technical electric field

$$\vec{F} = q\,\mathcal{E}\,. \tag{2.47}$$

Magnetic fields can not increase the particle energy but only deviate their paths due to the cross product in (2.46). Hence the particle motion considered here is limited only to deflections of the particle path. Such a deflection requires a centripetal force F_{zp}. Its strength is given by

$$\left|F_{\mathrm{zp}}\right| = \frac{m\left|v^2\right|}{R} \tag{2.48}$$

with m the mass of the particle, v its velocity and R the orbit's radius.

2.2.1 Particle Motion in a Cyclotron

Cyclotrons are accelerators and the particles gain energy within the gap of its electrodes. The treatise focuses on magnetic fields and thus only on particles with constant energy; therefore the energy gain of the particles is neglected. So here a cyclotron can be considered as a dipole magnet whose field area is so large that the whole particle path is covered. The equilibrium path is given if the forces are balanced by

$$\frac{m\,{v_s}^2}{r} = F_{\mathrm{zp}} = F_{\mathrm{L}} = q\,v_s\,B_y, \tag{2.49}$$

with v_s the particle velocity in direction of the particle path s. Using the relation $p = m v_s$ yields

$$\frac{1}{R} = \frac{q}{p} B_y\,. \tag{2.50}$$

If the field of the cyclotron is not totally uniform one obtains as local equilibrium

$$\frac{1}{R(x, y, s)} = \frac{q}{p} B_y(x, y, s). \tag{2.51}$$

2.2.2 Paraxial Approximations

Amongst other things the path of a particle in a cyclotron is defined by the particle velocity. Its local radius depends on the particular position (see (2.51)) [6, 7]. The relation (2.51) is generally valid. In synchrotrons or storage rings the magnetic field strength is adjusted to the particle's momentum, because otherwise the particles will hit the aperture and get lost.

The ideal orbit is the trajectory of the particle for which the accelerator is designed. Most of the particles in a bunch do not have the initial conditions for this orbit; but one assumes that their deviations from the ideal orbit are small. Different particles in an accelerator bunch are assumed to have only small deviations from the ideal orbit. Therefore one can study the field influence considering only local derivatives. Further the magnet fields are assumed to be independent of the longitudinal coordinate s.

Using a coordinate system, which follows the ideal particle path (i.e. Frenet-Serret coordinates, see Sect. 3.4) one can substitute $r = R + x$, with R the radius of the foreseen orbit of the ideal particle and x the horizontal offset from this ideal orbit. Then the force F_x is given by

$$F_x(r) = \frac{mv^2}{\underbrace{R + x}_{=r}} - evB_y(R + x)\,. \tag{2.52}$$

Given that the local field change is small, the field variation can be described by

$$B_y(r) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{\partial^2 B_y}{\partial x^2} x^2 + \dots \tag{2.53}$$

This allows describing the dependence of the field on the offset of the particle from the ideal orbit using

$$B_y(x) = B_{y0} \left[1 + \frac{R}{B_{y0}} \frac{\partial B_y}{\partial x} \frac{x}{R} \right] . \tag{2.54}$$

In textbooks on beam dynamics (e.g. [7]) the derivatives in (2.53) are approximated using the coefficients of the power series (2.42) [4]:

$$\left. \frac{\partial^{n-1} B_y}{\partial x^{n-1}} \right|_{x=0,\, y=0} = \frac{(n-1)!}{R_{\text{Ref}}^{n-1}} \operatorname{Re}[\mathbf{C_n}] = \frac{(n-1)!}{R_{\text{Ref}}^{n-1}} B_n , \tag{2.55}$$

$$\left. \frac{\partial^{n-1} B_x}{\partial x^{n-1}} \right|_{x=0,\, y=0} = \frac{(n-1)!}{R_{\text{Ref}}^{n-1}} \operatorname{Im}[\mathbf{C_n}] = \frac{(n-1)!}{R_{\text{Ref}}^{n-1}} A_n . \tag{2.56}$$

This approach neglects the curvature of the particle trajectory within e.g. a long dipole magnet as the whole theory assumes that the "lenses" (i.e. the magnets) are short, similar to the "thin lens" approximation in geometric optics.

2.2.3 Summary

Particle accelerators use magnetic fields to guide the particles around the ideal path. The aperture of these magnets is evacuated, thus material free, and the ramp rates of the magnetic fields are small enough so that the magnetic quasistatic approximation can be used. As the aperture is material and current free, the potential of the magnetic field is described by Laplace equation (2.13).

The transverse offset of the particles from the ideal orbit is small, therefore beam dynamics of synchrotrons and colliders uses the paraxial approximation to describe the effect of the magnetic fields on the particle beam. Commonly approximate circular multipoles are used for that purpose, even if the particle path is significantly bent within the magnet (e.g. in a dipole in a small synchrotron).

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