Chapter 2
Relation algebras

The theory of relation algebras is an abstract algebraic version of the calculus of relations that was outlined in Chapter 1. There are four new components to the theory. First, ten simple laws from the calculus of relations have been singled out as axioms of the theory; in particular, the theory is axiomatic in nature. Second, each of the axioms has the form of an equation, and the only rules of inference that are permitted are the rule of substitution and the rule of replacement of equals by equals, both familiar from high school algebra. Derivations of laws are therefore no longer set-theoretical arguments in which one shows that a pair of elements from the universe belongs to the relation on one side of an equation if and only if it belongs to the relation on the other side; rather, they are equational derivations from the ten axioms. Third, the equational setting implies that one can apply standard algebraic methods—such as the formation of homomorphic images, direct products, and subalgebras—to analyze the models of the theory, that is to say, to analyze relation algebras, and to construct new relation algebras from old ones. This leads to a rich algebraic theory that is similar in spirit to the algebraic theories of groups (an abstraction of the theory of permutations), rings (an abstraction of the theory of polynomials), and Boolean algebras (an abstraction of the calculus of classes). Fourth, although the intended models of the theory are algebras of relations (to be defined in Chapter 3), there are other kinds of algebras in which the axioms of the theory are all valid. This generality leads to a variety of beautiful and unexpected models, and to interconnections with other important mathematical domains such as group theory and projective geometry. It also leads to a host of interesting questions that otherwise would not arise.

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2.1 Fundamental notions and axioms

The notion of a relation algebra is a special case of a more general notion of algebra from the theory of general algebraic structures. An *algebra* is a system consisting of a non-empty set called the *universe* of the algebra, and a family of operations on the universe. These operations are called the *fundamental operations* of the algebra.

Definition 2.1. A *relation algebra* is an algebra

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,;\,,\,{}^{\smile}\,,\,1')$$

in which $+$ and $;$ are binary operations on the universe A, while $-$ and ${}^{\smile}$ are unary operations on A, and 1' is a distinguished constant of A, such that the following axioms are satisfied for all elements r, s, and t in A.

(R1) $r + s = s + r$.
(R2) $r + (s + t) = (r + s) + t$.
(R3) $-(-r + s) + -(-r + -s) = r$.
(R4) $r\,;(s\,;t) = (r\,;s)\,;t$.
(R5) $r\,;1' = r$.
(R6) $r^{\smile\smile} = r$.
(R7) $(r\,;s)^{\smile} = s^{\smile}\,;r^{\smile}$.
(R8) $(r + s)\,;t = r\,;t + s\,;t$.
(R9) $(r + s)^{\smile} = r^{\smile} + s^{\smile}$.
(R10) $r^{\smile}\,;-(r\,;s) + -s = -s$.

The set A is called the *universe* of $\mathfrak{A}$. The *Boolean* operations $+$ and $-$ are called (*Boolean*) *addition* and *complement* (or *complementation*) respectively. The *Peircean* operations $;$ and ${}^{\smile}$ are called *relative multiplication* and *converse* (or *conversion*) respectively. The distinguished Peircean constant 1' is called the *identity element*. □

The axioms in the preceding definition are commonly referred to using the following names: (R1) is the *commutative law for addition*, (R2) is the *associative law for addition*, (R3) is *Huntington's law*, (R4) is the *associative law for relative multiplication*, (R5) is the (right-hand) *identity law for relative multiplication*, (R6) is the *first involution law*, (R7) is the *second involution law*, (R8) is the (right-hand) *distributive law for relative multiplication*, (R9) is the *distributive law for converse*, and (R10) is *Tarski's law*.

As the notation of the preceding definition implies, we shall use upper case (German) fraktur letters to refer to relation algebras (and, much later on, to relational structures). When referring to other algebraic or geometric structures such as groups, Boolean algebras, or projective geometries, we shall usually use the upper case italic letter that denotes the universe of the structure. This will simplify the notation and make it more readable. We shall adopt one more simplification of notation. In order to distinguish carefully between the operations of different relation algebras $\mathfrak{A}$ and $\mathfrak{B}$, one should employ different notations to distinguish the operations of the two algebras, for example, by using superscripts such as

$$\mathfrak{A} = (A\,,\, +^{\mathfrak{A}}\,,\, -^{\mathfrak{A}}\,,\, ;^{\mathfrak{A}}\,,\, {}^{\smile\mathfrak{A}}\,,\, 1'^{\mathfrak{A}})$$

In practice, the context usually makes clear when the operation symbols in question refer to the operations of $\mathfrak{A}$ and when they refer to the operations of $\mathfrak{B}$; so we shall always omit such superscripts when no confusion can arise.

It is common practice in algebra to identify the universe of an algebra with the algebra itself, therefore to speak of algebra as if one were speaking of universe, and vice versa. We shall follow this practice to a certain extent. For example, we shall often speak about elements in $\mathfrak{A}$ and subsets of $\mathfrak{A}$, instead of elements in, and subsets of, the universe of $\mathfrak{A}$. We shall also speak about the *cardinality* of $\mathfrak{A}$, by which is meant the cardinality, or size, of the universe of $\mathfrak{A}$.

The conventions regarding the order in which operations are to be performed when parentheses are omitted are the same as those mentioned in Section 1.2 for the calculus of relations: unary operations take precedence over binary operations, and among binary operations, multiplications take precedence over additions and subtractions. For example, in fully parenthesized form, Axioms (R7), (R8), and (R10) might be written as

$$(r\,;s)^{\smile} = (s^{\smile})\,;(r^{\smile}), \qquad (r+s)\,;t = (r\,;t) + (s\,;t),$$

and

$$((r^{\smile})\,;(-(r\,;s))) + (-s) = -s.$$

Axioms (R1)–(R3) imply that the *Boolean part* of a relation algebra $\mathfrak{A}$, namely the algebra $(A\,,\,+\,,\,-)$, is a Boolean algebra. In particular, the notions and laws from the theory of Boolean algebras apply

to relation algebras. For example, the binary operations of (*Boolean*) *multiplication*, *difference* or *subtraction*, and *symmetric difference* are respectively defined by

$$r \cdot s = -(-r + -s), \quad r - s = r \cdot -s, \quad r \ominus s = (r - s) + (s - r)$$

for all r and s in $\mathfrak{A}$. The dual use of the symbol $-$ to denote the unary operation of complement, and the binary operation of Boolean difference should cause no confusion. The same dual use of this symbol occurs in the arithmetic of the integers. The context always makes clear whether the unary or the binary operation is intended. Regarding the order of operations when parentheses are omitted, multiplications take precedence over differences. For example, $r\,;s - r\,;t$ is to be understood as $(r\,;s) - (r\,;t)$. The Boolean constants *zero* and *one* (or the *unit*) are respectively defined by

$$0 = -(-1' + 1') \qquad \text{and} \qquad 1 = -1' + 1'.$$

Note that when the terms "addition" and "multiplication", or their analogues "sum" and "product", are used without any modifier, they refer to the Boolean operations and not the Peircean operations, unless the context makes another intention clear.

It is well known and easy to check that the operation $\ominus$ of symmetric difference is associative and commutative in the sense that the equations

$$r \ominus (s \ominus t) = (r \ominus s) \ominus t \qquad \text{and} \qquad r \ominus s = s \ominus r$$

always hold. Zero is the identity element for this operation because

$$r \ominus 0 = 0 \ominus r = r,$$

and every element is its own inverse with respect to this operation because $r \ominus r = 0$. Conclusion: the elements of a relation algebra form a Boolean group (that is to say, a group in which each element is its own inverse) under the operation of symmetric difference.

A *partial order* $\leq$ is defined on the universe of a relation algebra by

$$r \leq s \quad \text{if and only if} \quad r + s = s.$$

This definition makes clear that every inequality $r \leq s$ may be viewed as an equation, namely the equation $r + s = s$ (or, equivalently, the equation $r \cdot s = r$). We write $r < s$ if $r \leq s$ and $r \neq s$.

The *sum*, or *supremum*, of a set X of elements in a relation algebra is defined to be the least upper bound of X, provided such a least upper bound exists. In other words, r is the supremum of X if r is an upper bound for X in the sense that $s \leq r$ for every s in X, and if r is below every other upper bound t of X in the sense that $r \leq t$. Of course, the supremum of an infinite set of elements may not exist. The *product*, or *infimum*, of the set X is defined to be the greatest lower bound of X, provided such a greatest lower bound exists. In other words, r is the infimum of X if r is a lower bound for X in the sense that $r \leq s$ for every s in X, and if r is above every other lower bound t of X in the sense that $t \leq r$. We shall denote the sum and product of X, when they exist, by $\sum X$ and $\prod X$ respectively. Notice that if X is a finite set, say

$$X = \{r_0, \ldots, r_{n-1}\},$$

then

$$\sum X = r_0 + \cdots + r_{n-1} \qquad \text{and} \qquad \prod X = r_0 \cdot \cdots \cdot r_{n-1}.$$

The sum of a system of elements $(r_i : i \in I)$ is defined to be the supremum of the set $\{r_i : i \in I\}$. We shall write $\sum_{i \in I} r_i$ to denote this sum (when it exists), and if the context makes clear what index set I is intended, then we shall often omit any reference to it and write simply $\sum_i r_i$, or even just $\sum r_i$. Analogous remarks apply to the product of a system of elements.

The supremum of the empty subset is, by convention, 0. Indeed, 0 is an upper bound of the empty subset (it is vacuously above every element in the empty set), and it is obviously the least such upper bound. Similarly, the infimum of the empty subset is, by convention, 1. Indeed, 1 is a lower bound of the empty subset (it is vacuously below every element in the empty set), and it is obviously the greatest such lower bound.

A relation algebra is said to be *complete* if every subset of the universe has a supremum and an infimum. As with Boolean algebras, it suffices to require the existence of the supremum of every subset; the existence of the infimum of every subset then follows easily. A relation algebra is said to be *countably complete*, or *σ-complete*, if every countable subset of the universe has a supremum and an infimum. Again, it suffices to require the existence of the supremum of every countable subset; the existence of the infimum of every countable subset then follows.

An *atom* in a relation algebra is defined to be a minimal, non-zero element. In other words, r is an atom if $r \neq 0$, and if $s \leq r$ always implies that either $s = 0$ or $s = r$. A relation algebra is said to be *atomic* if every non-zero element is above an atom, and *atomless* if it contains no atoms at all. Warning: the relation algebra with just one element in its universe, namely zero, is both atomic and atomless. It is called the *degenerate* relation algebra. A non-degenerate relation algebra may be atomic, or atomless, or neither. However, a finite relation algebra is necessarily atomic, and therefore a non-degenerate atomless relation algebra is necessarily infinite.

Two elements r and s in a relation algebra are said to be *disjoint* if their product $r \cdot s$ is 0. More generally, a set of elements is said to be disjoint if any two distinct elements in the set are disjoint. Similarly, a system $(r_i : i \in I)$ of elements is called disjoint if $i \neq j$ always implies that r_i and r_j are disjoint. A *partition* of an element r is a disjoint set, or system, of elements that has r as its supremum. With a few exceptions (that will be explicitly pointed out), it is always assumed that the elements in a partition are all non-zero.

A *Boolean homomorphism* from a relation algebra $\mathfrak{A}$ to a relation algebra $\mathfrak{B}$ is a mapping φ from the universe of $\mathfrak{A}$ to the universe of $\mathfrak{B}$ that preserves the Boolean operations of addition and complement in the sense that

$$\varphi(r + s) = \varphi(r) + \varphi(s) \qquad \text{and} \qquad \varphi(-r) = -\varphi(r)$$

for all elements r and s in $\mathfrak{A}$. It is easy to check that a Boolean homomorphism must also preserve the defined operations of multiplication, subtraction, and symmetric difference, and it must map zero and one to zero and one respectively. If a Boolean homomorphism φ is one-to-one, or onto, or both, then φ is called a Boolean *monomorphism*, *epimorphism*, or *isomorphism* respectively. A Boolean isomorphism that maps $\mathfrak{A}$ to itself is called a Boolean *automorphism* of $\mathfrak{A}$.

Axioms (R4)–(R7) imply that the *Peircean part* of a relation algebra $\mathfrak{A}$, namely the algebra $(A\,,\,;\,,\,\breve{}\,,\,1')$, is a monoid with involution. In other words, it is a semigroup under the operation $;$, with an identity element 1', and with a unary operation $\breve{}$ that satisfies the two involution laws. In this respect, the Peircean part of a relation algebra is somewhat similar in nature to a group (which is also a monoid with an involution). In relation algebras, however, elements do not in general have inverses with respect to the operation of relative multiplication. Nevertheless, the arithmetic of relation algebras is a curious

and fascinating blend of the laws of Boolean algebra and of group theory. The distributivity axioms (R8) and (R9) ensure that a relation algebra is a Boolean algebra with operators (see below), so the entire theory of Boolean algebras with operators can be applied to relation algebras.

A binary Peircean operation $\dot{+}$ of *relative addition* and a distinguished Peircean constant 0' called the *diversity element* are defined by

$$r \dot{+} s = -(-r\,;-s) \qquad \text{and} \qquad 0' = -1'$$

respectively.

Tarski's law, that is to say, Axiom (R10), is the real workhorse of the theory, and most of the important laws are directly or indirectly derived with its help. It is clear from the definition of the partial order $\leq$ that (R10) is just an equational form of the inequality

$$r^{\smile}\,;-(r\,;s) \leq -s.$$

In the presence of (R1)–(R3) and (R6)–(R9), this inequality is equivalent to the implication

$$\text{if} \quad (r\,;s)\cdot t = 0, \quad \text{then} \quad (r^{\smile}\,;t)\cdot s = 0 \tag{R11}$$

(the details are left as an exercise). In fact, in the presence of the other axioms, (R10) is equivalent to the following *De Morgan-Tarski laws*:

$$\begin{aligned}
(r\,;s)\cdot t = 0 \qquad &\text{if and only if} \qquad (r^{\smile}\,;t)\cdot s = 0,\\
&\text{if and only if} \qquad (t\,;s^{\smile})\cdot r = 0,\\
&\text{if and only if} \qquad (s^{\smile}\,;r^{\smile})\cdot t^{\smile} = 0,\\
&\text{if and only if} \qquad (t^{\smile}\,;r)\cdot s^{\smile} = 0,\\
&\text{if and only if} \qquad (s\,;t^{\smile})\cdot r^{\smile} = 0
\end{aligned}$$

(see Lemma 4.8 and Corollary 4.9). As one passes from one of these equations to another, the variables are permuted in a cyclic fashion. For this reason, the De Morgan-Tarski laws are sometimes called the cycle laws. We shall use the term *cycle law* to refer to (R11) alone.

2.2 Boolean algebras with operators

A number of the deepest and most important results about relation algebras hold for a much broader class of algebras called Boolean al-

gebras with operators. In this section, we take a preliminary look at these algebras.

Consider an arbitrary Boolean algebra $(A, +, -)$. A binary operation $;$ on the universe A is said to be *distributive* (over addition), or *additive*, if it is distributive in each argument in the sense that for any elements r, s, and t in A,

$$r ; (s + t) = r ; s + r ; t \qquad \text{and} \qquad (r + s) ; t = r ; t + s ; t.$$

The operation is said to satisfy the *general finite distributivity law* (over addition) if for all finite, non-empty subsets X and Y of A,

$$(\sum X) ; (\sum Y) = \sum\{r ; s : r \in X \text{ and } s \in Y\}$$

It is a straightforward matter to extend the preceding definitions to operations on A of arbitrary rank $n > 0$. A word about terminology: when speaking of the distributivity of an operation, we shall always mean its distributivity over addition, unless explicitly stated otherwise.

Lemma 2.2. *An operation on a Boolean algebra is distributive if and only if it satisfies the general finite distributivity law.*

Proof. Obviously, an operation that satisfies the general finite distributivity law must be distributive (in each argument). To establish the reverse implication, consider the case of a binary distributive operation $;$. A straightforward argument by induction on natural numbers $n > 0$ shows that

$$p ; (s_0 + \cdots + s_{n-1}) = p ; s_0 + \cdots + p ; s_{n-1}, \tag{1}$$

$$(r_0 + \cdots + r_{n-1}) ; q = r_0 ; q + \cdots + r_{n-1} ; q, \tag{2}$$

whenever the elements involved belong to the given Boolean algebra. Suppose now that X and Y are finite non-empty sets of elements in the Boolean algebra, and write

$$p = \sum X \qquad \text{and} \qquad q = \sum Y.$$

Use the preceding definitions of p and q, the equations in (1) and (2), and the associative and commutative laws for addition to arrive at

$$\begin{aligned} p ; q &= p ; (\sum Y) = \sum\{p ; s : s \in Y\} = \sum\{(\sum X) ; s : s \in Y\} \\ &= \sum\{\sum\{r ; s : r \in X\} : s \in Y\} = \sum\{r ; s : r \in X \text{ and } s \in Y\}. \end{aligned}$$

The proof for operations of ranks different from two is completely analogous. □

A binary operation ; on a Boolean algebra A is said to be *monotone* if it is monotone in each argument in the sense that $s \leq t$ implies

$$r\,;s \leq r\,;t \qquad \text{and} \qquad s\,;r \leq t\,;r$$

for all elements r, s, and t in A. A monotone operation ; clearly satisfies the following *general monotony law*:

$$r \leq u \quad \text{and} \quad s \leq v \quad \text{implies} \quad r\,;s \leq u\,;v.$$

This definition and observation easily extend to operations of other ranks. Distributive operations are always monotone.

Lemma 2.3. *A distributive operation on a Boolean algebra is monotone.*

Proof. Consider the case of a distributive binary operation ; on a Boolean algebra. If r, s, and t are elements in the given algebra, and if $s \leq t$, then $s + t = t$, by the definition of $\leq$, and therefore

$$r\,;(s+t) = r\,;t \qquad \text{and} \qquad (s+t)\,;r = t\,;r.$$

Since

$$r\,;(s+t) = r\,;s + r\,;t \qquad \text{and} \qquad (s+t)\,;r = s\,;r + t\,;r,$$

it may be concluded that

$$r\,;s + r\,;t = r\,;t \qquad \text{and} \qquad s\,;r + t\,;r = t\,;r.$$

Consequently, $r\,;s \leq r\,;t$ and $s\,;r \leq t\,;r$, by the definition of $\leq$. □

Distributive laws for operations on Boolean algebras have infinitary versions as well. A binary operation ; on a Boolean algebra A is said to be *quasi-completely distributive*, or *quasi-complete* for short, if for all elements r in A and all non-empty subsets X of A, the existence of the sum $\sum X$ implies that the sums

$$\sum\{r\,;s : s \in X\} \quad \text{and} \quad \sum\{s\,;r : s \in X\}$$

exist, and that

$$r\,;(\textstyle\sum X) = \sum\{r\,;s : s \in X\} \quad \text{and} \quad (\sum X)\,;r = \sum\{s\,;r : s \in X\}.$$

The operation ; is said to satisfy the *general quasi-complete distributivity law* if for all non-empty subsets X and Y of A, the existence of the suprema $\sum X$ and $\sum Y$ in A implies that the supremum of the set

$$\{r\,;s : r \in X \text{ and } s \in Y\}$$

exists, and (as in the case of finite subsets) that

$$(\textstyle\sum X)\,;(\sum Y) = \sum\{r\,;s : r \in X \text{ and } s \in Y\}.$$

If, in the preceding two definitions, the sets X and Y are allowed to be empty, then the operation ; is said to be *completely distributive*, or *complete* for short, and is said to satisfy the *general complete distributivity law* respectively. Warning: some authors use the term "completely distributive" to refer to what we have called "quasi-completely distributive".

The preceding definitions are easily extended to operations on A of arbitrary rank $n > 0$. Nullary operations are completely distributive by convention. As in the case of finite distributivity, unless explicitly stated otherwise, it will always be understood that the phrases "quasi-complete distributivity" and "complete distributivity" refer to distributivity over addition.

Lemma 2.4. *An operation on a Boolean algebra is quasi-completely distributive if and only if it satisfies the general quasi-complete distributivity law, and analogously for complete distributivity.*

Proof. If an operation on a Boolean algebra satisfies the general quasi-complete distributivity law, then it is certainly quasi-completely distributive (in each argument). To establish the reverse implication, consider the case of a binary operation ; that is quasi-completely distributive. Let X and Y be non-empty subsets of the given Boolean algebra such that the sums

$$p = \textstyle\sum X \qquad \text{and} \qquad q = \sum Y \tag{1}$$

exist. The assumption of quasi-complete distributivity and the second equation in (1) imply that

$$p\,;q = p\,;(\textstyle\sum Y) = \sum\{p\,;s : s \in Y\}. \tag{2}$$

In other words, the sum on the right exists and is equal to $p\,;q$. Similarly, the assumption of quasi-complete distributivity and the first equation in (1) imply that for each element s in Y,

$$p\,;s = (\sum X)\,;s = \sum\{r\,;s : r \in X\}. \tag{3}$$

In other words, for each element s in Y, the sum on the right side exists and is equal to $p\,;s$. Combine these observations to arrive at

$$\begin{aligned} p\,;q = \sum\{p\,;s : s \in Y\} &= \sum\{\sum\{r\,;s : r \in X\} : s \in Y\} \\ &= \sum\{r\,;s : r \in X \text{ and } s \in Y\}, \end{aligned}$$

by (2), (3), and the infinite associative law for addition.

A completely analogous argument applies to operations of ranks different from two. The proof that an operation is completely distributive if and only if it satisfies the general complete distributivity law can be obtained from the preceding proof by omitting everywhere the words "non-empty" and "quasi". □

It is not difficult to see that complete operations are just quasi-complete operations that are *normal* in the sense that their value on a sequence of elements is 0 whenever one of the arguments is 0. In the case of a binary operation $;$, this means that

$$r\,;0 = 0 \qquad \text{and} \qquad 0\,;r = 0$$

for every element r in the algebra.

Lemma 2.5. *An operation on a Boolean algebra is complete if and only if it is quasi-complete and normal.*

Proof. Focus on the case of a binary operation $;$. The definitions of complete distributivity and quasi-complete distributivity for this operation differ only in one point: the set X involved in the defining equations

$$r\,;(\sum X) = \sum\{r\,;s : s \in X\}, \quad (\sum X)\,;r = \sum\{s\,;r : s \in X\} \tag{1}$$

is allowed to be empty in the former definition, but not in the latter. Consequently, in order to prove the lemma it suffices to show that the operation $;$ is normal if and only if the equations in (1) hold when the set X is empty.

Assume that X is empty, and observe that the sets

$$\{r\,;s : s \in X\} \qquad \text{and} \qquad \{s\,;r : s \in X\}$$

are then also empty, so that

$$\sum X = \sum\{r\,;s : s \in X\} = \sum\{s\,;r : s \in X\} = 0. \tag{2}$$

If ; is normal, then

$$r\,;(\textstyle\sum X) = r\,;0 = 0 = \sum\{r\,;s : s \in X\} \qquad \text{and}$$
$$(\textstyle\sum X)\,;r = 0\,;r = 0 = \sum\{s\,;r : s \in X\},$$

by (2) and the assumption that ; is normal, so the equations in (1) hold. On the other hand, if the equations in (1) hold, then

$$r\,;0 = r\,;(\textstyle\sum X) = \sum\{r\,;s : s \in X\} = 0 \qquad \text{and}$$
$$0\,;r = (\textstyle\sum X)\,;r = \sum\{s\,;r : s \in X\} = 0,$$

by (1) and (2), so the operation ; is normal. □

An element in a Boolean algebra is called a *quasi-atom* if it is either an atom or zero. The set of quasi-atoms in a Boolean algebra is just the set of atoms together with the zero element. A binary operation ; on an atomic Boolean algebra A is said to be *quasi-completely distributive for quasi-atoms*, or *quasi-complete for quasi-atoms* for short, if for any non-empty sets X and Y of quasi-atoms in A, the existence of the suprema $\sum X$ and $\sum Y$ in A implies that

$$(\textstyle\sum X)\,;(\sum Y) = \sum\{r\,;s : r \in X \text{ and } s \in Y\}.$$

In other words, existence of the sums $\sum X$ and $\sum Y$ implies that the sum on the right side of the preceding equation exists and is equal to the element on the left side.

Lemma 2.6. *An operation on an atomic Boolean algebra is quasi-completely distributive if and only if it is quasi-completely distributive for quasi-atoms.*

Proof. A quasi-completely distributive operation is obviously quasi-completely distributive for quasi-atoms. To prove the reverse implication, consider the case of a binary operation ; on an atomic Boolean algebra A, and assume that ; is quasi-completely distributive for quasi-atoms. Let X and Y be non-empty subsets of A such that the suprema

$$p = \textstyle\sum X \qquad \text{and} \qquad q = \sum Y$$

exist in A. It is to be shown that $p\,;q$ is the supremum of the set

$$Z = \{r\,;s : r \in X \text{ and } s \in Y\}. \tag{1}$$

For each element t in A, write A_t for the set of quasi-atoms in A that are below t. The set A_t is never empty, because it contains 0. Every element in an atomic Boolean algebra is the supremum of the set of quasi-atoms that it dominates, so t is the supremum of A_t. Observe that

$$A_p = \bigcup_{r\in X} A_r \qquad \text{and} \qquad A_q = \bigcup_{s\in Y} A_s. \tag{2}$$

For instance, if u is in A_r for some r in X, then $u \leq r \leq p$, and therefore u is in A_p. On the other hand, if u is in A_p, then

$$u \leq p = \sum X,$$

so $u \leq r$ for some r in X, because u is a quasi-atom and the set X is not empty. Therefore, u is in A_r for some r in X, by the definition of the set A_r.

Write

$$U_{rs} = \{u\,;v : u \in A_r \text{ and } v \in A_s\} \tag{3}$$

for each r in X and s in Y, and write

$$W = \{u\,;v : u \in A_p \text{ and } v \in A_q\}. \tag{4}$$

It follows from (2)–(4) that

$$W = \bigcup\{U_{rs} : r \in X \text{ and } s \in Y\}. \tag{5}$$

Use (4), the assumed quasi-complete distributivity of ; for quasi-atoms, and the fact that each element in A is the sum of the quasi-atoms that it dominates, to obtain

$$\begin{aligned} p\,;q &= (\textstyle\sum A_p)\,;(\sum A_q) \\ &= \textstyle\sum\{u\,;v : u \in A_p \text{ and } v \in A_q\} = \sum W. \end{aligned} \tag{6}$$

A similar argument, using (3) instead of (4), implies that

$$\begin{aligned} r\,;s &= (\textstyle\sum A_r)\,;(\sum A_s) \\ &= \textstyle\sum\{u\,;v : u \in A_r \text{ and } v \in A_s\} = \sum U_{rs} \end{aligned} \tag{7}$$

for each r in X and s in Y.

Since $\sum W$ exists, by (6), and $\sum U_{rs}$ exists for each r in X and s in Y, by (7), it follows from (5) and the infinite associative law for addition that

$$\sum W = \sum\{\sum U_{rs} : r \in X \text{ and } s \in Y\}. \tag{8}$$

In other words, the sum on the right exists and is equal to the sum on the left. Combine (6), (7), and (8) with (1) to arrive at the desired conclusion:

$$\begin{aligned} p\,;q = \sum W &= \sum\{\sum U_{rs} : r \in X \text{ and } s \in Y\} \\ &= \sum\{r\,;s : r \in X \text{ and } s \in Y\} = \sum Z. \end{aligned}$$

The argument for operations of any finite non-zero rank is entirely analogous. □

A binary operation ; on an atomic Boolean algebra A is said to be *completely distributive for atoms*, or *complete for atoms* for short, if for any sets X and Y of atoms in A, the existence of the suprema $\sum X$ and $\sum Y$ in A implies that

$$(\sum X)\,;(\sum Y) = \sum\{r\,;s : r \in X \text{ and } s \in Y\}.$$

In other words, existence of the sums $\sum X$ and $\sum Y$ implies that the sum on the right side of the preceding equation exists and is equal to the element on the left side. Notice that the sets X and Y are allowed to be empty.

Lemma 2.7. *An operation on an atomic Boolean algebra is completely distributive if and only if it is completely distributive for atoms.*

Proof. One approach to proving the lemma is to imitate the proof of Lemma 2.6, deleting the term "quasi-" everywhere, and allowing the sets X and Y to be empty. Another approach is to derive the lemma directly from Lemma 2.6. We take the latter approach.

A completely distributive operation is obviously completely distributive for atoms. To prove the reverse implication, consider the case of a binary operation ; on an atomic Boolean algebra A, and assume that ; is completely distributive for atoms. Notice that ; must be a normal operation. For example, for any element p in A, take X to be the set of atoms below p, and Y to be the empty set. Clearly,

$$p = \sum X \qquad \text{and} \qquad 0 = \sum Y, \tag{1}$$

by the assumption that A is atomic. Consequently,

$$p\,;0 = (\sum X)\,;(\sum Y) = \sum\{r\,;s : r \in X \text{ and } s \in Y\} = \sum \varnothing = 0,$$

by (1), the assumed complete distributivity of the operation ; for atoms, and the assumption that the set Y is empty. An analogous argument shows that $0\,;r = 0$.

To prove that ; is quasi-completely distributive for quasi-atoms, consider non-empty sets X and Y of quasi-atoms in A such that the suprema

$$p = \sum X \qquad \text{and} \qquad q = \sum Y \tag{2}$$

exist in A. It is to be shown that $p\,;q$ is the supremum of the set

$$Z = \{r\,;s : r \in X \text{ and } s \in Y\}. \tag{3}$$

Take X_0 and Y_0 be the sets of atoms in X and Y respectively, and put

$$Z_0 = \{r\,;s : r \in X_0 \text{ and } s \in Y_0\}. \tag{4}$$

The sets X_0 and Y_0 differ from X and Y only in that they cannot contain 0 and may therefore be empty. Clearly,

$$p = \sum X_0 \qquad \text{and} \qquad q = \sum Y_0, \tag{5}$$

by (2). The assumption that the operation ; is completely distributive for atoms implies that

$$p\,;q = \sum Z_0, \tag{6}$$

by (4) and (5).

If 0 does not belong to either of the sets X and Y, then

$$X = X_0, \qquad Y = Y_0, \qquad \text{and} \qquad Z = Z_0,$$

by (3) and (4). If 0 does belong to at least one of the sets X and Y, then 0 belongs to the set Z, by (3), because the operation ; is normal; and therefore $Z = Z_0 \cup \{0\}$. In either case, $\sum Z = \sum Z_0$, so

$$p\,;q = \sum Z_0 = \sum Z,$$

by (6). Consequently, ; is quasi-completely distributive for quasi-atoms, as claimed.

Apply Lemma 2.6 to conclude that ; is quasi-completely distributive. Since ; is also normal, it follows by Lemma 2.5 that ; is completely distributive. The argument for operations of any finite non-zero rank is entirely analogous. □

An operation on a Boolean algebra that is distributive (in each argument) is traditionally called an *operator*. Distinguished constants are vacuously seen to be operators. A Boolean algebra that has been expanded by adjoining to it a system of operators as fundamental operations is called a *Boolean algebra with operators*. If each of the operators of rank at least one is normal, then one speaks of a *Boolean algebra with normal operators*. Similarly, if each of the operators of rank at least one is quasi-complete, or complete, then one speaks of a *Boolean algebra with quasi-complete operators*, or *complete operators*, respectively.

In order to keep the notation as simple as possible when considering Boolean algebras with operators, we shall restrict our attention to algebras of the form

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,;\,,\,^\smile\,,\,1'),$$

where $+$ and $;$ are binary operations, while $-$ and $^\smile$ are unary operations, and 1' is a distinguished constant. Such algebras are said to be *similar* to, or of the *same similarity type* as, relation algebras. When speaking of a Boolean algebra with operators, *we shall always assume that the algebra in question has the preceding form*, so for us, a Boolean algebra with operators is an algebra $\mathfrak{A}$ of the same similarity type as a relation algebra, in which Axioms (R1)–(R3), (R8), the dual of (R8), and (R9) hold in $\mathfrak{A}$. It is very important to note, however, that all definitions and results given in this work for such Boolean algebras with operators can, with minor and obvious modifications in the statements and proofs, be extended to Boolean algebras with operators of an arbitrary similarity type.

In referring to the operations and distinguished constants of a relation algebra (or a Boolean algebra with operators), we shall usually use the same symbol to refer to the corresponding operations or distinguished constants of different algebras. For instance, we shall usually use the symbol $+$ to refer to the operation of addition in relation algebras $\mathfrak{A}$ and $\mathfrak{B}$, even when speaking about both algebras at the same time. Similarly, we shall usually use the symbol 1' to refer to the identity element in $\mathfrak{A}$ and in $\mathfrak{B}$, even when speaking about both algebras at the same time. This convention is very common in mathematical practice; it simplifies the notation, renders the notation easier to read, and usually does not lead to confusion because the context makes clear whether the symbol being used is referring to an operation (or distinguished constant) in $\mathfrak{A}$ or in $\mathfrak{B}$. (For a concrete example

from ordinary mathematics, the symbol $+$ is used to denote addition of natural numbers, of rational numbers, of real numbers, of complex numbers, of elements in an arbitrary abelian group, and of elements in an arbitrary Boolean algebra.) In ambiguous cases, it is possible to clarify the intended use of the symbol explicitly, either in words or with the help of some sort of symbolism. For example, we could use notation such as $+^{\mathfrak{A}}$ and $+^{\mathfrak{B}}$ to distinguish the operations of addition in $\mathfrak{A}$ and in $\mathfrak{B}$.

2.3 Verifying axioms

The process of verifying that a given algebra satisfies the axioms of relation algebra is often a rather tedious task. If the given algebra is constructed from a Boolean algebra—for example, from a Boolean algebra of subsets of a given set—and if that Boolean algebra is atomic, then the process can usually be streamlined. It may only be necessary to check the validity of a few of the relation algebraic axioms, and then only with respect to the atoms of the algebra. The next theorem gives an example of this phenomenon.

Theorem 2.8. *Suppose* $(A\,,\,+\,,\,-)$ *is an atomic Boolean algebra. If* $;$ *and* $^\smile$ *are, respectively, binary and unary operations on* A *that are completely distributive for atoms, and if* 1' *is an element in* A*, then*

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,;\,,\,^\smile\,,\,1')$$

is a relation algebra if and only if (R4)–(R7) *and* (R11) *hold for all atoms* r, s, *and* t *in* $\mathfrak{A}$.

Proof. The implication from left to right is trivially true. To establish the reverse implication, it must be shown that that if (R4)–(R7) and (R11) are true for all atoms in $\mathfrak{A}$, then they are true for all elements in $\mathfrak{A}$. Notice that (R1)–(R3) hold by assumption, and (R8) and (R9) hold by Lemma 2.7 and the assumption that $;$ and $^\smile$ are completely distributive for atoms.

As an example, here is the verification of (R11). Assume (R11) holds for atoms, and consider arbitrary elements r, s, and t in $\mathfrak{A}$. It is to be shown that (R11) holds for r, s, and t. Let U, V, and W be the sets of atoms in $\mathfrak{A}$ that are below these elements respectively. Every element

in an atomic Boolean algebra is the supremum of the set of atoms that it dominates, so

$$r = \sum U, \qquad s = \sum V, \qquad t = \sum W, \tag{1}$$

and therefore

$$(r\,;s)\cdot t = [(\textstyle\sum U)\,;(\sum V)]\cdot(\sum W) \tag{2}$$

and

$$(r^\smile\,;t)\cdot s = [(\textstyle\sum U)^\smile\,;(\sum W)]\cdot(\sum V). \tag{3}$$

The operations ; and $^\smile$ are assumed to be completely distributive for atoms, so they are completely distributive, by Lemma 2.7; and the Boolean operation of multiplication is also completely distributive. The sets

$$X = \{(u\,;v)\cdot w : u\in U, v\in V, w\in W\} \tag{4}$$

and

$$Y = \{(u^\smile\,;w)\cdot v : u\in U, v\in V, w\in W\} \tag{5}$$

therefore have suprema in $\mathfrak{A}$, and in fact

$$\textstyle\sum X = [(\sum U)\,;(\sum V)]\cdot(\sum W) \tag{6}$$

and

$$\textstyle\sum Y = [(\sum U)^\smile\,;(\sum W)]\cdot(\sum V), \tag{7}$$

by (1).

The hypothesis of (R11) is that $(r\,;s)\cdot t = 0$. From this equation, together with (2) and (6), it follows that $\sum X = 0$. Consequently,

$$(u\,;v)\cdot w = 0$$

for all elements u, v, and w in U, V, and W respectively, by (4) and Boolean algebra. Apply (R11) for atoms to obtain

$$(u^\smile\,;w)\cdot v = 0$$

for all u, v and w in U, V, and W respectively. Therefore, $\sum Y = 0$, by (5). Combine this equation with (7) to arrive at

$$[(\sum U)^\smile ; (\sum W)] \cdot (\sum V) = 0.$$

In view of (3), it may be concluded that $(r^\smile ; t) \cdot s = 0$, as desired.

The verification of (R4)–(R7) can be handled in an entirely analogous fashion. The details are left as an exercise. □

There are several key points that allow the preceding proof to go through. First, the only operations that occur in (R4)–(R7) and (R11) are operations that are completely distributive. In particular, the operation of complement does not occur in these axioms. Second, each of these axioms has the form of an equation or of an implication between equations. Third, in each equation a given variable occurs at most once on each side of the equation. Finally, it is necessary to assume complete distributivity for atoms instead of quasi-complete distributivity for atoms because it is necessary to take into account the cases when one or more of the elements r, s, and t are zero.

Theorem 2.8 is quite useful, but it has a drawback: the verification of (R4), (R5), and (R7) for atoms may involve computations of relative products and converses of elements that are not atoms. For example, in order to verify (R4) for atoms r, s, and t, one must first compute the relative products $r ; s$ and $s ; t$. Neither of these relative products need be an atom, so in computing $(r ; s) ; t$ and $r ; (s ; t)$, one is computing relative products of atoms with elements that may not be atoms. It is sometimes more advantageous to use a form of Theorem 2.8 that involves computations of relative products and converses for atoms only.

Theorem 2.9. *Suppose* $(A , + , -)$ *is an atomic Boolean algebra. If* $;$ *and* $^\smile$ *are, respectively, binary and unary operations on* A *that are completely distributive for atoms, and if* 1' *is an element in* A*, then*

$$\mathfrak{A} = (A , + , - , ; , ^\smile , 1')$$

is a relation algebra if and only if the converse of every atom is an atom, and the following conditions hold for all atoms p*,* r*,* s*, and* t *in* $\mathfrak{A}$*.*

(i) *If* $p \leq r ; q$ *for some atom* $q \leq s ; t$ *then* $p \leq q ; t$ *for some atom* $q \leq r ; s$*.*

(ii) $r\,;s = 0$ *or* $r\,;s = r$ *whenever* s *is an atom below* 1', *and* $r\,;s = r$ *for at least one such atom* s.
(iii) *If* $t \leq r\,;s$, *then* $t^\smile \leq s^\smile\,;r^\smile$.
(iv) *If* $t \leq r\,;s$, *then* $s \leq r^\smile\,;t$.

Proof. We begin with two preliminary observations. First, under the hypothesis that $\mathfrak{A}$ is an atomic relation algebra, the operation of converse in $\mathfrak{A}$ must map the set of atoms bijectively to itself (see Lemma 4.1(vii)) and Axiom (R6) must hold for all atoms, that is to say,

$$r^{\smile\smile} = r \tag{1}$$

for every atom r in $\mathfrak{A}$. Second, under the hypothesis that $\mathfrak{A}$ is an algebra satisfying the conditions of the theorem, the operation $^\smile$ must map the set of atoms bijectively to itself, and (1) must hold for all atoms. To prove this second observation, fix an atom r in $\mathfrak{A}$, and notice that $r^\smile$ and $r^{\smile\smile}$ are also atoms, by the assumption that $^\smile$ maps atoms to atoms. There is an atom s below 1' such that $r = r\,;s$, by the second part of (ii). Apply condition (iv) twice, first with r in place of t to obtain $s \leq r^\smile\,;r$, and then with $r^\smile$, r, and s in place of r, s, and t respectively to obtain $r \leq r^{\smile\smile}\,;s$. The product $r^{\smile\smile}\,;s$ is either 0 or $r^{\smile\smile}$, by the first part of condition (ii), and it cannot be 0, because the product is above the atom r. Consequently, $r^{\smile\smile}\,;s = r^{\smile\smile}$ and therefore $r \leq r^{\smile\smile}$. The validity of (1) for r follows at once from this last inequality and the fact that r and $r^{\smile\smile}$ are atoms.

The validity of (1) for atoms easily implies that the operation $^\smile$ maps the set of atoms bijectively to itself. Indeed, if r and s are atoms such that $r^\smile = s^\smile$, then $r^{\smile\smile} = s^{\smile\smile}$, and therefore $r = s$, by (1). Also, every atom r is the image under $^\smile$ of an atom, namely the atom $r^\smile$.

On the basis of the observations of the preceding paragraphs—namely that under either hypothesis, the operation $^\smile$ maps the set of atoms bijectively to itself, and (1) holds for atoms—we shall show that conditions (ii), (iii), and (iv) are respectively equivalent to the validity of (R5), (R7), and (R11) for atoms, and on the basis of (iii) and (1), condition (i) is equivalent to the validity of (R4) for atoms. The theorem then follows from Theorem 2.8. In the arguments below, the variables p, q, r, s, and t range over the set of atoms in $\mathfrak{A}$.

If condition (iv) holds for all atoms r, s, and t, then by replacing r with $r^\smile$ and interchanging s and t in (iv), and applying (1), we arrive at the implication

$$s \leq r^\smile ; t \qquad \text{implies} \qquad t \leq r ; s,$$

which is just the contrapositive of (R11) for atoms. On the other hand, if (R11) holds for all atoms, then by replacing r with $r^\smile$ and interchanging s and t in (R11), and applying (1), we arrive at the contrapositive of condition (iv), namely

$$(r^\smile ; t) \cdot s = 0 \qquad \text{implies} \qquad (r ; s) \cdot t = 0.$$

Thus, condition (iv) is equivalent to the validity of (R11) for atoms.

Turn next to the proof that condition (iii) is equivalent to the validity of (R7) for atoms. Assume first that condition (iii) holds for all atoms r, s, and t. Take atoms $s^\smile$, $r^\smile$, and $t^\smile$ for r, s, and t respectively in condition (iii), and then invoke (1), to obtain

$$t^\smile \leq s^\smile ; r^\smile \qquad \text{implies} \qquad t \leq r ; s.$$

This implication, together with condition (iii) and the fact that $^\smile$ is a bijection of the set of atoms, shows that $^\smile$ maps the set of atoms below $r ; s$ bijectively to the set of atoms below $s^\smile ; r^\smile$. Consequently,

$$\{t^\smile : t \leq r ; s\} = \{q : q \leq s^\smile ; r^\smile\}. \tag{2}$$

Since $\mathfrak{A}$ is atomic, every element is the sum of the atoms it dominates. In particular,

$$r ; s = \sum\{t : t \leq r ; s\} \qquad \text{and} \qquad s^\smile ; r^\smile = \sum\{q : q \leq s^\smile ; r^\smile\}. \tag{3}$$

The operation $^\smile$ is assumed to be completely distributive for atoms, so the supremum of the set $\{t^\smile : t \leq r ; s\}$ exists, and in fact

$$\sum\{t^\smile : t \leq r ; s\} = (\sum\{t : t \leq r ; s\})^\smile = (r ; s)^\smile, \tag{4}$$

by the first equation in (3). Combine (4) with (2) and the second equation in (3) to arrive at

$$(r ; s)^\smile = s^\smile ; r^\smile. \tag{5}$$

Thus, (R7) holds for atoms.

To prove the reverse implication, assume (R7) holds for all atoms. In particular, (5) holds for two given atoms r and s. The hypothesis that the operation $^\smile$ is completely distributive for atoms implies that it is completely distributive, by Lemma 2.7, and therefore monotone, by Lemma 2.3. Consider now an arbitrary atom t in $\mathfrak{A}$. If $t \leq r ; s$,

then $t^\smile \leq (r\,;s)^\smile$, by the monotony of $^\smile$, and therefore $t^\smile \leq s^\smile\,;r^\smile$, by (5). Thus, condition (iii) holds.

The remaining two arguments are similar in spirit to the preceding ones, so we shall be briefer in our presentation. Consider first the equivalence of condition (i) with (R4) for atoms. On the basis of (1) and condition (iii), condition (i) implies its own converse. Indeed, suppose condition (i) holds for all atoms p, r, s, and t. To establish the converse, assume that

$$p \leq q\,;t \qquad \text{for some} \qquad q \leq r\,;s. \tag{7}$$

As was shown in the preceding paragraphs, on the basis of (1), condition (iii) implies (R7) for atoms. Therefore, by applying the operation $^\smile$ to both inequalities in (7), and using the monotony of $^\smile$, together with (R7) for atoms, we obtain

$$p^\smile \leq t^\smile\,;q^\smile \qquad \text{for some} \qquad q^\smile \leq s^\smile\,;r^\smile. \tag{8}$$

If the atoms p, q, r, s, and t in condition (i) are respectively replaced by the atoms $p^\smile$, $q^\smile$, $t^\smile$, $s^\smile$, and $r^\smile$, then the hypothesis of condition (i) assumes the form of (8), and the conclusion of the condition assumes the form

$$p^\smile \leq q^\smile\,;r^\smile \qquad \text{for some} \qquad q^\smile \leq t^\smile\,;s^\smile. \tag{9}$$

Apply $^\smile$ to the inequalities in (9), and use the monotony of $^\smile$, (R7) for atoms, and (1) to conclude that

$$p \leq r\,;q \qquad \text{for some} \qquad q \leq s\,;t. \tag{10}$$

Thus, the hypothesis (7) implies the conclusion (10), which is just what the converse of condition (i) says.

The equality

$$r\,;(s\,;t) = (r\,;s)\,;t \tag{11}$$

holds for atoms r, s, and t just in case every atom p below the left side is also below the right side, and vice versa. The complete distributivity of the operation $;$ for atoms implies that p will be below the left side just in case there is an atom q below $s\,;t$ such that $p \leq r\,;q$. Similarly, p will be below the right side just in case there is an atom $q \leq r\,;s$ such that $p \leq q\,;t$. Consequently, equation (11) holds atoms just in

case condition (i) and its converse hold. Since condition (i) implies its converse, by the observations of the preceding paragraph, it follows that condition (i) is equivalent to the validity of (R4) for atoms.

It remains to treat condition (ii). The element 1' is the sum of the atoms in $\mathfrak{A}$ that it dominates, so

$$r\,;1' \leq r \qquad \text{if and only if} \qquad r\,;s \leq r \tag{12}$$

for every atom $s \leq 1'$, by the complete distributivity of the operation ; for atoms. Since r is an atom, the inequality on the right side of (12) is equivalent to saying that $r\,;s = 0$ or $r\,;s = r$ for every atom $s \leq 1'$. Consequently, the equality $r\,;1' = r$ is equivalent to the validity of the inequality on the right side of (12) for all atoms $s \leq 1'$, together with the requirement that $r\,;s = r$ for some atom $s \leq 1'$. □

The preceding proof establishes more than is claimed in the statement of the theorem. It shows that if (R6) is valid in an atomic Boolean algebra with completely distributive operators $\mathfrak{A}$, then (R5) is valid in $\mathfrak{A}$ if and only if condition (ii) of the theorem holds for all atoms, (R7) is valid in $\mathfrak{A}$ if and only if condition (iii) holds for all atoms, and (R11) is valid in $\mathfrak{A}$ if and only if condition (iv) holds for all atoms. It also shows that if (R6) and (R7) are valid in $\mathfrak{A}$, then (R4) is valid in $\mathfrak{A}$ if and only if condition (i) holds for all atoms. Since (R10) is equivalent to (R11) in any Boolean algebra with operators in which (R6) and (R7) are valid (see the relevant remarks at the end of Section 2.1), it follows that if (R6) and (R7) are valid in $\mathfrak{A}$, then (R10) is valid in $\mathfrak{A}$ if and only if condition (iv) holds for all atoms. These consequences are particularly helpful when verifying that some, but not necessarily all, of (R4)–(R7) and (R10) hold in a Boolean algebra with completely distributive operators.

2.4 Language of relation algebras

For the most part, the development of the theory of relation algebras can proceed in an informal fashion, without any reference to logic—just as is the case with most algebraic theories such as group theory and ring theory. At times, however, notions and methods of first-order logic do play an important role. The purpose of this section is to provide a brief introduction to the relevant logical notions and notation that we shall need.

The *first-order language of relation algebras*—also called the *elementary language of relation algebras*—is a language $\mathcal{L}$ with two types of symbols, logical and non-logical. The logical symbols of $\mathcal{L}$ include a countably infinite sequence of variables

$$v_0, v_1, v_2, \ldots, v_n, \ldots,$$

and we occasionally write $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ for the first three of these variables. There is also a binary relation symbol $=$ denoting the relation of equality between individuals. In addition, there are two sentential connectives, namely a symbol $\neg$ for *negation* and a symbol $\rightarrow$ for *implication*, there is a symbol $\forall$ for *universal quantification*, and there are parentheses (and).

The non-logical symbols of $\mathcal{L}$ consist of an individual constant symbol 1', two binary operation symbols $+$ and $;$, and two unary operation symbols $-$ and $^\smile$. Formally speaking, we should use some notational device to distinguish between the operation symbols of $\mathcal{L}$ and the operations of a relation algebra $\mathfrak{A}$, for example by writing the former in boldface and the latter in lightface, or by writing both in lightface but adding a superscript $\mathfrak{A}$ to the latter, as in $+^{\mathfrak{A}}$. In practice, there is usually little danger of confusion, so the cumbersome additional notation may for the most part be omitted. We employ it only when it seems necessary.

An *expression* is any string of symbols in $\mathcal{L}$. Terms and formulas are special kinds of expressions. Terms are built up from variables and the individual constant symbol 1' by an inductive process using the operation symbols: variables and the individual constant symbol 1' are *atomic terms* (this is the *base case*, or *base clause*, of the definition); if σ and τ are terms, then so are

$$(\sigma + \tau), \qquad (-\sigma), \qquad (\sigma\,;\tau), \qquad \text{and} \qquad (\sigma^\smile)$$

(this is the *induction case*, or the *induction clause*, of the definition); and an expression is a *term* if and only if it can be shown to be a term by a finite number of applications of the base clause and the induction clause. In order to simplify notation, we follow the standard conventions regarding the way in which terms are to be read when parentheses are omitted: unary operation symbols take precedence over binary operation symbols, and among binary operation symbols, multiplication symbols take precedence over addition symbols (and outer parentheses are always omitted). For example,

$$v_0^\smile + -v_1 \,;\, v_2^\smile \qquad \text{abbreviates} \qquad ((v_0^\smile) + ((-v_1) \,;\, (v_2^\smile)))$$

The notation $\sigma(v_0, \ldots, v_{n-1})$ indicates that σ is a term and the variables occurring in σ form a subset of $\{v_0, \ldots, v_{n-1}\}$.

The inductive definition of a term carries with it a method for proving that all terms possess a given property: one shows first that all variables and the individual constant symbol 1' possess the property (this is called the *base case* of the proof), and then one shows that if terms σ and τ possess the property, then so do the terms obtained from σ and τ by the induction clause of the definition (this is called the *induction step* of the proof). This method of proof is called *proof by induction on terms*. The definition of a term also carries with it an analogous method for defining notions that apply to terms; such a definition is called a *definition by induction on terms*.

Formulas are also defined by an inductive process, using sentential connectives and quantifiers. *Atomic formulas* are *equations* $\sigma = \tau$, where σ and τ are terms (this is the *base case*, or *base clause*, of the definition); if Γ and Δ are formulas, then so are

$$(\Gamma \to \Delta), \qquad (\neg\Gamma), \qquad \text{and} \qquad (\forall v \Gamma)$$

for every variable v (this is the *induction case*, or the *induction clause*, of the definition); and an expression is a *formula* if and only if it can be shown to be a formula by a finite number of applications of the base clause and the induction clause. The symbols $\vee$, $\wedge$, $\leftrightarrow$, and $\exists$ for *disjunction*, *conjunction*, *equivalence* (or *bi-implication*), and *existential quantification* respectively, are introduced as abbreviations in the usual manner:

$$\begin{array}{lll}
(\Gamma \vee \Delta) & \text{abbreviates} & ((\neg\Gamma) \to \Delta), \\
(\Gamma \wedge \Delta) & \text{abbreviates} & (\neg(\Gamma \to (\neg\Delta))), \\
(\Gamma \leftrightarrow \Delta) & \text{abbreviates} & (\neg((\Gamma \to \Delta) \to (\neg(\Delta \to \Gamma)))), \\
(\exists v_i \Gamma) & \text{abbreviates} & (\neg(\forall v_i(\neg\Gamma))).
\end{array}$$

We follow the standard conventions regarding the omission of parentheses when writing formulas: $\neg$ has priority over $\vee$ and $\wedge$, which in turn have priority over $\to$ and $\leftrightarrow$ (and outer parentheses are always omitted). For example, the formula

$$\Gamma_0 \wedge \cdots \wedge \Gamma_{n-1} \to \Gamma_n$$

is to be understood as follows: the conjunction of formulas $\Gamma_0, \ldots, \Gamma_{n-1}$ implies the formula Γ_n.

Just as with the definition of a term, the inductive definition of a formula carries with it a method for proving that all formulas possess a given property: one shows first that all atomic formulas possess the property (this is the *base case* of the proof), and then one shows that if formulas Γ and Δ possess the property, then so do the formulas obtained from Γ and Δ by the induction clause of the definition (this is the *induction step* of the proof). This method of proof is called *proof by induction on formulas*. The definition of a formula also carries with it an analogous method for defining notions that apply to formulas; such a definition is called a *definition by induction on formulas*.

Here is an example: the definition of the notion of a variable occurring free in a formula proceeds by induction on formulas. Base clause: a variable v_i *occurs free* in an atomic formula Γ if v_i is one of the symbols in Γ. Induction clause: the variable v_i occurs free in $\neg\Gamma$ if it occurs free in Γ, it occurs free in $\Gamma \to \Delta$ if it occurs free in Γ or in Δ, and it occurs free in $\exists v_j \Gamma$ if it occurs free in Γ and is not equal to v_j. Warning: a variable v_i may occur free in a formula Γ, and still be *bound* in some subformula of Γ in the sense that v_i occurs in the subformula but is not free in that subformula. The notation $\Gamma(v_0, \ldots, v_{n-1})$ indicates that Γ is a formula and the variables occurring free in Γ form a subset of $\{v_0, \ldots, v_{n-1}\}$.

Various special kinds of formulas are of particular importance in the study of relation algebras. Here are some examples. A *sentence* is a formula in which no variables occur free. A *quantifier-free formula*, or an *open formula*, is a formula in which no quantifiers occur at all. A *universal formula* is a formula of the form $\forall v_{i_0} \ldots \forall v_{i_{n-1}} \Delta$, where Δ is a quantifier-free formula, and an *existential formula* is defined analogously, with $\forall$ replaced by $\exists$; if the formula is in fact a sentence, then we speak of a *universal sentence* or an *existential sentence* respectively. A *positive formula* is a formula built up from equations using only conjunction, disjunction, and existential and universal quantification, and a *positive sentence* is a positive formula that is also a sentence. A *conditional equation* is a quantifier-free formula of the form

$$(\varepsilon_0 \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_{n-1}) \to \varepsilon_n,$$

where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ are equations. A *universal Horn formula* is a universal formula in which the quantifier-free part Δ is either a disjunction of negations of equations or else a conditional equation; equivalently, Δ

is a disjunction consisting of negations of equations and at most one unnegated equation; if the formula is in fact a sentence, then we speak of a *universal Horn sentence*, and if it is quantifier-free, then we speak of a (*basic*, or *open*) *Horn formula*. An *identity* is a universal sentence in which the quantifier-free part is a single equation.

Associated with the language of relation algebras are a series of semantic notions that we shall need. Fix an algebra

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,;\,,\,\breve{}\,,\,1')$$

of the same similarity type as relation algebras, and write A^n for the set of sequences of the form

$$r = (r_0, \ldots, r_{n-1}),$$

where $r_0, \ldots, r_{n-1}$ are all in $\mathfrak{A}$.

Every term $\gamma(v_0, \ldots, v_{n-1})$ (whose variables are, by convention, among $v_0, \ldots, v_{n-1}$) determines an operation of rank n on the universe of $\mathfrak{A}$. This operation is called the *polynomial* (*of rank n*) *induced by* γ, and it is denoted by $\gamma^{\mathfrak{A}}$. It is defined by induction on terms. The base clause of the definition concerns variables and constants. If γ is a variable v_i (for some $i < n$), then $\gamma^{\mathfrak{A}}$ is the ith *projection* (*of rank n*), that is to say, it is the operation of rank n that maps each sequence r in A^n to its ith coordinate r_i. If γ is the constant symbol 1', then $\gamma^{\mathfrak{A}}$ is the constant operation of rank n that maps each sequence r in A^n to the distinguished constant 1' in $\mathfrak{A}$. For the induction clause of the definition, suppose that γ has one of the forms

$$\sigma + \tau, \qquad -\sigma, \qquad \sigma\,;\tau, \qquad \sigma^{\breve{}},$$

where σ and τ are terms for which the polynomials $\sigma^{\mathfrak{A}}$ and $\tau^{\mathfrak{A}}$ of rank n have already been defined. The polynomial $\gamma^{\mathfrak{A}}$ is defined by specifying that for each sequence r in A^n, the value of $\gamma^{\mathfrak{A}}(r)$ is

$$\sigma^{\mathfrak{A}}(r) + \tau^{\mathfrak{A}}(r), \qquad -\sigma^{\mathfrak{A}}(r), \qquad \sigma^{\mathfrak{A}}(r)\,;\tau^{\mathfrak{A}}(r), \qquad \sigma^{\mathfrak{A}}(r)^{\breve{}}$$

respectively. (The first occurrences of $+$, $-$, $;$, and $\breve{}$ above refer to symbols in $\mathcal{L}$, while the second occurrences refer to the corresponding operations in $\mathfrak{A}$.) When the context makes clear which algebra $\mathfrak{A}$ is intended, we shall usually omit the superscript and write simply $\gamma(r_0, \ldots, r_{n-1})$ or $\gamma(r)$ for the value of $\gamma^{\mathfrak{A}}$ on r, and we shall even refer to $\gamma(r)$ as the value of the *term* γ on r.

There is an ambiguous point in the preceding definition that should be clarified. A term γ with variables among $v_0, \ldots, v_{m-1}$ also has variables among $v_0, \ldots, v_{n-1}$ for each natural number $n \geq m$, and therefore γ induces a polynomial of rank n in $\mathfrak{A}$ for each such n. For example, the term $v_0 + v_2$ induces a polynomial of rank 3 in $\mathfrak{A}$ that maps each triple (r_0, r_1, r_2) to the sum $r_0 + r_2$, and it also induces a polynomial of rank 4 that maps each quadruple (r_0, r_1, r_2, r_3) to the sum $r_0 + r_2$. The notation $\gamma(v_0, \ldots, v_{n-1})$ is used to indicate that the intended polynomial induced by γ is the one of rank n.

The notion of a sequence of elements satisfying a formula in $\mathfrak{A}$ is defined by induction on formulas. Let Γ be a formula whose free and bound variables are among $v_0, \ldots, v_{n-1}$, and let r be a sequence of n elements in $\mathfrak{A}$. The base clause of the definition says that if Γ is an equation of the form $\sigma = \tau$, then r *satisfies* Γ in $\mathfrak{A}$ just in case the values of σ and τ on r—that is to say, the elements $\sigma(r)$ and $\tau(r)$ in $\mathfrak{A}$—are the same. The induction clause splits into three cases: if Γ is the formula $\neg\Delta$, then r satisfies Γ (in $\mathfrak{A}$) just in case r does not satisfy Δ; if Γ is the formula $\Delta \to \Omega$, then r satisfies Γ just in case r satisfies Ω whenever it satisfies Δ; and if Γ is the formula $\forall v_i \Delta$, then r satisfies Γ just in case every sequence s obtained from r by replacing r_i with an arbitrary element in $\mathfrak{A}$ satisfies the formula Δ. It is easy to check that r satisfies a disjunction $\Delta \vee \Omega$ just in case r satisfies at least one of Δ and Ω, and r satisfies a conjunction $\Delta \wedge \Omega$ just in case r satisfies both Δ and Ω. Also, r satisfies the formula $\exists v_i \Delta$ just in case there is a sequence s obtained from r by replacing r_i with some element in $\mathfrak{A}$ such that s satisfies Δ. Notice that only the free variables occurring in Γ really matter in the definition of satisfaction. In other words, if r and t are two n-termed sequences of elements in $\mathfrak{A}$ such that $r_i = t_i$ whenever v_i occurs free in Γ, then either r and t both satisfy Γ, or neither of them satisfies Γ. For that reason, we shall usually not bother to refer to the bound variables of Γ in the future.

The semantical notions of truth and model are defined in terms of satisfaction. A formula Γ with free variables among $v_0, \ldots, v_{n-1}$ is *true* in $\mathfrak{A}$ if it is satisfied by every n-termed sequence $(r_0, \ldots, r_{n-1})$ of elements in $\mathfrak{A}$, and Γ *fails* in $\mathfrak{A}$ if there is some n-termed sequence of elements in $\mathfrak{A}$ that does not satisfy Γ. If Γ is true in $\mathfrak{A}$, then $\mathfrak{A}$ is called a *model of* Γ. These notions are extended from formulas to sets of formulas, and from individual algebras to classes of algebras, in the obvious way: a set of formulas $\mathcal{S}$ is true in $\mathfrak{A}$ if every formula in $\mathcal{S}$ is true in $\mathfrak{A}$, and in this case $\mathfrak{A}$ is called a *model* of $\mathcal{S}$. More generally, $\mathcal{S}$

is true in a class of algebras K if $\mathcal{S}$ is true in every algebra in K. The class of all models of a set of formulas $\mathcal{S}$ is denoted by $\mathsf{Mo}(\mathcal{S})$.

The language $\mathcal{L}$ is also provided with a deductive apparatus: a set of logical axioms and a set of finitary rules of inference (such as modus ponens and the rule of substitution). The precise nature of this deductive apparatus is not important for our development. What does matter is that it is strong enough to prove the *Completeness Theorem* for first-order logic, which says that a formula Γ is derivable from a set of formulas $\mathcal{S}$ with the help of the logical axioms and rules of inference of $\mathcal{L}$ if and only if every algebra $\mathfrak{A}$ that is a model of $\mathcal{S}$ is also a model of Γ. A consequence of the Completeness Theorem is the *Compactness Theorem*. It says that a set of formulas $\mathcal{S}$ in $\mathcal{L}$ has a model whenever every finite subset of $\mathcal{S}$ has a model.

A set of formulas $\mathcal{T}$ in $\mathcal{L}$ is called a *first-order theory*, or an *elementary theory*, if it is closed under the *relation of provability*, that is to say, if a formula Γ is provable from a set of formulas in $\mathcal{T}$ using the deductive apparatus of $\mathcal{L}$, then Γ belongs to $\mathcal{T}$. An *axiomatization*, or a *set of axioms*, of a theory $\mathcal{T}$ is a set of formulas $\mathcal{S}$ such that $\mathcal{T}$ is the set of formulas in $\mathcal{L}$ that are provable from $\mathcal{S}$ using the deductive apparatus of $\mathcal{L}$.

Certain kinds of theories play an important role in the study of relation algebras. A theory is said to be *universal* if it has an axiomatization consisting of universal formulas. Similarly, a theory is said to be *universal Horn*, *conditional equational*, or *equational*, if it has an axiomatization consisting of universal Horn formulas, conditional equations, or equations respectively. The set of all formulas that are true in a class K of algebras is easily seen to be a theory. It is called the *elementary theory*, or the *first-order theory*, of K, and it is denoted by $Th(\mathsf{K})$.

Sometimes the word "theory" is used in a looser sense to refer to an arbitrary set of first-order formulas, or to a set of first-order formulas of some specific type. For example, the set of all universal formulas that are true in K is called the *universal theory* of K, the set of all universal Horn formulas that are true in K is called the *universal Horn theory* of K, the set of all conditional equations that are true in K is called the *conditional equational theory* of K, and the set of all equations that are true in K is called the *equational theory* of K. The latter is denoted by $Eq(\mathsf{K})$.

The theory of relation algebras is equational in nature: its axioms all have the form of equations, and the basic laws that are studied

have the form of equations or implications between equations. The deductive apparatus, and in particular the set of rules of inference, required for deriving equations from equations is much more elementary than the one required for deriving arbitrary first-order formulas. In fact, the following rules suffice. (1) *Tautology Rule*: every equation of the form $\sigma = \sigma$ is provable. (2) *Symmetry Rule*: from the equation $\sigma = \tau$, the equation $\tau = \sigma$ is provable. (3) *Transitivity Rule*: from equations $\sigma = \tau$ and $\tau = \gamma$, the equation $\sigma = \gamma$ is provable. (4) *Replacement Rule*: from equations $\sigma = \tau$ and $\gamma = \delta$, each of the equations

$$\sigma + \gamma = \tau + \delta, \qquad -\sigma = -\tau, \qquad \sigma\,;\gamma = \tau\,;\delta, \qquad \sigma^{\smile} = \tau^{\smile}$$

is provable. (5) *Substitution Rule*: from an equation $\varepsilon(v_0, \ldots, v_{n-1})$, each substitution instance $\varepsilon(\gamma_0, \ldots, \gamma_{n-1})$ obtained by simultaneously replacing each variable v_i with a term γ_i is provable.

In addition to its use in describing a certain type of first-order theory, the phrase *equational theory* is also used in a more restrictive sense to describe a set of equations that is closed under the relation of provability using the rules of inference (1)–(5). The sufficiency of these rules is captured in the *Completeness Theorem* for equational logic, which says that an equation ε is derivable from a set of equations $\mathcal{E}$ using the rules of inference (1)–(5) if and only if every model of $\mathcal{E}$ is also a model of ε.

2.5 Language of relations

There is another first-order language that is often used in the study of relation algebras. We call it the (*first-order* or *elementary*) *language of the theory of* (*binary*) *relations*, or more simply, the *language of relations*, and we denote it by $\mathcal{L}^*$. The logical symbols of $\mathcal{L}^*$ are the same as those of $\mathcal{L}$. The non-logical symbols of $\mathcal{L}^*$ consist of an unspecified number of binary relation symbols. For the purpose of illustration, it may be assumed for now that these relation symbols are enumerated in a sequence indexed by ordinal numbers: $\boldsymbol{R}_0, \boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$. Occasionally, it is also advantageous to include a sequence of individual constant symbols (also called *operation symbols of rank zero*). We shall always state explicitly when such symbols are assumed to be present in the language. Again, for the purpose of illustration, it may be supposed

that these individual constant symbols, if any, are enumerated in a sequence indexed by ordinal numbers: $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$. There are no operation symbols of rank higher than zero in $\mathcal{L}^*$.

The terms of $\mathcal{L}^*$ are the variables and the individual constant symbols, if any. The atomic formulas of $\mathcal{L}^*$ are the expressions of the form $\boldsymbol{R}_i\sigma\tau$ and the equations $\sigma = \tau$, where σ and τ are terms. Arbitrary formulas of $\mathcal{L}^*$ are built up from atomic formulas in exactly the same way as in $\mathcal{L}$.

A *relational structure* appropriate for this language is a system

$$\mathfrak{U} = (U,\, R_0,\, R_1,\, R_2, \ldots,\, \alpha_0,\, \alpha_1,\, \alpha_2, \ldots),$$

where R_i is a distinguished binary relation on the universe U for each index i, and α_j is a distinguished constant (that is to say, an element) in U for each index j. These relations and constants are the *interpretations* in $\mathfrak{U}$ of the relation symbols $\boldsymbol{R}_i$ and individual constant symbols $\boldsymbol{\alpha}_j$ in $\mathcal{L}^*$. Occasionally, we shall suppress explicit mention of the distinguished constants, and write simply

$$\mathfrak{U} = (U,\, R_0,\, R_1,\, R_2, \ldots).$$

The definition for $\mathcal{L}^*$ of the value of a term γ on a sequence r of n elements in $\mathfrak{U}$ is a simplified form of the definition for $\mathcal{L}$, as only a base clause is needed: the value of $\gamma(r)$ in $\mathfrak{U}$ is r_i when γ is a variable v_i, and it is the distinguished constant α_j when γ is the individual constant symbol $\boldsymbol{\alpha}_j$. The definition for $\mathcal{L}^*$ of the notion of a sequence r satisfying a formula Γ with variables among $v_0, \ldots, v_{n-1}$ in $\mathfrak{U}$ is very close to the definition for $\mathcal{L}$. Only the base clauses of the definition need to be extended in the following way: if Γ is an atomic formula of the form $\boldsymbol{R}_i\sigma\tau$, then the sequence r *satisfies* Γ in $\mathfrak{U}$ just in case the pair $(\sigma(r), \tau(r))$ belongs to the distinguished relation R_i. The remaining semantical notions such as truth and model are defined for $\mathcal{L}^*$ just as they are for $\mathcal{L}$.

2.6 Historical remarks

The idea of an axiomatic approach to the calculus of relations via a finite set of equational axioms goes back to Tarski [104]. In writing about a set-theoretical approach to the foundations of the theory of relations (see Chapter 1), he said the following.

> The above [set-theoretical] way of constructing the elementary theory of relations will probably seem quite natural to anyone who is familiar with modern mathematical logic. If, however, we are interested not in the whole theory of relations but merely in the calculus of relations, we must admit that this method has certain defects from the point of view of simplicity and elegance. We obtain the calculus of relations in a very roundabout way, and in proving theorems of this calculus we are forced to make use of concepts and statements which are outside the calculus. It is for this reason that I am going to outline another method of developing this calculus.
>
> In constructing the calculus of relations according to the second method we use only one kind of variables, namely relation variables, and we use the same constants as in the first method, with the exception of the quantifiers. [Tarski permitted the use of sentential connectives in [104], but avoided this usage in his later constructions of the calculus of relations.] From these constants and variables we construct relation designations exactly as before. In the construction of sentences, however, certain modifications are necessary on account of the absence of individual variables and quantifiers. As elementary sentences we take only sentences of the form 'R=S', where 'R' and 'S' stand for relation designations; and we form compound sentences from simpler ones by means of the connectives of the sentential calculus.
>
> Moreover we single out certain sentences which we call axioms. . . .

The set of axioms given in [104] is somewhat different from the one given in Definition 2.1. In particular, it is not equational in nature since sentential connectives are used. However, the possibility of giving an equational axiomatization of the calculus of relations is explicitly mentioned on p. 87 of [104]. The axiomatization given in Definition 2.1 was worked out by Tarski some time during the period 1942–1944, and was explicitly used by him in an unpublished manuscript dating from that period. A minor variant of this axiomatization was published in 1951 in Chin-Tarski [23].

The theory of Boolean algebras with operators was developed by Bjarni Jónsson and Tarski [54], and Lemmas 2.2–2.4 and 2.6 (in the more general forms given in Exercises 2.12–2.14 and 2.16) are due to them. Lemmas 2.5 and 2.7 (in the more general forms given in Exercises 2.15 and 2.17) are due to Givant. Also, Theorems 2.8 and 2.9 in their present form are due to Givant, but variants of them may have been known to others as well. There are, of course, versions of these theorems that apply to more general classes of Boolean algebras with complete operators. Also, there is a very close connection between Theorem 2.9 and a corresponding result regarding an axiomatization

of the class of atom structures of atomic relation algebras (see Chapter 19, in particular Theorems 19.12, 19.16, and 19.30).

The observation in Exercise 2.7 is explicitly made in Givant [34] (see Lemma 1.2 of that work). The results in Exercises 2.8 and 2.9 are due to Tarski; see Theorem 2.2 of [23]. The results in Exercises 2.18 and 2.19 are formulated in Theorem 1.5, and the subsequent remark, of Jónsson-Tarski [54]; but the proof given there for the case when the rank of the operation is greater than one is incorrect, as was pointed out by Richard Kramer. The correct proof given in the solutions section is due to Kramer.

Exercises

2.1. Prove that in any Boolean algebra (possibly with additional operations), if the supremum of every subset exists, then the infimum of every subset exists as well. Prove that, dually, if the infimum of every subset exists, then the supremum of every subset exists as well.

2.2. Prove that the following infinite associative laws hold in an arbitrary Boolean algebra (possibly with additional operations). Consider a system of index sets $(I_j : j \in J)$ with union I, and for each i in I, suppose that r_i is an element in the given Boolean algebra.

(i) If the sum $s_j = \sum_{i \in I_j} r_i$ exists for each index j in J, and if the sum $\sum_{j \in J} s_j$ exists, then the sum $\sum_{i \in I} r_i$ exists and

$$\sum_{i \in I} r_i = \sum_{j \in J} s_j.$$

In other words, under the given hypotheses, we have

$$\sum_{i \in I} r_i = \sum_{j \in J} \left(\sum_{i \in I_j} r_i \right).$$

(ii) If the sum $s_j = \sum_{i \in I_j} r_i$ exists for each index j in J, and if the sum $\sum_{i \in I} r_i$ exists, then the sum $\sum_j s_j$ exists and

$$\sum_{j \in J} s_j = \sum_{i \in I} r_i.$$

In other words, under the given hypotheses, we have

$$\sum_{j \in J} \left(\sum_{i \in I_j} r_i \right) = \sum_{i \in I} r_i.$$

2.3. Prove that the following conditions on an element r in an arbitrary Boolean algebra (possibly with additional operations) are equivalent.

(i) r is an atom.
(ii) For every element s, either $r \leq s$ or $r \cdot s = 0$, but not both.
(iii) For every element s, either $r \leq s$ or $r \leq -s$, but not both.
(iv) $r \neq 0$, and whenever $r \leq s + t$, we have $r \leq s$ or $r \leq t$.
(v) $r \neq 0$, and whenever $r \leq \sum_i s_i$, we have $r \leq s_i$ for some i.

2.4. Prove that the following conditions on a Boolean algebra (possibly with additional operations) are equivalent.

(i) The algebra is atomic;
(ii) Every element in the algebra is the supremum of the set of atoms that it dominates;
(iii) The unit in the algebra is the supremum of the set of all atoms.

2.5. Prove that a non-degenerate atomless Boolean algebra (possibly with additional operations) is necessarily infinite.

2.6. The purpose of this and the next exercise is to show that (R10) and (R11) are equivalent on the basis of (R1)–(R3) and (R6)–(R9). Prove that the left-hand distributive law for relative multiplication,

$$r\,;(s+t) = r\,;s + r\,;t.$$

follows from (R6)—(R9). Use this law to derive the right-hand monotony law for relative multiplication,

$$s \leq t \qquad \text{implies} \qquad r\,;s \leq r\,;t.$$

2.7. Prove that (R10) is equivalent to the implication (R11) on the basis of the Boolean axioms (R1)–(R3) and the left-hand distributive law (or even just the right-hand monotony law) for relative multiplication (see Exercise 2.6).

2.8. Prove that (R10) is equivalent to the De Morgan-Tarski laws in the presence of the other axioms of relation algebra.

2.9. Prove that an algebra $\mathfrak{A}$ (of the same similarity type as relation algebras) is a relation algebra if and only if the axioms (R1)–(R5) and the De Morgan-Tarski laws hold in $\mathfrak{A}$. This gives an alternative (non-equational) axiomatization of the theory of relation algebras.

2.10. Complete the proof of Theorem 2.8 by treating each of the axioms (R4)–(R7).

2.11. Assuming the hypotheses of Theorem 2.8 on an algebra $\mathfrak{A}$ (of the same similarity type as relation algebras), give a simple set of necessary and sufficient conditions for $\mathfrak{A}$ to be a relation algebra when $\smile$ is the identity function on the set of atoms. Do the same when the operation ; is commutative on the set of atoms. What if $\smile$ is the identity function, and ; is commutative, on the set of atoms?

2.12. Prove Lemma 2.2 for an operation of arbitrary rank $n > 0$.

2.13. Prove Lemma 2.3 for an operation of arbitrary rank $n > 0$.

2.14. Prove Lemma 2.4 for an operation of arbitrary rank $n > 0$.

2.15. Prove Lemma 2.5 for an operation of arbitrary rank $n > 0$.

2.16. Prove Lemma 2.6 for an operation of arbitrary rank $n > 0$.

2.17. Prove Lemma 2.7 for an operation of arbitrary rank $n > 0$.

2.18. Let $(A, +, -)$ be a Boolean algebra.

(i) Prove that a unary operation f on A is distributive (or quasi-completely distributive) if and only if there is a normal and distributive (or quasi-completely distributive) operation g on A and an element t in A such that

$$f(r) = g(r) + t$$

for all elements r in A.

(ii) Prove that a binary operation f on A is distributive (or quasi-completely distributive) if and only if there is a normal and distributive (or quasi-completely distributive) operation g on A, two unary normal and distributive (or quasi-completely distributive) operations h and k on A, and an element t in A such that

$$f(r, s) = g(r, s) + h(r) + k(s) + t$$

for all elements r and s in A.

2.19. Formulate and prove a version of Exercise 2.18 that applies to distributive (or quasi-completely distributive) operations of arbitrary ranks $n \geq 1$.

2.20. Prove that Theorem 2.8 continues to hold if references to completely distributivity and to atoms are replaced by references to quasi-completely distributivity and to quasi-atoms respectively.

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