Chapter 2
Buckling and Post-buckling of Beams

Abstract This chapter presents buckling and post-buckling analysis of straight beams under thermal and mechanical loads. The Euler and Timoshenko beam theories are considered and buckling and postbuckling behaviors are discussed. The buckling analysis of beams with piezoelectric layers is presented and the effect of piezo-control on the beam stability is analyzed. The vibration of thermo-electrically excited beams in the state of buckling and post-buckling is discussed and the chapter concludes with the thermal dynamic analysis of beams. The beam material in this chapter is assumed to be functionally graded, where the presented formulations may be simply reduced to the beams with isotropic/homogeneous material.

2.1 Introduction

Beams are the basic elements of many structural systems and design problems. Under mechanical forces or thermal stresses, beams may become structurally unstable. The mechanical or thermal stresses may be induced by the static or dynamic loads, providing a static or dynamic stability problem. The buckling and post-buckling behavior of beams should be essentially known for a structural design problem. This chapter presents the basic governing equations for the stability analysis of beams. Static and dynamic buckling and post-buckling problems of beams of functionally graded materials, piezo-control of buckling and post-buckling, beams on elastic foundation, and dynamic buckling of beams are discussed in detail (Fig. 2.1).

2.2 Kinematic Relations

The strain-displacement relations for straight beams under loading conditions that produce axial or lateral deflections are derived in this section. Different types of beam theories, from the Euler beam theory to the more sophisticated higher order beam
theory may be considered for the analysis of beams. In this section, the analysis of beams may be based on the first order shear deformation theory using the Timoshenko assumptions. According to this theory, the displacement field of the beam is assumed to be

\[ \bar{u}(x, z) = u + z \varphi \]
\[ \bar{w}(x, z) = w \]  

(2.2.1)

where \( \bar{u}(x, z) \) and \( \bar{w}(x, z) \) are displacements of an arbitrary point of the beam along the \( x \) and \( z \)-directions, respectively. Here, \( u \) and \( w \) are the displacement components of middle surface and \( \varphi \) is the rotation of the beam cross-section, which is function of \( x \) only. The strain-displacement relations for the beam are given in the form

\[ \varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \left( \frac{\partial \bar{w}}{\partial x} \right)^2 \]
\[ \gamma_{xz} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} \]  

(2.2.2)

where \( \varepsilon_{xx} \) and \( \gamma_{xz} \) are the axial and shear strains. Substituting Eq. (2.2.1) into (2.2.2) give

\[ \varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} \]
\[ \gamma_{xz} = \varphi + \frac{dw}{dx} \]  

(2.2.3)

The constitutive law for a material, using the linear thermoelasticity assumption, is given by [3]

\[ \sigma_{xx} = E [\varepsilon_{xx} - \alpha(T - T_0)] \]
\[ \sigma_{xz} = \frac{E}{2(1 + \nu)} \gamma_{xz} \]  

(2.2.4)
In Eq. (2.2.4), $\sigma_{xx}$ and $\sigma_{xz}$ are the axial and shear stresses, $T_0$ is the reference temperature, and $T$ is the absolute temperature distribution through the beam. Equations (2.2.2) and (2.2.4) are combined to give the axial and shear stresses in the beam in terms of the middle surface displacements as

$$
\sigma_{xx} = E \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} - \alpha (T - T_0) \right)
$$

$$
\sigma_{xz} = \frac{E}{2(1+\nu)} \left( \varphi + \frac{dw}{dx} \right)
$$

The stress resultants of the beam expressed in terms of the stresses through the thickness, according to the first order theory, are

$$
N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dz
$$

$$
M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xx} dz
$$

$$
Q_{xz} = K_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz
$$

where $K_s$ is the shear correction factor. The values of 5/6 or $\pi^2/12$ may be used as an approximate value for the composite and FGM beams with rectangular cross section. The shear correction factor is taken as $K_s = \pi^2/12$ for the FGM beam in this section.

We assume that the beam material is made of functionally graded materials (FGMs). For an FGM that is made of two constituent materials, ceramic and metal may be assumed as the constituent materials. If the volume fraction of ceramic is $V_c$ and that of metal is $V_m$, then a power law distribution of the constituents across the beam thickness may be assumed of the form [3]

$$
V_c + V_m = 1, \quad V_c = \left( \frac{1}{2} + \frac{z}{h} \right)^k
$$

where $h$ is total height of beam’s cross section and $z$ is the coordinate measured from the middle surface of the beam ($-h/2 \leq z \leq h/2$), $k$ is the power law index which has the value equal or greater than zero. Variation of $V_c$ with $k$ and $z/h$ is shown in Fig. 2.2. The value of $k$ equal to zero represents a fully ceramic beam ($V_c = 1$) and $k$ equal to infinity represents a fully metallic beam ($V_c = 0$). We assume that the mechanical and thermal properties of the FGM beam are distributed based on Voigt’s rule [4]. Thus, the property variation of a functionally graded material using Eq. (2.2.7) is given by
Fig. 2.2 Variation of ceramic volume fraction with power law index and thickness coordinate

\[ P(z) = P_m + P_{cm} \left( \frac{1}{2} + \frac{z}{h} \right)^k \]  

where \( P_{cm} = P_c - P_m \), and \( P_m \) and \( P_c \) are the corresponding properties of the metal and ceramic, respectively. In this analysis the material properties, such as Young’s modulus \( E(z) \), coefficient of thermal expansion \( \alpha(z) \), and the thermal conductivity \( K(z) \) may be expressed by Eq. (2.2.8), where Poisson’s ratio \( \nu \) is assumed to be constant across the beam thickness due to its small variations for the constituent materials [3].

Using Eqs. (2.2.5), (2.2.6), and (2.2.8) and noting that \( u, w, \) and \( \varphi \) are only functions of \( x \), the expressions for \( N_x, M_x, \) and \( Q_{xz} \) are obtained as

\[ N_x = E_1 \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) + E_2 \frac{d\varphi}{dx} - N^T \]
\[ M_x = E_2 \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) + E_3 \frac{d\varphi}{dx} - M^T \]
\[ Q_{xz} = \frac{E_1 K_s}{2(1 + \nu)} \left( \varphi + \frac{dw}{dx} \right) \]  

where \( E_1, E_2, \) and \( E_3 \) are constants and \( N^T \) and \( M^T \) are thermal force and thermal moment resultants, which are calculated using the following relations

\[ E_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)dz = h \left( E_m + \frac{E_{cm}}{k+1} \right) \]
\[ E_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} zE(z)dz = h^2 E_{cm} \left( \frac{1}{k+2} - \frac{1}{2k+2} \right) \]
2.2 Kinematic Relations

\[ E_3 = \int_{-\frac{b}{2}}^{\frac{b}{2}} z^2 E(z) dz = h^3 \left( \frac{1}{12} E_m + E_{cm} \left( \frac{1}{k + 3} - \frac{1}{k + 2} + \frac{1}{4k + 4} \right) \right) \]

\[ N^T = \int_{-\frac{b}{2}}^{\frac{b}{2}} E(z) \alpha(z)(T - T_0) dz \]

\[ M^T = \int_{-\frac{b}{2}}^{\frac{b}{2}} z E(z) \alpha(z)(T - T_0) dz \]  

(2.2.10)

Note that to find the thermal force and moment resultants, the temperature distribution through the beam should be known.

2.3 Equilibrium Equations

Equilibrium equations of an FGM beam under thermal loads may be obtained through the static version of virtual displacement principle. According to this principle, assuming that the external load is absent, an equilibrium position occurs when the first variation of strain energy function vanishes. Thus, one may write

\[ \delta U = \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[ \sigma_{xx} \delta \varepsilon_{xx} + K_s \sigma_{xz} \delta \gamma_{xz} \right] dz dy dx = 0 \]  

(2.3.1)

With the aid of definitions for stress resultants given by Eq. (2.2.6), and performing the integration by part technique to relieve the displacement gradients, the following system of equilibrium equations are obtained

\[ \frac{dN_x}{dx} = 0 \]

\[ \frac{dM_x}{dx} - Q_{xz} = 0 \]

\[ \frac{dQ_{xz}}{dx} + N_x \frac{d^2 w}{dx^2} = 0 \]  

(2.3.2)

The boundary conditions for each side of the beam may be set by the force type or kinematical type, as

\[ N_x \quad \text{or} \quad u \]

\[ M_x \quad \text{or} \quad \varphi \]

\[ Q_{xz} + N_x \frac{dw}{dx} \quad \text{or} \quad w \]  

(2.3.3)

In each set, the first condition is the natural boundary condition and the second one stands as the essential type of boundary condition. Based on the above boundary
conditions, following cases are possible in thermal stability analysis of a beam

\[
\text{Clamped (C)}: u = w = \varphi = 0 \\
\text{Simply-Supported (S)}: u = w = M_x = 0 \\
\text{Roller (R)}: u = \varphi = Q_{xz} + N_x \frac{dw}{dx} = 0 \quad (2.3.4)
\]

### 2.4 Stability Equations

To derive the stability equations, the adjacent-equilibrium criterion is used. Assume that the equilibrium state of a functionally graded beam is defined in terms of the displacement components \(u_0, w_0, \varphi_0\). The displacement components of a neighboring stable state differ by \(u_1, w_1, \varphi_1\) with respect to the equilibrium position. Thus, the total displacements of a neighboring state are [1]

\[
\begin{align*}
    u &= u_0 + u_1 \\
    w &= w_0 + w_1 \\
    \varphi &= \varphi_0 + \varphi_1
\end{align*}
\quad (2.4.1)
\]

Similar to the displacements, the force and moment resultants of a neighboring state may be related to the state of equilibrium as

\[
\begin{align*}
    N_x &= N_{x0} + N_{x1} \\
    M_x &= M_{x0} + M_{x1} \\
    Q_{xz} &= Q_{xz0} + Q_{xz1}
\end{align*}
\quad (2.4.2)
\]

Here, stress resultants with subscript 1 represent the linear parts of the force and moment resultant increments corresponding to \(u_1, \varphi_1, \) and \(w_1\). The stability equations may be obtained by substituting Eqs. (2.4.1) and (2.4.2) into (2.3.2). Upon substitution, the terms in the resulting equations with subscript 0 satisfy the equilibrium conditions and therefore drop out of the equations. Also, the nonlinear terms with subscript 1 are ignored because they are small compared to the linear terms. The remaining terms form the stability equations as

\[
\begin{align*}
    E_1 \frac{d^2 u_1}{dx^2} + E_2 \frac{d^2 \varphi_1}{dx^2} &= 0 \\
    E_2 \frac{d^2 u_1}{dx^2} + E_3 \frac{d^2 \varphi_1}{dx^2} - \frac{E_1 K_x}{2(1+\nu)} \left( \varphi_1 + \frac{dw_1}{dx} \right) &= 0 \\
    \frac{E_1 K_x}{2(1+\nu)} \left( \frac{d \varphi_1}{dx} + \frac{d^2 w_1}{dx^2} \right) + N_{x0} \frac{d^2 w_1}{dx^2} &= 0
\end{align*}
\quad (2.4.3)
\]
Combining Eq. (2.4.3) by eliminating $u_1$ and $\varphi_1$ provides an ordinary differential equation in terms of $w_1$, which is the stability equation of an FGM beam under transverse thermal loadings

$$\frac{d^4 w_1}{dx^4} + \mu^2 \frac{d^2 w_1}{dx^2} = 0$$

with

$$\mu^2 = \frac{E_1 N_x^T}{(E_1 E_3 - E_2^2) \left(1 - 2N_T \frac{1+\nu}{E_1 K_s}\right)}$$

The stress resultants with subscript 1 are linear parts of resultants that correspond to the neighboring state. Using Eqs. (2.2.9) and (2.4.3), the expressions for $Q_{xz1}$, $N_{x1}$, and $M_{x1}$ become

$$N_{x1} = E_1 \frac{du_1}{dx} + E_2 \frac{d\varphi_1}{dx}$$
$$M_{x1} = E_2 \frac{du_1}{dx} + E_3 \frac{d\varphi_1}{dx}$$
$$Q_{xz1} = \frac{E_1 K_s}{2(1+\nu)} \left(\varphi_1 + \frac{dw_1}{dx}\right)$$

### 2.5 Thermal Buckling of FGM Beams

#### 2.5.1 Introduction

The mechanical and thermal buckling of beams, as a major solid structural component, have been the topic of many researches for a long period of time. Development of the new materials, such as the functionally graded materials (FGMs), have necessitated more researches in this area. Huang and Li [5] obtained an exact solution for mechanical buckling of FGM columns subjected to uniform axial load based on various beam theories. Zhao et al. [6] studied the post-buckling of simply supported rod made of functionally graded materials under uniform thermal load and nonlinear temperature distribution across the beam thickness using the numerical shooting method. They found that under the same temperature condition, the deformation of immovably simply supported FGM rod is smaller than those of the two homogeneous material rods. Also end constrained force of FGM rod is smaller than the corresponding values of the two homogeneous material rods with small deformation. Accordingly, the stability of FGM rod is higher than those of two homogeneous material rods when there is a temperature difference. Li et al. [7] presented the post-buckling behavior of fixed-fixed FGM beams based on the Timoshenko beam theory.
under nonlinear temperature loading. They found the effect of transverse shear deformation on the critical buckling temperature of beams and used the shooting method to predict the post-buckling behavior of beams. It is found that the non-dimensional thermal axial force increases along with the increase of the power law index, as the increment of metal constituent can produce more thermal expansion of the beam under the same value of thermal load. Kiani and Eslami [8] discussed the buckling of functionally graded material beams under three types of thermal loadings through the thickness. A semi inverse method to study the instability and vibration of FGM beams is carried out by Aydogdu [9]. Ke et al. [10] presented the post-buckling of a cracked FGM beam for hinged-hinged and clamped-hinged edge conditions based on the Timoshenko beam theory. Also, Ke et al. [11] presented the free vibration and mechanical buckling of cracked FGM beams using the first order shear deformation beam theory for three types of boundary conditions. They found that the FGM beams with a smaller slenderness ratio and a lower Young’s modulus ratio are much more sensitive to the edge crack. Ma and Lee [12] discussed the nonlinear behavior of FGM beams under in-plane thermal loading by means of first order shear deformation theory of beams. Derivation of the equations is based on the concept of neutral surface and numerical shooting method is used to solve the coupled nonlinear equations. Their study concluded that when a clamped-clamped FGM beam is subjected to uniform thermal loading it follows the bifurcation-type buckling, while the simply-supported beams do not. Most recently, post-buckling path of an Euler–Bernoulli beam under the action of in-plane thermal loading is investigated in [13] using the energy-based Ritz method. A full analytical method is presented in [14], which accurately predicts the temperature-deflection path of clamped-clamped and hinged-hinged FGM beams.

In this section, buckling analysis of FGM beams subjected to thermal loading is analyzed based on the Timoshenko beam theory [8]. Five possible types of boundary conditions are assumed and the existence of bifurcation type buckling is examined for each case. Based on the static version of virtual displacements, three coupled differential equations are obtained as equilibrium equations. The beam is assumed under three types of thermal loads and closed form approximate solutions are obtained to evaluate the critical buckling temperatures.

### 2.5.2 Functionally Graded Timoshenko Beams

Consider a beam made of FGMs with rectangular cross section [8]. It is assumed that length of the beam is \( L \), width is \( b \), and the height is \( h \). Rectangular Cartesian coordinates is used such that the \( x \) axis is at the left side of the beam on its middle surface and along the length and \( z \) is measured from the middle surface and is positive upward, as shown in Fig. 2.1. The analysis of beam is based on the first order shear deformation beam theory using the Timoshenko assumptions. The kinematic relations, equilibrium equations, and the stability equations for the first order shear
deformation theory of beams were established in the previous sections. We first examine the condition where the beam should follow the bifurcation path.

### 2.5.3 Existence of Bifurcation Type Buckling

Consider a beam made of FGMs subjected to transverse temperature distribution. When axial deformation is prevented in the beam, an applied thermal load may produce an axial load. Only perfectly flat pre-buckling configurations are considered in the present case, which lead to bifurcation type buckling. Now, based on Eq. (2.3.2) and in the prebuckling state, when beam is completely undeformed, the generated pre-buckling force through the beam is equal to

\[ N_{x0} = -N^T \]  

Here, a subscript 0 is adopted to indicate the pre-buckling state. Also, according to Eq. (2.3.2), an extra moment is produced through the beam which is equal to

\[ M_{x0} = -M^T \]  

In general, this extra moment may cause deformation through the beam, except when it is vanished for some especial types of thermal loading or when boundary conditions are capable of handling the extra moments. The clamped and roller boundary conditions are capable of supplying the extra moments on the boundaries, while the simply-supported edge does not. Therefore, the \( C-C \) and \( C-R \) FGM Timoshenko beams remain un-deformed prior to buckling, while the other types of beams with at least one simply supported edge, commence to deflect. Also, an isotropic homogeneous beam under simply supported boundary conditions remains un-deformed when it is subjected to uniform temperature rise, because thermal moment vanishes through the beam. Therefore, bifurcation type buckling exists for the \( C-C \) and \( C-R \) FGM beams subjected to arbitrary transverse thermal loading. The same is true for the isotropic homogeneous beams subjected to uniform temperature rise.

### 2.5.4 Thermal Buckling

When temperature distribution through the beam is along the thickness direction only, the parameter \( \mu \) of Eq. (2.4.5) is constant. In this case, the analytical solution of Eq. (2.4.3) is [8]

\[ w_1(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x) + C_3 x + C_4 \]  

(2.5.3)
Table 2.1  Boundary conditions for the FGM Timoshenko beams. Here, C indicates clamped, S shows simply-supported, and R is used for roller edge [8]

<table>
<thead>
<tr>
<th>Edge supports</th>
<th>B.Cs at x = 0</th>
<th>B.Cs at x = L</th>
</tr>
</thead>
<tbody>
<tr>
<td>C – C</td>
<td>$u_1 = w_1 = \varphi_1 = 0$</td>
<td>$u_1 = w_1 = \varphi_1 = 0$</td>
</tr>
<tr>
<td>S – S</td>
<td>$u_1 = w_1 = M_{x1} = 0$</td>
<td>$u_1 = w_1 = M_{x1} = 0$</td>
</tr>
<tr>
<td>C – S</td>
<td>$u_1 = w_1 = \varphi_1 = 0$</td>
<td>$u_1 = w_1 = M_{x1} = 0$</td>
</tr>
<tr>
<td>C – R</td>
<td>$u_1 = w_1 = \varphi_1 = 0$</td>
<td>$u_1 = \varphi_1 = Q_{xz1} + N_x0 \frac{dw_1}{dx} = 0$</td>
</tr>
<tr>
<td>S – R</td>
<td>$w_1 = u_1 = M_{x1} = 0$</td>
<td>$u_1 = \varphi_1 = Q_{xz1} + N_x0 \frac{dw_1}{dx} = 0$</td>
</tr>
</tbody>
</table>

Using Eqs. (2.4.3), (2.4.6), and (2.5.3) the expressions for $u_1$, $\varphi_1$, $N_{x1}$, $M_{x1}$, and $Q_{xz1}$ become

$$
\varphi_1(x) = -S(\mu)(C_1 \cos(\mu x) - C_2 \sin(\mu x)) - C_3
$$

$$
u_1(x) = \frac{E_2}{E_1} S(\mu)(C_1 \cos(\mu x) - C_2 \sin(\mu x)) + C_3 x + C_6
$$

$$
M_{x1}(x) = \mu S(\mu) \left( \frac{E_1 E_3 - E_2^2}{E_1} \right)(C_1 \sin(\mu x) + C_2 \cos(\mu x)) + E_2 C_5
$$

$$
Q_{xz1}(x) = \frac{E_1 K_s}{2(1 + \nu)}(\mu - S(\mu))(C_1 \cos(\mu x) - C_2 \sin(\mu x))
$$

$$
N_{x1}(x) = E_1 C_5
$$

(2.5.4)

with

$$
S(\mu) = \frac{\mu}{1 + 2(1 + \nu) \mu^2 \frac{E_1 E_3 - E_2^2}{E_1^2 K_s}}
$$

(2.5.5)

The constants of integration $C_1$ to $C_6$ are obtained using the boundary conditions of the beam. Also, the parameter $\mu$ must be minimized to find the minimum value of $N_{x1}^T$ associated with the thermal buckling load. Five types of boundary conditions are assumed for the FGM beam with combination of the roller, simply supported, and clamped edges. Boundary conditions in each case are listed in Table 2.1. Let us consider a beam with both edges clamped. Using Eqs. (2.5.3) and (2.5.4), the constants $C_1$ to $C_6$ must satisfy the system of equations

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\sin(\mu L) & \cos(\mu L) & L & 1 & 0 & 0 \\
-S(\mu) & 0 & -1 & 0 & 0 & 0 \\
-S(\mu) \cos(\mu L) & S(\mu) \sin(\mu L) & -1 & 0 & 0 & 0 \\
\frac{E_2}{E_1} S(\mu) & 0 & 0 & 0 & 0 & 1 \\
\frac{E_2}{E_1} S(\mu) \cos(\mu L) - \frac{E_2}{E_1} S(\mu) \sin(\mu L) & 0 & 0 & L & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
$$

(2.5.6)
Table 2.2 Constants of formula (2.5.8) which are related to boundary conditions [8]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>C − C</th>
<th>S − S</th>
<th>C − S</th>
<th>S − R</th>
<th>C − R</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>78.95684</td>
<td>19.73920</td>
<td>44.41969</td>
<td>4.93480</td>
<td>19.73920</td>
</tr>
</tbody>
</table>

To have a nontrivial solution, the determinant of coefficient matrix must be set to zero, which yields

\[ S(\mu)L(2 - 2\cos(\mu L) - LS(\mu)\sin(\mu L)) = 0 \] (2.5.7)

The smallest positive value of \( \mu \) which satisfies Eq. (2.5.7) is \( \mu_{\text{min}} = \frac{6.28319}{L} \). It can be seen that for the other types of boundary conditions, except \( C − S \) case, the nontrivial solution leads to an exact parameter for \( \mu \). Using an approximate solution given in [15] for the critical axial force of \( C − S \) beams, the critical thermal force for an FGM Timoshenko beam with arbitrary boundary conditions can be expressed as

\[
N_{cr}^T = \frac{pL}{1 + q\frac{k\nu}{K_sL^2}\left(\frac{E_3}{E_1} - \left(\frac{E_2}{E_1}\right)^2\right)}
\] (2.5.8)

where \( p \) and \( q \) are constants depending upon the boundary conditions and are listed in Table 2.2. The critical thermal buckling force for an FGM beam based on the classical beam theory may be derived by omitting the shear term in Eq. (2.5.8).

For an isotropic homogeneous beam, setting \( k = 0 \) in Eq. (2.5.8), the critical buckling force is

\[
N_{cr}^T = \frac{pEh^3}{12L^2 + qh^2\frac{1+\nu}{K_s}}
\] (2.5.9)

which is a well-known relation for the critical axial buckling load of the Timoshenko beam given in [15].

2.5.5 Types of Thermal Loads

Uniform Temperature Rise (UTR)

Consider a beam which is at reference temperature \( T_0 \). When the axial displacement is prevented, the uniform temperature may be raised to \( T_0 + \Delta T \) such that the beam buckles. Substituting \( T = T_0 + \Delta T \) into Eq. (2.2.10) gives [8]
Using Eq. (2.5.8), the critical buckling temperature difference \( \Delta T_{cr}^{Uni} \) is expressed in the form

\[
\Delta T_{cr}^{Uni} = \frac{h}{\Delta_1} \frac{T_m - T_0}{E_m \alpha_m + E_{cm} \alpha_m + E_m \alpha_{cm}} \left[ \frac{h}{L} \right]^2 \frac{F(k, \xi)}{G(k, \xi, \zeta)} \left[ 1 + q \frac{1 + \frac{E_{cm} \alpha_{cm}}{E_m \alpha_m}}{k} \left( \frac{h}{L} \right)^2 E(k, \xi) \right]
\]

where \( \xi = \frac{E_m \alpha_m}{E_{cm} \alpha_{cm}} \) and \( \zeta = \frac{E_{cm} \alpha_{cm}}{E_m \alpha_m} \). Also, the functions \( E(k, \xi) \), \( F(k, \xi) \), and \( G(k, \xi, \zeta) \) are defined as

\[
F(k, \xi) = \frac{1}{12} + \frac{\xi (k^2 + k + 2)}{4(k + 1)(k + 2)(k + 3)} - \frac{\xi^2 k^2}{4(k + 1)(k + 2)^2(k + 1 + \xi)}
\]

\[
E(k, \xi) = \frac{k + 1}{12(k + 1 + \xi)} + \frac{\xi (k^2 + k + 2)}{4(k + 1)(k + 2)(k + 3)(k + 1 + \xi)} - \frac{\xi^2 k^2}{4(k + 2)^2(k + 1 + \xi)^2}
\]

\[
G(k, \xi, \zeta) = 1 + \frac{\xi + \zeta}{k + 1} + \frac{\xi \zeta}{2k + 1}
\]

### Linear Temperature Through the Thickness (LTD)

Consider a thin FGM beam which the temperature in ceramic-rich and metal-rich surfaces are \( T_c \) and \( T_m \), respectively. The temperature distribution for the given boundary conditions is obtained by solving the heat conduction equation across the beam thickness. If the beam thickness is thin enough, the temperature distribution is approximated linear through the thickness. So the temperature as a function of thickness coordinate \( z \) can be written in the form [8]

\[
T = T_m + (T_c - T_m) \left( \frac{1}{2} + \frac{z}{h} \right)
\]

Substituting Eq. (2.5.13) into (2.2.10) gives the thermal force as

\[
N^T = \frac{h}{\Delta_1} \frac{T_m - T_0}{E_m \alpha_m + E_{cm} \alpha_m + E_m \alpha_{cm}} \left( \frac{h}{L} \right)^2 \left( \frac{E_m \alpha_m}{2} + \frac{E_{cm} \alpha_m + E_m \alpha_{cm}}{k + 2} + \frac{E_{cm} \alpha_{cm}}{2k + 1} \right)
\]

where \( \Delta T = T_c - T_m \). Combining Eqs. (2.5.8) and (2.5.14) gives the final form of the critical buckling temperature difference through the thickness as
\[ \Delta T_{cr}^{Linear} = \frac{\frac{p}{\alpha_m} \left( \frac{h}{L} \right)^2 F(k, \xi)}{H(k, \xi, \zeta) \left( 1 + q \frac{1 + \nu}{K_s} \left( \frac{h}{L} \right)^2 E(k, \xi) \right)} - (T_m - T_0) \frac{G(k, \xi, \zeta)}{H(k, \xi, \zeta)} \] (2.5.15)

Here, the functions \( E(k, \xi) \), \( F(k, \xi) \), and \( G(k, \xi, \zeta) \) are defined in Eq. (2.5.12) and function \( H(k, \xi, \zeta) \) is defined as given below

\[ H(k, \xi, \zeta) = \frac{1}{2} + \frac{\xi + \zeta}{k + 2} + \frac{\xi \zeta}{2k + 2} \] (2.5.16)

**Nonlinear Temperature Through the Thickness (NLTD)**

Assume an FGM beam where the temperature in ceramic-rich and metal-rich surfaces are \( T_c \) and \( T_m \), respectively. The governing equation for the steady-state one-dimensional heat conduction equation, in the absence of heat generation, becomes [8]

\[ \frac{d}{dz} \left( K(z) \frac{dT}{dz} \right) = 0 \]

\[ T \left( \frac{h}{2} \right) = T_c \]

\[ T \left( -\frac{h}{2} \right) = T_m \] (2.5.17)

where \( K(z) \) is given by Eq. (2.2.8). Solving this equation via polynomial series and taking the sufficient terms to assure the convergence, yields the temperature distribution across the beam thickness as

\[ T = T_m + \frac{D}{N_T} \left( T_c - T_m \right) \left\{ \sum_{i=0}^{N_T} \frac{(-1)^i}{i k + 1} \left( \frac{K_{cm}}{K_m} \right)^i \left( \frac{1}{2} + \frac{z}{h} \right)^{i+1} \right\} \] (2.5.18)

with

\[ D = \sum_{i=0}^{N_T} \frac{(-1)^i}{i k + 1} \left( \frac{K_{cm}}{K_m} \right)^i \] (2.5.19)

Evaluating \( N_T \) and solving for \( \Delta T \) gives the critical buckling value of the temperature difference as
Table 2.3 Temperature dependent coefficients for $SU S304$ and $Si3N4$ [16]

<table>
<thead>
<tr>
<th>Material</th>
<th>Properties</th>
<th>$P_{-1}$</th>
<th>$P_0$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU S304$</td>
<td>$\alpha [K^{-1}]$</td>
<td>0</td>
<td>12.33e−6</td>
<td>8.086e−4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$E [Pa]$</td>
<td>0</td>
<td>201.04e+9</td>
<td>3.079e−4</td>
<td>$-6.534e−7$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K [Wm^{-1}K^{-1}]$</td>
<td>0</td>
<td>15.379</td>
<td>$-1.264e−3$</td>
<td>$-2.092e−6$</td>
<td>$-7.223e−10$</td>
</tr>
<tr>
<td></td>
<td>$\nu$</td>
<td>0</td>
<td>0.28</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rho [kgm^{-3}]$</td>
<td>0</td>
<td>2170</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Si3N4$</td>
<td>$\alpha [K^{-1}]$</td>
<td>0</td>
<td>5.8723e−6</td>
<td>9.095e−4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$E [Pa]$</td>
<td>0</td>
<td>348.43e+9</td>
<td>$-3.07e−4$</td>
<td>2.16e−7</td>
<td>$-8.946e−11$</td>
</tr>
<tr>
<td></td>
<td>$K [Wm^{-1}K^{-1}]$</td>
<td>0</td>
<td>13.723</td>
<td>$-1.032e−3$</td>
<td>5.466e−7</td>
<td>$-7.876e−11$</td>
</tr>
<tr>
<td></td>
<td>$\nu$</td>
<td>0</td>
<td>0.28</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rho [kgm^{-3}]$</td>
<td>0</td>
<td>8166</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\Delta T_{cr}^{N\text{LTD}} = \frac{P_{-1}}{\alpha_m} \left( \frac{h}{L} \right)^2 F(k, \xi) \frac{I(k, \xi, \zeta, \gamma)}{\left(1 + q \frac{1+\nu}{K_m} \left( \frac{h}{L} \right)^2 E(k, \xi) \right)} - (T_m - T_0) \frac{G(k, \xi, \zeta)}{I(k, \xi, \zeta, \gamma)} \quad (2.5.20)
\]

In this relation $\gamma = \frac{K_m}{K_w}$ and the function $I(k, \xi, \zeta, \gamma)$ is defined as

\[
I(k, \xi, \zeta, \gamma) = \frac{1}{D} \sum_{i=0}^{N} \frac{(-1)^i}{ik+1} \left[ \frac{1}{ik+2} + \frac{\zeta + \eta}{ik+2} + \frac{\zeta \eta}{ik+2} \right] \quad (2.5.21)
\]

### 2.5.6 Results and Discussion

Consider a ceramic-metal functionally graded beam [8]. The combination of materials consist of Silicon-Nitride as ceramic and stainless steel as metal. The elasticity modulus, the thermal expansion coefficient, and the thermal conductivity coefficient for these constituents are highly dependent to the temperature. The temperature dependency of the material properties are assumed to follow the Touloukian model as [16]

\[
P = P_0 \left( P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3 \right) \quad (2.5.22)
\]

where $P_{-1}$, $P_0$, $P_1$, $P_2$, and $P_3$ are constants and unique to each constituent. For the constituents of this study, these constants are given in Table 2.3. The temperature independent case ($TID$) describes the condition where properties are evaluated at reference temperature $T_0 = 300$ K. The case $TD$, on the other hand, represents the conditions where properties are calculated at current temperature based on the Touloukian model described by Eq. (2.5.22).
To validate the formulation of this section, the effect of transverse shear deformation on critical buckling temperature difference of a homogeneous beam is plotted in Fig. 2.3. For this purpose, the results are compared between the Euler and Timoshenko beam theories. The beam is under the uniform temperature rise. Nondimensional critical buckling temperature is defined by \( \lambda_{cr} = \frac{\alpha \Delta T_{cr}^{Uni}}{(L/h)^2} \). It is apparent that the critical buckling temperature difference for beams with \( L/h \) ratio more than 50 is identical between the two theories. But, for \( L/h \) ratio less than 50, the difference between the two theories become larger and it will become more different for \( L/h \) values less than 20. The same graph is reported in [7] based on the numerical shooting method.

In Fig. 2.4, the critical buckling temperature difference of an FGM beam under the uniform temperature rise loading is depicted [8]. Both edges are clamped. The TD case indicates that properties are temperature dependent, whereas the TID indicates that properties are evaluated at reference temperature. As seen, when the power law index increases, the critical buckling temperature decreases permanently. When it is compared to the TD case, the TID case overestimates the buckling temperatures. Difference between the TID and TD cases is more pronounced at higher temperatures. As seen for the constituents of this study, an increase in the power law index results in less critical buckling temperature difference.

In Fig. 2.5, two other cases of thermal loadings are compared with respect to each other. As seen, in both of these cases an increase in the power law index results in lower buckling temperature. The LTD case, as an approximate solution for the NLTD case, underestimates the critical buckling temperatures except for the case of reduction of an FGM beam to the associated homogeneous case. This is expected, as in this case the analytical solution of the heat conduction equation is also linear.
Fig. 2.4 Effect of temperature dependency of the constituents on $\Delta T_{cr}$

Fig. 2.5 Effect of temperature dependency of the constituents on $\Delta T_{cr}$ subjected to linear and nonlinear heat conduction across the thickness

The influence of boundary conditions on buckling temperature difference is plotted in Fig. 2.6. The uniform temperature rise case of loading is assumed and properties are assumed to be TD. The case of a homogeneous beam is chosen. As expected, the higher buckling temperature belongs to a beam with both edges clamped and the lower one is associated with a beam with one side simply supported and the other one roller. The critical buckling temperature of the S-S and C-R cases are the same.
2.6 Thermo-Electrical Buckling of Beams

2.6.1 Introduction

Smart materials belong to a class of advanced materials which are used widely in structural engineering. As a branch of smart materials, the piezoelectric materials are used extensively due to their applications in controlling the deformation, vibration, and instability of solid structures. Many studies are reported on behavior of structures integrated with the piezoelectric layers. Kapuria et al. [17] developed an efficient coupled zigzag theory for electro-thermal stress analysis of hybrid piezoelectric beams. Control and stability analysis of a composite beam with piezoelectric layers subjected to axial periodic compressive loads is reported by Chen et al. [18]. In their study, by employing the Euler beam theory and nonlinear strain-displacement, Hamilton’s principle is used to obtain the dynamic equation of the beams integrated with the piezoelectric layers.

Piezoelectric FGM structures have the advantages of functionally graded materials and piezoelectric materials linked together. Bian et al. [19] presented an exact solution based on the state space formulation to study the functionally graded beams integrated with surface bounded piezoelectric actuators and sensors. Alibeigloo [20] reported an analytical solution for thermoelasticity analysis of the FGM beams integrated with piezoelectric layers. By assuming the simply-supported edge conditions, he used the state space method in conjunction with the Fourier series in longitudinal direction to obtain an analytical solution. An analytical method for deflection control of the FGM beams containing two piezoelectric layers is reported by Gharib et al. [21]. Following the Timoshenko beam theory and the power law form of material property distribution, three coupled ordinary differential equations are derived as equilibrium equations of an FGM beam integrated with orthotropic piezoelectric
layers. Vibration of thermally post-buckled functionally graded material beams with surface-bonded piezoelectric layers subjected to both thermal and electrical loads is carried out by Li et al. [22]. Buckling and postbuckling analysis of the FGM beams with general boundary conditions is reported by Kiani et al. [23, 24] and [25] based on the Euler and Timoshenko beam theories.

The instability problem of piezoelectric FGM beams subjected to thermal load and applied constant voltage is discussed in this section [24]. Three types of thermal loads and five types of boundary conditions are assumed for the beam. Based on the Timoshenko beam theory and power law assumption for property distribution, the equilibrium and stability equations for the beam are derived and the eigenvalue solution is carried out to obtain the critical buckling temperature.

### 2.6.2 Piezoelectric FGM Beam

Consider a beam with rectangular cross section made of an FGM substrate of thickness $h$, width $b$, length $L$, and piezoelectric films of thickness $h_p$ that are perfectly bonded on its top and bottom surfaces as actuators. No adhesive layer is assumed to exist between the smart layers and the FGM media. Due to the asymmetrically mid-plane configuration of the FGM beam, total structure acts as an asymmetrical three-layered media.

### 2.6.3 Governing Equations

Material properties of the FGM substrate are distributed based on a power law form function described by Eq. (2.2.8). The rectangular Cartesian coordinates is used such that the $x$ axis is along the length of the beam on its middle surface and $z$ is measured from the middle surface and is positive upward, as shown in Fig. 2.7. Analysis of the beam is developed based on Timoshenko’s beam theory [24]. Axial and lateral components of displacement field through the beam are given in Eq. (2.2.1).

If the applied voltage $V_p$ to the piezoelectric layers is across the thickness, then the electrical field is generated only in the $z$—direction and is denoted by $E_z$, which is equal to [26–29]

$$E_z = \frac{V_p}{h_p} \quad (2.6.1)$$

It should be pointed out that, since the electrical field is equal to the negative gradient of electrical potential, the electric potential at the top surface of the top smart layer $z = +h/2 + h_p$ and bottom surface of the bottom layer $z = -h/2 - h_p$ are equal to $-V_p$ and $V_p$, respectively. Within the framework of linear thermoelasticity of a medium, stress-strain relations are
\[ \sigma_{xx} = E \left[ \varepsilon_{xx} - \alpha (T - T_0) \right] \]
\[ \sigma_{xz} = \frac{E}{2(1 + \nu)} \gamma_{xz} \] (2.6.2)

and for the piezoelectric layers [26]
\[ \sigma^p_{xx} = E_a \left[ \varepsilon_{xx} - \alpha_p (T - T_0) - d_{31} E_z \right] \]
\[ \sigma^p_{xz} = \frac{E_p}{2(1 + \nu_p)} \gamma_{xz} \]
\[ D_z = E_p d_{31} \varepsilon_{xx} + k_{33} E_z + \nu_p (T - T_0) \] (2.6.3)

In the above equations, \( \sigma_{xx} \) and \( \sigma_{xz} \) are the axial and shear stresses through the FGM layer and \( \sigma^p_{xx} \) and \( \sigma^p_{xz} \) are the axial and shear stresses through the piezoelectric layers. Also, \( \nu \) and \( \nu_p \) are Poisson’s ratios for the FGM beam and piezoelectric layers, respectively. Here, \( T_0 \) is the reference temperature and \( T \) is the temperature distribution through the beam. Also, \( E_p, D_z, d_{31}, \) and \( k_{33} \) are the elasticity modules, electric displacement, piezoelectric constant, pyroelectric constant, and the dielectric permittivity coefficient for the piezoelectric layers, respectively. Equations (2.2.1), (2.6.2), and (2.6.3) are combined to give the axial and shear stresses in terms of the middle surface displacements as

\[ \sigma_{xx} = E \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} - \alpha (T - T_0) \right) \]
\[ \sigma^p_{xx} = E_p \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} - \alpha_p (T - T_0) - \frac{V_p}{h_p} \frac{d_{31}}{d_3} \right) \]
\[ \sigma_{xz} = \frac{E}{2(1 + \nu)} \left( \varphi + \frac{dw}{dx} \right) \]
\[ \sigma^p_{xz} = \frac{E_p}{2(1 + \nu_p)} \left( \varphi + \frac{dw}{dx} \right) \] (2.6.4)
The force and moment resultants in the beam expressed in terms of the stresses through the thickness, according to the Timoshenko beam theory, are

\[
N_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{xx} dz + \int_{\frac{1}{2}}^{\frac{3}{2}} \sigma_{xx}^{p} dz + \int_{-\frac{1}{2}}^{-\frac{3}{2}} \sigma_{xx}^{p} dz \\
M_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} z\sigma_{xx} dz + \int_{\frac{1}{2}}^{\frac{3}{2}} z\sigma_{xx}^{p} dz + \int_{-\frac{1}{2}}^{-\frac{3}{2}} z\sigma_{xx}^{p} dz \\
Q_{xz} = K_s \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{xz} dz + \int_{\frac{1}{2}}^{\frac{3}{2}} \sigma_{xz}^{p} dz + \int_{-\frac{1}{2}}^{-\frac{3}{2}} \sigma_{xz}^{p} dz \right) \tag{2.6.5}
\]

Here, \( K_s \) is the shear correction factor which is chosen as \( K_s = 5/6 \).

Using Eqs. (2.6.4) and (2.6.5) and noting that \( u \) and \( w \) are functions of \( x \) only, \( N_x, M_x, \) and \( Q_{xz} \) are obtained as

\[
N_x = (E_1 + 2h_p E_p) \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) + E_2 \frac{d\varphi}{dx} - N^T - 2V_p E_p d_{31} \\
M_x = E_2 \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) + (E_3 + H E_p) \frac{d\varphi}{dx} - M^T \\
Q_{xz} = K_s \left( \frac{E_1}{2(1 + \nu)} + \frac{h_p E_p}{1 + \nu_p} \right) \left( \varphi + \frac{dw}{dx} \right) \tag{2.6.6}
\]

Constants \( E_i \) are defined in Eq. (2.2.10). Besides, \( N^T \) and \( M^T \) are thermal force and thermal moment resultants through the piezo-FGM beam and are

\[
H = \frac{2}{3} \left( \frac{h_p^3}{2} + \frac{3}{2} h_p h + \frac{3}{4} h_p h^2 \right) \\
N^T = \int_{-\frac{1}{2}}^{\frac{1}{2}} E(z)\alpha(z)(T - T_0) dz \\
M^T = \int_{-\frac{1}{2}}^{\frac{1}{2}} z E(z)\alpha(z)(T - T_0) dz \tag{2.6.7}
\]

The total potential energy \( U \) for a piezoelectric FGM beam under thermal loads is defined as the sum of total potential energies for piezoelectric layers \( U_p \) and the potential energy of the FGM beam \( U_b \) as

\[
U_b = \frac{1}{2} \int_x \int_y \int_z \left( \sigma_{xx} (\varepsilon_{xx} - \alpha(T - T_0)) + K_s \sigma_{xz} \varepsilon_{xz} \right) dz dy dx + \\
U_p = \frac{1}{2} \int_x \int_y \int_z \left( \sigma_{xx}^{p} (\varepsilon_{xx} - \alpha_p(T - T_0)) \right) K_s \sigma_{xz}^{p} \varepsilon_{xz} - E_z D_z) dz dy dx \tag{2.6.8}
\]
where in definition of $U_b$, $z \in [\frac{-h}{2}, \frac{h}{2}]$, and in definition of $U_p$, $z \in [\frac{-h}{2} - h_p, \frac{-h}{2}] \cup \left[\frac{h}{2}, h_p + \frac{h}{2}\right]$. Deriving the first variation of total potential energy function and performing the integration by part to relieve the displacement gradients, yields the equilibrium equations and associated boundary conditions given by Eqs. (2.3.2) and (2.3.3).

### 2.6.4 Existence of Bifurcation Type Buckling

Consider a beam made of FGMs with piezoelectric layers subjected to the transverse temperature distribution and applied actuator voltage. When the axial deformation is prevented in the beam, an applied thermal load and external voltage may produce an axial load. Only perfectly flat pre-buckling configurations are considered in the present work, which lead to bifurcation type buckling. Now, based on Eq. (2.6.6) and in the prebuckling state, when beam is completely undeformed and axial elongation is prevented at boundaries, the generated pre-buckling force through the beam is equal to [24]

$$N_{x0} = -N^T - 2V_p E_p d_{31}$$

(2.6.9)

Here, a subscript 0 is adopted to indicate the pre-buckling state deformation. Also, according to Eq. (2.6.6), an extra moment is produced through the beam which is equal to

$$M_{x0} = -M^T$$

(2.6.10)

In general, this extra moment may cause deformation through the beam, except when it is vanished for some especial types of thermal loading or when boundary conditions are capable of handling the extra moments. The clamped and roller boundary conditions are capable of supplying the extra moments on the boundaries, while the simply-supported edge does not. Therefore, the $C - C$ and $C - R$ piezoelectric FGM Timoshenko beams remain undeformed prior to buckling, while for the other types of beams with at least one simply supported edge, beam commence to deflect. Also, symmetrically mid-plane beam remains undeformed when it is subjected to uniform temperature rise, because thermal moment vanishes through the beam. Therefore, bifurcation type buckling exists for the $C - C$ and $C - R$ piezoelectric FGM beams subjected to arbitrary transverse thermal loading and constant voltage. The same is true for the beams with isotropic homogeneous core and simply supported boundary conditions subjected to the combined action of uniform temperature rise and constant voltage.
### 2.6.5 Stability Equations

To derive the stability equations, the adjacent-equilibrium criterion is used. Assume that the equilibrium state of a beam is defined in terms of the displacement components \( u_0, w_0, \) and \( \varphi_0 \). The displacement components of a neighboring stable state differ by \( u_1, w_1, \) and \( \varphi_1 \) with respect to the equilibrium position. The incremental stress resultants are obtained using Eq. (2.4.2). Since thermal resultant and in-plane electrical force are constant, Eq. (2.4.4) describes the stability equation of the beam, where a new definition for parameter \( \mu \) is needed as follows

\[
\mu^2 = \frac{(E_1 + 2h_p E_p)(N^T + 2V_p E_p d_{31})}{((E_3 + HE_p)(E_1 + 2h_p E_p) - E_2^2) \left(1 - \frac{N^T + 2V_p E_p d_{31}}{K_s \left( \frac{E_1}{1+\nu_p} + \frac{h_p E_p}{1+\nu_p} \right)} \right)} \tag{2.6.11}
\]

Besides, definition of perturbed displacements and stress resultants are given as

\[
w_1(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x) + C_3 x + C_4 \\
\varphi_1(x) = -S(\mu)(C_1 \cos(\mu x) - C_2 \sin(\mu x)) - C_3 \\
u_1(x) = \frac{E_2}{E_1 + 2h_p E_p} S(\mu)(C_1 \cos(\mu x) - C_2 \sin(\mu x)) + C_5 x + C_6 \\
M_{x1}(x) = \mu S(\mu) \frac{((E_3 + HE_p)(E_1 + 2h_p E_p) - E_2^2)}{E_1 + 2h_p E_p} (C_1 \sin(\mu x) + C_2 \cos(\mu x)) + E_2 C_5 \\
Q_{xz1}(x) = K_s \left( \frac{E_1}{2(1+\nu)} + \frac{h_p E_p}{1+\nu_p} \right) (\mu - S(\mu))(C_1 \cos(\mu x) - C_2 \sin(\mu x)) \\
N_{x1}(x) = (E_1 + 2h_p E_p) C_5 \tag{2.6.12}
\]

with

\[
S(\mu) = \frac{\mu}{1 + \mu^2 \frac{(E_3 + HE_p)(E_1 + 2h_p E_p) - E_2^2}{K_s (E_1 + 2h_p E_p) \left( \frac{E_1}{1+\nu_p} + \frac{h_p E_p}{1+\nu_p} \right)}} \tag{2.6.13}
\]

The constants of integration \( C_1 \) to \( C_6 \) are obtained using the boundary conditions of the beam. The parameter \( \mu \) has to be minimized to find the minimum value of \( N^T \) associated with the thermal buckling load. Five types of boundary conditions are considered for the ends of the beam. Mathematical expression of edge conditions are given in Table 2.1. Similar to the process developed in the previous section, the critical thermal force through the beam is obtained as

\[
N_{ct}^T = \frac{p \left( (E_3 + HE_p)(E_1 + 2h_p E_p) - E_2^2 \right)}{L^2 \left( (E_1 + 2h_p E_p) + \frac{q\left((E_3 + HE_p)(E_1 + 2h_p E_p) - E_2^2\right)}{K_s \left( \frac{E_1}{1+\nu_p} + \frac{h_p E_p}{1+\nu_p} \right) L^2} \right)} - 2V_p E_p d_{31} \tag{2.6.14}
\]
Neglecting the term produced by shear deformation, gives the critical thermal load of piezoelectric FGM beams based on the Euler–Bernoulli beam theory as

\[ N_{cr}^T = \frac{p}{L^2} \left( E_3 + HE_p - \frac{E_2^2}{E_1 + 2h_p E_p} \right) - 2V_p E_p d_{31} \] (2.6.15)

Also, Eq. (2.6.14) may be reduced to thermal buckling force of an FGM beam without piezoelectric layers, when both \( h_p \) and \( V_p \) tend to zero. In this case, \( N_{cr}^T \) becomes

\[ N_{cr}^T = \frac{p}{L^2} \left( E_3 - \frac{E_2^2}{E_1} \right) \] (2.6.16)

### 2.6.6 Types of Thermal Loads

**Uniform Temperature Rise (UTR)**

Consider a beam at reference temperature \( T_0 \). In such a case, the uniform temperature may be raised to \( T_0 + \Delta T \) such that the beam buckles. Evaluating the thermal force resultant for the case of uniform temperature rise loading, using Eq. (2.6.7), reaches us to [24]

\[ \Delta T_{cr} = \frac{p}{L^2 Q_1} \left\{ \left( E_1 + 2h_p E_p \right) \left( E_3 + HE_p \right) - E_2^2 \right\} \] (2.6.17)

\[ - \frac{2V_p E_p d_{31}}{Q_1} \]

with

\[ Q_1 = h \left( \alpha_m E_m + \frac{\alpha_m E_{cm} + \alpha_{cm} E_m}{n + 1} + \frac{\alpha_{cm} E_{cm}}{2n + 1} \right) + 2h_p \alpha_p E_p \] (2.6.18)

**Nonlinear Temperature Through the Thickness (NLTD)**

To calculate the critical buckling temperature for the case of gradient through the thickness of beam, the one-dimensional equation of heat conduction in the \( z \) direction must be solved. In the FGM media, heat conduction equation for the steady state one-dimensional case, in the absence of heat generation, becomes [24]

\[ \frac{d}{dz} \left( K(z) \frac{dT}{dz} \right) = 0 \] (2.6.19)

with the boundary conditions
\[ T \left( \frac{+h}{2} \right) = T_c \quad T \left( \frac{-h}{2} \right) = T_m \] (2.6.20)

The solution of heat conduction equation along with the thermal boundary conditions is obtained via the power-series solution as

\[ T(z) = T_m + (T_c - T_m) \sum_{i=0}^{N} \frac{1}{ik+1} \left( -\frac{K_{cm}}{K_m} \right)^i \left( \frac{1}{2} + \frac{z}{h} \right)^{i+k+1}, \quad z \in [-\frac{h}{2}, +\frac{h}{2}] \] (2.6.21)

where \( N \) is the number of expanded terms and should be chosen appropriately to assure convergence of the solution.

Considering temperature \( T_t \) at top surface of the beam and \( T_b \) at bottom surface of the beam, the temperature boundary conditions become

\[ T \left( \frac{h + h_p}{2} \right) = T_t, \quad T \left( \frac{-h + h_p}{2} \right) = T_b \] (2.6.22)

and the temperature distribution through each of the piezoelectric layers is

\[ T(z) = \frac{2}{h_p} \left( T_t (z - \frac{h}{2}) - T_c (z - \frac{h + h_p}{2}) \right), \quad z \in \left[ \frac{h}{2}, +\frac{h + h_p}{2} \right] \]

\[ T(z) = \frac{2}{h_p} \left( -T_b (z + \frac{h}{2}) + T_m (z + \frac{h + h_p}{2}) \right), \quad z \in \left[ -\frac{h + h_p}{2}, -\frac{h}{2} \right] \] (2.6.23)

Here, \( T_c \) and \( T_m \) are obtained in terms of \( T_t \) and \( T_b \) when continuity conditions of temperature and thermal charge are applied to the bonded surfaces of piezoelectric layers and FGM media as

\[ T_m = \frac{T_b + (T_t + T_b) \frac{h_pK_m}{2QzhK_p}}{1 + \frac{h_pK_m}{QzhK_p}} \]

\[ T_c = \frac{T_t + (T_t + T_b) \frac{h_pK_m}{2QzhK_p}}{1 + \frac{h_pK_m}{QzhK_p}} \] (2.6.24)

in which

\[ Q_2 = \sum_{i=0}^{N} \frac{1}{ik+1} \left( -\frac{K_{cm}}{K_m} \right)^i \] (2.6.25)

Evaluating the thermal force resultant through three layers and solving for \( \Delta T_{cr} = T_t - T_b \) reaches us to
\[ \Delta T_{cr} = \left(1 + \frac{h_p K_m}{Q_2 h K_p}\right) \left\{ \frac{N^T_{cr}}{Q_3} - \frac{(T_b - T_0)Q_1}{Q_3 + \frac{h_p K_w}{2Q_2 h K_p} Q_4 + \frac{h_p E_p \alpha_p}{2}(1 + \frac{h_p K_w}{Q_2 h K_p})} \right\} \]  

(2.6.26)

where

\[ Q_3 = \frac{h}{Q_2} \left\{ E_m \alpha_m \sum_{i=0}^{N} \frac{(-\frac{K_m}{K_m})^i}{(ik + 1)(ik + 2)} + (E_c \alpha_m + E_m \alpha_c) \sum_{i=0}^{N} \frac{(-\frac{K_m}{K_m})^i}{(ik + 1)(ik + k + 2)} + E_c \alpha_c \sum_{i=0}^{N} \frac{(-\frac{K_c}{K_m})^i}{(ik + 1)(ik + 2k + 2)} \right\} \]

\[ Q_4 = h \left( \alpha_m E_m + \frac{\alpha_m E_c + \alpha_c E_m}{k + 1} + \frac{\alpha_c E_c}{2k + 1} \right) \]  

(2.6.27)

For the case when middle layer is homogeneous, the temperature distribution is linear through-the-thickness. In this case Eq. (2.6.26) simplifies to

\[ \Delta T_{cr} = 2 \frac{N^T_{cr}}{Q_{Hom}} - 2(T_b - T_0) \]  

(2.6.28)

2.6.7 Results and Discussion

Consider a piezoelectric FGM beam. The combination of materials consists of aluminum and alumina for the FGM substrate and PZT-5A for piezoelectric layers. The actuator layer thickness, unless otherwise stated, is \( h_p = 0.001 \) m. Young’s modules, coefficient of thermal expansion, and conductivity for aluminum are \( E_m = 70 \) GPa, \( \alpha_m = 23 \times 10^{-6} / ^\circ \) C and \( K_m = 204 \) W/mK, and for alumina are \( E_c = 380 \) GPa, \( \alpha_c = 7.4 \times 10^{-6} / ^\circ \) C and \( K_c = 10.4 \) W/mK, respectively. The PZT-5A properties are \( E_p = 63 \) GPa, \( \nu_p = 0.3 \) and \( d_{31} = 2.54 \times 10^{-10} \) m/V [23, 24, 26].

Figure 2.8 depicts the critical buckling temperature difference versus \( h \) for a piezoelectric/ceramic/piezoelectric beam for various types of boundary conditions subjected to the uniform temperature rise, when power law index is chosen \( k = 0 \) [24]. It is apparent that by increasing \( h \), \( \Delta T_{cr} \) becomes larger. Also, the critical buckling temperature for the \( S - S \) and \( C - R \) types of boundary conditions are identical and lower than the \( C - C \) and \( C - S \) beams, but larger than the value related to the \( S - R \) beams.

The influence of beam geometry on \( \Delta T_{cr} \), for various power law indices, when the applying voltage is \( V_p = +200 \) V, is illustrated in Fig. 2.9 for the uniform temperature
rise and the $C - R$ boundary conditions. As the thickness increases, the critical buckling temperature increases. Also it may be concluded that the critical temperature for the given constituents decreases for $k < 2$, then increases for $2 < k < 10$ and finally decreases for $k > 10$.

The buckling temperature difference $\Delta T_{cr}$ for a $C - C$ piezoelectric FGM beam ($L = 0.25 \text{ m}, h = 0.01 \text{ m}$) that is subjected to uniform temperature rise and constant voltage is calculated and presented in Table 2.4. Five cases of electrical loadings are considered $V_p = 0, \pm 200 \text{ V}, \pm 500 \text{ V}$. Here, $V_p = 0 \text{ V}$ denotes a grounding condition. The results show that for this type of piezoelectric layer, the critical buckling temperature difference decreases with the increase of the applied voltage. The changes are, however, small. It should be mentioned that increasing or decreasing the critical
2.6 Thermo-Electrical Buckling of Beams

Table 2.4  Effect of applied voltage on buckling temperature difference of the $C-C$ piezo-FGM beams subjected to uniform temperature rise ($L = 0.25$ m, $h = 0.01$ m) [24]

<table>
<thead>
<tr>
<th>$V_p (V)$</th>
<th>Theory</th>
<th>$k = 0$</th>
<th>$k = 0.5$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+500</td>
<td>Euler</td>
<td>796.606</td>
<td>478.876</td>
<td>413.243</td>
<td>389.489</td>
<td>420.050</td>
</tr>
<tr>
<td></td>
<td>Timoshenko</td>
<td>782.473</td>
<td>470.930</td>
<td>406.483</td>
<td>382.738</td>
<td>411.168</td>
</tr>
<tr>
<td>+200</td>
<td>Euler</td>
<td>796.948</td>
<td>479.174</td>
<td>413.561</td>
<td>389.851</td>
<td>420.491</td>
</tr>
<tr>
<td></td>
<td>Timoshenko</td>
<td>782.814</td>
<td>471.229</td>
<td>406.801</td>
<td>383.100</td>
<td>411.609</td>
</tr>
<tr>
<td>0</td>
<td>Euler</td>
<td>797.175</td>
<td>479.373</td>
<td>413.773</td>
<td>390.092</td>
<td>420.785</td>
</tr>
<tr>
<td></td>
<td>Timoshenko</td>
<td>783.042</td>
<td>471.428</td>
<td>407.013</td>
<td>383.341</td>
<td>411.903</td>
</tr>
<tr>
<td>−200</td>
<td>Euler</td>
<td>797.403</td>
<td>479.572</td>
<td>413.985</td>
<td>390.333</td>
<td>421.079</td>
</tr>
<tr>
<td></td>
<td>Timoshenko</td>
<td>783.270</td>
<td>471.627</td>
<td>407.225</td>
<td>383.582</td>
<td>412.197</td>
</tr>
<tr>
<td>−500</td>
<td>Euler</td>
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<td>479.870</td>
<td>414.303</td>
<td>390.695</td>
<td>421.520</td>
</tr>
<tr>
<td></td>
<td>Timoshenko</td>
<td>783.611</td>
<td>471.925</td>
<td>407.544</td>
<td>383.944</td>
<td>412.638</td>
</tr>
<tr>
<td>Without layers</td>
<td>Euler</td>
<td>711.323</td>
<td>403.017</td>
<td>330.461</td>
<td>292.974</td>
<td>302.254</td>
</tr>
<tr>
<td></td>
<td>Timoshenko</td>
<td>699.680</td>
<td>397.133</td>
<td>325.896</td>
<td>288.874</td>
<td>297.174</td>
</tr>
</tbody>
</table>

temperature difference by applying voltage in comparison with the grounding condition depends on both sign of the applied voltage and the sign of piezoelectric constant. For the piezoelectric layers used in this study, the piezoelectric constant $d_{31}$ is positive and it can be seen that the critical buckling temperature decreases by increasing the voltage. For a piezoelectric material, when the piezoelectric constant is negative (such as PZT-4 or PZT-5H), when voltage increases the critical buckling temperature difference becomes larger.

2.7 Postbuckling of Piezo-FGM Timoshenko Beams

2.7.1 Introduction

The post-buckling analysis of beams subjected to either mechanical or thermal loads is a complex nonlinear problem. Buckling and post-buckling behavior of elastic rods subjected to thermal load is presented by Li et al. [30]. They achieved the results by solving the nonlinear equilibrium equations of the slender pinned-fixed Euler–Bernoulli beams via the shooting method. Librescu et al. [31] studied the behavior of thin walled beams made of FGMs that operate at high temperatures. The study includes the vibration and instability analysis with the effects of the volume fraction and temperature gradients. Employing the finite element method, Bhangale and Ganesan [32] investigated the thermoelastic buckling and vibration behavior of an FGM sandwich beam. Literature on the analytical solution for the post-buckling of FGM beams under thermal loads is limited to a few published articles [14, 33].
This section presents an analytical solution for the post-buckling response of the FGM Timoshenko beam with and without layers of piezoelectric actuator using the mid-plane based concept [34]. The governing equations for the static behavior of FGM beam with two piezoelectric layers under thermo-electrical load are derived. The three nonlinear equilibrium equations are reduced to a fourth order uncoupled equation in terms of the lateral deflection. With the analytical solution of this equation, any type of boundary condition may be considered to be used. It is seen that considering the clamped-clamped boundary condition leads to an eigenvalue problem. For the simply supported-simply supported edge conditions, however, the boundary conditions are non-homogeneous and the response of the beam is of the nonlinear bending type. Numerical results are presented for a beam made of SUS304 as metal constituent and Si$_3$N$_4$ as ceramic constituent.

2.7.2 Governing Equations

Assume a beam made of FGMs with rectangular cross section $b \times h$ and length $L$ bonded with two identical piezoelectric layers at the top and bottom surfaces of the beam. Thickness of each piezoelectric layer is $h_p$. The schematic and coordinate system of the beam are shown in Fig. 2.7.

Thermo-mechanical properties are graded across the thickness, where their patterns may be expressed by any arbitrary mathematical function. Since the volume fraction of each phase gradually varies in the gradation direction, the mechanical properties of FGMs vary across this direction. Here, we assume a continuous alteration of the volume fraction of ceramic from ceramic-rich surface to the metal-rich surface. The gradation profile is assumed of the form given by Eqs. (2.2.7) and (2.2.8).

For the piezoelectric layers, the stress-strain relation is updated to account for the electrical effects such that

$$
\sigma_x = E_p \left[ \varepsilon_x - \alpha_p(T - T_0) - d_p E_z \right]
$$

$$
\sigma_{xz} = G_p \gamma_{xz}
$$

(2.7.1)

In the above equations, $\sigma_x$ and $\sigma_{xz}$ are the axial and transversal components of the stress tensor, $E$ and $G$ are the elasticity and shear modulus, and $E_z$ and $d$ are the electric field and piezoelectric constant, respectively. Besides, a subscript $p$ indicates that characteristics belong to the piezoelectric layers.

For the case when piezoelectric layers are thin enough, applying the electrical voltage to the top and bottom surfaces induces the electrical field where only the through-the-thickness one may be assumed to be dominant. When only the reverse effect of piezoelectric layers is considered, we may write [22–24, 33]
Since the electrical field is negative gradient of the electrical potential, the applied actuator voltage to the top of the top piezoelectric layer \((z = h/2 + h_p)\) and to the bottom of the bottom piezoelectric layer \((z = -h/2 - h_p)\) are \(-V_p\) and \(+V_p\), respectively.

Relations between stress components and stress resultants, within the framework of the Timoshenko beam theory, are \([24, 35]\)

\[
N_x = \int_{-\frac{h}{2} - h_p}^{\frac{h}{2} + h_p} \sigma_x dz \\
M_x = \int_{-\frac{h}{2} - h_p}^{\frac{h}{2} + h_p} z \sigma_x dz \\
Q_{xz} = \int_{-\frac{h}{2} - h_p}^{\frac{h}{2} + h_p} K_s \sigma_{xz} dz
\]  

(2.7.3)

where \(K_s\) is called the shear correction factor and it depends upon the geometry, boundary conditions, and loading type. Determination of the shear coefficient is not straightforward. Normally \(K_s = 5/6\) is used for a rectangular section. Substituting \(\sigma_x\) and \(\sigma_{xz}\) into Eq. (2.7.3) and integrating with respect to the \(z\) coordinate, result in

\[
N_x = (E_1 + 2E_p h_p) \left\{ \frac{du_0}{dx} + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right\} + E_2 \frac{d\varphi}{dx} - N^T - N^E \\
M_x = E_2 \left\{ \frac{du_0}{dx} + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right\} + (E_3 + H_3 E_p) \frac{d\varphi}{dx} - M^T \\
Q_{xz} = K_s (G_1 + 2G_p h_p) \left( \varphi + \frac{dw_0}{dx} \right)
\]  

(2.7.4)

where \(M^T\), \(N^T\), and \(N^E\) are the thermal moment, thermal force, and electrical force resultants, respectively. It is worth mentioning that no electrical moment is induced in the structure since the electrical loading and electrical properties are symmetrical with respect to the mid-plane. Besides, \(E_1\), \(E_2\), and \(E_3\) are stretching, coupling stretching-bending, and the bending stiffnesses, respectively, and are obtained as

\[
E_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz = h \left( E_c + \frac{E_{mc}}{k + 1} \right) \\
E_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} z E(z) dz = h^2 E_{mc} \left( \frac{1}{2k + 2} - \frac{1}{k + 2} \right)
\]
\[ E_3 = \int_{-h/2}^{h/2} z^2 E(z)dz = h^3 \left[ \frac{E_c}{12} + E_{mc} \left( \frac{0.25}{k+1} - \frac{1}{k+2} + \frac{1}{k+3} \right) \right] \]

\[ H_3 = 2 \int_{h/2}^{h+p+h/2} z^2 E_p dz = \frac{2}{3} \left( h^3_p + \frac{3}{2} h^2_p h + \frac{3}{4} h_p h^2 \right) \]

\[ N^T = \int_{-h/2}^{h/2} E(z) \alpha(z) (T - T_0) dz + 2 h_p E_p \alpha_p (T - T_0) \]

\[ N^E = 2 E_p d_p V_p \]

\[ M^T = \int_{-h/2}^{h/2} z E(z) \alpha(z) (T - T_0) dz \]  \hspace{1cm} (2.7.5)

With the absence of external applied loads, the total virtual potential energy of the beam is equal to the virtual strain energy of the beam under thermal and electrical loads, which is equal to

\[ \delta U = \int_0^L \int_{-h/2}^{h/2-h_p} \int_0^h (\sigma_x \delta \varepsilon_x + K_s \sigma_{xz} \delta \gamma_{xz} - D_z \delta \varepsilon_z) dydzdx \]  \hspace{1cm} (2.7.6)

The equilibrium equations of FGM Timoshenko beam with piezoelectric layers are obtained according to the virtual work principle [35]. Integrating Eq. (2.7.6) by part, with the consideration of Eqs. (2.7.4) and (2.7.5), result in the followings nonlinear equilibrium equations [34]

\[ \delta u_0 : \frac{d N_x}{dx} = 0 \]

\[ \delta w_0 : \frac{d Q_{xz}}{dx} + \frac{d}{dx} \left( N_x \frac{d w_0}{dx} \right) = 0 \]

\[ \delta \varphi : Q_{xz} - \frac{d M_x}{dx} = 0 \]  \hspace{1cm} (2.7.7)

Due to this integration process, natural and essential boundary conditions are obtained as

\[ N_x = 0 \text{ or } u_0 = \text{known} \]

\[ Q_{xz} + N_x \frac{d w_0}{dx} = 0 \text{ or } w_0 = \text{known} \]

\[ M_x = 0 \text{ or } \varphi = \text{known} \]  \hspace{1cm} (2.7.8)

For the sake of simplicity and generality, the following non-dimensional variables are introduced and are used in the rest of this work
\[ \xi = \frac{x}{L}, \quad U^* = \frac{u}{L}, \quad W^* = \frac{w}{h}, \quad \delta = \frac{h}{L}, \quad \mu = \frac{h_p}{h} \]

\[ f_1 = \frac{E_1 + 2E_p h_p}{E_m h}, \quad f_2 = \frac{E_2}{E_m h L}, \quad f_3 = \frac{E_3 + H_3 E_p}{E_m h L^2} \]

\[ g_1 = \frac{K_s (G_1 + 2G_p h_p)}{E_m h} \]

\[ N^{T*} = \frac{N_T}{E_m h}, \quad N^{E*} = \frac{N_E}{E_m h} \]

\[ M^{T*} = \frac{M_T}{E_m h L} \]  

(2.7.9)

where \( E_m \) is the elasticity modulus of metal constituent. Substitution of the above non-dimensional parameters into Eq. (2.7.7) and utilizing Eq. (2.7.4) give the governing equations of the beam in dimensionless forms as [34]

\[ \frac{f_1}{d^2 U^*}{d \xi^2} + f_1 \delta^2 \frac{d W^*}{d \xi} + f_2 = 0 \]  

(2.7.10a)

\[ \left\{ f_1 \left( \frac{d U^*}{d \xi} + \frac{1}{2} \delta^2 \left( \frac{d W^*}{d \xi} \right)^2 \right) + f_2 \frac{d \varphi}{d \xi} - N^{T*} - N^{E*} \right\} \delta \frac{d^2 W^*}{d \xi^2} + g_1 \left( \frac{d \varphi}{d \xi} + \delta \frac{d^2 W^*}{d \xi^2} \right) = 0 \]  

(2.7.10b)

\[ g_1 \left( \varphi + \delta \frac{d W^*}{d \xi} \right) - f_2 \left( \frac{d^2 U^*}{d \xi^2} + \delta \frac{d^2 W^*}{d \xi^2} \right) + f_3 \frac{d^2 \varphi}{d \xi^2} = 0 \]  

(2.7.10c)

Solution

At the first glance it seems to be difficult to solve Eq. (2.7.10) analytically due to the strong non-linearity and the included coupling in the ordinary differential equations. To obtain an analytical solution, first these equations are uncoupled. Based on the first equilibrium equations (2.7.7), the axial force resultant is constant through the span at each temperature. Since \( N^{T*} \) and \( N^{E*} \) are both constants, we may write [34]

\[ f_1 \left( \frac{d U^*}{d \xi} + \frac{1}{2} \delta^2 \left( \frac{d W^*}{d \xi} \right)^2 \right) + f_2 \frac{d \varphi}{d \xi} = -N^{M*} \]  

(2.7.11)

where \( N^{M*} \) is constant along the span but varies at each load step. Substitution of Eq. (2.7.11) into (2.7.10b) leads to

\[ g_1 \left( \frac{d \varphi}{d \xi} + \delta \frac{d^2 W^*}{d \xi^2} \right) + (-N^{M*} - N^{T*} - N^{E*}) \delta \frac{d^2 W^*}{d \xi^2} = 0 \]  

(2.7.12)

and by substituting Eq. (2.7.10a) into (2.7.10c) we have
\[
\left( \frac{f_3 f_1 - f_2^2}{f_1} \right) \frac{d^2 \varphi}{d \xi^2} - g_1 \left( \varphi + \delta \frac{d W^*}{d \xi} \right) = 0 \quad (2.7.13)
\]

By differentiating Eq. (2.7.12) and using Eq. (2.7.13), the following equations are obtained

\[
\left( \frac{P_1 F - P_1 P_2}{P_2} \right) \delta \frac{d^3 W^*}{d \xi^3} - P_2 \delta \frac{d W^*}{d \xi} = P_2 \varphi \quad (2.7.14a)
\]

\[
\left( \frac{P_1 F - P_1 P_2}{P_2} \right) \delta \frac{d^4 W^*}{d \xi^4} - F \delta \frac{d^2 W^*}{d \xi^2} = 0 \quad (2.7.14b)
\]

Here, we have set

\[
P_1 = \frac{f_3 f_1 - f_2^2}{f_1}, \quad P_2 = g_1, \quad F = N^{M*} + N^{T*} + N^{E*} \quad (2.7.15)
\]

As seen, Eq. (2.7.10) are changed to new decoupled equations. Since \( F - P_2 \) is negative and both \( P_1 \) and \( F \) are positive, the solution of Eq. (2.7.14b) may be written as

\[
W^* = C_1 \sin(a \xi) + C_2 \cos(a \xi) + C_3 \xi + C_4 \quad (2.7.16)
\]

in which we have set

\[
a = \sqrt{\frac{FP_2}{P_1 P_2 - P_1 F}} \quad (2.7.17)
\]

and \( C_1, C_2, C_3, \) and \( C_4 \) are constants that depends on the boundary conditions on both sides. Based on Eq. (2.7.14a), we have the following closed-form solution for \( \varphi \)

\[
\varphi = -\delta a C_1 \left( 1 - \frac{F}{P_2} \right) \cos(a \xi) + \delta a C_2 \left( 1 - \frac{F}{P_2} \right) \sin(a \xi) - \delta C_3 \quad (2.7.18)
\]

We may now considers two types of boundary conditions; simply-supported-simply-supported \((S - S)\) and clamped-clamped \((C - C)\). Mathematical expressions for these classes of edge supports are

**Clamped(C)**: \( U^* = W^* = \varphi = 0 \)

**Simply-supported(S)**: \( U^* = W^* = M^*_x = 0 \)

where

\[
M^*_x = f_2 \left( \frac{d U^*}{d \xi} + \frac{1}{2} \delta^2 \left( \frac{d W^*}{d \xi} \right)^2 \right) + f_3 \frac{d \varphi}{d \xi} - M^T* \quad (2.7.19)
\]
2.7.3 **Clamped–Clamped Boundary Conditions**

For the case of a beam with both edges clamped, both slopes and deflections vanish at both edges of the beam. Recalling Eqs. (2.7.16) and (2.7.18), we have the following system of homogeneous equations [34]

\[
\begin{align*}
C_2 + C_4 &= 0 \\
-\delta a \left(1 - \frac{F}{P_2}\right) C_1 - \delta C_3 &= 0 \\
\sin(a) C_1 + \cos(a) C_2 + C_3 + C_4 &= 0 \\
-\delta a \left(1 - \frac{F}{P_2}\right) \cos(a) C_1 + \delta a \left(1 - \frac{F}{P_2}\right) \sin(a) C_2 - \delta C_3 &= 0
\end{align*}
\]  

(2.7.20)

which result in

\[
\begin{align*}
C_1 &= \left(\frac{\cos(a) - 1}{a (P_2 - F) - P_2 \sin(a)}\right) P_2 C \\
C_2 &= C \\
C_3 &= -a (1 - \frac{F}{P_2}) \left(\frac{\cos(a) - 1}{a (P_2 - F) - P_2 \sin(a)}\right) P_2 C \\
C_4 &= -C
\end{align*}
\]  

(2.7.21)

Notice that due to the homogeneous boundary conditions for the \(C-C\) edge supports, solution of the equations is obtained as an eigen-value problem. When the constants \(C_i, i = 1, 2, 3, 4\) are inserted into the last of Eq. (2.7.20), one may reach to the following transcendental equation

\[
\sin(a) - \frac{(1 - \cos(a))^2 P_2}{(P_2 - F) a - P_2 \sin(a)} = 0
\]  

(2.7.22)

Equation (2.7.22) has to be solved with respect to the parameter \(a\). For the buckling mode, which is associated with the minimum positive root of the above equation, we have

\[
a = 2\pi
\]  

(2.7.23)

Recalling Eq. (2.7.15) along with the substitution of Eq. (2.7.23) into (2.7.17), and considering the fact that in the prebuckling state \(N^{M*} = 0\), the critical thermal force of the beam is obtained as

\[
N_{cr}^{T*} = \frac{4\pi^2 P_1 P_2}{4\pi^2 P_1 + P_2} - N^{E*}
\]  

(2.7.24)

To trace the post-buckling equilibrium path of the beam, dependency of the lateral deflection to the temperature rise should be extracted. For the first buckling mode,
\( C_1 = C_3 = 0 \) and therefore the post-buckling deflection of the beam simplifies to
\[
W^*(\xi) = C [\cos(2\pi \xi) - 1] \quad (2.7.25)
\]
where \( C \) is a constant and has to be obtained considering the immovability conditions on both edges.

Substitution of Eq. (2.7.25) into (2.7.11) and integrating with respect to \( \xi \) along the beam length, we arrive at
\[
f_1 \int_0^1 \frac{dU^*}{d\xi} d\xi + \frac{\delta^2}{2} \int_0^1 \left( \frac{dW^*}{d\xi} \right)^2 d\xi + f_2 \int_0^1 \frac{d\phi}{d\xi} d\xi = - \int_0^1 N^M^* d\xi \quad (2.7.26)
\]
Here, the first integral on the left-hand side vanishes, because of the immovable condition on both sides. The third one also vanishes due to the clamping condition, which does not accept any rotation at the edge. Substituting Eq. (2.7.25) into (2.7.26) and using Eq. (2.7.17) and considering the boundary conditions, the following equation for \( C \) is obtained
\[
C = \sqrt{N^T^* + N^E^* - \frac{4\pi^2 P_1 P_2}{4\pi^2 P_1 + P_2}} \quad (2.7.27)
\]
It should be emphasized that parameter \( C \) is associated with deflection of the beam. Substituting \( \xi = 1/2 \) into Eq. (2.7.25) reaches us to
\[
C = -\frac{1}{2} W^*(1/2) \quad (2.7.28)
\]
Therefore, Eq. (2.7.27) presents the temperature-deflection path of the beam. In order to find the axial force \( N_x^* \) as a temperature rise function, definition of \( C \) from Eq. (2.7.27) is substituted into the result of Eq. (2.7.26) which gives us
\[
N^M^* = - f_1 \pi^2 \delta^2 C^2 \quad (2.7.29)
\]
It is seen that in the pre-buckling regime, which is free of lateral deflection, \( N^M^* = 0 \). Recalling Eq. (2.7.15), along with the substitution of Eq. (2.7.27) into (2.7.29), result in the total axial force resultants as
\[
N_x^* = -N^T^* - N^E^* \quad \text{in prebuckling regime}
\]
\[
N^*_x = -\frac{4\pi^2 P_1 P_2}{4\pi^2 P_1 + P_2} \quad \text{in postbuckling regime} \quad (2.7.30)
\]
As seen from the above equations, in both pre- and post-buckling regimes, the axial force is independent of beam position. Besides, in pre-buckling state the axial force
varies linearly with respect to temperature rise while in post-buckling regime it is independent of temperature rise parameter.

### 2.7.4 Simply Supported-Simply Supported Boundary Conditions

For a beam which is simply-supported at both edges, deflection and bending moment should be vanished at both ends. Therefore, the following system of equations is obtained in this case [34]

\[
C_2 + C_4 = 0 \\
\sin(a)C_1 + \cos(a)C_2 + C_3 + C_4 = 0 \\
\delta F C_2 = M^T + \frac{f_2}{f_1} N^M \\
\delta F \left( \sin(a)C_1 + \cos(a)C_2 \right) = M^T + \frac{f_2}{f_1} N^M 
\]  

(2.7.31)

The above equations are solved for the coefficients and written in terms of a constant \( C \)

\[
C_1 = \tan \left( \frac{a}{2} \right) C \\
C_2 = C \\
C_3 = 0 \\
C_4 = -C 
\]  

(2.7.32)

where

\[
C = \frac{M^T + \frac{f_2}{f_1} N^M}{\delta F} 
\]  

(2.7.33)

The relation between \( C \) and \( a \) from Eqs. (2.7.15), (2.7.17), and (2.7.33) can be derived as

\[
C = \frac{a^2 P_1 + P_2}{\delta a^2 P_1 P_2} \left( M^T + \frac{f_2}{f_1} \left( T^T + N^E \right) \right) + \frac{f_2}{\delta f_1} 
\]  

(2.7.34)

With the aid of Eqs. (2.7.16) and (2.7.32), substituting them into Eq. (2.7.11) and integrating with respect to \( \xi \), we have

\[
f_1 \int_0^1 \frac{dU^*}{d\xi} d\xi + f_1 \frac{\delta^2}{2} \int_0^1 \left( \frac{dW^*}{d\xi} \right)^2 d\xi + f_2 \int_0^1 \frac{d\varphi}{d\xi} d\xi + f_2 \int_0^1 N^M d\xi = - \int_0^1 N^M d\xi 
\]  

(2.7.35)
Because of the immovable boundary conditions for $U^*$, the first integral in Eq. (2.7.35) vanishes. The second and third integrals may be derived as

\[
\int_0^1 \left( \frac{dW^*}{d\xi} \right)^2 d\xi = C^2 \left( \frac{a^2 - a \sin(a)}{1 + \cos(a)} \right)
\]

\[
\int_0^1 \frac{d\varphi}{d\xi} d\xi = \frac{2aC\delta}{1 + \frac{P_1}{P_2}a^2} \left( \frac{1 - \cos(a)}{\sin(a)} \right)
\]

(2.7.36)

Substitution of Eqs. (2.7.36) into (2.7.35), performing the proper simplifications on the right-hand side integral of Eq. (2.7.35) according to Eqs. (2.7.33) and (2.7.34) along with defining the parameter $B$ as

\[
B = \left( \frac{MT^* - \frac{f_2}{f_1} (N^T + N^E)}{\delta a^2 P_1} \right) \left( 1 + \frac{P_1}{P_2}a^2 \right) + \frac{f_2}{\delta f_1}
\]

(2.7.37)

transforms Eq. (2.7.35) into the form

\[
-a^3 \frac{N^T + N^E}{P_1} \left( 1 + \frac{P_1}{P_2}a^2 \right) \left( 1 + \cos(a) \right) + \frac{f_1 B^2}{2P_2} \left( 1 + \frac{P_1}{P_2}a^2 \right)^3 (a - \sin(a))
\]

\[+ \frac{f_2}{2f_1 P_2} a^4 \left( 1 + \frac{P_1}{P_2}a^2 \right) (a - \sin(a)) + \frac{f_2 B}{P_2} a^2 \left( 1 + \frac{P_1}{P_2}a^2 \right) (a - \sin(a))
\]

\[+ \frac{2f_2 B}{P_2} a^2 \left( 1 + \frac{P_1}{P_2} \right) \sin(a) + \frac{2f_2^2}{f_1 P_2} a^4 \sin(a) + a^5 (1 + \cos(a)) = 0
\]

(2.7.38)

Equation (2.7.38) exhibits the relation between load parameters and $a$ in which $a \neq 0, 2(m - 1)\pi$ and $m = 1, 2, \ldots$. It is seen that depending on the load level, $a$ may take on multiple values for a given load. Furthermore, unlike the case of an FGM beam with both ends clamped, in which the load-deflection is presented in a closed-form solution, here the deflection of the beam at each load step should be extracted from a transcendental equation.

Notice that for the especial cases of homogeneity, i.e. when power law index is equal to zero or infinity, both stretching-bending coupling stiffness and thermal moment vanish. In this case the response of the beam is obtained from an eigen-value problem, since the system of equations (2.7.31) reduces to a homogeneous one. Same as the process developed for the case of an FGM beam with both edges clamped, it is seen that the characteristic equation of the beam is equal to

\[
\sin(a) = 0
\]

(2.7.39)
which has the minimum value \( a = \pi \). In such case, all constants \( C_2, C_3, \) and \( C_4 \) are equal to zero and the deflection equation of the beam is obtained as

\[
W^*(\xi) = C \sin(\pi \xi) \quad (2.7.40)
\]

Similar to the process used for the \( C - C \) type of edge supports, the constant \( C \) as a function of load parameter is obtained as

\[
C = \sqrt{\frac{N_T^* + N_E^* - \frac{\pi^2 P_1 P_2}{\pi^2 P_1 + P_2}}{0.25 f_1 \delta^2 \pi^2}} \quad (2.7.41)
\]

in which based on Eq. (2.7.40) we have

\[
C = W^*(1/2) \quad (2.7.42)
\]

and the total axial load, as a function of temperature parameter, is equal to

\[
N_x^* = -N_T^* - N_E^* \quad \text{in prebuckling regime}
\]

\[
N_x^* = -\frac{\pi^2 P_1 P_2}{\pi^2 P_1 + P_2} \quad \text{in postbuckling regime} \quad (2.7.43)
\]

Since the bifurcation point is the junction of primary and secondary equilibrium paths, the critical buckling thermal load is obtained when \( N_x^* \) is removed between Eq. (2.7.43)

\[
N_{cr}^T = \frac{\pi^2 P_1 P_2}{P_1 \pi^2 + P_2} - N_E^* \quad (2.7.44)
\]

2.7.5 Results and Discussion

In this part, a functionally graded material beam made of \( SUS304 \) as metal constituent and \( Si_3N_4 \) as ceramic constituent along with \( G1195N \) as piezoelectric layers are considered. Beam is under uniform temperature rise loading. The material properties of constituents are given in Table 2.5. While the presented method is obtained by analytical methods, for the sake of comparison, the critical buckling temperature difference of a beam with two piezoelectric layers is compared with the previous results of References [23, 24, 33]. A clamped beam made of Alumina with two bonded piezoelectric layers is assumed to be under uniform temperature rise loading. Piezoelectric layers are made of \( G1195N \). Properties of the host layer are \( E = 380 \text{Gpa}, \alpha = 7.4 \times 10^{-6} \text{K}^{-1} \) and \( \nu = 0.3 \). Five cases of applied actuator voltages are studied. It is seen that the results of this study are less than those reported by
Table 2.5  Material properties for $SU S304$, $Si_3N_4$ and $G1195N$ [14, 33]

<table>
<thead>
<tr>
<th>Material</th>
<th>Property</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU S304$</td>
<td>$\alpha_m$[K$^{-1}$]</td>
<td>$15.3210 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>$E_m$[Pa]</td>
<td>$207.79 \times 10^9$</td>
</tr>
<tr>
<td></td>
<td>$\nu$</td>
<td>0.28</td>
</tr>
<tr>
<td>$Si_3N_4$</td>
<td>$\alpha_c$[K$^{-1}$]</td>
<td>$7.4746 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>$E_c$[Pa]</td>
<td>$322.27 \times 10^9$</td>
</tr>
<tr>
<td></td>
<td>$\nu$</td>
<td>0.28</td>
</tr>
<tr>
<td>$G1195N$</td>
<td>$\alpha_p$[K$^{-1}$]</td>
<td>$0.9 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>$E_p$[Pa]</td>
<td>$63 \times 10^9$</td>
</tr>
<tr>
<td></td>
<td>$d_p$[m/V]</td>
<td>$2.54 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>$\nu_p$</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Table 2.6  A comparison on the effect of applied actuator voltage on the $\Delta T_{cr}$[K] for an Alumina beam with two smart layers [34]. Geometry of the beam are $h_p = 0.001$ m, $h = 0.01$ m, $L = 0.25$ m

<table>
<thead>
<tr>
<th>Source</th>
<th>$V_p = -500$ V</th>
<th>$V_p = -200$ V</th>
<th>$V_p = 0$</th>
<th>$V_p = +200$ V</th>
<th>$V_p = +500$ V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kiani et al. [23]</td>
<td>797.744</td>
<td>797.403</td>
<td>797.175</td>
<td>796.948</td>
<td>796.606</td>
</tr>
<tr>
<td>Kiani et al. [24]</td>
<td>783.611</td>
<td>783.270</td>
<td>783.042</td>
<td>782.814</td>
<td>782.473</td>
</tr>
<tr>
<td>Fu et al. [33]</td>
<td>$-$</td>
<td>797.403</td>
<td>797.180</td>
<td>796.950</td>
<td>$-$</td>
</tr>
<tr>
<td>Present study</td>
<td>780.645</td>
<td>780.304</td>
<td>780.077</td>
<td>779.850</td>
<td>779.510</td>
</tr>
</tbody>
</table>

Kiani et al. [23] and Fu et al. [33], which is expected since their results are developed within the framework of Euler beam theory. Relative difference between these results and those of [23] and [33] is at most 2.5 percent. It is worth mentioning that results of this study has a smaller difference with the results of Kiani et al. [24] which is also developed based on the Timoshenko beam theory. The small deviation arises from the assumption considered in [24], where the effect of piezoelectric layer thickness in thermal force resultant calculation is neglected. As seen, the assumption of Kiani et al. [24] is valid and relative differences is at most 0.4 percent (Table 2.6).

Numerical results presented in Figs. 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16 cover the case of an FGM beam without piezoelectric layers and Figs. 2.17 and 2.18 are associated with the case of hybrid FGM beams [34].

In Fig. 2.10 the critical buckling temperature difference of FGM beam without piezoelectric layers with respect to slenderness ratio and power law index for the $C-C$ case is presented. As expected, higher $L/h$ ratio results into lower $\Delta T_{cr}$, which is due to the lower flexural rigidity of the beam. For the constituents of this study, as the power law index increases the critical buckling temperature increases permanently, which is due to the less thermal expansion coefficient of ceramic constituent.
The effect of power law index on temperature-deflection equilibrium path of the FGM beams with various power law indices are given in Fig. 2.11. As the graphs exhibit, the load-deflection path of FGM beams for each value of power law index is of the bifurcation-type buckling. For the constituents of this study, behavior of the FGM beam stands between two associated homogeneous cases. For a prescribed temperature parameter, an increase in the power law index results in less deflection.

The effect of slenderness ratio for an FGM beam is plotted in Fig. 2.12. As expected, the thicker beams result into higher critical buckling temperatures. For a prescribed temperature parameter, the higher values of $L/h$ ratio produce higher midpoint deflection of the beam.

For the studied cases of Fig. 2.12, the total in-plane load parameter as a function of temperature rise is plotted in Fig. 2.13. It is seen that similar to Fig. 2.12, the
bifurcation temperatures are detectable from the force-deflection paths. As theoretically proved, in the pre-buckling state, the total in-plane force varies linearly with respect to the uniform temperature rise parameter, while it remains independent of temperature rise in post-buckling equilibrium path.

For the case of a linearly graded FGM beam with both edges simply-supported, the temperature-deflection equilibrium path for various slenderness ratios are plotted in Fig. 2.14. As it is seen, the temperature-deflection equilibrium path of FGM beams are not of the bifurcation-type buckling. Through the studied range, the load-deflection path of FGM beam with both ends simply-supported is unique and stable. A comparison of Figs. 2.12 and 2.14 reveals that the mid-plane deflection of the $S - S$ beams is larger than that of the $C - C$ beams, since in the latter case beam remains flat, while the $S - S$ beam initially starts lateral deflection. Besides, after
the bifurcation point in which the $C - C$ beam starts to have lateral deflection, the ability of edges to sustain the moment at boundaries results in less lateral deflection in comparison with the $S - S$ case.

For various values of the power law index, the load-deflection equilibrium path of FGM beams with both edges simply-supported is presented in Fig. 2.15. It is worth mentioning that, except for the two especial cases of homogeneity i.e. $k = 0, \infty$, the load-deflection path of the beam is unique and stable. On the other hand, the response for $k = 0, \infty$ cases is of the primary-secondary equilibrium path. Due to the unsymmetrical distribution of properties with respect to the mid-plane of simply-supported FGM beam, lateral deflection occurs at the onset of thermal loading. It is interesting to note that, unlike the case of $C - C$ beams, the response of $S - S$ FGM beam does not stand between the two associated homogeneous cases.
For the studied cases of Fig. 2.15, in Fig. 2.16 the total in-plane force is plotted as a function of uniform temperature rise parameter. As seen, unlike the $C - C$ case of boundary conditions, there is no branching point in the curves. The total force as a function of temperature rise is completely smooth.

To investigate the effect of applied actuator voltage on the post-buckling equilibrium path of $C - C$ and $S - S$ FGM beams, temperature-deflection curves are plotted in Figs. 2.17 and 2.18 for linearly graded $C - C$ and $S - S$ FGM beams, respectively. As seen, the effect of applied actuator voltage to the smart layers is negligible. For the piezoelectric layers used in this study, in comparison to the grounding case condition, applying the negative magnitude of $V_p$ postpones the bifurcation point of the $C - C$ case. Consequently, within the post-buckling range, applying a prescribed temperature rise results in less deflection. The latter case is true for the $S - S$ case too. However, as pointed-out, the effect of the applied actuator voltage is hardly distinguishable.

2.8 Vibration of Thermo-Electrically Post-buckled FGM Beams

2.8.1 Introduction

The ability of piezoelectric materials to surpass the vibrational motion, shape control, and delay the buckling is reported in literature. This ability is documented by some valuable books on the subject, such as Tzou’s one [36], or those reported by Yang [37, 38].
Fig. 2.17 Temperature-deflection equilibrium path of a hybrid $C - C$ FGM beam ($\mu = 0.01$)

For the case when beam has a rectangular cross-shape, Yang and his co-authors [39–41] have analyzed the three-dimensional behavior of electroelastic beams. In these works, extensional and transversal motions are studied. Double power-series solutions are developed in thickness and width directions. Wang and Queck [42, 43] analyzed the free vibration problem of a beam integrated with piezoelectric layer(s) based on the classical beam theory. Both open and closed-circuit electrical states are examined and the effect of electrical boundary conditions on free vibration motion is investigated. Most recently, Ke et al. [44] and Ke and Wang [45] analyzed the free vibration problem of a piezoelectric beam including Eringen’s nonlocal effects in thermo-electro-mechanical field.
Pradhan and Murmu have analyzed the free vibration of FGM sandwich beam including thickness variations in thermal field [46]. Based on the first order shear deformation beam theory, Xiang and Yang [47] examined the transverse heat conduction effects on small free vibrations of symmetrically laminated FGM beams. Using an improved perturbation technique and based on a higher-order shear deformation theory, Xia and Shen [48] investigated the small and large-amplitude vibration analysis of compressively and thermally post-buckled sandwich plates with functionally graded material (FGM) face sheets in thermal environments. The results of this paper show that as the volume fraction index increases, the fundamental frequency increases in the pre-buckling region, while in the post-buckling regime the behavior is vice versa. The free vibration analysis of an elastic rod around its post-buckled equilibrium state is addressed in the work of Neukirch et al. [49]. They employed both analytical and numerical schemes to conclude the results, before and after the buckling point.

The FGM structures when are incorporated with the piezoelectric layers are called hybrid FGMs. Vibrations of a Timoshenko beam with surface bonded piezoelectric layers in both pre/post-buckling states is studied by Li et al. [22]. In this work shooting method is implemented to solve the post-buckling and free vibration problems of a hybrid FGM beam, clamped at both ends. Recently, the free vibration of a clamped hybrid FGM beam under in-plane thermal loading is investigated by Fu et al. [33]. In this work, a fully analytical method is developed to analyze the post-buckling equilibrium path and large amplitude vibrations of the beam.

Researches on the analysis of FGPM structures, FGM structures and piezoelectric smart layers combined together, under thermo-electro mechanical loadings are limited in number. Besides, among these investigations, most of them analyze the geometrically linear response of the graded actuators. In many studies, variation of material properties in a specific direction is assumed to follow a prescribed distributed function. Huang et al. [50] developed a solution based on the two-dimensional theory of elasticity for the response of an FGPM beam with arbitrary through-the-thickness distribution of material properties. Introducing a stress function and an electrical displacement function, the equilibrium and Maxwell electrical equations are satisfied. Solution of stress function and electrical displacement function, however, are assumed to be quadratic through the span. Shi [51], Shi and Chen [52], and Liu and Shi [53] performed a series of investigations on orthotropic FGPM beams. In reference [51], Shi reported various analytical solutions for a cantilever beam where density varies as a cubic polynomial across the thickness. Shi and Chen in [52] consider the case of quadratic and cubic variations of elastic property and density of the beam across the thickness. With consideration of linearly graded piezoelectric parameter through-the-thickness, Liu and Shi [53] obtained the response of an FGPM beam based on the definition of stress function.

Kruusing [54] obtained an analytical solution for a cantilever Euler–Bernoulli FGPM beam under the action of a shear force at the tip. When an FGPM beam is subjected to electrical or electro-thermal loading, Joshi et al. [55, 56] developed the bending response of the structure. It is concluded that the behavior of an FGPM beam is largely affected by the composition rule of the constituents. Based on a
layer-wise formulation, Lee [57, 58] developed a finite elements method to investigate the response of the FGPM beam subjected to the combined action of thermal and electrical loads. Yang and Xiang [59] performed a comprehensive study on the static, dynamic, and free vibration behavior of the FGPM Timoshenko beams under the action of thermal, mechanical, and electrical excitations. In this work, three mechanical equations and the Maxwell-type electrical equation are solved simultaneously, employing the differential quadrature (DQ) method. Employing the classical, first order, and third order shear deformation beam theories, Komeili et al. [60] developed the finite elements and finite Fourier formulations to study the bending response of a monomorph FGPM beam under various types of loading. Dynamic response of the beam employing the Galerkin-based finite elements formulation is reported by Doroushi et al. [61] based on the third order shear deformation theory.

The present section implements the Ritz finite elements method to discrete the governing equations associated with the post-buckling of FGPM beams [62]. Furthermore, the vibration behavior of the beam in pre- and post-buckling regimes is analyzed. The established equations are nonlinear due to the presence of von-Karman’s geometrical non-linearity in strain components. The solution is divided into static and dynamic responses. Static response of the beam is the study of postbuckling equilibrium path under the in-plane thermoelectrical loading. The Newton–Raphson method is implemented to solve the nonlinear system of equations, iteratively. The dynamic response is the study of small free vibration in thermoelectrically pre/post-buckled states via the linear eigenvalue analysis. The variation of fundamental frequency in thermal field reveals that the behavior of a structure, depending on boundary conditions and the applied loads, may be of the bifurcation or critical point responses. It is shown that applying the appropriate external voltage, the buckling phenomenon of an FGPM structure is controlled and postponed within a noticeable range.

### 2.8.2 Governing Equations

Consider a beam made of functionally graded piezoelectric materials (FGPMs) of length $L$, width $b$, and thickness $h$. The beam is subjected to a mechanical distributed load $q$, temperature rise $\Delta T$, and applied voltage $V_0$, as shown in Fig. 2.19.

It is considered that the material properties vary continuously across the thickness direction according to the power law distribution given by Eq. (2.2.8). In this section, the Timoshenko beam theory is used with the following displacement field

$$
U(X, Z, t) = U_0(X, t) + Z \Psi(X, t)
$$

$$
W(X, Z, t) = W_0(X, t)
$$

(2.8.1)

where $(U_0, W_0)$ are the displacement components of a point on the mid-plane of the beam along axial and thickness coordinates, respectively, and $\Psi$ stands as the rotation of the cross-section.
Wang and Queck [63] performed a two-dimensional elasticity solution to obtain the distribution of electrical potential across the thickness, when beam is subjected to a constant uniform mechanical load. Results of this study reveals that for the case where a simply-supported beam is closed circuit at both top and bottom surfaces, the analytical distribution of electrical potential is obtained in a parabolic form, where the peak point stands at the middle. This type of distribution, also, has been used for the other types of boundary conditions, loading, and material property distribution. Also, some authors used the trigonometric functions along the thickness direction to satisfy the closed-circuit electrical conditions at the top and bottom layers [59, 64, 65].

In this part, considering both reverse and direct effects of a piezoelectric layer, the electric potential $V$ is assumed to obey the following distribution [44, 66]

$$V(X, Z, t) = \cos(\beta Z)\Phi(X, t) + \frac{V_0}{h}Z$$

(2.8.2)

where $\beta = \pi/h$ and $\Phi(X, t)$ is spatial function of the electric potential and the second term denotes the external electric voltage applied to beam’s electrodes.

The constitutive equations for the FGPM beam under thermo-electro-mechanical loads may be expressed as follow [61]

$$\sigma_X = Q_{11}\varepsilon_X - Q_{11}\alpha_1\Delta T - e_{31}E_Z$$
$$\tau_{XZ} = Q_{55}\gamma_{XZ} - e_{15}E_X$$
$$D_X = e_{15}\gamma_{XZ} + k_{11}E_X$$
$$D_Z = e_{31}\varepsilon_X + k_{33}E_Z + p_3\Delta T$$

(2.8.3)

where $\varepsilon_X$, $\gamma_{XZ}$, $\sigma_X$, $\tau_{XZ}$, $D_i$, and $E_i$ represent the axial strain, shear strain, axial stress, shear stress, dielectric displacements, and the corresponding electric field components, respectively. Here, $e_{ij}$, $k_{ij}$, $\alpha_1$, $p_3$ are the piezoelectric, dielectric, thermal expansion, and pyroelectric coefficients, respectively, and $Q_{11}$ and $Q_{55}$ are the elastic stiffness coefficients.
The von-Karman type nonlinear strain-displacement relations can be obtained using Eq. (2.8.1) as

\[
\varepsilon_x = U_{0,x} + \frac{1}{2} W_{0,x}^2 + Z \Psi_x \\
\gamma_{XZ} = W_{0,x} + \Psi
\]  

(2.8.4)

Since the electric field vector is negative gradient of the total potential function, using Eq. (2.8.2), the electric field components are

\[
E_x = -V_x = -\cos(\beta Z) \Phi_x \\
E_Z = -V_Z = \beta \sin(\beta Z) \Phi - E_0
\]  

(2.8.5)

where we have set \( E_0 = \frac{V_0}{h} \).

The governing equations may be derived on the basis of Hamilton’s principle. According to this principle, equations of motion are obtained when the following equality holds

\[
\delta \int (K - H + R) dt = 0
\]  

(2.8.6)

where the variation of the electric enthalpy \( \delta H \) and the variation of the kinetic energy \( \delta K \) are, respectively [59]

\[
\delta H = b \int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} (\sigma_x \delta \varepsilon_x + K_s \tau_{XZ} \delta \gamma_{XZ} - D_x \delta E_x - D_Z \delta E_Z) dZ dX
\]  

(2.8.7)

\[
\delta K = b \int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} \rho (U_{,t} \delta U_{,t} + W_{,t} \delta W_{,t}) dZ dX
\]  

(2.8.8)

In definition of \( \delta T \), \( K_s \) is the shear correction factor and is taken as \( K_s = \pi^2 / 12 \).

The virtual work \( \delta R \) due to the out-of-plane mechanical load \( q \) is

\[
\delta R = b \int_0^L q \delta W dX
\]  

(2.8.9)

Using Eqs. (2.8.3), (2.8.4), and (2.8.5), the stress resultants of the Timoshenko beam theory are

\[
N_x = A_{11}(U_{0,x} + \frac{1}{2} W_{0,x}^2) + B_{11} \Psi_x - N_{X}^T - A_{31}^e \Phi + D_{31}^e E_0 \\
M_x = B_{11}(U_{0,x} + \frac{1}{2} W_{0,x}^2) + D_{11} \Psi_x - M_{X}^T - B_{31}^e \Phi + E_{31}^e E_0 \\
Q_x = K_s A_{55}(W_{0,x} + \Psi) - K_s D_{15}^e \Phi_x
\]  

(2.8.10)
where $N_X^T$ and $M_X^T$ are the thermal force and moment resultants that are defined as

$$ (N_X^T, M_X^T) = \int_{-\frac{h}{2}}^{+\frac{h}{2}} Q_{11} \alpha_1 \Delta T (1, Z) dZ $$

(2.8.11)

Other quantities that are not specified, are given in [61].

The inertia terms are defined as

$$ (I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \rho (1, Z, Z^2) dZ $$

(2.8.12)

For the sake of generality and simplicity, the following non-dimensional parameters [59] are introduced and used in the rest of this work

$$ x = \frac{X}{L}, \quad \Phi_0 = \sqrt{\left( \frac{A_{11}}{F_{33}^e} \right)}, \quad u = \frac{U_0}{h}, \quad w = \frac{W_0}{h}, $$

$$ \psi = \Psi, \quad \phi = \frac{\Phi}{\Phi_0}, \quad \lambda_q = \frac{qL^2}{A_{11}h}, \quad \lambda_T = \frac{D_Z^T}{F_{33}^e \Phi_0}, $$

$$ \lambda_V = \frac{H_{33}^e E_0}{F_{33}^e \Phi_0}, \quad \lambda_{VT} = \frac{(D_{31}^e E_0 - N_X^T)L}{A_{11}h}, \quad \lambda_{VT} = \frac{(E_{31}^e E_0 - M_X^T)L}{D_{11}} $$

(2.8.13)

where $D_Z^T = \int_{-\frac{h}{2}}^{+\frac{h}{2}} p_3 \Delta T \beta \sin(\beta Z) dZ$, and the quantities that are not introduced are given in [61].

Substituting Eqs. (2.8.4) and (2.8.5) into Eq. (2.8.6), then integrating in thickness direction with consideration of Eq. (2.8.10), and applying the fundamental lemma of calculus, the weak-formulation of the governing equations in dimensionless form are obtained as [62]

$$ \int_0^1 \left[ u_{,x} + \frac{1}{2} \gamma_{12} w_{,x}^2 + \gamma_{13} \psi_{,x} - \gamma_{14} \phi + \lambda_{VT} \right] \delta u_{,x} dx = $$

$$ - \int_0^1 \left[ \eta_{11} u_{,tt} + \eta_{13} \psi_{,tt} \right] \delta u dx $$

(2.8.14)

$$ \int_0^1 \left[ \left[ u_{,x} + \frac{1}{2} \gamma_{12} w_{,x}^2 + \gamma_{13} \psi_{,x} - \gamma_{14} \phi + \lambda_{VT} \right] \gamma_{12} w_{,x} ight. $$

$$ + \gamma_{22} w_{,x} + \frac{\gamma_{22}}{\gamma_{12}} \psi - \gamma_{24} \phi_{,x} \left] \delta w_{,x} dx - \int_0^1 \lambda_q \delta w dx = $$

$$ - \int_0^1 \left[ \eta_{22} w_{,tt} \right] \delta w dx $$

(2.8.15)
\[ \int_0^1 \left[ \left\{ \gamma_{31} \left( u_{,x} + \frac{1}{2} \gamma_{12} w_{,x}^2 \right) + \psi_{,x} - \gamma_{34} \delta \phi + \lambda_{VT} \right\} \right] \delta \psi_{,x} \]

\[ + \left\{ \gamma_{32} w_{,x} + \frac{\gamma_{32}}{\gamma_{12}} \psi_{,x} + \left( \gamma_{34} - \gamma_{34} \right) \phi_{,x} \right\} \delta \psi \right] dx =

\[-\int_0^1 \left( \eta_{31} u_{,tt} + \eta_{33} \psi_{,tt} \right) \delta \psi dx \] (2.8.16)

\[ \int_0^1 \left[ \left\{ \gamma_{42} w_{,x} + \gamma_{43} \psi + \gamma_{44} \phi_{,x} \right\} \right] \delta \phi_{,x} +

\left\{ \gamma_{41} \left( u_{,x} + \frac{1}{2} \gamma_{12} w_{,x}^2 \right) - \left( \gamma_{43} - \frac{\gamma_{42}}{\gamma_{12}} \right) \psi_{,x} + \phi + \lambda_T - \lambda_V \right\} \delta \phi \right] dx = 0 \] (2.8.17)

where the constants appeared in the above equations are defined in [61].

Using integration by parts in Eqs. (2.8.14)–(2.8.17), the boundary conditions become

\[ N_X = 0 \quad \text{or} \quad u = 0, \]
\[ \gamma_{12} N_X w_{,x} + Q_X = 0 \quad \text{or} \quad w = 0, \]
\[ M_X = 0 \quad \text{or} \quad \psi = 0, \]
\[ \gamma_{42} w_{,x} + \gamma_{43} \psi + \gamma_{44} \phi_{,x} = 0 \quad \text{or} \quad \phi = 0, \] (2.8.18)

where the latter one is the electrical boundary condition, and the first three are the mechanical ones. It is noted that in the solution process of this work, the natural electrical boundary condition is considered for each of the edge supports.

In the stability analysis of an FGPM beam, the boundary conditions may be assumed to be immovable simply supported or clamped. Mathematical expressions for each of these edges are

Simply-supported \( (S) \) : \( u = w = M_X = 0 \)
Clamped \( (C) \) : \( u = w = \psi = 0 \) (2.8.19)

2.8.3 Finite Elements Model

The Ritz-based finite element method is used to solve the weak forms of the governing equations. The variables are approximated as [62, 67]
$$u(x, t) = \sum_{j=1}^{l} u_j^e(t) \Psi_1^j(x)$$

$$w(x, t) = \sum_{j=1}^{m} w_j^e(t) \Psi_2^j(x)$$

$$\psi(x, t) = \sum_{j=1}^{n} \psi_j^e(t) \Psi_3^j(x)$$

$$\phi(x, t) = \sum_{j=1}^{p} \phi_j^e(t) \Psi_4^j(x)$$

(2.8.20)

where \( \Psi_i^\alpha(x) (\alpha = 1, 2, 3, 4) \) are the Lagrange interpolation functions of degree \((l - 1), (m - 1), (n - 1), \text{ and } (p - 1)\), respectively. Using Eq. (2.8.20), the virtual displacements are

$$\delta u = \Psi_1^1, \quad \delta w = \Psi_2^2, \quad \delta \psi = \Psi_3^3, \quad \delta \phi = \Psi_4^4$$

(2.8.21)

In this work, the quadratic interpolation functions are used to approximate the variables in the elements. Substitution of Eqs. (2.8.20) and (2.8.21) into Eqs. (2.8.14)–(2.8.17), yield the following finite element model [62]

$$\sum_{j=1}^{l} M_{ij}^{11}(u_j^e)_{,tt} + \sum_{j=1}^{m} M_{ij}^{12}(w_j^e)_{,tt} + \sum_{j=1}^{n} M_{ij}^{13}(\psi_j^e)_{,tt} + \sum_{j=1}^{p} M_{ij}^{14}(\phi_j^e)_{,tt} = F_i^1, \quad (i = 1, \ldots, l)$$

(2.8.22)

$$\sum_{j=1}^{l} K_{ij}^{11} u_j^e + \sum_{j=1}^{m} K_{ij}^{12} w_j^e + \sum_{j=1}^{n} K_{ij}^{13} \psi_j^e + \sum_{j=1}^{p} K_{ij}^{14} \phi_j^e = F_i^1, \quad (i = 1, \ldots, m)$$

(2.8.23)

$$\sum_{j=1}^{l} M_{ij}^{21}(u_j^e)_{,tt} + \sum_{j=1}^{m} M_{ij}^{22}(w_j^e)_{,tt} + \sum_{j=1}^{n} M_{ij}^{23}(\psi_j^e)_{,tt} + \sum_{j=1}^{p} M_{ij}^{24}(\phi_j^e)_{,tt} = F_i^2, \quad (i = 1, \ldots, l)$$

(2.8.24)
\[ \sum_{j=1}^{l} M_{ij}^{41} (u_j^e)_{,tt} + \sum_{j=1}^{m} M_{ij}^{42} (w_j^e)_{,tt} + \sum_{j=1}^{n} M_{ij}^{43} (\psi_j^e)_{,tt} + \sum_{j=1}^{p} M_{ij}^{44} (\phi_j^e)_{,tt} + \]
\[ \sum_{j=1}^{l} K_{ij}^{41} u_j^e + \sum_{j=1}^{m} K_{ij}^{42} w_j^e + \sum_{j=1}^{n} K_{ij}^{43} \psi_j^e + \sum_{j=1}^{p} K_{ij}^{44} \phi_j^e = F_4^i, \quad (i = 1, \ldots, p) \] (2.8.25)

Definitions of the elements of \( M_{ij}, K_{ij}, \) and \( F_i \) are given in [61].

The element equations (2.8.22)–(2.8.25) can be expressed in a compact form as

\[ [M] \{ \ddot{\Delta} \} + ([K_L] + [K_{NL1}] + [K_{NL2}]) \{ \Delta \} = \{ F_m \} + \{ F_e \} + \{ F_T \} \] (2.8.26)

where \([M]\) is the matrix of inertia, and \([K_L], [K_{NL1}], \) and \([K_{NL2}]\) are the linear, first order nonlinear, and second order nonlinear stiffness matrices, respectively, and \([F_m], [F_e], \) and \([F_T]\) are the mechanical, electrical, and thermal force vectors, respectively. Besides \([\Delta] = \{ [u], [\phi], [w], [\Psi] \}^T\), is the matrix of nodal values.

To study the vibration of a beam in pre/post-buckling states, the solution of the governing equation (2.8.26) is assumed as [68]

\[ \{ \Delta \} = \{ \Delta_s \} + \{ \Delta_d \} \] (2.8.27)

where \([\Delta_s]\) is the time-independent particular solution denoting the large displacements and is implemented to study the pre-buckling and post-buckling regimes of the beam. Besides, \([\Delta_d]\) is the time-dependent solution with small magnitude which is used to study the free vibration analysis of a beam in the pre/post-buckling configurations.

Substituting Eq. (2.8.27) into the finite element equation (2.8.26), results to the following set of equations [62]

\[ ([K_L] + [K_{NL1}] + [K_{NL2}]) \{ \Delta_s \} = \{ F_m \} + \{ F_e \} + \{ F_T \} \] (2.8.28)

\[ [M] \{ \ddot{\Delta}_d \} + (2[K_L] + 3[K_{NL1}] + 4[K_{NL2}]) \{ \Delta_d \} = \{ 0 \} \] (2.8.29)

Equation (2.8.28) is for the post-buckling analysis, and Eq. (2.8.29) is associated with the vibration analysis of the buckled structure.

Due to the nonlinear effects in stiffness matrices of the above equations, an iterative method has to be used for each load step. Two commonly-used iterative schemes are the Picard iteration procedure and the Newton–Raphson method. The details of these methods are available in [69]. Both direct iteration and Newton–Raphson methods are examined to solve the nonlinear finite element equation (2.8.28). It is important to note that for the cases in which there exists a rapid change in the graph trend of load-deflection path, the direct iterative procedure (Picard method) does not converge within the reasonable iteration steps. This feature occurs due to dependency of the solutions to converged magnitudes of the previous load step in each load increment.
Table 2.7  Thermo-electro-mechanical properties of PZT-4 and PZT-5H [61]

<table>
<thead>
<tr>
<th></th>
<th>PZT − 4</th>
<th>PZT − 5H</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{11}$[GPa]</td>
<td>81.3</td>
<td>60.6</td>
</tr>
<tr>
<td>$Q_{55}$[GPa]</td>
<td>25.6</td>
<td>23.0</td>
</tr>
<tr>
<td>$e_{31}$[Cm$^{-2}$]</td>
<td>−10.0</td>
<td>−16.604</td>
</tr>
<tr>
<td>$e_{15}$[Cm$^{-2}$]</td>
<td>40.3248</td>
<td>44.9046</td>
</tr>
<tr>
<td>$k_{11}$[C$^2$m$^{-2}$N$^{-1}$]</td>
<td>$0.6712 \times 10^{-8}$</td>
<td>$1.5027 \times 10^{-8}$</td>
</tr>
<tr>
<td>$k_{33}$[C$^2$m$^{-2}$N$^{-1}$]</td>
<td>$1.0275 \times 10^{-8}$</td>
<td>$2.554 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\alpha_1$[K$^{-1}$]</td>
<td>$2 \times 10^{-6}$</td>
<td>$10 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\rho_3$[Cm$^{-2}$K$^{-1}$]</td>
<td>$2.5 \times 10^{-5}$</td>
<td>$0.548 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\rho$[kg.m$^{-3}$]</td>
<td>7500</td>
<td>7500</td>
</tr>
</tbody>
</table>

The Newton–Raphson method, however, seems to be more rapid-convergent. In this section, therefore, only the Newton–Raphson method is considered to obtain the results. Using the converged magnitudes of the nodal parameters, obtained through the iterative procedure for each load step in Eq. (2.8.28), the free vibration response of the post-buckled actuator is analyzed using Eq. (2.8.29) as the updated static equilibrium position of each load increment.

### 2.8.4 Result and Discussions

To assess the nonlinear pre/post-buckling free vibration behavior of an FGPM beam, a monomorph FGPM beam made of PZT-4 and PZT-5H piezoelectric materials is considered. Top surface of the beam is PZT-4 rich, while the bottom one is PZT-5H rich. Table 2.7 represents the thermo-electro-mechanical properties of these constituents. In all the rest, thickness is assumed to be $h = 0.001$ m, unless otherwise stated. Here, the dimensionless natural frequency is assumed to be $\Omega = \omega h \sqrt{(\rho / Q_{11})_{PZT-4}}$.

The temperature-deflection path of FGPM beam with $L/h = 25$ is depicted in Fig. 2.20 [62]. It is seen that, due to the non-symmetrical distribution of material properties across the thickness, the behavior of an FGPM beam under in-plane thermal loading is not of the bifurcation-type buckling, except for the case where the FGPM beam is reduced to a fully homogeneous one ($k = 0$). On the other hand, the equilibrium path of monomorph $S − S$ FGPM beam is unique and stable. Furthermore, due to the higher coefficient of thermal expansion near the bottom surface, beam bends downward. The power law index of composition rule plays an important role on the magnitude of lateral deflection. For the constituents of this study, as the power law index decreases, structure becomes stiffer and experiences less deflection.

The end-shortening force of an FGPM beam with the $S − S$ boundary conditions is plotted in Fig. 2.21 for three different values of power law index when beam is subjected only to temperature rise. Apparently, for the case of $k = 0$, when beam is
2.8 Vibration of Thermo-Electrically Post-buckled FGPM Beams

**Fig. 2.20** Effect of the volume fraction index on the thermal post-buckling equilibrium paths of immovable $S - S$ FGPM beams ($V_0 = 0, L/h = 40$)

![Graph showing effect of volume fraction index on equilibrium paths](image)

**Fig. 2.21** The end-shortening force of an $S - S$ FGPM beam for different volume fraction indices ($V_0 = 0, L/h = 40$)

![Graph showing end-shortening force vs. temperature](image)

fully homogeneous, end-shortening force varies linearly up to the bifurcation-point. The reason is that in pre-buckling state beam does not undergo any rotation, axial displacement, lateral deflection, and electrical potential. For the case of non-zero power law indices, however, end-shortening force varies nonlinearly with respect to temperature during the thermal loading process.

The effect of slenderness on load-deflection path of isotropic/homogeneous beams is depicted in **Fig. 2.22**. Since beam is subjected to only uniform temperature rise loading, thermal load-deflection paths are of the symmetrical primary-secondary path, where the post-buckling branch is stable. As expected, the higher $L/h$ ratio results in more magnitude of deflection. Consequently, the bifurcation temperatures
are postponed when beam becomes thicker. This is due to the stiffness loss caused by increasing the \( L/h \) ratio.

In Fig. 2.23 the temperature-deflection equilibrium paths of the FGPM beams are plotted for various power law indices. Similar to the case of \( S-S \) beams, when \( k = 0 \), that is the reduction of an FGPM beam to a fully homogeneous one, the problem is posed as a bifurcation-type buckling. For the case when distribution of properties is described with a nonzero volume law index, problem is not of the bifurcation-type buckling. However, the behavior of the beam is totally different from those observed for the \( S-S \) beams in Fig. 2.20. In the case of \( C-C \) FGPM beams, thermal moments are handled by the edge supports while due to the pyroelectric effect, at the onset of thermal loading, beam experiences lateral deflection. The magnitude of this deflection in initial levels of loading, however, is very small. The load-deflection path of \( C-C \) FGPM beams is unique and stable in the studied range. As seen in this figure, for each volume law index, there exist a unique temperature in which the deflection changes significantly with small amount of temperature increase. These points may be called the critical points, since they have high importance for design purposes.

The temperature-deflection path of \( C-C \) FGPM beams for three values of slenderness ratio is given in Fig. 2.24, when beam is subjected to both external voltage and temperature rise loading. Due to the presence of both external voltage effect and pyroelectric effect in the Maxwell equation (denoted respectively by \( \lambda_V \) and \( \lambda_T \)), problem is not of the bifurcation-type buckling. However, in each load-deflection path there exist a critical temperature in which deflection changes significantly with a small amount of increase in temperature.

Boundary conditions effect the load-deflection path of the \( S-S \), \( C-S \), and \( C-C \) FGPM beams and are shown in Fig. 2.25. Due to the inability of simply-supported edge in supplying the extra moment, the load-deflection paths of \( S-S \) and \( C-S \) are
Fig. 2.23 Effect of the volume fraction index on the thermal post-buckling equilibrium paths of immovable $C-C$ FGPM beams ($V_0 = 0, L/h = 60$).

completely smooth, unique, and stable. For the $C-C$ case, however, critical point temperature is observed in load-deflection path. As expected, the $C-C$ is the most stiff case and $S-S$ is the least one.

The effect of external actuator voltage on load-deflection equilibrium path of the FGPM beams is revealed in Fig. 2.26. It is seen that applying the negative voltage to the actuator electrodes increases the critical point temperature. This feature is valid for the constituents of this study, since the induced in-plane force in the beam may be of the compressive or tensile type, depending on the signs of piezoelectric coefficients. The effect of applied external voltage is negligible in pre-critical state, while it is more pronounced in post-critical phase.

The curves of the fundamental frequency versus the applied temperature rise and the fundamental frequency versus the mid-point nonlinear deflection of the $S-S$
Fig. 2.25 A comparison on boundary conditions effect on thermal load-deflection path of a FGPM beam ($V_0 = 50\, \text{V}, L/h = 35, k = 5$)

Fig. 2.26 The effect of the applied actuator voltage on the load-deflection path of a $C-C$ FGPM beam ($L/h = 50, k = 10$)

FGPM beams with ($L/h = 25, V_0 = 0\, \text{V}$) in pre/post-buckling regimes, are depicted in Figs. 2.27 and 2.28, respectively, for different values of the volume fraction indices. As discussed previously, buckling temperature differences (bifurcation points) are distinguishable from the fundamental frequency-temperature curves. Since, for the structures of the present study the buckling phenomenon occurs in the first mode of instability, in the bifurcation temperature state the fundamental frequency of the beam has to be equal to zero. It is seen that, due to the non-symmetrical distribution of material properties across the thickness, the behavior of an FGPM beam under in-plane thermal loading is not of the bifurcation-type buckling, except for the case when the FGPM beam is reduced to a fully homogeneous one ($k = 0$) [70]. Apparently, volume fraction index of composition rule plays an important effect on the free
2.8 Vibration of Thermo-Electrically Post-buckled FGPM Beams

**Fig. 2.27** Effect of the volume fraction index on the pre/post buckling fundamental frequency of immovable $S - S$ FGPM beams ($V_0 = 0, L/h = 25$)

**Fig. 2.28** The dimensionless fundamental frequency versus the mid-point dimensionless deflection of $S - S$ FGPM beam for different volume fraction indices ($V_0 = 0, L/h = 25$)

vibration behavior of the FGPM actuators. The associated load-deflection path for each case of positive power law index is unique and stable.

In Fig. 2.29 the temperature rise-fundamental frequency curves of the $C - C$ FGPM beams with $L/h = 60, V_0 = 0$ V are plotted for various power law indices. Similar to the case of $S - S$ beams, when $k = 0$, that is the reduction of an FGPM beam to a fully homogeneous one, the problem is posed as a bifurcation-type buckling. For the case when distribution of properties is described with a non-zero volume law index, problem is not of the bifurcation-type buckling. However, the behavior of beam is totally different from those observed for the $S - S$ beams in Fig. 2.27. In the case of $C - C$ FGPM beams, thermal moments are handled by the edge
supports, while due to the pyroelectric effect, at the onset of thermal loading beam experiences lateral deflection. The magnitude of this deflection in initial levels of loading, however, is so small but is not equal to zero. As seen in this figure, for each volume law index, there exist a unique temperature in which the magnitude of the fundamental frequency is very close to zero, and changes significantly with a small amount of temperature rise. These points may be called the critical points, since they have high importance for design purposes. However, the points can not be considered as the bifurcation points and the nonlinear behaviors of these structures are not of the primary-secondary equilibrium types.

2.9 Vibration of Thermally Post-buckled Beams on Elastic Foundation

2.9.1 Introduction

Thermal stability analysis of isotropic homogeneous beam-like structures and the vibration analysis in thermal field of beams with/without elastic foundation are conventional topics in structural mechanics. Li et al. [30] analyzed the buckling and postbuckling behavior of elastic rods subjected to thermal load. They achieved the results by solving the nonlinear equilibrium equations of the slender pinned-fixed Euler–Bernoulli beams via the shooting method. Li et al. [71] employed the shooting method for solving the equations related to buckling and postbuckling behavior of fixed-fixed elastic beam subjected to transversally non-uniform temperature loading. Li et al. [72] studied the natural frequency of slender Euler beams in thermal field with various boundary conditions. Thermal stability analysis of the Euler–Bernoulli
beams resting on a two-parameters nonlinear elastic foundation is studied by Song and Li [73] and Li and Batra [74]. In these studies ability of the Winkler foundation on mode alternation of buckling configuration of a pinned-fixed and pinned-pinned beams is examined. In all of the above-mentioned works, material properties are considered to be independent of temperature.

Stability analysis of the FGM beams that are in contact with an elastic foundation are limited in number. Sahraee and Saidi [75] applied the differential quadrature method and then analyzed the buckling and vibration of a deep FGM beam-columns resting on a Pasternak-type elastic foundation. Most recently, Fallah and Aghdam [76, 77] studied the nonlinear vibration and postbuckling behavior of functionally graded material beams resting on a nonlinear elastic foundation subjected to axial thermal [76] or mechanical [77] forces. Single mode Galerkin-based method is adopted to deduce the critical buckling and post-critical state of the beams. In this analysis, properties are assumed to be temperature independent and the response of the structure is confined to its first mode. However, as reported by Hetenyi [78], the Winkler elastic foundation largely affects the buckled shape of the beam, and therefore confining the buckled-shape of an in-contact beam similar to its contact-less shape causes the overestimation of both critical buckling temperature and post-buckling shape.

The problem of small amplitude vibration of beams under in-plane thermal or mechanical loadings is investigated employing various solution methods. Finite element formulation of Komijani et al. [62], shooting method solution of Li et al. [22], variational iteration method (VIM) solution of Fallah and Aghdam [77], single-term Galerkin solution of Wang et al. [33], differential quadrature solution of Pradhan and Murmu [46], and the analytical solution of Emam and Nayfeh [79] are some of the methods used to solve the resulting governing equations.

In this section, buckling, thermal post-buckling, and small amplitude free vibration of the FGM beams in thermal field are investigated [80]. The beam is analyzed under two types of thermal loads namely; uniform temperature rise and heat conduction across the thickness. Various combinations of clamped, simply-supported, and roller (sliding support) are considered as the edge supports of the structure. Properties of the graded medium are distributed across the thickness based on the power law model, where for each constituent they are functions of temperature. The general differential quadrature (GDQ) method is adopted to discretize the equation. The effects of various involved parameters are examined on the response of the structure.

### 2.9.2 Governing Equations

Consider a beam made of ceramic-metal FGMs with rectangular cross section $b \times h$ and length $L$ resting on a hardening three-parameters nonlinear elastic foundation, as shown in Fig. 2.30 [80].

Thermo-mechanical properties are graded across the thickness based on the power law form Eq. (2.2.7). Effective thermo-mechanical properties of the beam are
considered to follow the Voigt rule of mixture given by Eq. (2.2.8), except Poisson’s ratio \( \nu \), that is assumed to be constant across the thickness since it varies in a small range.

The analysis in this section is based on the Timoshenko beam theory assumption. Therefore, basic equations are the same with those used in the second section through Eqs. (2.2.1)–(2.2.10).

The equations of motion of FGM beams may be derived by applying the principle of virtual displacements

\[
\int_0^T (\delta U_s + \delta U_f - \delta T) \, dt = 0 \tag{2.9.1}
\]

where the total virtual strain energy of the beam \( \delta U_s \) can be written as

\[
\delta U_s = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \left( \sigma_{xx} \delta \varepsilon_{xx} + K_s \sigma_{xz} \delta \gamma_{xz} \right) dy dz dx \tag{2.9.2}
\]

The virtual strain energy of the nonlinear elastic foundation \( \delta U_f \) is expressed as

\[
\delta U_f = \int_0^L \int_0^b \left( K_w w_0 \delta w_0 + K_g \frac{\partial w_0}{\partial x} \delta \left( \frac{\partial w_0}{\partial x} \right) + K_{NL} w_0^3 \delta w_0 \right) dy dx \tag{2.9.3}
\]

In which, the linear Winkler stiffness, the shear layer stiffness, and the nonlinear Winkler stiffness are indicated as \( K_w, K_g, \) and \( K_{NL} \), respectively. Also, the kinetic energy \( \delta T \) is given by

\[
\delta T = \int_0^L \int_0^b \left( I_1 \frac{\partial u_0}{\partial t} \frac{\partial \delta u_0}{\partial t} + I_2 \frac{\partial \varphi}{\partial t} \frac{\partial \delta u_0}{\partial t} + I_3 \frac{\partial \varphi}{\partial t} \frac{\partial \delta \varphi}{\partial t} + I_4 \frac{\partial w_0}{\partial t} \frac{\partial \delta w_0}{\partial t} \right) dy dx \tag{2.9.4}
\]
where $I_1$, $I_2$, and $I_3$ are constants to be derived by utilizing Eq. (2.2.8) as

$$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z) dz = h \left( \rho_m + \frac{\rho_{cm}}{k+1} \right)$$

$$I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \rho(z) dz = h^2 \rho_{cm} \left( \frac{1}{k+1} - \frac{1}{2k+2} \right)$$

$$I_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \rho(z) dz = h^3 \left( \frac{1}{12} \rho_m + \rho_{cm} \left( \frac{1}{k+1} - \frac{1}{k+2} + \frac{1}{4k+4} \right) \right)$$

The equations of motion of an in-contact FGM Timoshenko beam are obtained according to the virtual work principle [35]. Integrating Eq. (2.9.1) by part, with the consideration of Eqs. (2.7.4) and (2.7.5), results in the following equations of motion [80]

$$\delta u_0 : \frac{\partial N_x}{\partial x} = I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \varphi}{\partial t^2}$$

$$\delta w_0 : \frac{\partial Q_{xz}}{\partial x} + \frac{\partial}{\partial x} \left( N_x \frac{\partial w_0}{\partial x} \right) - K_w w_0 + K_g \frac{\partial^2 w_0}{\partial x^2} - K_{NL} w_0^3 = I_1 \frac{\partial^2 w_0}{\partial t^2}$$

$$\delta \varphi : Q_{xz} - \frac{\partial M_x}{\partial x} = I_2 \frac{\partial^2 u_0}{\partial t^2} + I_3 \frac{\partial^2 \varphi}{\partial t^2}$$

Due to the integration process, the natural and essential boundary conditions are obtained as

$$N_x = 0 \quad \text{or} \quad u_0 = \text{known}$$

$$Q_{xz} + (K_g + N_x) \frac{\partial w_0}{\partial x} = 0 \quad \text{or} \quad w_0 = \text{known}$$

$$M_x = 0 \quad \text{or} \quad \varphi = \text{known}$$

For generalizing the subsequent results, the following non-dimensional variables are introduced and are used in the text

$$\xi = \frac{x}{L}, \quad U^* = \frac{u}{L}, \quad \delta = \frac{h}{L}, \quad W^* = \frac{w}{h}, \quad \kappa = \frac{K_s}{2(1+\nu)}$$

$$e_1 = \frac{E_1}{E_c^{\text{ref}} \ h}, \quad e_2 = \frac{E_2}{E_c^{\text{ref}} \ h^2}, \quad e_3 = \frac{E_3}{E_c^{\text{ref}} \ h^3}$$

$$K_w^* = \frac{K_w L^4}{E_c^{\text{ref}} I_0}, \quad K_g^* = \frac{K_g L^2}{E_c^{\text{ref}} I_0}, \quad K_{NL}^* = \frac{K_{NL} h^2 L^4}{E_c^{\text{ref}} I_0}$$

$$N_x^* = \frac{NL^2}{E_c^{\text{ref}} I_0}, \quad Q_{xz}^* = \frac{Q_{xz} L^2}{E_c^{\text{ref}} I_0}, \quad M_x^* = \frac{MT L}{E_c^{\text{ref}} I_0}, \quad N_T^* = \frac{N_T L^2}{E_c^{\text{ref}} I_0}, \quad M_T^* = \frac{M_T L}{E_c^{\text{ref}} I_0}$$
\[ \omega^* = \omega h \sqrt{\frac{\rho_c}{E_c^{ref}}} \]

\[ \eta = \frac{t}{h} \sqrt{\frac{E_c^{ref}}{\rho_c}} \]

\[ \Lambda_1 = \frac{I_1}{h \rho_c^{ref}}, \quad \Lambda_2 = \frac{I_2}{h^2 \rho_c^{ref}}, \quad \Lambda_3 = \frac{I_3}{h^3 \rho_c^{ref}} \]

(2.9.8)

where \( I_0 \) is the moment of inertia of the cross section and \( E_c^{ref} \) and \( \rho_c^{ref} \) are the elasticity modulus and density of the ceramic constituent at reference temperature.

Substitution of the above non-dimensional parameters into Eq. (2.7.7) and utilizing Eq. (2.7.4) give the governing equations of the beam in dimensionless forms as [80]

\[ \epsilon_1 \frac{\partial^2 U^*}{\partial \xi^2} + \epsilon_1 \delta \frac{\partial W^*}{\partial \xi} \frac{\partial^2 W^*}{\partial \xi^2} + \delta \frac{\partial^2 \phi}{\partial \xi^2} = \frac{\Lambda_1 \partial^2 U^*}{\partial \eta^2} + \frac{\Lambda_2 \partial^2 \phi}{\partial \eta^2} \]

\[ \kappa \epsilon_1 \left( \frac{\partial \phi}{\partial \xi} + \delta \frac{\partial^2 W^*}{\partial \xi^2} \right) + \left[ \epsilon_1 \left( \frac{\partial U^*}{\partial \xi} + \frac{1}{2} \delta \frac{\partial (\partial W^*)^2}{\partial \xi} \right) + \epsilon_2 \frac{\partial \phi}{\partial \xi} - \frac{N_{T^*}}{12 \delta} \right] \frac{\partial^2 W^*}{\partial \xi^2} \]

\[ - \frac{1}{12} K_w \delta^3 W^* + \frac{1}{12} K_g \delta^3 \frac{\partial^2 W^*}{\partial \xi^2} - \frac{1}{12} \delta^3 K_{NL} W^3 = \frac{\Lambda_1 \partial^2 W^*}{\partial \eta^2} \]

(2.9.9)

Five possible types of boundary conditions as the combinations of the clamped, roller, and simply supported edges are considered. Mathematical expressions for these classes of edge supports are

Clamped (C) : \( U^* = W^* = \phi = 0 \)

Simply supported (S) : \( U^* = W^* = M_x^* = 0 \)

Roller (R) : \( U^* = \phi = Q_{xz}^* + (K_g^* + N_x^*) \delta \frac{dW^*}{d\xi} = 0 \) (2.9.10)

where

\[ N_x^* = \frac{12}{\delta^2} \epsilon_1 \left( \frac{dU^*}{d\xi} + \frac{1}{2} \delta^2 \left( \frac{dW^*}{d\xi} \right)^2 \right) + \frac{12}{\delta} \epsilon_2 \frac{d\phi}{d\xi} - N_{T^*} \]

\[ M_x^* = \frac{12}{\delta} \epsilon_2 \left( \frac{dU^*}{d\xi} + \frac{1}{2} \delta^2 \left( \frac{dW^*}{d\xi} \right)^2 \right) + 12 \epsilon_3 \frac{d\phi}{d\xi} - M_{T^*} \]

\[ Q_{xz}^* = \frac{12}{\delta^2} \kappa \epsilon_1 \left( \phi + \delta \frac{dW^*}{d\xi} \right) \]

(2.9.11)
Solution Procedures

The solution of equations of motion (2.9.9) is divided into two regimes. Part of
the time-independent solution related to thermal post-buckling analysis with large
magnitude and part of dynamic solution for free vibration with small magnitude that
is time-dependent. Thus, the total solutions of Eq. (2.9.9) are [80]

\[ U^*(\xi, \eta) = U^*_s(\xi) + U^*_d(\xi, \eta) \]
\[ W^*(\xi, \eta) = W^*_s(\xi) + W^*_d(\xi, \eta) \]
\[ \varphi(\xi, \eta) = \varphi_s(\xi) + \varphi_d(\xi, \eta) \] (2.9.12)

Substituting Eq. (2.9.12) into (2.9.9) and collecting the static parts result in the fol-
lowing time-independent equations which describe the nonlinear stability behavior
of a beam under in-plane thermal load

\[ e_1 \frac{d^2 U^*_s}{d\xi^2} + e_1 \delta^2 \frac{dW^*_s}{d\xi} \frac{d^2 W^*_s}{d\xi^2} + \delta e_2 \frac{d^2 \varphi_s}{d\xi^2} = 0 \]
\[ \kappa_1 \left( \frac{d\varphi_s}{d\xi} + \delta \frac{d^2 W^*_s}{d\xi^2} \right) + \left[ e_1 \frac{dU^*_s}{d\xi} + \frac{1}{2} \delta^2 \left( \frac{dW^*_s}{d\xi} \right)^2 \right] + \delta e_2 \frac{d\varphi_s}{d\xi} - \frac{N_{T_s} \delta}{12} = \delta \frac{d^2 W^*_s}{d\xi^2} \]
\[ -\frac{1}{12} K^*_w \delta^3 W^*_s + \frac{1}{12} K^*_w \delta^3 \frac{d^2 W^*_s}{d\xi^2} - \frac{1}{12} \delta^3 K_{NL} W^*_s = 0 \]
\[ \kappa_e \left( \varphi_s + \delta \frac{dW^*_s}{d\xi} \right) - e_2 \left( \frac{d^2 U^*_s}{d\xi^2} + \delta^2 \frac{dW^*_s}{d\xi} \frac{d^2 W^*_s}{d\xi^2} \right) - \delta^2 e_3 \frac{d^2 \varphi_s}{d\xi^2} = 0 \] (2.9.13)

and linearizing the remaining part about the static equilibrium position with a small
amplitude dynamic response reaches us to

\[ e_1 \frac{\partial^2 U^*_s}{\partial \xi^2} + e_1 \delta^2 \frac{\partial W^*_s}{\partial \xi} \frac{\partial^2 W^*_s}{\partial \xi^2} + e_1 \delta^2 \frac{\partial W^*_s}{\partial \xi} \frac{\partial^2 W^*_d}{\partial \xi^2} + \delta e_2 \frac{\partial^2 \varphi_d}{\partial \xi^2} = \frac{\Lambda_1}{\delta^2} \frac{\partial^2 U^*_d}{\partial \eta^2} + \frac{\Lambda_2}{\delta} \frac{\partial^2 \varphi_d}{\partial \eta^2} \]
\[ \kappa_e \left( \frac{\partial \varphi_d}{\partial \xi} + \delta \frac{\partial^2 W^*_d}{\partial \xi^2} \right) + \left[ e_1 \left( \frac{\partial U^*_d}{\partial \xi} + \frac{1}{2} \delta^2 \left( \frac{\partial W^*_s}{\partial \xi} \right)^2 \right) + \delta e_2 \frac{\partial \varphi_d}{\partial \xi} - \frac{N_{T_d} \delta}{12} \right] \frac{\partial^2 W^*_d}{\partial \xi^2} \]
\[ + \left[ e_1 \left( \frac{\partial U^*_d}{\partial \xi} + \delta^2 \frac{\partial W^*_s}{\partial \xi} \frac{\partial W^*_d}{\partial \xi} \right) + e_2 \delta \frac{\partial \varphi_d}{\partial \xi} \right] \frac{\partial^2 W^*_s}{\partial \xi^2} + \left[ e_1 \left( \frac{\partial U^*_d}{\partial \xi} + \delta^2 \frac{\partial W^*_s}{\partial \xi} \frac{\partial W^*_d}{\partial \xi} \right) + e_2 \delta \frac{\partial \varphi_d}{\partial \xi} \right] \frac{\partial^2 W^*_d}{\partial \xi^2} \]
\[ + \delta^2 \frac{\partial W^*_s}{\partial \xi} \frac{\partial^2 W^*_d}{\partial \xi^2} + e_2 \delta^2 \frac{\partial \varphi_d}{\partial \xi^2} \frac{\partial^2 W^*_d}{\partial \xi^2} \right] \frac{\partial W^*_d}{\partial \xi} - \frac{1}{12} K^*_w \delta^3 W^*_d + \frac{1}{12} K^*_w \delta^3 \frac{\partial^2 W^*_d}{\partial \xi^2} - \frac{1}{4} \delta^3 K_{NL} W^*_s W^*_d = \frac{\Lambda_1}{\delta} \frac{\partial^2 W^*_s}{\partial \eta^2} \]
\[ \kappa e_1 \left( \varphi_d + \delta \frac{\partial W_d}{\partial \xi} \right) - e_2 \left( \delta \frac{\partial^2 U^*_d}{\partial \xi^2} + \delta^3 \frac{\partial W^*_d}{\partial \xi^2} + \delta^3 \frac{\partial^2 W^*_s}{\partial \xi^2} \right) - \delta^2 e_3 \frac{\partial^2 \varphi_d}{\partial \xi^2} = \Lambda_2 \frac{\partial^2 U^*_d}{\partial \eta^2} + \Lambda_3 \frac{\partial^2 \varphi_d}{\partial \eta^2} \]  

(2.9.14)

The analytical solution of Eq. (2.9.13) is complicated, due to the strong non-linearity and the included couplings in the partial differential equations. Therefore, to seek for a numerical solution to the problem, the GDQ method is employed. The ability of GDQ method to handle the nonlinear stability problems is exhibited by many authors [81]. A brief overview of the GDQ method is presented in Appendix A.

Utilizing the GDQ discretization to the dimensionless governing Eq. (2.9.13), one may reach to discretized form of the equations governing the pre/post-buckling equilibrium path of the beam. Equations and the associated boundary conditions are given in Appendix B.

The system of algebraic equations and associated boundary conditions presented in Appendix B may be written in the form

\[ [K_s]_{3N \times 3N} \{X_s\}_{3N \times 1} = \{F\}_{3N \times 1} \]  

(2.9.15)

where \([K_s]_{3N \times 3N}\) is the nonlinear stiffness matrix which depends on both unknown variable vector \([X_s]_{3N \times 1}\) and temperature. It must be noted that the right hand side of Eq. (2.9.15) may be different for each set of boundary conditions. The force matrix \([F]_{3N \times 1}\) is obtained through the thermally induced stress resultants and bending moments for the simply supported boundary conditions at the ends of the beam (\(\xi = 0, 1\)) and vanishes when the beam is clamped or roller at the ends (see the definition of thermal moment in Eq. 2.9.11). Thermal buckling, without consideration of the magnitude of the temperature difference, occurs only when \([F]_{3N \times 1} = 0\). Otherwise, lateral deflection occurs when \([F]_{3N \times 1} \neq 0\). The numerical algorithm to solve the postbuckling behavior in each case is given by Liew et al. [82]. The solution method of this section is the same with the process used by Komijani et al. [62].

When the solution of static phase is accomplished, small free vibration analysis is followed. The discrete form of the governing equations along with the associated boundary conditions are given in Appendix C. The presented equations are linear with respect to the dynamic variables denoted by a subscript \(d\). Solution of this system is obtained as an eigenvalue problem. The eigenvalues of the established system of equations present the non-dimensional frequency of the beam defined as \(\omega^*\).

### 2.9.3 Types of Thermal Loading

Two distinct types of thermal loadings are considered for the beam. Uniform temperature rise (\(UTR\)) and nonlinear temperature across the thickness (\(NLTD\)). Details may be found in Sect. 2.6.
2.9.4 Results and Discussion

As stated earlier, for the FGM beams when thermal force or thermal moment resultants are absent in the force vector, problem may be posed as a bifurcation-type buckling. Conditions for an FGM beam to remain flat under in-plane thermal loadings with general boundary conditions are studied by Kiani et al. [23, 24] for various beam theories. Apparently, for isotropic homogeneous beams that are subjected to the \textit{UTR} loading, bifurcation-type of instability occurs. In this section, an FGM beam made of \textit{SU304/Si3N4} is considered. Properties of these constituents are highly temperature dependent based on the well-known Touloukian model. The dependency is demonstrated in Eq. (2.5.22). Desired constants for \textit{SU304} and \textit{Si3N4} are given in Table 2.3.

In Table 2.8 the effects of temperature dependency, the Pasternak foundation, and the edge supports are examined on $\Delta T_{cr}$ [80]. As one may obtain from this Table, for beams without any foundation contact conditions, the critical buckling temperature of the $C - R$ and $S - S$ cases of edge supports are the same. Besides, in this case the \textit{TID} condition under-estimates the critical buckling temperatures. As expected, the $S - R$ case of edge supports has the lowest buckling temperature and the $C - C$ has the highest. For beams that are in-contact with the Winkler elastic foundation $K_w^* = 100$, the $S - R$ presents a stiffer configuration compared to the $S - S$ boundary condition. It should be mentioned that, however, the critical buckling temperature differences of contact-less $C - R$ and $S - S$ beams are the same. In the case of in-contact beams, the $C - R$ exhibits a stiffer configuration compared to the $S - S$ beam. Furthermore, the ability of elastic foundation in postponing the bifurcation point of the structure is significant. For the foundations that are stiff enough, this ability may be accompanied with the increase of critical points. For instance, as indicated in Table 2.8, for the $C - C$ and $S - S$ beams that are resting on a stiff foundation, the critical buckling temperature difference is associated with an antisymmetric mode shape. The presented results for the Winkler and Pasternak foundations with $K_w^* = 500$ show that the \textit{TID} case can not predict the correct buckling mode-shape of the beam. In this case, the buckling mode shape is predicted to be symmetric. The real state of the beam with \textit{TID} properties predicts the antisymmetrical buckling state for the above-mentioned case.

In Tables 2.9 and 2.10 the critical buckling temperature differences of an FGM beam with two types of boundary conditions are examined. Beam is subjected to the \textit{UTR} loading. As one may conclude, the $C - C$ type of boundary condition is stiffer than the $C - R$ type. Besides, for both types of edge supports, \textit{TID} case reveals the more precise approximation of $\Delta T_{cr}$, where in \textit{TID} case the critical buckling temperature is over-estimated. The effect of temperature dependency is more pronounced when beam is in contact with stiffer foundations. Furthermore, a comparison between the results of two isotropic homogeneous cases, i.e. $k = 0$ and $k = \infty$, show that \textit{Si3N4} is more sensitive to temperature than the \textit{SU304}. Therefore, as the power law index increases, since the properties of FG media tends to a metallic beam, temperature dependency is less pronounced. Since each constant of elastic foundation...
Table 2.8 $\Delta T_{cr}[K]$ for various boundary conditions of isotropic homogeneous (SU.S304) Timoshenko beam with various parameters of elastic foundation subjected to UTR loading. The value of $\delta = 0.04$ is considered. Antisymmetrical buckled shapes are indicated with a superscript * [80]

<table>
<thead>
<tr>
<th>$(K_w^<em>, K_s^</em>)$</th>
<th>C – C</th>
<th>C – R</th>
<th>S – S</th>
<th>S – R</th>
<th>C – S</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0) TID</td>
<td>337.94</td>
<td>85.53</td>
<td>85.53</td>
<td>21.45</td>
<td>174.08</td>
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<td>TD</td>
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<td>81.24</td>
<td>21.16</td>
<td>157.86</td>
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<td>(100, 0) TID</td>
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<td>326.88</td>
<td>222.94</td>
<td>252.24</td>
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<td>282.23</td>
<td>201.81</td>
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</tr>
<tr>
<td>(100, 10) TID</td>
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<td>387.22</td>
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<td>TD</td>
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<td>394.13</td>
<td>318.22</td>
<td>335.39</td>
<td>362.17</td>
</tr>
<tr>
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<td>396.19</td>
<td>359.05</td>
<td>313.02</td>
<td>381.19</td>
</tr>
<tr>
<td>TD</td>
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<td>335.43</td>
<td>319.78</td>
<td>273.45</td>
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<td>422.32*</td>
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<tr>
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<td>686.40</td>
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<tr>
<td>TD</td>
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<td>593.91</td>
<td>543.52*</td>
<td>566.97</td>
<td>606.16</td>
</tr>
</tbody>
</table>

Table 2.9 $\Delta T_{cr}[K]$ for the C – C FGM beams with $\delta = 0.04$, various power law indices and contact conditions subjected to UTR loading [80]

<table>
<thead>
<tr>
<th>$(K_w^<em>, K_s^</em>)$</th>
<th>k = 0</th>
<th>k = 0.5</th>
<th>k = 1</th>
<th>k = 2</th>
<th>k = 5</th>
<th>k = 10</th>
<th>k = $\infty$</th>
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<tbody>
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<td>(0,0) TID</td>
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<td>509.89</td>
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<td>394.39</td>
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<td>474.94</td>
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<td>(500,0) TID</td>
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<td>833.94</td>
<td>870.60</td>
<td>901.92</td>
<td>775.19</td>
</tr>
</tbody>
</table>

cause the elastic stiffness of the structure to be increased, an increase in the Winkler or shear constant of elastic foundation postpones the branching point of the beam.

In Tables 2.11 and 2.12 the critical buckling temperature differences of $C – C$ and $C – R$ types of boundary conditions are examined for various power law indices
and contact conditions. Beam is assumed to be under the \(NLTD\) case of thermal loading. To account for the temperature dependency of the constituents, the algorithm utilized by Shen [83] is studied herein. The bottom surface of the beam is kept at a constant temperature, i.e. \(T_m = 305K\). Similar to the results of Shen [83] for the case of a clamped shell under heat conduction, both \(TD\) and \(TID\) cases of \(\Delta T_r\) are the same for \(k = \infty\). In the case of \(NLTD\) type of thermal loading, temperature dependency is more pronounced for lower values of the power law index. Due to the resistance of elastic foundation against deflection of the beam, as each constant of elastic foundation increases bifurcation point of the beam increases too.

For the \(C - C\) and \(S - S\) types of FGM beams, load-deflection path of a contact-less FGM beam is depicted in Figs. 2.31 and 2.32 [80]. Apparently, the response of the \(C - C\) beams is of the bifurcation-type buckling for an arbitrary value of the power law index. For the FG beam with \(S - S\) edge supports, except for the case of reduction of an FGM beam to an isotropic homogeneous one, problem can not be considered as a primary-secondary equilibrium path. The response of a \(C - C\) beam is of the bifurcation-type buckling since edges are capable of supplying the arbitrary amount of the extra moment prior to buckling. The response of \(S - S\) isotropic homogeneous beam is of the same type, since thermal moment vanishes through the beam for the isotropic homogeneous structure. For the case of an FGM beam with finite positive value of power law index, problem is not of the bifurcation-type and within the studied range, the load-deflection path is unique and stable. As one may conclude, temperature dependency is more pronounced in the post-buckling regime. In the post-buckling regime, since the \(SU\) 304 is less sensitive to temperature compared to \(Si_3N_4\), as temperature rises the \(SU\) 304/\(Si_3N_4\) beam bends downward. This is due

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Table 2.11 $\Delta T_{cr}[K]$ for the $C - C$ FGM beams with various power law indices and contact conditions subjected to $NLTD$ loading [80]. The value of $\delta = 0.025$ is considered

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Table 2.12 $\Delta T_{cr}[K]$ for the $C - R$ FGM beams with various power law indices and contact conditions subjected to $NLTD$ loading [80]. The value of $\delta = 0.025$ is considered

<table>
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<td>594.95</td>
<td>571.07</td>
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</table>

to the higher coefficient of thermal expansion near the top surface in $SU S304/SI_3 N_4$ beam.

The influence of three-parameters elastic foundation on the $S - S$ and $C - C$ beams subjected to $UTR$ type of thermal loading is depicted in Figs. 2.33 and 2.34, respectively. Only $TD$ case of material properties is considered. Linearly graded
2.9 Vibration of Thermally Post-buckled Beams on Elastic Foundation

**Fig. 2.31** Influences of the power law index and temperature dependency on load deflection path of contact-less $C-C$ FGM beams with $\delta = 0.04$ subjected to $UTR$ loading

**Fig. 2.32** Postbuckling equilibrium path of contact-less $S-S$ FGM beams with $\delta = 0.04$ subjected to $UTR$ loading with various power law indices. Only $TD$ case is addressed.

properties of the constituents is assumed. The behavior of in-contact $C-C$ beams is of the bifurcation-type buckling. The nonlinear constant of elastic foundation has no effect on the critical buckling temperature difference, while it largely affects the post-buckling resistance of the beam. This effect is more pronounced in deep post-buckling regime. The Winkler and shear layer of the Pasternak elastic foundation deeply affect both critical buckling temperature and post-buckling equilibrium path of the beam. For the $S-S$ FGM beams, the load-deflection path is not of the bifurcation-type of instability. Apparently in this case beam initially starts to deflect laterally, since its material property is not symmetrical with respect to the mid-plane and edges can not supply the additional moment to retain the beam in its flat conditions. Each constant of
elastic foundation increases the elastic stiffness of the structure. Therefore in-contact beams has highly-raised post-buckling path when is compared with its contact-less state. Unlike the $C-C$ case, in the case of $S-S$ FGM beams the influence of nonlinear elastic foundation in load-deflection path initiates at the onset of thermal loading.

The end-shortening force of $C-C$ and $S-S$ FGM beams with various contact conditions is depicted in Figs. 2.35 and 2.36, respectively. Both $TD$ and $TID$ cases of material properties are addressed. As one may see, the bifurcation point for $C-C$ case of edge supports can be extracted from the force-temperature graph. Prior to stability loss, end-shortening force varies linearly with respect to temperature. This
Fig. 2.35 Influences of three-parameters elastic foundation \((K^w_*, K^g_*, K^NL_*)\) and temperature dependency on end-shortening force of a linearly graded \(C - C\) FGM beam with \(\delta = 0.025\) subjected to \(UTR\) loading.

Fig. 2.36 Influences of three-parameters elastic foundation \((K^w_*, K^g_*, K^NL_*)\) and temperature dependency on end-shortening force of a linearly graded \(S - S\) FGM beam with \(\delta = 0.04\) subjected to \(UTR\) loading.

correlation is also reported by Kiani et al. [24] based on the linear pre-buckling analysis. It is worth-mentioning that when properties are assumed to be \(TID\), in post-buckling regime end-shortening force is constant and is equal to the buckling force. While in the \(TD\) case, as the real state of the structure, end-shortening force diminishes as a function of temperature in the thermal loading process. This is due to the fact that constituents are temperature dependent and as temperature rises, beam loses stiffness. For the case of \(S - S\) edge supports, there is not a sharp change in force-temperature graph which accepts the uniqueness of a stable load-deflection path. From both of these figures, one may conclude that the stiffer the elastic foundation is, the more the axial end-shortening force is for a prescribed amount of temperature. Furthermore, since the load-deflection path of \(C - C\) beams...
Fig. 2.37 Influences of three-parameters elastic foundation \((K_w, K_y, K_{NL})\) and temperature dependency on moment of a linearly graded \(C - C\) FGM beam with \(\delta = 0.025\) subjected to \(UTR\) loading is of the bifurcation point, nonlinear constant of elastic foundation has no effect on pre-buckling axial end-shortening force.

The magnitude of moment at mid-point of the FGM beam is depicted for both \(S - S\) and \(C - C\) edge supports, when temperature distribution of the beam is raised uniformly through the beam. In Fig. 2.37, the bifurcation points are observed through the moment-temperature response of the \(C - C\) beam. Prior to buckling, moment varies linearly with respect to temperature. This is formerly reported by Kiani et al. [24] based on the linear pre-buckling analysis of beams. In the post-buckling regime, however, moment changes significantly and alters nonlinearly with respect to temperature. Similar to the end-shortening force, nonlinear constant of elastic foundation has no influence on prebuckling moment at mid-point. For the \(S - S\) beams, as seen in Fig. 2.38, however, there is no bifurcation point through the moment-temperature path of the beam. In both \(C - C\) and \(S - S\) cases, the effect of temperature dependency is more pronounced for stiffer elastic foundations.

The effect of elastic foundation on buckling and post-buckling resistance of both \(C - C\) and \(S - S\) beams is studied in Figs. 2.39 and 2.40, respectively. The \(NLTD\) case of thermal loading is considered. Metal rich surface of the beam is kept at \(T_m = 305\) K. Only \(TD\) case of material properties is addressed. The behavior of a simply-supported beam, even for the case of a homogeneous isotropic one, is not of the bifurcation-type buckling. It should be emphasized that, even for the case of fully isotropic homogeneous beams, the \(S - S\) beams start gain lateral deflection at the onset of loading. This is due to the inability of edges to retain the beam flat at initial steps of thermal loading. Similar to the \(UTR\) loading, for the \(NLTD\) case of temperature loading nonlinear constant of elastic foundation has no influence on critical-buckling temperature. Post-buckling resistance of the beam, however, is highly affected by this constant.
Fig. 2.38 Influences of three-parameters elastic foundation ($K^*_w$, $K^*_g$, $K^*_NL$) and temperature dependency on moment of a linearly graded $S - S$ FGM beam with $\delta = 0.04$ subjected to $UTR$ loading.

Fig. 2.39 Effect of three-parameters elastic foundation ($K^*_g$, $K^*_w$, $K^*_NL$) on load deflection path of a linearly graded $C - C$ FGM beams with $\delta = 0.04$ subjected to the $NLTD$ loading.

As a benchmark study, the first three frequencies of the FGM beams with various boundary conditions are presented in Tables 2.13, 2.14, 2.15, 2.16 and 2.17. It is seen that for the constituents of this study, as the power law index increases, the natural frequency of the system decreases. For each case of edge supports, an increase in the Winkler or Pasternak constants of elastic foundation results in higher natural frequency. This is due to the higher elastic stiffness of the beam when in-contact with foundation. As expected, for a prescribed contact condition and power law index, the $C - C$ beam has the highest natural frequency and $S - R$ has the least one.

For a contact-less beam, the fundamental frequency parameter as a function of temperature rise is depicted in Figs. 2.41 and 2.42 for the $S - S$ and $C - C$ cases of
Fig. 2.40  Effect of three-parameters elastic foundation \((K_g^*, K_w^*, K_{NL}^*)\) on load deflection path of a linearly graded \(S - S\) FGM beams with \(\delta = 0.04\) subjected to the \(NLTD\) loading

boundary conditions, respectively. For the case of a beam with both edges clamped, it is seen that before a prescribed temperature, i.e. the bifurcation point temperature, as temperature increases the frequency parameter diminishes. This is due to the decrease in total stiffness of the beam, since geometrical stiffness diminishes as temperature rises. Near the bifurcation point, frequency approaches to zero. After the bifurcation point, an increase in temperature results in higher frequency. This feature refers to the higher elastic stiffness of the beam created from the von-Karman non-linearity [72]. It is seen that temperature dependency of the constituents leads to more accurate results, where with the assumption of constant material properties, bifurcation points are exaggerated. Besides, in pre-buckling range, with the assumption of temperature dependency, the predicted frequency is less than the one obtained with the temperature independent assumption. This is due to the lower elasticity modulus of the constituents in \(TD\) case. A comparison of Figs. 2.41 and 2.42 reveals that the behavior of an FGM beam with the \(S - S\) boundary conditions is totally different from those with the \(C - C\) boundary conditions. For the FGM beam with both edges simply-supported, frequency does not approaches to zero, which somehow proves the non-existence of bifurcation type buckling. This is expected since a simply supported edge does not handle the moment and the total bending moment is affected by the temperature loading. Since the statement of bending moment is non-homogeneous in terms of \(u, w,\) and \(\varphi,\) the resulting system of equations can not be posed as an eigen-value problem and the load-path of the beam within the studied range is unique and stable. It should be mentioned that, however, load-deflection path of the \(S - S\) beams is free of bifurcation-point, but similar to the \(C - C\) case, frequency decreases up to a definite temperature and then increases.

The influence of elastic foundation on fundamental frequency of an FGM beam for the \(C - C\) and \(S - S\) boundary conditions are depicted in Figs. 2.43 and 2.44, respectively. As previously discussed, the \(TD\) case results in more accurate conclusions
Table 2.13 The first three natural frequencies of lateral vibration for the $C-C$ FGM beams with $\delta = 0.04$, various power law indices, and contact conditions [80]

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<td>16.2358</td>
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<td>75.3195</td>
<td>67.7579</td>
<td>61.7221</td>
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and therefore in the following discussion only this case is addressed. It is seen that an increase in the Winkler or Pasternak constants of elastic foundation results in higher stiffness and therefore fundamental frequency and critical buckling temperature are increased. For the case of $C-C$ beams, the nonlinear coefficient of elastic foundation has no effect on frequency parameter of the beam prior to buckling. This is expected since the pre-buckling deformation of the beam is linear. In contrast, in the $S-S$ beams nonlinear coefficient of elastic foundation affects the fundamental frequency with the initiation of temperature loading. This effect, however, is negligible.

The effects of various boundary conditions on frequency parameter of a beam subjected to uniform temperature rise loading is shown in Fig. 2.45. It is seen that responses of the $C-C$ and $C-R$ beams are totally different from the other three types. In the $C-C$ and $C-R$ cases, since edges are capable of supplying the extra moment, beam remains flat until a prescribed temperature in which frequency approaches to zero. After that, frequency increases monolithically as beam deflects more. For three other cases, however, the behavior is slightly different, since the beam initially starts lateral deflection at the onset of thermal loading.
The basic concept of the GDQ method is to find the derivatives of a function at a sample point to be approximated as a weighted linear summation of the value of the function in the whole domain. The governing differential equations have been reduced to a set of algebraic equations by this approximation. The number of algebraic equations depend upon the number of grid points. The mth. order derivative
Table 2.15  The first three natural frequencies of lateral 0771 vibration for the \( S - S \) FGM beams with \( \delta = 0.04 \), various power law indices, and contact conditions \[80\]

<table>
<thead>
<tr>
<th>((K_w^<em>, K_y^</em>))</th>
<th>( k = 0 )</th>
<th>( k = 0.5 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 5 )</th>
<th>( k = 10 )</th>
<th>( k = \infty )</th>
</tr>
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<tbody>
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<td>5.9860</td>
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<td>87.7174</td>
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<td>47.5561</td>
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<tr>
<td>(100, 0)</td>
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<td>14.0452</td>
<td>10.0771</td>
<td>8.9933</td>
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<td>7.5340</td>
<td>7.2501</td>
</tr>
<tr>
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<td>88.2918</td>
<td>60.9140</td>
<td>53.4103</td>
<td>47.9581</td>
<td>43.5890</td>
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<td>64.8502</td>
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<td>46.8536</td>
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<td>12.5052</td>
<td>11.2217</td>
<td>10.2509</td>
<td>9.4719</td>
<td>9.1447</td>
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<td>61.3698</td>
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<td>65.2584</td>
<td>57.0464</td>
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<tr>
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<td>90.5533</td>
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<td>58.6391</td>
<td>52.8650</td>
<td>48.2517</td>
<td>46.1358</td>
</tr>
</tbody>
</table>

of a function \( f(x) \) with respect to \( x \) at a sample point \( x_i \) is approximated by linear summation of all functional values at all grid points \[84\]. The mathematical expression is

\[
\frac{d^n f(x)}{dx^n} \bigg|_{x_i} \approx \sum_{j=1}^{N} C_{ij}^{(m)} \times f(x_j)
\]

where \( N \) is the number of grid points, \( x_i \) is the location of grid points, \( f(x_j) \) is the function value at \( x_j \), and \( C_{ij}^{(m)} \)'s are the weighting coefficients corresponding to the \( m \)th order derivative. Quan et al. \[85\] suggested a Lagrangian interpolation polynomial to overcome the numerical ill-conditions in determining the weighting coefficients \( C_{ij}^{(m)} \)

\[
f(x) = \sum_{i=1}^{N} \frac{M(x)}{(x - x_i)M^{(1)}(x_i)} f(x_i)
\]

where
Table 2.16  The first three natural frequencies of lateral vibration for the $C - S$ FGM beams with $\delta = 0.04$, various power law indices, and contact conditions [80]

<table>
<thead>
<tr>
<th>$(K^<em>_w, K^</em>_q)$</th>
<th>$k = 0$</th>
<th>$k = 0.5$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 5$</th>
<th>$k = 10$</th>
<th>$k = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0) First</td>
<td>15.3433</td>
<td>10.5716</td>
<td>9.2703</td>
<td>8.3268</td>
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<tr>
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<td>70.3097</td>
<td>61.5994</td>
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<td>(100, 0) First</td>
<td>18.3185</td>
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<td>11.4453</td>
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<tr>
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<td>70.7051</td>
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<td>50.5426</td>
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<tr>
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<td>54.4712</td>
<td>37.0606</td>
<td>33.5672</td>
<td>30.2882</td>
<td>27.6720</td>
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<tr>
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<td>107.0224</td>
<td>74.2916</td>
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<td>54.7495</td>
<td>52.3173</td>
<td>48.6660</td>
</tr>
</tbody>
</table>

\[
M(x) = \prod_{j=1}^{N} (x - x_j)
\]

\[
M^{(1)}(x_i) = \prod_{j=1}^{N} (x_i - x_j) \quad \text{for} \quad i = 1, 2, 3, \ldots, N
\]

By combining the above equation, one may reach to

\[
C_{ij}^{(1)} = \sum_{i=1}^{N} \frac{M^{(1)}(x_j)}{(x_j - x_i)M^{(1)}(x_j)} \quad \text{for} \quad i, j = 1, 2, 3, \ldots, N \quad \text{and} \quad i \neq j
\]

\[
C_{ii}^{(1)} = - \sum_{j=1, j \neq i}^{N} C_{ij}^{(1)} \quad \text{for} \quad i = 1, 2, 3, \ldots, N
\]
Table 2.17 The first three natural frequencies of lateral vibration for the $S - R$ FGM beams with $\delta = 0.04$, various power law indices, and contact conditions [80]

<table>
<thead>
<tr>
<th>$(K^<em>_w, K^</em>_y)$</th>
<th>$k = 0$</th>
<th>$k = 0.5$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
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<td>42.1266</td>
<td>36.9185</td>
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<td>30.1256</td>
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<td>10.3013</td>
<td>7.6167</td>
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<td>35.7448</td>
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<td>32.0377</td>
</tr>
</tbody>
</table>

The coefficients of the first order weighting matrix may be obtained using the above equations. Higher order coefficient matrices may be expressed as follow

\[
C^{(2)}_{ij} = \sum_{k=1}^{N} C_{ik}^{(1)} C_{kj}^{(1)} \quad for \quad i, j = 1, 2, 3, \ldots, N
\]

\[
C^{(3)}_{ij} = \sum_{k=1}^{N} C_{ik}^{(1)} C_{kj}^{(2)} \quad for \quad i, j = 1, 2, 3, \ldots, N
\]

\[
C^{(4)}_{ij} = \sum_{k=1}^{N} C_{ik}^{(1)} C_{kj}^{(3)} \quad for \quad i, j = 1, 2, 3, \ldots, N
\]

Various types of grid distributions which provide acceptable results have been introduced. However, in this section we use the normalized Chebyshev–Gauss–Lobatto grid points that are
Fig. 2.41 Effect of temperature dependency and power law index on the first mode frequency of $S - S$ FGM beams with $\delta = 0.04$ subjected to $UTR$ loading.

\[ x_i = \frac{1}{2} \left[ 1 - \cos \left( \pi \times \frac{i - 1}{N - 1} \right) \right] \quad \text{for } i = 1, 2, 3, \ldots, N \]

For more details about the GDQ and method of distribution of grid points, one may refer to [86, 87].
Fig. 2.43  Influences of three-parameters nonlinear elastic foundation \((K^*_w, K^*_g, K^*_{NL})\) on the first mode frequency of the linearly graded \(C - C\) FGM beam with \(\delta = 0.04\) subjected to \(UTR\) loading.

Fig. 2.44  Influences of three-parameters nonlinear elastic foundation \((K^*_w, K^*_g, K^*_{NL})\) on the first mode frequency of the linearly graded \(S - S\) FGM beam with \(\delta = 0.04\) subjected to \(UTR\) loading.

**Appendix B**

The governing equations and the associated equations for the pre/post-buckling equilibrium states of the beam are

\[
e_1 \sum_{j=1}^{N} C_{ij}^{(2)} U_{sj}^* + e_1 \delta^2 \left( \sum_{j=1}^{N} C_{ij}^{(1)} W_{sj}^* \right) \sum_{j=1}^{N} C_{ij}^{(2)} W_{sj}^* + \delta e_2 \sum_{j=1}^{N} C_{ij}^{(2)} \varphi_{sj} = 0
\]
Fig. 2.45  Effect of various boundary conditions of linearly graded FGM beam on the dimensionless frequency and deflection with $\delta = 0.04$ subjected to $UTR$ loading

![Graph showing the effect of various boundary conditions of linearly graded FGM beam on the dimensionless frequency and deflection with $\delta = 0.04$ subjected to $UTR$ loading.]

Fig. 2.46  Influences of various power law indices and temperature dependency on the first frequency of the $C \rightarrow C$ FGM beam with $\delta = 0.04$ subjected to $NLT D$ loading

![Graph showing the influences of various power law indices and temperature dependency on the first frequency of the $C \rightarrow C$ FGM beam with $\delta = 0.04$ subjected to $NLT D$ loading.]

$$\kappa e_1 \left( \sum_{j=1}^{N} C_{ij}^{(1)} \varphi_{sj} + \delta \sum_{j=1}^{N} C_{ij}^{(2)} W_{sj}^* \right) + \left\{ e_1 \left[ \sum_{j=1}^{N} C_{ij}^{(1)} U_{sj}^* + \frac{1}{2} \delta^2 \left( \sum_{j=1}^{N} C_{ij}^{(1)} W_{sj}^* \right) \right] \right. \right.$$  

$$+ \delta e_2 \sum_{j=1}^{N} C_{ij}^{(1)} \varphi_{sj} - \frac{1}{12} \delta^2 \sum_{j=1}^{N} C_{ij}^{(0)} N_j T_*^* \left. \right\} \delta \sum_{j=1}^{N} C_{ij}^{(2)} W_{sj}^* - \frac{1}{12} K_{w}^* \delta^3 \sum_{j=1}^{N} C_{ij}^{(0)} W_{sj}^*$$
Fig. 2.47  Effect of various power law indices and temperature dependency on the $S - S$ FGM beams with $\delta = 0.04$ subjected to NLTD loading

\[ + \frac{1}{12} K^*_g \delta^3 \sum_{j=1}^{N} C^{(2)}_{ij} W^*_s j - \frac{1}{12} \delta^3 K^*_{NL} \left( \sum_{j=1}^{N} C^{(0)}_{ij} W^*_s j \right)^3 = 0 \]

\[ \kappa e_1 \left( \sum_{j=1}^{N} C^{(0)}_{ij} \varphi_s j + \delta \sum_{j=1}^{N} C^{(1)}_{ij} W^*_s j \right) - \delta^2 e_3 \sum_{j=1}^{N} C^{(2)}_{ij} \varphi_s j \]

\[ - \delta e_2 \left[ \sum_{j=1}^{N} C^{(2)}_{ij} U^*_s j + \delta^2 \left( \sum_{j=1}^{N} C^{(1)}_{ij} W^*_s j \right) \sum_{j=1}^{N} C^{(2)}_{ij} W^*_s j \right] = 0 \quad i = 1, 2, 3, \ldots, N \]

Here, $C^{(0)}_{ij}$ is the Kronecker delta which is equal to one, when $i = j$, otherwise is equal to zero. Also, $C^{(1)}_{ij}$ and $C^{(2)}_{ij}$ are the weighting coefficient matrices of first and second order differentiations, respectively. Besides, subscript 's' indicates the static displacement. The beam is divided into $N$ grid points which indicate the number of nodes in the $\xi$ direction.

The boundary conditions at edge points ($i = 1, N$) may be written as

For the clamped end:

\[ U^*_s i = W^*_s i = \varphi^*_s i = 0 \]

For the simply supported edge:

\[ U^*_s i = W^*_s i = M^*_{x,s i} = 0 \]
For the roller edge:

\[ U_{si}^* = \varphi_{si} = Q_{x,z,si}^* + (K_0 + N_{x,si}) \delta \frac{dW_{si}^*}{dx} = 0 \]

Appendix C

The governing equations and the associated boundary conditions for the small-scale vibrations of a beam in pre/post-buckling regimes are

\[
e_1 \sum_{j=1}^{N} C_{ij}^{(2)} U_{dj}^* + e_1 \delta^2 \frac{\partial^2 W_{si}^*}{\partial \xi^2} \sum_{j=1}^{N} C_{ij}^{(1)} W_{dj}^* + e_1 \delta^2 \frac{\partial W_{si}^*}{\partial \xi} \sum_{j=1}^{N} C_{ij}^{(2)} W_{dj}^* + \delta e_2 \sum_{j=1}^{N} C_{ij} \varphi_{dj} = 0
\]

\[
- \frac{\delta^2}{12} N \kappa^* \left[ \delta \sum_{j=1}^{N} C_{ij}^{(2)} W_{dj}^* + \left[ e_1 \left( \sum_{j=1}^{N} C_{ij}^{(1)} U_{dj}^* + \delta^2 \frac{\partial W_{si}^*}{\partial \xi} \sum_{j=1}^{N} C_{ij}^{(1)} W_{dj}^* \right) + e_2 \sum_{j=1}^{N} C_{ij}^{(2)} \varphi_{dj} \right] \frac{\partial^2 W_{si}^*}{\partial \xi^2} \right]
\]

For the small amplitude free vibration analysis one may write

\[
\frac{\partial^2}{\partial \eta^2} < U_{dj}^*, W_{dj}^*, \varphi_{dj} > = -\omega^2 < U_{dj}^*, W_{dj}^*, \varphi_{dj} > .
\]
The boundary conditions at edge points \((i = 1, N)\) may be written as

For the clamped edge:
\[
U_{di}^* = W_{di}^* = \varphi_{di} = 0
\]

For the simply supported edge:
\[
U_{di}^* = W_{di}^* = M_{x,di}^* = 0
\]

For the roller edge:
\[
U_{di}^* = \varphi_{di} = Q_{xz,di}^* + (K_g + N_{x,si})\delta \frac{d W_{di}^*}{d \xi} + N_{x,di} \delta \frac{d W_{si}^*}{d \xi} = 0
\]

2.10 FGM Beams, Thermal Dynamic Buckling

Dynamic buckling is a complicated behavior which should be explored through the response of non-linear equations of motion of a structure. Definition of a dynamically buckled structure strongly depends upon the selected criterion. A wealth review on the concept of dynamic buckling and its applications to solid structures is reported in a review paper by Simitses [89] and also documented in a book by Simitses [90]. Among the most well-known and suitable criteria, the equation of motion criterion of Budiansky–Roth [91] (which is also known as the Budiansky–Hutchinson for initially imperfect structures [92]), the phase-plane approach of Hoff-Hsu [93], the modified total potential energy approach of Hoff-Simitses [94], displacement control approach of Volmir [95], quasi-bifurcation dynamic buckling of Kleiber et al. [96] and the criterion of Kubiak [97] or Kounadis [98] are the most frequently used ones. Each criterion has its own advantages and shortcomings. Meanwhile, the Budiansky–Roth criterion is the most popular one since is easy to be used in computer programming and has no limitation in structural analysis [89].

Referring to the thermal dynamic buckling, the Budiansky-Roth criterion is applied successfully to cylindrical shells [99–104], plates [105, 106] and spherical caps [107, 108]. However, thermal dynamic buckling of beams made of FGMs or even homogeneous materials based on the Budiansky–Roth criterion is not frequent in literature. The thermal dynamic buckling of FGM beams under rapid heating is reported recently in [88]. This research, however, is developed based on the Hoff–Simitses criterion. As known, the Hoff-Simitses criterion yields only the magnitude of critical temperature in which dynamic buckling phenomenon occurs and does not establish the dynamic sense of the structure prior or at the onset of buckling.

This section examines the thermal dynamic buckling and imperfection sensitivity of the FGM beams subjected to uniform rapid heating [111]. Temperature dependency, initial imperfections, and contact of a three-parameter conventional non-linear elastic foundation are also taken into account. The Timoshenko beam theory, geometrical non-linearity in the von-Karman sense, and uncoupled thermoelastic
constitutive law of a continuum medium are incorporated together to establish the Hamiltonian of the system. The conventional multi-term Ritz method is applied to the Hamiltonian of the system to establish the matrix representation of the non-linear equations of motion. To solve the highly coupled non-linear equations in time and space domains, a hybrid Newmark–Newton–Raphson method is applied to the governing equations which traces the temporal evolution of beam deformations. Solution method is general and may be used for arbitrary grading profile and edge supports. The Budiansky-Roth criterion is applied successively to the equations of motion. It is shown that the FGM beams do not undergo any type of thermal dynamic buckling in the Budiansky-Roth sense. However, a sufficiently stiff non-linear softening elastic foundation violates the response of the beam and results in the unbounded motion type of dynamic buckling.

### 2.10.1 Fundamental Equations of the FGM Beam

A beam with length \( L \) and thickness \( h \) is considered in the conventional Cartesian coordinate system \((x, z)\), as shown in Fig. 2.30 [111]. Material properties are assumed to be temperature dependent obeying the power law distribution given by Eqs. (2.2.8) and (2.5.22).

In this study, it is assumed that the displacement field is expressed based on the first order shear deformation theory (FSDT) consistent with the Timoshenko assumptions. According to this theory, the displacement components of a generic point of the beam can be written in terms of the mid-plane displacement components \((u_0, w_0)\) given by Eq. (2.2.1) such that [80]

\[
\begin{align*}
u(x, z, t) &= u_0(x, t) + z\phi(x, t) \\
w(x, z, t) &= w_0(x, t)
\end{align*}
\]  

(2.10.1)

where in the above equation \( \phi \) denotes the transverse normal rotation about \( x \) axis.

Considering the von-Karman type of geometrical non-linearity, consistent with the small strains and moderate rotations, the strain-displacement relations may be written in terms of the mid-plane displacement components as

\[
\begin{align*}
\varepsilon_{xx} &= u_{0,x} + \frac{1}{2}w_{0,x}^2 + w_{0,x}w_{0,x}^\ast + z\phi \\
\gamma_{xz} &= w_{0,x} + \phi
\end{align*}
\]  

(2.10.2)

In which a comma indicates the partial derivative with respect to \( x \)-direction. Besides, \( w^\ast \) is the initial imperfection function through the beam which demonstrates a deviation with respect to the flat condition.

Under the uncoupled thermoelastic assumptions, the constitutive law for the linear thermoelastic FGM beam exposed to thermal loadings will be
\[ \sigma_{xx} = E(z, T)\varepsilon_{xx} - E(z, T)\alpha(z, T)(T - T_0) \]
\[ \sigma_{xz} = G(z, T)\gamma_{xz} \]  

(2.10.3)

In the above equation, \( \sigma_{xx} \) and \( \sigma_{xz} \) are the axial and through-the-thickness shear stresses, respectively, and \( E(z) \) and \( G(z) \) are the elastic and shear modules. Furthermore, \( T \) and \( T_0 \) denote the temperature distribution and the initial temperature, respectively.

### 2.10.2 Governing Equations

The governing equations of the FGM Timoshenko beam exposed to sudden uniform temperature rise may be obtained based on the concept of Hamilton’s principle. This principle is the dynamic form of the virtual displacement principle and may be written as

\[ \delta \int_{t_1}^{t_2} [T - (V + U)] dt = 0 \]  

(2.10.4)

In which \( \delta T \) and \( \delta (U + V) \) represent the virtual kinetic energy and the virtual total potential energy, respectively.

The virtual kinetic energy of the beam per width is equal to summation of the virtual kinetic energy in longitudinal and transversal directions. Accordingly, one may write

\[ \int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \int_0^L \int_{-0.5h}^{+0.5h} \rho(z)(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}) \, dz \, dx \, dt \]  

(2.10.5)

By substituting Eq. (2.10.1) into (2.10.5) and performing some proper mathematical operations, the virtual kinetic energy of beam per unit width in terms of the mid-plane displacement components for the FGM beam becomes

\[ \int_{t_1}^{t_2} \delta T dt = -\int_{t_1}^{t_2} \int_0^L \{ I_1(\ddot{u}_0\delta u_0 + \ddot{w}_0\delta w_0) + I_2(\ddot{\phi}\delta u_0 + \ddot{u}_0\delta\phi) + I_3\ddot{\phi}\delta\phi \} \, dx \, dt \]  

(2.10.6)

In which the inertia resultants \( I_1, I_2, \) and \( I_3 \) are defined by

\[ (I_1, I_2, I_3) = \int_{-0.5h}^{+0.5h} \rho(z)(1, z, z^2) \, dz \]  

(2.10.7)

The virtual total potential energy of the FG beam is equal to the sum of the virtual energy of external applied loads, which is absent in this study, the virtual strain energy of the beam, and the virtual energy of elastic foundation. Thus, the virtual potential
energy of the beam per unit width is equal to

$$\int_{t_1}^{t_2} \delta(U + V)dt = \int_{t_1}^{t_2} \left\{ \int_0^L \int_{-0.5h}^{+0.5h} \left( \sigma_{xx} \delta \varepsilon_{xx} + \sigma_{xz} \delta \gamma_{xz} \right) dxdz + \int_0^L \left( K_w w \delta w + K_s w_{,x} \delta w_{,x} + K_{nl} w^3 \delta w \right) dx \right\} dt \quad (2.10.8)$$

where in the above equation, $K_w$, $K_s$, and $K_{nl}$ are the Winkler, Pasternak, and nonlinear constants of elastic foundation, respectively. Positive values of $K_{nl}$ indicate softening nonlinear elastic medium whereas the negative values of $K_{nl}$ are associated with the softening non-linear elastic medium. By substituting Eq. (2.10.1) into (2.10.8) and accomplishing some mathematical operations, the virtual total potential energy per unit width in terms of the mid-plane displacement components for the FGM beam may be written in the following form [111]

$$\int_{t_1}^{t_2} \delta(U + V)dt = \int_{t_1}^{t_2} \left\{ E_1 \left( u_{0,x} + \frac{1}{2} w_{0,x}^2 + w_{0,x} w_{,x}^* \right) + E_2 \phi_{,x} \right\} \delta u_{0,x} + \left\{ E_2 \left( u_{0,x} + \frac{1}{2} w_{0,x}^2 + w_{0,x} w_{,x}^* \right) + E_3 \phi_{,x} - M^T \right\} \delta \phi_{,x} + G_1 (\phi + w_{0,x}) \delta \phi + \left\{ E_1 \left( u_{0,x} + \frac{1}{2} w_{0,x}^2 + w_{0,x} w_{,x}^* \right) + E_2 \phi_{,x} - N^T \right\} \left( w_{0,x} + w_{,x}^* \right) \delta w_{0,x} + (G_1 (\phi + w_{0,x}) + K_s w_{0,x}) \delta w_{0,x} + (K_w w_{0} \pm K_{nl} w_{0}^3) \delta w_{0} \right\} dxdt \quad (2.10.9)$$

where in the above equation $E_1$, $E_2$, and $E_3$ are the stretching, coupling bending-stretching, and bending stiffness, respectively, and $G_1$ is the shear stiffness which are defined by

$$(E_1, E_2, E_3, G_1) = \int_{-0.5h}^{+0.5h} (E(z), zE(z), z^2E(z), G(z))dz \quad (2.10.10)$$

Besides, $N^T$ and $M^T$ are, respectively, the thermal force and moment resultants generated in derivation of Eq. (2.10.9) as

$$(N^T, M^T) = \int_{-0.5h}^{+0.5h} (1, z)E(z, T)\alpha(z, T)(T - T_0)dz \quad (2.10.11)$$

**Solution Method**

At this stage, to accomplish the spatial approximation, the displacement field is expressed in terms of the proper shape functions based on the well-known Ritz method as follows [111]
### Table 2.18  Appropriate $p$-Ritz shape functions associated with the boundary conditions (2.10.13) [111]

<table>
<thead>
<tr>
<th>$p$-Ritz functions</th>
<th>$C - C$</th>
<th>$S - C$</th>
<th>$S - S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_m^u$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
</tr>
<tr>
<td>$N_m^w$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
</tr>
<tr>
<td>$N_m^{\phi}$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
<td>$(\frac{x}{L})^m (1 - \frac{x}{L})$</td>
<td>$(\frac{x}{L})^{m-1}$</td>
</tr>
</tbody>
</table>

In Eq. (2.10.12) $M$ is a required number to assure the convergence of the series. Besides, $N_m^u$, $N_m^w$, and $N_m^{\phi}$ are the Ritz approximation functions which should be chosen according to the essential type of boundary conditions. Two types of edge supports, including immovable simply-supported ($S$) and immovable clamped ($C$), are considered. Mathematical interpretation of these supports are

\[
S : u_0 = w_0 = M_{xx} = 0 \\
C : u_0 = w_0 = \phi = 0
\] (2.10.13)

Since the adoption of shape functions depends only on the essential type of boundary conditions [35], various functions may be chosen as the shape functions. In this study, polynomial type of shape functions are considered as the Ritz approximation functions. Table 2.18 presents these admissible shape functions for three types of boundary conditions namely; simply supported-simply supported ($S - S$), clamped-simply supported ($C - S$), and clamped-clamped ($C - C$).

It is to be noticed that the expressed shape functions in Table 2.18 are adopted according to the boundary conditions which are described by Eq. (2.10.13).

Substitution of the series expansion (2.10.12) into the virtual energies (2.10.6) and (2.10.9) and subsequently substitution of the results into the Hamilton principle (2.10.4) leads to the matrix representation of the equations of motion as

\[
\begin{bmatrix}
M_{uu} & M_{uw} & M_{u\phi} \\
M_{wu} & M_{ww} & M_{w\phi} \\
M_{\phi u} & M_{\phi w} & M_{\phi\phi}
\end{bmatrix}
\begin{bmatrix}
\ddot{U} \\
\ddot{W} \\
\ddot{\phi}
\end{bmatrix}
+ \begin{bmatrix}
K_{uu} & K_{uw} & K_{u\phi} \\
K_{wu} & K_{ww} & K_{w\phi} \\
K_{\phi u} & K_{\phi w} & K_{\phi\phi}
\end{bmatrix}
\begin{bmatrix}
U \\
W \\
\phi
\end{bmatrix}
= \begin{bmatrix}
F_u \\
F_w \\
F_{\phi}
\end{bmatrix}
\]

(2.10.14)

For the interest of brevity, elements of the generalized mass matrix, stiffness matrix, and force vector are given at the end of this section.
In a compact form, Eq. (2.10.14) may be written as

\[
[M(T)] \{\ddot{X}\} + [K(T, X)] \{X\} = \{F(T)\} \tag{2.10.15}
\]

It is noticed that due to accountancy of the von-Karman type of geometrical non-linearity, the generalized stiffness matrix is a function of unknown time-dependent nodal vector \{X\}.

There are several available numerical methods to approximate the second-order time derivatives and convert the differential equations into the algebraic equations. Among them, the constant acceleration method of time-approximation schemes is widely used in structural dynamics [69]. Subsequently, here, following the Newmark method, temporal approximation is done. By utilizing this method, Eq. (2.10.15) can be reduced to

\[
[\hat{K}(T, X)] \{X\}_{j+1} = \{\hat{F}(T)\}_{j,j+1} \tag{2.10.16}
\]

where

\[
[\hat{K}(T, X)] = [K(T, X)] + a_0 [M(T)]
\]

\[
\{\hat{F}(T)\} = \{F(T)\}_{j+1} + [M(T)] \left( a_0 \{X\}_j + a_1 \{\dot{X}\}_j + a_2 \{\ddot{X}\}_j \right) \tag{2.10.17}
\]

and

\[
a_0 = \frac{1}{\beta \Delta t^2}, \quad a_1 = \frac{1}{\beta \Delta t}, \quad a_2 = \frac{1 - 2\beta}{2\beta} \tag{2.10.18}
\]

Once the solution \{X\} is known at \(t_{j+1} = (j + 1) \Delta t\), the first and second derivatives of \{X\} at \(t_{j+1}\) can be computed from

\[
\{\ddot{X}\}_{j+1} = a_0 ((X)_{j+1} - (X)_j) - a_1 \{\dot{X}\}_j - a_2 \{\dddot{X}\}_j
\]

\[
\{\dot{X}\}_{j+1} = \{\dot{X}\}_j + a_3 \{\ddot{X}\}_j + a_4 \{\dddot{X}\}_{j+1} \tag{2.10.19}
\]

and

\[
a_3 = (1 - \alpha) \Delta t, \quad a_4 = \alpha \Delta t \tag{2.10.20}
\]

The resulting equations are solved at each time step using the information known from the preceding time step solution. At time \(t = 0\), the initial values of \{X\}, \{\dot{X}\}, and \{\ddot{X}\} are known or obtained by solving Eq. (2.10.15) at time \(t = 0\) and are used to initiate the time marching procedure. Since the beam is initially at rest, the initial values of \{X\} and \{\dot{X}\} are assumed to be zero. An iterative scheme should be applied to Eq. (2.10.15) to solve the resulting highly non-linear algebraic equations. In this section, the well-known Newton–Raphson iterative scheme is used in which the tangent stiffness matrix is evaluated based on the developed method in [69].
2.10.3 Numerical Investigation

The procedure outlined in the previous section is used herein to study the dynamic unbounded motion and imperfection sensitivity of the FGM beams under sudden thermal loading. Beam is resting on an elastic foundation. Constants of elastic foundation are normalized as given below [111]

\[
(k_w, k_s, k_{nl}) = \left( \frac{12K_w L^4}{E_c^{ref} h^3}, \frac{12K_s L^2}{E_c^{ref} h^3}, \frac{12K_{nl} L^4 h^2}{E_c^{ref} h^3} \right)
\]  

(2.10.21)

where \(E_c^{ref}\) represents the ceramic elasticity modulus at the reference temperature.

Since only the clamped and simply supported edges are taken into consideration, the initial imperfection function is assumed as

\[
w^* = \mu h \sin \left( \frac{\pi x}{L} \right)
\]  

(2.10.22)

where \(\mu\) indicates the out of plane amplitude of imperfection with respect to the flatness condition.

In all presented examples of this section, Stainless Steel (SUS304) and Silicon Nitride (Si3N4) are considered as the combination of FGM material constituents. Temperature dependent coefficients of these materials are given in Table 2.3. In order to effectively model the material properties, the temperature dependency of the material should be taken into account. In all examples, beam thickness and length are set equal to \(h = 4\) cm and \(L / h = 25\).

Comparison Studies

To demonstrate the validity and accuracy of the proposed solution method and the obtained formulations, comparison studies are provided. In the first one, dynamic critical buckling temperature differences of this study for the clamped FGM beam resting on an elastic foundation are compared with those reported in [88], which is performed based on the Hoff-Simitsis criterion. The beam is subjected to sudden uniform temperature rise. It should be emphasized that softening constant of elastic foundation is chosen a sufficiently large number to ensure that post-buckling equilibrium path becomes unstable [88]. Besides, analysis is performed for the TD case of material properties. Thus, the results for several power law index values are reported in Table 2.19 with considering \(\mu = 0.01\). In each case, relative difference is also provided. The imperfection shape function of [88] differs with the one used in this section. However, for small amplitudes of imperfection, the imperfection shapes will be the same. As seen, the results of two studies are close with small differences. These differences may be due to the different criteria, different beam theories, and also different numerical solutions.
Table 2.19  Comparison of dynamic critical buckling temperature difference of the $C - C$ FGM beam with $\mu = 0.01$ resting on a softening elastic foundation ($k_w, k_s, k_{nl}) = (10, 10, -1000)$ subjected to sudden uniform temperature rise between the results of this section [111] and those reported by Ghiasian et al. [88]

<table>
<thead>
<tr>
<th></th>
<th>$k = 0$</th>
<th>$k = 0.5$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 5$</th>
<th>$k = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>589.2</td>
<td>478.1</td>
<td>446.1</td>
<td>424.6</td>
<td>406.9</td>
<td>366.3</td>
</tr>
<tr>
<td>Ghiasian et al. [88]</td>
<td>590.9</td>
<td>480.3</td>
<td>448.9</td>
<td>426.3</td>
<td>409.8</td>
<td>367.7</td>
</tr>
<tr>
<td>Difference (%)</td>
<td>0.288</td>
<td>0.458</td>
<td>0.624</td>
<td>0.399</td>
<td>0.708</td>
<td>0.381</td>
</tr>
</tbody>
</table>

Parametric Studies

Based on the Budiansky-Roth criterion, dynamic buckling analysis of an FGM Timoshenko beam with initial geometric imperfection resting on a three-parameter elastic foundation is studied. Thermal load is considered as uniform temperature rise which is applied suddenly to the beam and its temporal dependency is chosen as a unit step function. In the present parametric studies, essential conditions for occurrence of dynamic buckling phenomena are studied. In addition, time history of midspan displacement and phase-plane curves of the FGM beams are investigated under various values of the applied sudden thermal loads. According to these curves, dynamic thermal buckling load level of the system is recognized for different conditions. Subsequently, influences of temperature-dependency, imperfection amplitude, power law index, and boundary conditions on thermal dynamic buckling phenomenon are studied.

Based on the Budiansky-Roth criterion, a large increase occurs in the deflection amplitude when the nonlinear equations of motion of the system are solved for different load levels [109]. According to the Budiansky-Roth criterion, in this study, dynamic buckling load is detected via trail and error scheme. Based on this method, the equations of motion of the beam are solved for several values of sudden thermal loads starting from a small value and being gradually increased. The dynamic buckling phenomenon for the FGM beam is studied for two cases; the FGM beams with and without elastic foundation.

For the first case, the maximum transverse midspan displacement of the $C - C$ FGM Timoshenko beams ($w(L/2, t)$) versus thermal load is represented in Fig. 2.48a [111]. The imperfection amplitude is chosen to be $\mu = 0.1$. Furthermore, the constant values of softening elastic foundation are chosen as $(k_w, k_s, k_{nl}) = (10, 10, -1000)$. It is seen that the maximum displacement increases smoothly with the exposed thermal loading until an unbounded motion which occurs at a higher level of temperature. According to the Budiansky-Roth criterion, this load is introduced as dynamic buckling load level of the system. It is to be mentioned that the correspondent post-buckling equilibrium path is unstable under this condition (with considering softening elastic foundation), as formerly discussed in [88]. For the second case, the maximum transverse midspan displacement of a $C - C$ FGM beam without elastic foundation
versus the applied thermal load is represented in Fig. 2.48b. As seen, no large jump is observed in the maximum displacement as the applied thermal load increases. It should be emphasized that the correspondent post-buckling equilibrium path for this case is stable as reported in [88, 110]. So, it can be concluded that the occurrence of dynamic buckling phenomenon is possible for the FGM beam, just by making the post-buckling equilibrium path unstable. As known, nonlinear equilibrium path of the FGM beams subjected to uniform temperature rise may be unique and stable or of the bifurcation type of buckling with stable post-buckling branch. Therefore, thermal dynamic buckling phenomenon in the Budiansky-Roth sense does not occur for the contact-less FGM beams under sudden uniform heating. Dynamic buckling indeed occurs when beam is resting on a sufficiently stiff softening elastic medium. The phrase sufficiently stiff softening elastic medium for the foundation means that foundation changes the static equilibrium path from stable to unstable.

Figure 2.48c, d, e and f reveal the same results for the FGM beams exposed to sudden thermal loads in two cases, with and without elastic foundation for the $C - S$ and $S - S$ cases of boundary conditions [111]. Discussions in these cases are the same with Fig. 2.48a, b.

In the next sections, only the FGM beams resting on sufficiently softening elastic foundation ($(k_w, k_s, k_{nl}) = (10, 10, -1000)$) in which the possibility of dynamic buckling occurrence exists, are studied.

Dynamic buckling load may be detected by tracing the transversal displacement of the structure during a time span under different magnitudes of the applied load levels. Subsequently, here, the transverse midspan displacement of the FGM beam under four levels of thermal loads for three types of boundary conditions including $(C - C), (C - S)$, and $(S - S)$ are represented in Figs. 2.49a, c and 2.50e, respectively. It is observed that for each case of boundary condition, simple oscillations with finite amplitudes are occurred under the first three load levels. By increasing only 0.1 K in magnitude of thermal shock and applying the fourth level of the thermal loads, the beams undergo unbounded displacements. These loads are identified as thermal dynamic buckling load level of the beams since only 0.1 K increase in temperature results in severe change in displacement. It should be emphasized that the type of dynamic buckling is unbounded, since the associated post-buckling equilibrium path of the beam under the associated static load is of the upper limit load type of instability with completely softening post-limit load behavior.

The phase-plane curves corresponding to traverse midspan displacement of the FGM beam are also depicted in Fig. 2.49b, d, e, respectively, for the $(C - C), (C - S)$, and $(S - S)$ types of boundary conditions. As seen, for each case of boundary condition, three stable dynamic solutions with related closed form curves exist and these curves are associated with three load levels less than the dynamic buckling temperature. However, a diverged curve is observed for dynamic buckling temperature.

It is to be noticed that the fluctuations in the phase-plane curves for the $C - S$ and $S - S$ cases exist because of the coupling between in-plane and out of plane vibrations in these cases. For the $C - C$ case of FGM beams, the induced bending moments due to thermal loading and geometrical non-linearity are compensated at the edge supports, whereas in the $S - C$ and $S - S$ cases, such feature does not exist and
Fig. 2.48 Maximum non-dimensional deflection in temporal evolution of the midspan of the temperature dependent FGM beams for various edge supports.

coupling between in-plane and out-of-plane motions results in such chaotic phase planes.

Figure 2.50 is known as the dynamic imperfection sensitivity curve in which the influence of amplitude imperfection is investigated on thermal dynamic buckling load. An FGM beam with $k = 1$ resting on a softening elastic foundation is considered. This investigation is done for three types of boundary conditions includ-
Fig. 2.49 Characteristics of temperature dependent FGM beams resting on softening elastic foundation and subjected to sudden heating. Right ones: Temporal evolution of non-dimensional mid-span lateral deflection for various rapid heating values, Left ones: The associated phase-planes.

ing C – C, C – S, and S – S and two model of material properties i.e. the TD and TID. As seen from Fig. 2.50, the dynamic buckling temperature difference is decreased with the increase of imperfection amplitude for each case of boundary condition. Besides, this decrease is much more noticeable for the lower values of imperfection amplitude. It should be mentioned that in this case the structure may be
called imperfection sensitive in dynamic sense, since the dynamic limit load temperature decreases noticeably with the introduction of higher imperfection amplitude. Furthermore, considering temperature dependency leads to the underestimation of dynamic buckling temperature for all types of boundary conditions.

The comparison between the imperfection sensitivity curves associated with three types of boundary conditions reveals that for each value of imperfection parameter, the maximum dynamic buckling temperatures are obtained for the $C - C$ case of boundary condition and the minimum ones are related to the $S - S$ case.

The influence of power law index on dynamic buckling temperature difference for imperfect FGM beam resting on softening elastic foundation is exhibited in Fig. 2.51. The imperfection parameter of the beam is considered to be $\mu = 0.1$ and three types of boundary conditions, namely; $C - C$, $C - S$, and $S - S$ are investigated. Besides, results are presented for both $TID$ and $TD$ cases of material properties. As seen, for each type of boundary condition, dynamic buckling temperature differences are reduced by increasing the power law index. Specially, this reduction is much more profound for values of $k < 2$. In addition, results reveal the higher amount of $\Delta T_{cr}$ for the $TID$ case of material properties. In other words, $\Delta T_{cr}$ under $TID$ assumption stands as the upper bound for those obtained under the $TD$ case of material properties for all types of boundary conditions.

Similar to the previous section, by comparing results obtained for the three types of boundary conditions, it could be understood that for each value of power law index, the maximum thermal dynamic buckling load is achieved for the $C - C$ boundary condition and the minimum ones are associated with the $S - S$ case.

The elements of the stiffness matrix of Eq. (2.10.14) are [111]
Fig. 2.51 Influence of power law index on dynamic buckling temperature difference of the FGM beams resting on softening elastic foundation

\[ K_{uu}^{nm} = \int_0^L E_1 \frac{dN_m^u}{dx} \frac{dN_n^u}{dx} dx \]

\[ K_{uw}^{nm} = \frac{1}{2} \int_0^L \left( E_1 \frac{dN_m^u}{dx} \frac{dw_0}{dx} \frac{dN_n^u}{dx} + 2E_1 \frac{dN_m^w}{dx} \frac{dw^*}{dx} \frac{dN_n^u}{dx} \right) dx \]

\[ K_{w\phi}^{nm} = \int_0^L E_2 \frac{dN_m^\phi}{dx} \frac{dN_n^u}{dx} dx \]

\[ K_{uw}^{nm} = \int_0^L \left( E_1 \frac{dN_m^u}{dx} \frac{dw_0}{dx} \frac{dN_n^w}{dx} + E_1 \frac{dN_m^w}{dx} \frac{dw^*}{dx} \frac{dN_n^w}{dx} \right) dx \]

\[ \frac{1}{2} \int_0^L \left( \frac{1}{2} E_2 \frac{dN_m^u}{dx} \left( \frac{dw_0}{dx} \right)^2 \frac{dN_n^w}{dx} + \frac{3}{2} E_1 \frac{dN_m^w}{dx} \frac{dw_0}{dx} \frac{dw^*}{dx} \frac{dN_n^w}{dx} \right) dx \]

\[ + E_1 \frac{dN_m^w}{dx} \left( \frac{dw^*}{dx} \right)^2 \frac{dN_n^w}{dx} - N_T \frac{dN_m^w}{dx} \frac{dN_n^w}{dx} + G_1 \frac{dN_m^w}{dx} \frac{dN_n^w}{dx} \]

\[ + K_w N_m^w N_n^w + K_s \frac{dN_m^w}{dx} \frac{dN_n^w}{dx} + K_{nl} N_m^w N_n^w \left( \frac{dN_m^w}{dx} \right) dx \]

\[ K_{w\phi}^{nm} = \int_0^L \left( E_2 \frac{dN_m^\phi}{dx} \frac{dw_0}{dx} \frac{dN_n^\phi}{dx} + E_2 \frac{dN_m^\phi}{dx} \frac{dw^*}{dx} \frac{dN_n^\phi}{dx} + G_1 \frac{dN_m^\phi}{dx} \frac{dN_n^\phi}{dx} \right) dx \]

\[ K_{\phi u}^{nm} = \int_0^L E_2 \frac{dN_m^\phi}{dx} \frac{dN_n^\phi}{dx} dx \]

\[ K_{\phi w}^{nm} = \frac{1}{2} \int_0^L \left( E_2 \frac{dN_m^w}{dx} \frac{dN_n^\phi}{dx} + 2E_2 \frac{dN_m^w}{dx} \frac{dw^*}{dx} \frac{dN_n^\phi}{dx} + 2G_1 \frac{dN_m^w}{dx} \frac{dN_n^\phi}{dx} \right) dx \]

\[ K_{\phi\phi}^{nm} = \int_0^L \left( E_3 \frac{dN_m^\phi}{dx} \frac{dN_n^\phi}{dx} + G_1 \frac{dN_m^\phi}{dx} \frac{dN_n^\phi}{dx} \right) dx \]
The elements of the mass matrix are

\[
M_{nm}^{uu} = \int_0^L I_1 N_m^u N_n^u \, dx \\
M_{nm}^{uw} = 0 \\
M_{nm}^{u\phi} = \int_0^L I_2 N_m^\phi N_n^u \, dx \\
M_{nm}^{wu} = 0 \\
M_{nm}^{ww} = \int_0^L I_1 N_m^w N_n^w \, dx \\
M_{nm}^{w\phi} = 0 \\
M_{nm}^{\phi u} = \int_0^L I_2 N_m^{\phi u} N_n^\phi \, dx \\
M_{nm}^{w\phi} = 0 \\
M_{nm}^{\phi \phi} = \int_0^L I_3 N_m^{\phi \phi} N_n^{\phi} \, dx
\]

and the elements of the force vector are

\[
F_n^u = \int_0^L N^T \frac{dN_n^u}{dx} \, dx \\
F_n^w = \int_0^L N^T \frac{du^*}{dx} \frac{dN_n^w}{dx} \, dx \\
F_n^\phi = \int_0^L M^T \frac{dN_n^{\phi}}{dx} \, dx
\]

### 2.11 Problems

1. Use Eqs. (2.2.5), (2.2.6), (2.2.9) and (2.3.1) to derive the equilibrium equations (2.3.2).
2. Derive the stability equations (2.4.3) using Eqs. (2.4.1), (2.4.2) and (2.3.2).
3. Employing the stability equations (2.4.3) and by eliminating \( u_1 \) and \( \phi_1 \) arrive at (2.4.4).
4. Find the determinant of matrix equation (2.5.6) and prove that it yields Eq. (2.5.7).
5. Find the critical thermal force of a \( C - S \) type of boundary condition of the Timoshenko beam with isotropic material property of length \( L \) and the modulus of elasticity \( E \). Assume that the beam is under uniform temperature rise \( \Delta T \).
6. Obtain the parameter $\mu$ given by Eq. (2.6.11) for the piezo-FGM beams using Eq. (2.4.4).

7. What is the thermal buckling load for the same beam of Problem 6, when the beam is under uniform temperature rise?

8. Find the thermal buckling load of Problem 7, when the beam material is made of an isotropic metal. Find the same buckling load when the material is pure 8 ceramic.

9. Reconsider Problem 8 and check the delay of thermal buckling load when a $\pm 500$ Volts is applied to the beam.

References


References


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