Chapter 1
Tensor Notation

A Working Knowledge in Tensor Analysis

This chapter is not meant as a replacement for a course in tensor analysis, but it will provide a sufficient working background to tensor notation and algebra.

1.1 Cartesian Frame of Reference

Physical quantities encountered are either scalars (e.g., time, temperature, pressure, volume, density), or vectors (e.g., displacement, velocity, acceleration, force, torque), or tensors (e.g., stress, displacement gradient, velocity gradient, alternating tensors – we deal mostly with second-order tensors). These quantities are distinguished by the following generic notation:

s denotes a scalar (lightface italic)
$\mathbf{u}$ denotes a vector (boldface)
$\mathbf{F}$ denotes a tensor (boldface)

The distinction between vector and tensor is usually clear from the context. When they are functions of points in a three-dimensional Euclidean space $\mathbb{E}$, they are called **fields**. The set of all vectors (or tensors) form a normed vector space $\mathbb{U}$.

Distances and time are measured in the Cartesian frame of reference, or simply frame of reference, $\mathcal{F} = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, which consists of an origin O, a clock, and an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, see Fig. 1.1,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3 \tag{1.1}$$

where the Kronecker delta is defined as

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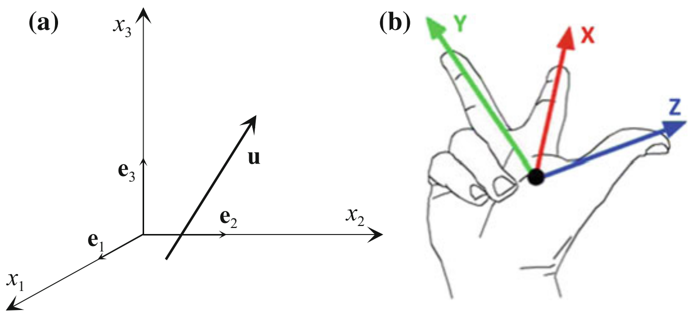

Fig. 1.1 **a** Cartesian frame of reference and **b** right hand rule

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.2}$$

We only deal with right-handed frames of reference (applying the right-hand rule, when the thumb is in direction 1, and the forefinger in direction 2, the middle finger lies in direction 3, or an even permutation of this), where $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1$.

The Cartesian components of a vector $\mathbf{u}$ are given by

$$u_i = \mathbf{u} \cdot \mathbf{e}_i \tag{1.3}$$

so that one may write

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i = u_i \mathbf{e}_i. \tag{1.4}$$

Here we have employed the *summation convention*, i.e., whenever there are repeated subscripts, a summation is implied over the range of the subscripts, from (1, 2, 3). For example,

$$A_{ij} B_{jk} = \sum_{j=1}^{3} A_{ij} B_{jk}. \tag{1.5}$$

This short-hand notation is due to Einstein (Fig. 1.2), who argued that physical laws must not depend on coordinate systems, and therefore must be expressed in tensorial format. This is the essence of the *Principle of Frame Indifference*, to be discussed later.

The *alternating tensor* is defined as

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1, & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ \;\;0, & \text{otherwise.} \end{cases} \tag{1.6}$$

Fig. 1.2 Albert Einstein (1879–1955) got the Nobel Prize in Physics in 1921 for his explanation in photoelectricity. He derived the effective viscosity of a dilute suspension of neutrally buoyant spheres, $\eta = \eta_s(1 + \frac{5}{2}\phi)$, η_s: the solvent viscosity, ϕ: the sphere volume fraction

1.1.1 Position Vector

In the frame $\mathcal{F} = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the position vector is denoted by

$$\mathbf{x} = x_i \mathbf{e}_i, \tag{1.7}$$

where x_i are the components of $\mathbf{x}$.

1.2 Frame Rotation

Consider the two frames of references, $\mathcal{F} = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathcal{F}' = \left\{O; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\right\}$, as shown in Fig. 1.3, one obtained from the other by a rotation. Hence,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad \mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}.$$

Define the cosine of the angle between $\left(\mathbf{e}_i, \mathbf{e}'_j\right)$ as

$$A_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j.$$

Thus A_{ij} can be regarded as the components of $\mathbf{e}'_i$ in $\mathcal{F}$, or the components of $\mathbf{e}_j$ in $\mathcal{F}'$. We write

$$\mathbf{e}'_p = A_{pi} \mathbf{e}_i \quad A_{pi} A_{qi} = \delta_{pq}.$$

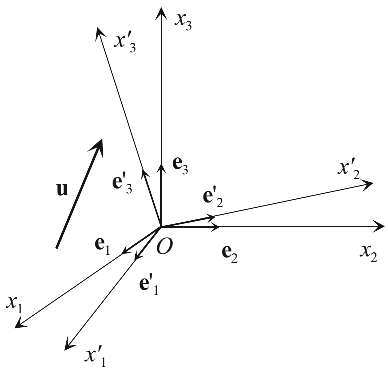

Fig. 1.3 Two frames of reference sharing a common origin

Similarly

$$\mathbf{e}_i = A_{pi}\mathbf{e}'_p \quad A_{pi}A_{pj} = \delta_{ij}.$$

1.2.1 Orthogonal Matrix

A matrix is said to be an orthogonal matrix if its inverse is also its transpose; furthermore, if its determinant is $+1$, then it is a proper orthogonal matrix. Thus $[\mathbf{A}]$ is a proper orthogonal matrix.

We now consider a vector $\mathbf{u}$, expressed in either frame $\mathcal{F}$ or $\mathcal{F}'$,

$$\mathbf{u} = u_i\mathbf{e}_i = u'_j\mathbf{e}'_j.$$

Taking scalar product with either base vector,

$$u'_i = \mathbf{e}'_i \cdot \mathbf{e}_j u_j = A_{ij}u_j,$$
$$u_j = \mathbf{e}_j \cdot \mathbf{e}'_i u'_i = A_{ij}u'_i.$$

In matrix notation,

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad [\mathbf{u}] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \left[\mathbf{u}'\right] = \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix},$$

we have

$$\begin{aligned} \left[\mathbf{u}'\right] &= [\mathbf{A}]\,.\,[\mathbf{u}]\,, \quad & [\mathbf{u}] &= [\mathbf{A}]^T \cdot \left[\mathbf{u}'\right], \\ u'_i &= A_{ij}u_j, & u_j &= A_{ij}u'_i. \end{aligned} \tag{1.8}$$

In particular, the position transforms according to this rule

$$\mathbf{x} = x_i' \mathbf{e}'_i = x_j \mathbf{e}_j \qquad x_i' = A_{ij} x_j \text{ or } x_j = A_{ij} x_i'.$$

1.2.2 Rotation Matrix

The matrix A is called a rotation – in fact a proper rotation ($det\mathbf{A} = +1$).

1.3 Tensors

1.3.1 Zero-Order Tensors

Scalars, which are invariant under a frame rotation, are said to be tensors of zero order.

1.3.2 First-Order Tensor

A set of three scalars referred to one frame of reference, written collectively as $\mathbf{v} = (v_1, v_2, v_3)$, is called a tensor of first order, or a vector, if the three components transform according to (1.8) under a frame rotation.
Clearly,

- If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $\mathbf{u} + \mathbf{v}$ is also a vector.
- If $\mathbf{u}$ is a vector, then $\alpha\mathbf{u}$ is also a vector, where α is a real number.

The set of all vectors form a vector space $\mathcal{U}$ under addition and multiplication. In this space, the usual scalar product can be shown to be an inner product. With the norm induced by this inner product, $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$, $\mathcal{U}$ is a normed vector space. We also refer to a vector $\mathbf{u}$ by its components, u_i.

1.3.3 Outer Products

Consider now two tensors of first order, u_i and v_i. The product $u_i v_j$ represents the outer product of $\mathbf{u}$ and $\mathbf{v}$, and written as (the subscripts are assigned from left to right by convention),

$$[\mathbf{uv}] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}.$$

In a frame rotation, from $\mathcal{F}$ to $\mathcal{F}'$, the components of this change according to

$$u_i' v_j' = A_{im} A_{jn} u_m v_n.$$

1.3.4 Second-Order Tensors

In general, a set of 9 scalars referred to one frame of reference, collectively written as $\mathbf{W} = [W_{ij}]$, transformed to another set under a frame rotation according to

$$W_{ij}' = A_{im} A_{jn} W_{mn}, \tag{1.9}$$

is said to be a second-order tensor, or a two-tensor, or simply a tensor (when the order does not have to be explicit). In matrix notation, we write

$$\left[\mathbf{W}'\right] = [\mathbf{A}]\,[\mathbf{W}]\,[\mathbf{A}]^T \text{ or } \mathbf{W}' = \mathbf{AWA}^T \text{ or } W_{ij}' = A_{ik} W_{kl} A_{jl}.$$

In the direct notation, we denote a tensor by a bold face letter (without the square brackets). This direct notation is intimately connected to the concept of a linear operator, e.g., Gurtin [34].

1.3.5 Third-Order Tensors

A set of 27 scalars referred to one frame of reference, collectively written as $\mathbf{W} = [W_{ijk}]$, transformed to another set under a frame rotation according to

$$W_{ijk}' = A_{il} A_{jm} A_{kn} W_{lmn}, \tag{1.10}$$

is said to be a third-order tensor.

Obviously, the definition can be extended to a set of $3n$ scalars, and $\mathbf{W} = [W_{i_1 i_2 \ldots i_n}]$ (n indices) is said to be an n -order tensor if its components transform under a frame rotation according to

$$W_{i_1 i_2 \ldots i_n}' = A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} W_{j_1 j_2 \ldots j_n}. \tag{1.11}$$

We will deal mainly with vectors and tensors of second order. Usually, a higher-order (higher than 2) tensor is formed by taking outer products of tensors of lower orders, for example the outer product of a two-tensor $\mathbf{T}$ and a vector $\mathbf{n}$ is a third-order tensor $\mathbf{T} \otimes \mathbf{n}$. One can verify that the transformation rule (1.11) is obeyed.

1.3.6 Transpose Operation

The components of the transpose of a tensor **W** are obtained by swapping the indices:

$$[\mathbf{W}]_{ij} = W_{ij}, \quad [\mathbf{W}]^T_{ij} = W_{ji}.$$

A tensor **S** is *symmetric* if it is unaltered by the transpose operation,

$$\mathbf{S} = \mathbf{S}^T, \qquad S_{ij} = S_{ji}.$$

It is *anti-symmetric* (or *skew*) if

$$\mathbf{S} = -\mathbf{S}^T, \quad S_{ij} = -S_{ji}.$$

An anti-symmetric tensor must have zero diagonal terms (when $i = j$).
Clearly

- If **U** and **V** are two-tensors, then **U** + **V** is also a two-tensor.
- If **U** is a two-tensor, then $\alpha\mathbf{U}$ is also a two-tensor, where α is a real number. The set of **U** form a vector space under addition and multiplication.

1.3.7 Decomposition

Any second-order tensor can be decomposed into symmetric and anti-symmetric parts:

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{W} + \mathbf{W}^T\right) + \frac{1}{2}\left(\mathbf{W} - \mathbf{W}^T\right),$$
$$W_{ij} = \frac{1}{2}\left(W_{ij} + W_{ji}\right) + \frac{1}{2}\left(W_{ij} - W_{ji}\right). \tag{1.12}$$

Returning to (1.9), if we interchange i and j, we get

$$W'_{ji} = A_{jm}A_{in}W_{mn} = A_{jn}A_{im}W_{nm}.$$

The second equality arises because m and n are dummy indices, mere labels in the summation. The left side of this expression is recognised as the components of the transpose of **W**. The equation asserts that the components of the transpose of **W** are also transformed according to (1.9). Thus, if **W** is a two-tensor, then its transpose is also a two-tensor, and the Cartesian decomposition (1.12) splits an arbitrary two-tensor into a symmetric and an anti-symmetric tensor (of second order).

We now go through some of the first and second-order tensors that will be encountered in this course.

1.3.8 Some Common Vectors

Position, displacement, velocity, acceleration, linear and angular momentum, linear and angular impulse, force, torque, are vectors. This is because the position vector transforms under a frame rotation according to (1.8). Any other quantity linearly related to the position (including the derivative and integral operation) will also be a vector.

1.3.9 Gradient of a Scalar

The gradient of a scalar is a vector. Let ϕ be a scalar, its gradient is written as

$$\mathbf{g} = \nabla\phi, \quad g_i = \frac{\partial \phi}{\partial x_i}.$$

Under a frame rotation, the new components of $\nabla\phi$ are

$$\frac{\partial \phi}{\partial x_i'} = \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = A_{ij}\frac{\partial \phi}{\partial x_j},$$

which qualifies $\nabla\phi$ as a vector.

1.3.10 Some Common Tensors

We have met a second-order tensor formed by the outer product of two vectors, written compactly as **uv**, with components (for vectors, the outer products is written without the symbol $\otimes$)

$$(\mathbf{uv})_{ij} = u_i v_j.$$

In general, the outer product of n vectors is an n-order tensor.

Unit Tensor. The Kronecker delta is a second-order tensor. In fact it is invariant in any coordinate system, and therefore is an *isotropic* tensor of second-order. To show that it is a second-order tensor, note that

$$\delta_{ij} = A_{ik}A_{jk} = A_{ik}A_{jl}\delta_{kl},$$

which follows from the orthogonality of the transformation matrix. δ_{ij} are said to be the components of the second-order unit tensor **I**. Finding isotropic tensors of arbitrary orders is not a trivial task.

Gradient of a Vector. The gradient of a vector is a two-tensor: if u_i and u'_i are the components of $\mathbf{u}$ in $\mathcal{F}$ and $\mathcal{F}'$,

$$\frac{\partial u'_i}{\partial x'_j} = \frac{\partial x_l}{\partial x'_j}\frac{\partial}{\partial x_l}(A_{ik}u_k) = A_{ik}A_{jl}\frac{\partial u_k}{\partial x_l}.$$

This qualifies the gradient of a vector as a two-tensor.

Velocity Gradient. If $\mathbf{u}$ is the velocity field, then $\nabla\mathbf{u}$ is the gradient of the velocity. Be careful with the notation here. By our convention, the subscripts are assigned from left to right, so

$$(\nabla\mathbf{u})_{ij} = \nabla_i u_j = \frac{\partial u_j}{\partial x_i}.$$

In most books on viscoelasticity including this, the term "velocity gradient" is taken to mean the second-order tensor $\mathbf{L} = (\nabla\mathbf{u})^T$ with components

$$L_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{1.13}$$

Strain Rate and Vorticity Tensors. The velocity gradient tensor can be decomposed into a symmetric part $\mathbf{D}$, called the strain rate tensor, and an anti-symmetric part $\mathbf{W}$, called the vorticity tensor:

$$\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \quad \mathbf{W} = \frac{1}{2}\left(\nabla\mathbf{u}^T - \nabla\mathbf{u}\right). \tag{1.14}$$

Stress Tensor and Quotient Rule. We are given that stress $\mathbf{T} = [T_{ij}]$ at a point $\mathbf{x}$ is defined by, (see Fig. 1.4),

$$\mathbf{t} = \mathbf{T}\mathbf{n}, \quad t_i = T_{ij}n_j, \tag{1.15}$$

where $\mathbf{n}$ is a normal unit vector on an infinitesimal surface ΔS at point $\mathbf{x}$, and $\mathbf{t}$ is the surface traction (force per unit area) representing the force the material on the positive side of $\mathbf{n}$ is pulling on the material on the negative side of $\mathbf{n}$. Under a frame rotation, since both $\mathbf{t}$ (force) and $\mathbf{n}$ are vectors,

$$\begin{aligned} &\mathbf{t}' = \mathbf{A}\mathbf{t}, \quad \mathbf{t} = \mathbf{A}^T\mathbf{t}' && \mathbf{n}' = \mathbf{A}\mathbf{n}, \quad \mathbf{n} = \mathbf{A}^T\mathbf{n}', \\ &\mathbf{A}^T\mathbf{t}' = \mathbf{t} = \mathbf{T}\mathbf{n} = \mathbf{T}\mathbf{A}^T\mathbf{n}' && \mathbf{t}' = \mathbf{A}\mathbf{T}\mathbf{A}^T\mathbf{n}'. \end{aligned}$$

From the definition of the stress, $\mathbf{t}' = \mathbf{T}'\mathbf{n}'$, and therefore

$$\mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^T.$$

So the stress is a second-order tensor.

In fact, as long as $\mathbf{t}$ and $\mathbf{n}$ are vector, the 9 components T_{ij} defined in the manner indicated by (1.15) form a second-order tensor. This is known as the *quotient rule*.

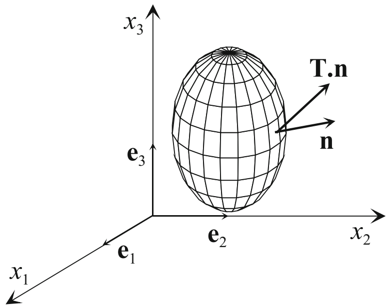

Fig. 1.4 Defining the stress tensor

1.4 Tensor and Linear Vector Function

L is a linear vector function on $\mathcal{U}$ if it satisfies

- $L(\mathbf{u}_1 + \mathbf{u}_2) = L(\mathbf{u}_1) + L(\mathbf{u}_2)$,
- $L(\alpha\mathbf{u}) = \alpha L(\mathbf{u})$, $\forall \mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$, $\forall \alpha \in \mathbb{R}$

1.4.1 Claim

Let $\mathbf{W}$ be a two-tensor, and define a vector-valued function through

$$\mathbf{v} = L(\mathbf{u}) = \mathbf{W}\mathbf{u},$$

then L is a linear function. Conversely, for any linear function on $\mathcal{U}$, there is a unique two-tensor $\mathbf{W}$ such that

$$L(\mathbf{u}) = \mathbf{W}\mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{U}.$$

The first statement can be easily verified. For the converse part, given the linear function, let define W_{ij} through

$$L(\mathbf{e}_i) = W_{ji}\mathbf{e}_j.$$

Now, $\forall \mathbf{u} \in \mathcal{U}$,

$$\begin{aligned} \mathbf{v} &= L(\mathbf{u}) = L(u_i\mathbf{e}_i) = u_i W_{ji}\mathbf{e}_j \\ v_j &= W_{ji}u_i. \end{aligned}$$

$\mathbf{W}$ is a second-order tensor because $\mathbf{u}$ and $\mathbf{v}$ are vectors. The uniqueness part of $\mathbf{W}$ can be demonstrated by assuming that there is another $\mathbf{W}'$, then

$$\left(\mathbf{W} - \mathbf{W}'\right)\mathbf{u} = 0, \quad \forall \mathbf{u} \in \mathcal{U},$$

which implies that $\mathbf{W}' = \mathbf{W}$.

In this connection, one can define a second-order tensor as a linear function, taking one vector into another. This is the direct approach, e.g., Gurtin [34], emphasising linear algebra. We use whatever notation is convenient for the purpose at hand. The set of all linear vector functions forms a vector space under addition and multiplication. The main result here is that

$$L(\mathbf{e}_i) = \mathbf{W}\mathbf{e}_i = W_{ji}\mathbf{e}_j \quad W_{ji} = \mathbf{e}_j \cdot (\mathbf{W}\mathbf{e}_i).$$

1.4.2 Dyadic Notation

Thus, one may write

$$\mathbf{W} = W_{ij}\mathbf{e}_i\mathbf{e}_j. \tag{1.16}$$

This is the basis for the *dyadic* notation, the $\mathbf{e}_i\mathbf{e}_j$ play the role of the basis "vectors" for the tensor $\mathbf{W}$.

1.5 Tensor Operations

1.5.1 Substitution

The operation $\delta_{ij}u_j = u_i$ replaces the subscript j by i – the tensor δ_{ij} is therefore sometimes called the substitution tensor.

1.5.2 Contraction

Given a two-tensor W_{ij}, the operation

$$W_{ii} = \sum_{i=1}^{3} W_{ii} = W_{11} + W_{22} + W_{33}$$

is called a contraction. It produces a scalar. The invariance of this scalar under a frame rotation is seen by noting that

$$W'_{ii} = A_{ik}A_{il}W_{kl} = \delta_{kl}W_{kl} = W_{kk}.$$

This scalar is also called the trace of $\mathbf{W}$, written as

$$\mathrm{tr}\,\mathbf{W} = W_{ii}. \tag{1.17}$$

It is one of the invariants of **W** (i.e., unchanged in a frame rotation). If the trace of **W** is zero, then **W** is said to be traceless. In general, given an n-order tensor, contracting any two subscripts produces a tensor of $(n-2)$ order.

1.5.3 Transpose

Given a two-tensor $\mathbf{W} = [W_{ij}]$, the transpose operation swaps the two indices

$$\mathbf{W}^T = \left(W_{ij}\mathbf{e}_i\mathbf{e}_j\right)^T = W_{ij}\mathbf{e}_j\mathbf{e}_i, \quad \left[\mathbf{W}^T\right]_{ij} = W_{ji}. \tag{1.18}$$

1.5.4 Products of Two Second-Order Tensors

Given two second-order tensors, **U** and **V**,

$$\mathbf{U} = U_{ij}\mathbf{e}_i\mathbf{e}_j, \quad \mathbf{V} = V_{ij}\mathbf{e}_i\mathbf{e}_j,$$

one can form different products from them, and it is helpful to refer to the dyadic notation here.

- The tensor product $\mathbf{U} \otimes \mathbf{V}$ is a 4th-order tensor, with component $U_{ij}V_{kl}$,

$$\mathbf{U} \otimes \mathbf{V} = U_{ij}V_{kl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l. \tag{1.19}$$

- The single dot product **U**.**V** is a 2nd-order tensor, sometimes written without the dot (the dot is the contraction operator),

$$\begin{aligned}\mathbf{U} \cdot \mathbf{V} = \mathbf{UV} &= \left(U_{ij}\mathbf{e}_i\mathbf{e}_j\right) \cdot (V_{kl}\mathbf{e}_k\mathbf{e}_l) \\ &= U_{ij}\mathbf{e}_i\delta_{jk}V_{kl}\mathbf{e}_l = U_{ij}V_{jl}\mathbf{e}_i\mathbf{e}_l,\end{aligned} \tag{1.20}$$

  with components $U_{ik}V_{kl}$, just like multiplying two matrices U_{ik} and V_{kj}. This single dot product induces a contraction of a pair of subscripts (j and k) in $U_{ij}V_{kl}$, and acts just like a vector dot product.
- The double dot (or scalar, or inner) product produces a scalar,

$$\begin{aligned}\mathbf{U} : \mathbf{V} &= \left(U_{ij}\mathbf{e}_i\mathbf{e}_j\right) : (V_{kl}\mathbf{e}_k\mathbf{e}_l) = \left(U_{ij}\mathbf{e}_i\right)\delta_{jk} \cdot (V_{kl}\mathbf{e}_l) \\ &= U_{ij}V_{kl}\delta_{jk}\delta_{il} = U_{ij}V_{ji}.\end{aligned} \tag{1.21}$$

The dot operates on a pair of base vectors until we run out of dots. The end result is a scalar (remember our summation convention). It can be shown that the scalar product is in fact an inner product.

- The norm of a two-tensor is defined from the inner product in the usual manner,

$$\|\mathbf{U}\|^2 = \mathbf{U}^T : \mathbf{U} = U_{ij}U_{ij} = \text{ tr}\left(\mathbf{U}^T\mathbf{U}\right). \tag{1.22}$$

The space of all linear vector functions therefore form a normed vector space.

- One writes $\mathbf{U}^2 = \mathbf{UU}$, $\mathbf{U}^3 = \mathbf{U}^2\mathbf{U}$, etc.
- A tensor $\mathbf{U}$ is invertible if there exists a tensor, $\mathbf{U}^{-1}$, called the inverse of $\mathbf{U}$, such that

$$\mathbf{UU}^{-1} = \mathbf{U}^{-1}\mathbf{U} = \mathbf{I} \tag{1.23}$$

One can also define the vector cross product between two second-order tensors (and indeed any combination of dot and cross vector products). However, we refrain from listing all possible combinations here.

1.6 Invariants

1.6.1 Invariant of a Vector

When a quantity is unchanged with a frame rotation, it is said to be invariant. From a vector, a scalar can be formed by taking the scalar product with itself, $v_i v_i = v^2$. This is of course the magnitude of the vector and it is the only independent scalar invariant for a vector.

1.6.2 Invariants of a Tensor

From a second-order tensor $\mathbf{S}$, there are three independent scalar invariants that can be formed, by taking the trace of $\mathbf{S}$, $\mathbf{S}^2$ and $\mathbf{S}^3$,

$$I = \text{tr}\mathbf{S} = S_{ii}, \quad II = \text{tr}\mathbf{S}^2 = S_{ij}S_{ji}, \quad III = \text{tr}\mathbf{S}^3 = S_{ij}S_{jk}S_{ki}.$$

However, it is customary to use the following invariants

$$I_1 = I, \quad I_2 = \frac{1}{2}\left(I^2 - II\right), \quad I_3 = \frac{1}{6}\left(I^3 - 3I\ II + 2III\right) = \det \mathbf{S}.$$

It is also possible to form ten invariants between two tensors (Gurtin [34]).

1.7 Decompositions

We now quote some of the well-known results without proof, some are intuitively obvious, others not.

1.7.1 Eigenvalue and Eigenvector

A scalar ω is an *eigenvalue* of a two-tensor $\mathbf{S}$ if there exists a non-zero vector $\mathbf{e}$, called the *eigenvector*, satisfying

$$\mathbf{Se} = \omega \mathbf{e}. \tag{1.24}$$

The characteristic space for $\mathbf{S}$ corresponding to the eigenvalue ω consists of all vectors in the eigenspace, $\{\mathbf{v} : \ \mathbf{Sv} = \omega \mathbf{v}\}$. If the dimension of this space is n, then ω is said to have geometric multiplicity of n. The *spectrum* of $\mathbf{S}$ is the ordered list $\{\omega_1, \omega_2, \ldots\}$ of all the eigenvalues of S.

A tensor $\mathbf{S}$ is said to be positive definite if it satisfies

$$\mathbf{S} : \mathbf{vv} > 0, \qquad \forall \mathbf{v} \neq \mathbf{0}. \tag{1.25}$$

We record the following theorems:

- The eigenvalues of a positive definite tensor are strictly positive.
- The characteristic spaces of a symmetric tensor are mutually orthogonal.
- Spectral decomposition theorem: Let $\mathbf{S}$ be a symmetric two-tensor. Then there is a basis consisting entirely of eigenvectors of $\mathbf{S}$. For such a basis, $\{\mathbf{e}_i, i = 1, 2, 3\}$, the corresponding eigenvalues $\{\omega_i, i = 1, 2, 3\}$ form the entire spectrum of $\mathbf{S}$, and $\mathbf{S}$ can be represented by the *spectral representation*, where

$$\begin{aligned} &\mathbf{S} = \sum_{i=1}^{3} \omega_i \mathbf{e}_i \mathbf{e}_i, \text{ when } \mathbf{S} \text{ has three distinct eigenvalues,} \\ &\mathbf{S} = \omega_1 \mathbf{ee} + \omega_2 (\mathbf{I} - \mathbf{ee}), \text{ when } \mathbf{S} \text{ has two distinct eigenvalues,} \\ &\mathbf{S} = \omega \mathbf{I}, \text{ when } \mathbf{S} \text{ has only one eigenvalue.} \end{aligned} \tag{1.26}$$

1.7.2 Square Root Theorem

Let $\mathbf{S}$ be a symmetric positive definite tensor. Then there is a unique positive definite tensor $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{S}$. We write

$$\mathbf{U} = \mathbf{S}^{1/2}.$$

The proof of this follows from the spectral representation of $\mathbf{S}$.

1.7.3 Polar Decomposition Theorem

For any given tensor $\mathbf{F}$, there exist positive definite tensors $\mathbf{U}$ and $\mathbf{V}$, and a rotation tensor $\mathbf{R}$, such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{1.27}$$

Each of these representations is unique, and

$$\mathbf{U} = \left(\mathbf{F}^T\mathbf{F}\right)^{1/2}, \quad \mathbf{V} = \left(\mathbf{F}\mathbf{F}^T\right)^{1/2}. \tag{1.28}$$

The first representation ($\mathbf{RU}$) is called the right, and the second ($\mathbf{VR}$) is called the left polar decomposition.

1.7.4 Cayley–Hamilton Theorem

The most important theorem is the Cayley–Hamilton theorem: Every tensor $\mathbf{S}$ satisfies its own characteristic equation

$$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3\mathbf{I} = \mathbf{0}, \tag{1.29}$$

where $I_1 = \mathrm{tr}\mathbf{S}$, $I_2 = \frac{1}{2}\left((\mathrm{tr}\mathbf{S})^2 - \mathrm{tr}\mathbf{S}^2\right)$, and $I_3 = \det\mathbf{S}$ are the three scalar invariants for $\mathbf{S}$, and $\mathbf{I}$ is the unit tensor in three dimensions.

In two dimensions, this equation reads

$$\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I} = \mathbf{0}, \tag{1.30}$$

where $I_1 = \mathrm{tr}\mathbf{S}$, $I_2 = \det\mathbf{S}$ are the two scalar invariants for $\mathbf{S}$, and $\mathbf{I}$ is the unit tensor in two dimensions.

Cayley–Hamilton theorem is used to reduce the number of independent tensorial groups in tensor-valued functions. We record here one possible use of the Cayley–Hamilton theorem in two dimensions. The three-dimensional case is reserved as an exercise.

Suppose $\mathbf{C}$ is a given symmetric positive definite tensor in 2-D,

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix},$$

and its square root $\mathbf{U} = \mathbf{C}^{1/2}$ is desired. From the characteristic equation for $\mathbf{U}$,

$$\mathbf{U} = I_1^{-1}(\mathbf{U})\left[\mathbf{C} + I_3(\mathbf{U})\,\mathbf{I}\right],$$

so if we can express the invariants of $\mathbf{U}$ in terms of the invariant of $\mathbf{C}$, we're done. Now, if the eigenvalues of $\mathbf{U}$ are λ_1 and λ_2, then

$$I_1(\mathbf{U}) = \lambda_1 + \lambda_2,\; I_2(\mathbf{U}) = \lambda_1\lambda_2,$$
$$I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2,\; I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2.$$

Thus

$$I_2(\mathbf{U}) = \sqrt{I_2(\mathbf{C})},$$
$$I_1^2(\mathbf{U}) = I_1(\mathbf{C}) + 2\sqrt{I_2(\mathbf{C})}.$$

Therefore

$$\mathbf{U} = \frac{\mathbf{C} + \sqrt{I_2(\mathbf{C})}\mathbf{I}}{\sqrt{I_1(\mathbf{C}) + 2\sqrt{I_2(\mathbf{C})}}}.$$

1.8 Derivative Operations

Suppose $\varphi(\mathbf{u})$ is a scalar-valued function of a vector $\mathbf{u}$. The derivative of $\varphi(\mathbf{u})$ with respect to $\mathbf{u}$ in the direction $\mathbf{v}$ is defined as the linear operator $D\varphi(\mathbf{u})[\mathbf{v}]$:

$$\varphi(\mathbf{u} + \alpha\mathbf{v}) = \varphi(\mathbf{u}) + \alpha D\varphi(\mathbf{u})[\mathbf{v}] + HOT,$$

where HOT are terms of higher order, which vanish faster than α. Also, the square brackets enclosing $\mathbf{v}$ are used to emphasise the linearity of in $\mathbf{v}$. An operational definition for the derivative of $\varphi(\mathbf{u})$ in the direction $\mathbf{v}$ is therefore,

$$D\varphi(\mathbf{u})[\mathbf{v}] = \frac{d}{d\alpha}\left[\varphi(\mathbf{u} + \alpha\mathbf{v})\right]_{\alpha=0}. \tag{1.31}$$

This definition can be extended verbatim to derivatives of a tensor-valued (of any order) function of a tensor (of any order). The argument $\mathbf{v}$ is a part of the definition. We illustrate this with a few examples.

Example 1 Consider the scalar-valued function of a vector, $\varphi(\mathbf{u}) = u^2 = \mathbf{u}\cdot\mathbf{u}$. Its derivative in the direction of $\mathbf{v}$ is

$$D\varphi(\mathbf{u})[\mathbf{v}] = \frac{d}{d\alpha}\varphi(\mathbf{u} + \alpha\mathbf{v})_{\alpha=0} = \frac{d}{d\alpha}\left[u^2 + 2\alpha\mathbf{u}\cdot\mathbf{v} + \alpha^2 v^2\right]_{\alpha=0}$$
$$= 2\mathbf{u}\cdot\mathbf{v}.$$

Example 2 Consider the tensor-valued function of a tensor, $\mathbf{G}(\mathbf{A}) = \mathbf{A}^2 = \mathbf{A}\mathbf{A}$. Its derivative in the direction of $\mathbf{B}$ is

$$
\begin{aligned}
D\mathbf{G}(\mathbf{A})[\mathbf{B}] &= \frac{d}{d\alpha}\left[\mathbf{G}(\mathbf{A}+\alpha\mathbf{B})\right]_{\alpha=0} \\
&= \frac{d}{d\alpha}\left[\mathbf{A}^2+\alpha(\mathbf{AB}+\mathbf{BA})+O\left(\alpha^2\right)\right]_{\alpha=0} \\
&= \mathbf{AB}+\mathbf{BA}.
\end{aligned}
$$

1.8.1 Derivative of det(A)

Consider the scalar-valued function of a tensor, $\varphi(\mathbf{A}) = \det\mathbf{A}$. Its derivative in the direction of $\mathbf{B}$ can be calculated using

$$
\begin{aligned}
\det(\mathbf{A}+\alpha\mathbf{B}) &= \det\alpha\mathbf{A}\left(\mathbf{A}^{-1}\mathbf{B}+\alpha^{-1}\mathbf{I}\right) = \alpha^3\det\mathbf{A}\det\left(\mathbf{A}^{-1}\mathbf{B}+\alpha^{-1}\mathbf{I}\right) \\
&= \alpha^3\det\mathbf{A}\left(\alpha^{-3}+\alpha^{-2}I_1\left(\mathbf{A}^{-1}\mathbf{B}\right)+\alpha^{-1}I_2\left(\mathbf{A}^{-1}\cdot\mathbf{B}\right)+I_3\left(\mathbf{A}^{-1}\mathbf{B}\right)\right) \\
&= \det\mathbf{A}\left(1+\alpha I_1\left(\mathbf{A}^{-1}\mathbf{B}\right)+O\left(\alpha^2\right)\right).
\end{aligned}
$$

Thus

$$
D\varphi(\mathbf{A})[\mathbf{B}] = \frac{d}{d\alpha}\left[\varphi(\mathbf{A}+\alpha\mathbf{B})\right]_{\alpha=0} = \det\mathbf{A}\,\mathrm{tr}\left(\mathbf{A}^{-1}\mathbf{B}\right).
$$

1.8.2 Derivative of tr(A)

Consider the first invariant $I(\mathbf{A}) = \mathrm{tr}\mathbf{A}$. Its derivative in the direction of $\mathbf{B}$ is

$$
\begin{aligned}
DI(\mathbf{A})[\mathbf{B}] &= \frac{d}{d\alpha}\left[I(\mathbf{A}+\alpha\mathbf{B})\right]_{\alpha=0} \\
&= \frac{d}{d\alpha}\left[\mathrm{tr}\mathbf{A}+\alpha\mathrm{tr}\mathbf{B}\right]_{\alpha=0} = \mathrm{tr}\mathbf{B} = \mathbf{I}:\mathbf{B}.
\end{aligned}
$$

1.8.3 Derivative of tr(A^2)

Consider the second invariant $II(\mathbf{A}) = \mathrm{tr}\mathbf{A}^2$. Its derivative in the direction of $\mathbf{B}$ is

$$
\begin{aligned}
DII(\mathbf{A})[\mathbf{B}] &= \frac{d}{d\alpha}\left[II(\mathbf{A}+\alpha\mathbf{B})\right]_{\alpha=0} \\
&= \frac{d}{d\alpha}\left[\mathbf{A}:\mathbf{A}+\alpha(\mathbf{A}:\mathbf{B}+\mathbf{B}:\mathbf{A})+O\left(\alpha^2\right)\right]_{\alpha=0} \\
&= 2\mathbf{A}:\mathbf{B}.
\end{aligned}
$$

1.9 Gradient of a Field

1.9.1 Field

A function of the position vector $\mathbf{x}$ is called a field. One has a scalar field, for example the temperature field $T(\mathbf{x})$, a vector field, for example the velocity field $\mathbf{u}(\mathbf{x})$, or a tensor field, for example the stress field $\mathbf{S}(\mathbf{x})$. Higher-order tensor fields are rarely encountered, as in the many-point correlation fields. Conservation equations in continuum mechanics involve derivatives (derivatives with respect to position vectors are called *gradients*) of different fields, and it is absolutely essential to know how to calculate the gradients of fields in different coordinate systems. We also find it more convenient to employ the dyadic notation at this point.

1.9.2 Cartesian Frame

We consider first a scalar field, $\varphi(\mathbf{x})$. The Taylor expansion of this about point x is

$$\varphi(\mathbf{x} + \alpha\mathbf{r}) = \varphi(\mathbf{x}) + \alpha r_j \frac{\partial}{\partial x_j}\varphi(\mathbf{x}) + O\left(\alpha^2\right).$$

Thus the gradient of $\varphi(\mathbf{x})$ at point $\mathbf{x}$, now written as $\nabla\varphi$, defined in (1.31), is given by

$$\nabla\varphi[\mathbf{r}] = \mathbf{r} \cdot \frac{\partial\varphi}{\partial\mathbf{x}}. \tag{1.32}$$

This remains unchanged for a vector or a tensor field.

Gradient Operator. This leads us to define the gradient operator as

$$\nabla = \mathbf{e}_j \frac{\partial}{\partial x_j} = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}. \tag{1.33}$$

This operator can be treated as a vector, operating on its arguments. By itself, it has no meaning; it must operate on a scalar, a vector or a tensor.

Gradient of a Scalar. For example, the gradient of a scalar is

$$\nabla\varphi = \mathbf{e}_j \frac{\partial\varphi}{\partial x_j} = \mathbf{e}_1 \frac{\partial\varphi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\varphi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\varphi}{\partial x_3}. \tag{1.34}$$

Gradient of a Vector. The gradient of a vector can be likewise calculated

$$\nabla\mathbf{u} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right)\left(u_j \mathbf{e}_j\right) = \mathbf{e}_i \mathbf{e}_j \frac{\partial u_j}{\partial x_i}. \tag{1.35}$$

In matrix notation,

$$[\nabla \mathbf{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}.$$

The component $(\nabla \mathbf{u})_{ij}$ is $\partial u_j/\partial x_i$; some books define this differently.

Transpose of a Gradient. The transpose of a gradient of a vector is therefore

$$\nabla \mathbf{u}^T = \mathbf{e}_i \mathbf{e}_j \frac{\partial u_i}{\partial x_j}. \tag{1.36}$$

In matrix notation,

$$[\nabla \mathbf{u}]^T = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}.$$

Divergence of a Vector. The divergence of a vector is a scalar defined by

$$\nabla \cdot \mathbf{u} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(u_j \mathbf{e}_j \right) = \mathbf{e}_i \cdot \mathbf{e}_j \frac{\partial u_j}{\partial x_i} = \delta_{ij} \frac{\partial u_j}{\partial x_i}$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{1.37}$$

The divergence of a vector is also an invariant, being the trace of a tensor.

Curl of a Vector. The curl of a vector is a vector defined by

$$\nabla \times \mathbf{u} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \right) \times \left(u_j \mathbf{e}_j \right) = \mathbf{e}_i \times \mathbf{e}_j \frac{\partial u_j}{\partial x_i} = \varepsilon_{kij} \mathbf{e}_k \frac{\partial u_j}{\partial x_i}$$

$$= \mathbf{e}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \mathbf{e}_2 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \mathbf{e}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right). \tag{1.38}$$

The curl of a vector is sometimes denoted by rot.

Divergence of a Tensor. The divergence of a tensor is a vector field defined by

$$\nabla \cdot \mathbf{S} = \left(\mathbf{e}_k \frac{\partial}{\partial x_k} \right) \cdot \left(S_{ij} \mathbf{e}_i \mathbf{e}_j \right) = \mathbf{e}_j \frac{\partial S_{ij}}{\partial x_i}. \tag{1.39}$$

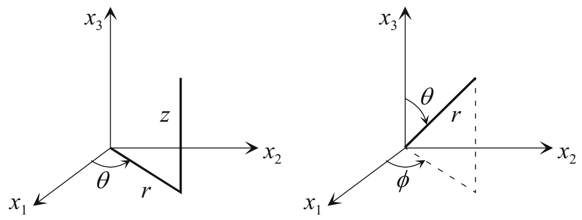

Fig. 1.5 Cylindrical and spherical frame of references

1.9.3 Non-Cartesian Frames

All the above definitions for gradient and divergence of a tensor remain valid in a non-Cartesian frame, provided that the derivative operation is also applied to the basis vectors as well. We illustrate this process in two important frames, cylindrical and spherical coordinate systems (Fig. 1.5); for other systems, consult Bird et al. [6].

Cylindrical Coordinates. In a cylindrical coordinate system (Fig. 1.5, left), points are located by giving them values to $\{r, \theta, z\}$, which are related to $\{x = x_1, y = x_2, z = x_3\}$ by

$$\begin{aligned} &x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \\ &r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\tfrac{y}{x}\right), \quad z = z \end{aligned}$$

The basis vectors in this frame are related to the Cartesian ones by

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, & \mathbf{e}_x &= \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta \\ \mathbf{e}_\theta &= -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y, & \mathbf{e}_y &= \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta \end{aligned}$$

Physical components. In this system, a vector $\mathbf{u}$, or a tensor $\mathbf{S}$, are represented by, respectively,

$$\begin{aligned} \mathbf{u} &= u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z, \\ \mathbf{S} &= S_{rr}\mathbf{e}_r\mathbf{e}_r + S_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + S_{rz}\mathbf{e}_r\mathbf{e}_z + S_{\theta r}\mathbf{e}_\theta\mathbf{e}_r \\ &\quad + S_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + S_{\theta z}\mathbf{e}_\theta\mathbf{e}_z + S_{zr}\mathbf{e}_z\mathbf{e}_r + S_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + S_{zz}\mathbf{e}_z\mathbf{e}_z. \end{aligned}$$

Gradient operator. The components expressed this way are called physical components. The gradient operator is converted from one system to another by the chain rule,

$$
\begin{aligned}
\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} &= (\cos\theta \mathbf{e}_r - \sin\theta \mathbf{e}_\theta)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) \\
&+ (\sin\theta \mathbf{e}_r + \cos\theta \mathbf{e}_\theta)\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}.
\end{aligned} \tag{1.40}
$$

When carrying out derivative operations, remember that

$$
\begin{array}{lll}
\dfrac{\partial}{\partial r}\mathbf{e}_r = 0, & \dfrac{\partial}{\partial r}\mathbf{e}_\theta = 0, & \dfrac{\partial}{\partial r}\mathbf{e}_z = 0 \\
\dfrac{\partial}{\partial \theta}\mathbf{e}_r = \mathbf{e}_\theta, & \dfrac{\partial}{\partial \theta}\mathbf{e}_\theta = -\mathbf{e}_r, & \dfrac{\partial}{\partial \theta}\mathbf{e}_z = 0 \\
\dfrac{\partial}{\partial z}\mathbf{e}_r = 0, & \dfrac{\partial}{\partial z}\mathbf{e}_\theta = 0, & \dfrac{\partial}{\partial z}\mathbf{e}_z = 0
\end{array} \tag{1.41}
$$

Gradient of a vector. The gradient of any vector is

$$
\begin{aligned}
\nabla \mathbf{u} &= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right)(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) \\
&= \mathbf{e}_r\mathbf{e}_r \frac{\partial u_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta \frac{\partial u_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_z \frac{\partial u_z}{\partial r} + \mathbf{e}_\theta\mathbf{e}_r \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \mathbf{e}_\theta\mathbf{e}_\theta \frac{u_r}{r} \\
&+ \mathbf{e}_\theta\mathbf{e}_\theta \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \mathbf{e}_\theta\mathbf{e}_r \frac{u_\theta}{r} + \mathbf{e}_\theta\mathbf{e}_z \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \mathbf{e}_z\mathbf{e}_r \frac{\partial u_r}{\partial z} + \mathbf{e}_z\mathbf{e}_\theta \frac{\partial u_\theta}{\partial z} \\
&+ \mathbf{e}_z\mathbf{e}_z \frac{\partial u_z}{\partial z} \\
\nabla \mathbf{u} &= \mathbf{e}_r\mathbf{e}_r \frac{\partial u_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta \frac{\partial u_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_z \frac{\partial u_z}{\partial r} + \mathbf{e}_\theta\mathbf{e}_r \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) \\
&+ \mathbf{e}_\theta\mathbf{e}_\theta \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \mathbf{e}_\theta\mathbf{e}_z \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \mathbf{e}_z\mathbf{e}_r \frac{\partial u_r}{\partial z} + \mathbf{e}_z\mathbf{e}_\theta \frac{\partial u_\theta}{\partial z} \\
&+ \mathbf{e}_z\mathbf{e}_z \frac{\partial u_z}{\partial z}.
\end{aligned} \tag{1.42}
$$

Divergence of a vector. The divergence of a vector is obtained by a contraction of the above equation:

$$
\nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}. \tag{1.43}
$$

1.9.4 Spherical Coordinates

In a spherical coordinate system (Fig. 1.5, right), points are located by giving them values to $\{r, \theta, \phi\}$, which are related to $\{x = x_1, y = x_2, z = x_3\}$ by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$
$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right).$$

The basis vectors are related by

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_1\sin\theta\cos\phi + \mathbf{e}_2\sin\theta\sin\phi + \mathbf{e}_3\cos\theta,\\
\mathbf{e}_\theta &= \mathbf{e}_1\cos\theta\cos\phi + \mathbf{e}_2\cos\theta\sin\phi - \mathbf{e}_3\sin\theta,\\
\mathbf{e}_\phi &= -\mathbf{e}_1\sin\phi + \mathbf{e}_2\cos\phi,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{e}_r\sin\theta\cos\phi + \mathbf{e}_\theta\cos\theta\cos\phi - \mathbf{e}_\phi\sin\phi,\\
\mathbf{e}_2 &= \mathbf{e}_r\sin\theta\sin\phi + \mathbf{e}_\theta\cos\theta\sin\phi + \mathbf{e}_\phi\cos\phi,\\
\mathbf{e}_3 &= \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta.
\end{aligned}$$

Gradient operator. Using the chain rule, it can be shown that the gradient operator in spherical coordinates is

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{1.44}$$

We list below a few results of interest.
Gradient of a scalar. The gradient of a scalar is given by

$$\nabla\varphi = \mathbf{e}_r\frac{\partial\varphi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}. \tag{1.45}$$

Gradient of a vector. The gradient of a vector is given by

$$\begin{aligned}
\nabla\mathbf{u} = {}& \mathbf{e}_r\mathbf{e}_r\frac{\partial u_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta\frac{\partial u_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_\phi\frac{\partial u_\phi}{\partial r} + \mathbf{e}_\theta\mathbf{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\\
&+ \mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) + \mathbf{e}_\phi\mathbf{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right)\\
&+ \mathbf{e}_\theta\mathbf{e}_\phi\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} + \mathbf{e}_\phi\mathbf{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r}\cot\theta\right)\\
&+ \mathbf{e}_\phi\mathbf{e}_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right).
\end{aligned} \tag{1.46}$$

Divergence of a vector. The divergence of a vector is given by

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}. \tag{1.47}$$

Divergence of a tensor. The divergence of a tensor is given by

$$\begin{aligned}\nabla \cdot \mathbf{S} = \mathbf{e}_r &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(S_{\theta r}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi}\right.\\ &\left. - \frac{S_{\theta\theta}+S_{\phi\phi}}{r}\right] + \mathbf{e}_\theta\left[\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 S_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(S_{\theta\theta}\sin\theta\right)\right.\\ &\left. + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}-S_{r\theta}-S_{\phi\phi}\cot\theta}{r}\right] + \mathbf{e}_\phi\left[\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 S_{r\phi}\right)\right.\\ &\left. + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(S_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}-S_{r\phi}+S_{\phi\theta}\cot\theta}{r}\right].\end{aligned} \tag{1.48}$$

1.10 Integral Theorems

1.10.1 Gauss Divergence Theorem

Various volume integrals can be converted to surface integrals by the following theorems, due to Gauss (Fig. 1.6):

$$\int_V \nabla\varphi dV = \int_S \varphi\mathbf{n} dS, \tag{1.49}$$

Fig. 1.6 Carl Friedrich Gauss (1777–1855) was a Professor of Mathematics at the University of Göttingen. He made several contributions to Number Theory, Geodesy, Statistics, Geometry, Physics. His motto was “few, but ripe” (*Pauca, Sed Matura*). He did not publish several important papers because they did not satisfy these requirements

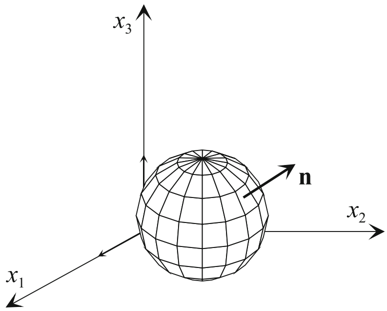

Fig. 1.7 A region enclosed by a closed surface with outward unit vector field

$$\int_V \nabla \cdot \mathbf{u} dV = \int_S \mathbf{n} \cdot \mathbf{u} dS, \tag{1.50}$$

$$\int_V \nabla \cdot \mathbf{S} dV = \int_S \mathbf{n} \cdot \mathbf{S} dS. \tag{1.51}$$

The proofs may be found in Kellogg [43]. In these, V is a bounded regular region, with bounding surface S and outward unit vector $\mathbf{n}$ (Fig. 1.7), φ, $\mathbf{u}$, and $\mathbf{S}$ are differentiable scalar, vector, and tensor fields with continuous gradients. Indeed the indicial version of (1.50) is valid even if u_i are merely three scalar fields of the required smoothness (rather than three components of a vector field).

1.10.2 Stokes Curl Theorem

Various surfaces integrals can be converted into contour integrals using the following theorems:

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{u})\, dS = \oint_C \mathbf{t} \cdot \mathbf{u} dC, \tag{1.52}$$

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{S})\, dS = \oint_C \mathbf{t} \cdot \mathbf{S} dC. \tag{1.53}$$

In these, $\mathbf{t}$ is a tangential unit vector along the contour C. The direction of integration is determined by the right-hand rule: thumb pointing in the direction of $\mathbf{n}$, fingers curling in the direction of C.

Fig. 1.8 Gottfried W. Leibniz (1646–1716) was a German philosopher and mathematician, who independently with Newton, laid the foundation for integral and differential calculus in 1675

1.10.3 Leibniz Formula

If φ is a field (a scalar, a vector, or a tensor) define on a region $V(t)$, which is changing in time, with bounding surface $S(t)$, also changing in time with velocity $\mathbf{u}_S$, then (Leibniz formula, Fig. 1.8)

$$\frac{d}{dt}\int_V \varphi dV = \int_V \frac{\partial\varphi}{\partial t} dV + \int_S \varphi \mathbf{u}_S \cdot \mathbf{n} dS. \tag{1.54}$$

Problems

Problem 1.1 The components of vectors **u**, **v**, and **w** are given by u_i, v_i, w_i. Verify that

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= u_i v_i,\\
\mathbf{u}\times\mathbf{v} &= \varepsilon_{ijk}\mathbf{e}_i u_j v_k,\\
(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= \varepsilon_{ijk} u_i v_j w_k,\\
(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}),\\
(\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u},\\
(\mathbf{u}\times\mathbf{v})^2 &= u^2 v^2 - (\mathbf{u}\cdot\mathbf{v})^2,
\end{aligned}$$

where $u^2 = |\mathbf{u}|^2$ and $v^2 = |\mathbf{v}|^2$.

Problem 1.2 Let **A** be a 3 × 3 matrix with entries A_{ij},

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Verify that

$$\begin{aligned} \det[\mathbf{A}] &= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}, \\ \varepsilon_{lmn} \det[\mathbf{A}] &= \varepsilon_{ijk} A_{il} A_{jm} A_{kn} = \varepsilon_{ijk} A_{li} A_{mj} A_{nk}, \\ \det[\mathbf{A}] &= \frac{1}{6} \varepsilon_{ijk} \varepsilon_{lmn} A_{il} A_{jm} A_{kn}. \end{aligned}$$

Problem 1.3 Verify that

$$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}.$$

Given that two 3 × 3 matrices of components

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad [\mathbf{B}] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

verify that if $[\mathbf{C}] = [\mathbf{A}] \cdot [\mathbf{B}]$, then the components of **C** are $C_{ij} = A_{ik} B_{kj}$. Thus if $[\mathbf{D}] = [\mathbf{A}]^T [\mathbf{B}]$, then $D_{ij} = A_{ki} B_{kj}$.

Problem 1.4 Show that, if $\left[A_{ij}\right]$ is a frame rotation matrix,

$$\begin{aligned} &\det\left[A_{ij}\right] = \left(\mathbf{e}'_1 \times \mathbf{e}'_2\right) \cdot \mathbf{e}'_3 = 1. \\ &[\mathbf{A}]^T [\mathbf{A}] = [\mathbf{A}] [\mathbf{A}]^T = [\mathbf{I}], \quad [\mathbf{A}]^{-1} = [\mathbf{A}]^T, \quad \det[\mathbf{A}] = 1. \end{aligned}$$

Problem 1.5 Verify that

$$\varepsilon_{ijk} u_i v_j w_k = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

Consider a second-order tensor W_{ij} and a vector $u_i = \varepsilon_{ijk} W_{jk}$. Show that if **W** is symmetric, **u** is zero, and if **W** is anti-symmetric the components of **u** are twice those of **W** in magnitude. This vector is said to be the axial vector of **W**.

Hence, show that the axial vector associated with the vorticity tensor of (1.14) is $-\nabla \times \mathbf{u}$.

Problem 1.6 If **D**, **S** and **W** are second-order tensors, **D** symmetric and **W** anti-symmetric, show that

$$\begin{aligned}&\mathbf{D}:\mathbf{S}=\mathbf{D}:\mathbf{S}^T=\mathbf{D}:\tfrac{1}{2}\left(\mathbf{S}+\mathbf{S}^T\right),\\&\mathbf{W}:\mathbf{S}=-\mathbf{W}:\mathbf{S}^T=\mathbf{W}:\tfrac{1}{2}\left(\mathbf{W}-\mathbf{W}^T\right),\\&\mathbf{D}:\mathbf{W}=0.\end{aligned}$$

Further, show that

$$\begin{aligned}&\text{if } \mathbf{T}:\mathbf{S}=0\ \ \forall\mathbf{S} \text{ then } \mathbf{T}=0,\\&\text{if } \mathbf{T}:\mathbf{S}=0\ \ \forall \text{ symmetric } \mathbf{S} \text{ then } \mathbf{T} \text{ is anti-symmetric},\\&\text{if } \mathbf{T}:\mathbf{S}=0\ \ \forall \text{ anti-symmetric } \mathbf{S} \text{ then } \mathbf{T} \text{ is symmetric}.\end{aligned}$$

Problem 1.7 Show that **Q** is orthogonal if and only if $\mathbf{H}=\mathbf{Q}-\mathbf{I}$ satisfies

$$\mathbf{H}+\mathbf{H}^T+\mathbf{H}\mathbf{H}^T=0, \quad \mathbf{H}\mathbf{H}^T=\mathbf{H}^T\mathbf{H}.$$

Problem 1.8 Show that, if **S** is a second-order tensor, then $I=\text{tr}\,\mathbf{S}$, $II=\text{tr}\,\mathbf{S}^2$, $III=\det\mathbf{S}$ are indeed invariants. In addition, show that

$$\det\left(\mathbf{S}-\omega\mathbf{I}\right)=-\omega^3+I_1\omega^2-I_2\omega+I_3.$$

If ω is an eigenvalue of **S** then $\det\left(\mathbf{S}-\omega\mathbf{I}\right)=0$. This is said to be the characteristic equation for **S**.

Problem 1.9 Apply the result above to find the square root of the Cauchy-Green tensor in a two-dimensional shear deformation

$$[\mathbf{C}]=\begin{bmatrix}1+\gamma^2 & \gamma\\ \gamma & 1\end{bmatrix}.$$

Investigate the corresponding formula for the square root of a symmetric positive definite tensor **S** in three dimensions.

Problem 1.10 Write down all the components of the strain rate tensor and the vorticity tensor in a Cartesian frame.

Problem 1.11 Given that $\mathbf{r}=x_i\mathbf{e}_i$ is the position vector, **a** is a constant vector, and $f(r)$ is a function of $r=|\mathbf{r}|$, show that

$$\nabla\cdot\mathbf{r}=3, \quad \nabla\times\mathbf{r}=\mathbf{0}, \quad \nabla\left(\mathbf{a}\cdot\mathbf{r}\right)=\mathbf{a}, \quad \nabla f=\frac{1}{r}\frac{df}{dr}\mathbf{r}.$$

Problem 1.12 Show that the divergence of a second-order tensor **S** in cylindrical coordinates is given by

$$\nabla \cdot \mathbf{S} = \mathbf{e}_r \left(\frac{\partial S_{rr}}{\partial r} + \frac{S_{rr} - S_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zr}}{\partial z} \right)$$
$$+ \mathbf{e}_\theta \left(\frac{\partial S_{r\theta}}{\partial r} + \frac{2S_{r\theta}}{r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{S_{\theta r} - S_{r\theta}}{r} \right)$$
$$+ \mathbf{e}_z \left(\frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta} + \frac{\partial S_{zz}}{\partial z} \right). \tag{1.55}$$

Problem 1.13 Show that, in cylindrical coordinates, the Laplacian of a vector $\mathbf{u}$ is given by

$$\nabla^2 \mathbf{u} = \mathbf{e}_r \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) \right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \right]$$
$$+ \mathbf{e}_\theta \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) \right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} \right]$$
$$+ \mathbf{e}_z \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right]. \tag{1.56}$$

Problem 1.14 Show that, in cylindrical coordinates,

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{e}_r \left[u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta u_\theta}{r} \right]$$
$$+ \mathbf{e}_\theta \left[u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} \right]$$
$$+ \mathbf{e}_z \left[u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right]. \tag{1.57}$$

Problem 1.15 The stress tensor in a material satisfies $\nabla \cdot \mathbf{S} = \mathbf{0}$. Show that the volume-average stress in a region V occupied by the material is

$$\langle \mathbf{S} \rangle = \frac{1}{2V}\int_S (\mathbf{xt} + \mathbf{tx})\, dS, \tag{1.58}$$

where $\mathbf{t} = \mathbf{n} \cdot \mathbf{S}$ is the surface traction. The quantity on the left side of (1.58) is called the *stresslet* (Batchelor [4]).

Problem 1.16 Calculate the following integrals on the surface of the unit sphere

$$\langle \mathbf{nn} \rangle = \frac{1}{S}\int_S \mathbf{nn} dS \tag{1.59}$$
$$\langle \mathbf{nnnn} \rangle = \frac{1}{S}\int_S \mathbf{nnnn} dS. \tag{1.60}$$

These are the averages of various moments of a uniformly distributed unit vector on a sphere surface.

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