Chapter 2
Isometric Approximation in Bounded Sets and Its Applications

Pekka Alestalo

Abstract We give a review of results related to the isometric approximation problem in bounded sets, and their application in the extension problems for bilipschitz and quasisymmetric maps. We also list several recent articles dealing with the approximation problem for mappings defined in the whole space.

Keywords Nearisometry • Quasisymmetric • Bilipschitz • Extension

Mathematics Subject Classification (2010) Primary 30C65; Secondary 46B20

2.1 Introduction

Definition 2.1 Let X and Y be metric spaces with distance written (in the Polish notation) as $|x - y|$, and let $\varepsilon \geq 0$. A mapping $f: X \to Y$ is an ε-nearisometry if

$$\big||f(x) - f(y)| - |x - y|\big| \leq \varepsilon$$

for all $x, y \in X$.

We remark that these mappings are often called ε-isometries, ε-quasi-isometries, etc., and that the condition is equivalent to

$$|x - y| - \varepsilon \leq |f(x) - f(y)| \leq |x - y| + \varepsilon.$$

The nearisometry condition does not imply continuity (unless $\varepsilon = 0$), but these maps are closely related to $(1 + \varepsilon)$-bilipschitz maps that satisfy

$$|x - y|/(1 + \varepsilon) \leq |f(x) - f(y)| \leq (1 + \varepsilon)|x - y|$$

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for all $x, y \in X$. In particular, if the diameter $d(X)$ is finite, then every $(1+\varepsilon)$-bilipschitz map $f: X \to Y$ is a $d(X)\varepsilon$-nearisometry.

Our starting point is the following theorem from [14] and [19].

Theorem 2.1 *Let E and F be real normed spaces and let $f: E \to F$ be a surjective ε-nearisometry with $f(0) = 0$. Then there is a surjective linear isometry $T: E \to F$ satisfying*

$$\|T - f\|_E \equiv \sup\{|Tx - f(x)| \mid x \in E\} \leq 2\varepsilon.$$

The original proof in [14] was for Hilbert spaces only, and with a constant 10ε. The bound 2ε, obtained in [19], is the best universal one, but it can be improved to $J(E)\varepsilon$ for Hilbert spaces, cf. [13]. Here $J(E)$ is the Jung's constant of the space E.

A comprehensive history of these developments, some counterexamples demonstrating the sharpness of the constants, and a survey of further progress up to c. 2002 can be found in the article [25], which I recommend to the interested reader. See also [22] for some updates. Additional surveys of these problems in [11] and [21] are also useful. Furthermore, many of the original proofs are reproduced in Chapter 13 of the monograph [16], which contains also other closely related material.

However, some important counterexamples related to the approximation problem in bounded subsets were discovered only after Väisälä's survey article appeared. In the following sections I will describe these developments and present applications of the results to extension problems for mappings that are, in a certain sense, close to either an isometry or a similarity.

To close this introduction, I remark that also the case of mappings defined in the whole space, but without the surjectivity assumption, has attracted a lot of interest and new results in the last couple of years. Since this is not my area of speciality and I want to concentrate in the approximation problem for bounded sets, I will only list here some of these references: [7–10, 12, 20, 27, 28].

2.2 Isometric Approximation in Bounded Sets

We start with the approximation of nearisometries in the closed unit ball $\mathbf{B}^n \subset \mathbb{R}^n$.

Theorem 2.2 *There is a universal constant $C > 0$ such that every ε-nearisometry $f: \mathbf{B}^n \to \mathbb{R}^n$ has an isometric approximation $T: \mathbb{R}^n \to \mathbb{R}^n$ satisfying*

$$\|T - f\|_{\mathbf{B}^n} \equiv \sup\{|Tx - f(x)| \mid x \in \mathbf{B}^n\} \leq C \log(n+1) \cdot \varepsilon.$$

History A similar result was proved by John [15] already in 1961, but with an error term $10n^{3/2}\varepsilon$. A more general formulation of this can also be found in the book [6, Theorem 14.11.]. The logarithmic upper bound was found in 2003 by Kalton in [17], whereas Matouvsková [18] constructed already in 2002 examples, where the error grows logarithmically. It follows that the logarithmic dependence on the dimension n is optimal.

We consider next the case, where the subset $A \subset \mathbf{B}^n$ is otherwise arbitrary, but contains the points $\overline{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n \in A$.

Theorem 2.3 *There is a universal constant $C > 0$ such that every ε-nearisometry $f : A \to \mathbb{R}^n$ has an isometric approximation $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfying*

$$\|T - f\|_A \equiv \sup\{|Tx - f(x)| \mid x \in A\} \leq C\, n \cdot \varepsilon.$$

History An upper bound $C\, n^{3/2}\, \varepsilon$ was obtained in [4, 3.12] using John's idea (see [6, Chapter 14]). In 2005, Vestfid [26] found the linear bound $C\, n\, \varepsilon$ and showed that the linear growth is optimal in n.

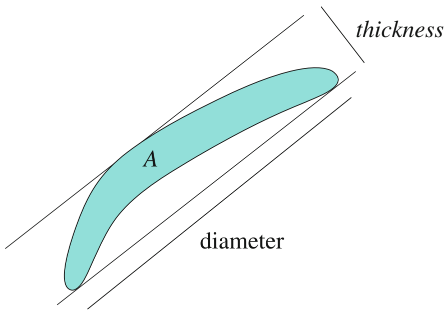
Before more general versions, we need a definition.

Definition 2.2 The **thickness** $\theta(A)$ of a set $A \subset \mathbb{R}^n$ is the infimum of numbers $t > 0$ such that A lies between two parallel hyperplanes with mutual distance t.

The inequality

$$0 \leq \theta(A) \leq d(A)$$

is always true, but the thickness $\theta(A)$ can be very small even if the diameter $d(A)$ is large. In particular, $\theta(A) = 0$ if and only if A is contained in some hyperplane.



The following theorem is from [4, 3.3].

Theorem 2.4 *Let $A \subset \mathbb{R}^n$ be a compact set such that*

$$\theta(A) \geq \frac{d(A)}{t}$$

for some $t \geq 1$, and let $f : A \to \mathbb{R}^n$ be an ε-nearisometry. Then there is an isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|T - f\|_A \equiv \sup\{|Tx - f(x)| \mid x \in A\} \leq C_n t\, \varepsilon.$$

Remark The upper bound is sharp with respect to the parameter t, but the asymptotic behaviour of C_n is unknown. Vestfrid's examples show that the growth of C_n is at least linear in n. On the other hand, an upper bound for C_n can be derived from the proof in [4]. The proof proceeds by induction on n, and the growth of C_n can be analysed from a system of recursion formulas. Numerical experiments for $n \le 50$ by the author (unpublished) indicate that

$$\lim_{n\to\infty} \frac{\log C_n}{3^n} \approx 0.5756.$$

(Curiously, this number seems to be equal to $(1/4)\log 10$ up to at least 10 decimal places, which I discovered by accident.) It follows from this that

$$C_n \lesssim 1.778^{3^n},$$

so there seems to be a huge gap between upper and lower estimates.

Without any restrictions on the geometry of the set A we obtained the following result in [4, 2.2].

Theorem 2.5 *Let $A \subset \mathbb{R}^n$ be a compact set and let $f: A \to \mathbb{R}^n$ be an $\varepsilon d(A)$-nearisometry. Then there is an isometry $T: \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\|T - f\|_A \equiv \sup\{|Tx - f(x)| \mid x \in A\} \le c_n d(A)\sqrt{\varepsilon}.$$

Remark Numerical estimation of c_n for large n using the proof seems difficult but should be possible: it leads to nested optimization problems for recursion formulas. However, I have calculated that our proofs give $c_3 = 19$, whereas $C_3 = 10^7$ for thick sets.

The following example shows that the $\sqrt{\varepsilon}$-term is essential in general.

Example 2.1 Let $f: A = \{-1, 0, 1\} \to \mathbb{R}^2$ be defined, using complex notation, by

$$f(x) = \begin{cases} x, & x = \pm 1 \\ i\sqrt{\varepsilon}, & x = 0. \end{cases}$$

Then f is $(1 + \varepsilon)$-bilipschitz and hence a 2ε-nearisometry, but for all isometric approximations $T: \mathbb{R}^2 \to \mathbb{R}^2$ the error is at least $\sqrt{\varepsilon}/2$. This follows easily by minimizing the distance from the set fA to the line $T\mathbb{R}$.

2.3 From Approximation to Bilipschitz Extension

In this and the following section we give some examples of extension results that can be proven by using the approximation results for bounded sets. The main problem is to extend a mapping $f: A \to \mathbb{R}^n$ to a mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ having similar properties as the original f.

The easiest case for bilipschitz extension occurs if the set A has thickness in all scales.

Definition 2.3 Let $c \geq 1$. A set $A \subset \mathbb{R}^n$ is c-uniformly thick if

$$\theta(A \cap B(a,r)) \geq 2r/c$$

for all $a \in A$ and $r > 0$.

Uniform thickness does not allow isolated points, but, on the other hand, extending a map from an isolated point (at least to its neighbourhood) is very easy. In order to obtain the most general setting for extension, we need a more general definition that does not rule out isolated points if there is enough thickness around them, in a larger scale related to the distance from an isolated point of A to the rest of A.

Definition 2.4 Let $A \subset \mathbb{R}^n$. For $a \in A$ we set $s(a) = d(a, A \setminus \{a\})$.

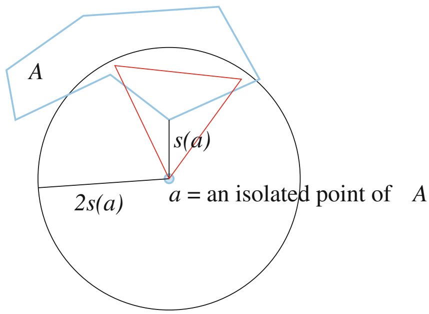
Then $s(a) > 0$ if and only if a is isolated in A.

Definition 2.5 Let $c \geq 1$. We say that the set $A \subset \mathbb{R}^n$ is c-sturdy if

(1) $\theta(A \cap B(a,r)) \geq 2r/c$ whenever $a \in A,\ r \geq cs(a),\ A \not\subset B(a,r)$,
(2) $\theta(A) \geq d(A)/c$.

If A is unbounded, we omit (2), and the condition $A \not\subset B(a,r)$ of (1) is unnecessary.

Examples of sturdy sets are $\mathbb{Z}^n \subset \mathbb{R}^n$, the Koch snowflake curve in the plane, bounded Lipschitz domains, and all uniformly thick sets.



intersection of A and $B(a,r)$ contains vertices of a thick triangle

The following extension theorem is from [5].

Theorem 2.6 *Let $A \subset \mathbb{R}^n$ be c-sturdy. Then there are $\delta = \delta(c,n)$ and $C = C(c,n)$ such that every $(1+\varepsilon)$-bilipschitz map $f\colon A \to \mathbb{R}^n$, with $\varepsilon \leq \delta$, extends to a $(1+C\varepsilon)$-bilipschitz map $F\colon \mathbb{R}^n \to \mathbb{R}^n$.*

The proof is based on approximation results for nearisometries and will be sketched in a more general setting in the next section.

Remark The converse result in $\mathbb{R}^2$ was proven in [1]: If a set $A \subset \mathbb{R}^2$ has the above extension property for $(1+\varepsilon)$-bilipschitz maps with a small ε, then it is sturdy, and there are quantitative relations between all constants involved.

2.4 From Approximation to Quasisymmetric Extension

In this section we consider a more general class of mappings, the quasisymmetric ones, and present the main extension result from [3].

Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism, called a growth function.

Definition 2.6 An injective map $f\colon A \to \mathbb{R}^n$ is η-quasisymmetric if the ratios of distances are changed in a controlled way:

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right)$$

for all distinct $x, y, z \in A$. If $\eta(t) = t$, then f is a similarity.

An L-bilipschitz map is quasisymmetric with $\eta(t) = L^2 t$, $t \ge 0$. Conversely, a quasisymmetric map with a linear growth $\eta(t) = Ct$ is always bilipschitz. However, since there is no general bound for the Lipschitz-constant of a similarity, we cannot say anything about the constant in this converse part.

It was proven in [24, 3.12] and [23, 6.5] that one can often replace the growth function η with a power form.

Theorem 2.7 *If A is relatively connected, then one can always choose*

$$\eta(t) = C \cdot \max\left(t^{\alpha}, t^{1/\alpha}\right), \ t \ge 0,$$

where $C \ge 1$ and $\alpha > 0$.

Here relative connectedness is much weaker than connectedness.

Definition 2.7 Let $M \ge 1$. A metric space X is M-relatively connected if, for all pairs of distinct points (x, y) and (w, z), there is a finite sequence $(x_0, x_1, \ldots, x_{k-1}, x_k)$ such that

$$x_0 = x,\ x_1 = y,\ x_{k-1} = w,\ x_k = z$$

and

$$\frac{1}{M} \le \frac{|x_{j+1}-x_j|}{|x_j - x_{j-1}|} \le M$$

for all $1 \le j \le k-1$.

Examples of relatively connected spaces include all connected ones, the Cantor middle-third set, etc.

In this light, the following condition seems natural and turns out to be the best way to measure how close a quasisymmetric mapping is from a similarity. Some problems with other possible approaches are considered in [2].

Definition 2.8 A mapping $f: A \to \mathbb{R}^n$ is ε-power-quasisymmetric if it is η-quasisymmetric with

$$\eta(t) = (1+\varepsilon) \cdot \max(t^{1+\epsilon}, t^{1/(1+\epsilon)}).$$

We remark that suitable radial stretching maps in $\mathbb{R}^n$ will satisfy this condition, but they are not bilipschitz.

Theorem 2.8 *Let $A \subset \mathbb{R}^n$ be c-sturdy. Then there are $\delta = \delta(c,n)$ and $C = C(c,n)$ such that every ε-power-quasisymmetric map $f: A \to \mathbb{R}^n$, with $\varepsilon \leq \delta$, extends to a $C\varepsilon$-power-quasisymmetric map $F: \mathbb{R}^n \to \mathbb{R}^n$.*

Main Steps of the Proof

- Show that ε-power-quasisymmetric maps can be well approximated by similarities in balls $A \cap B(a,r)$ if r is suitably chosen. This follows from sturdiness and the isometric approximation results of Alestalo et al. [4] by scaling.
- Show that one may assume A to be unbounded, so that sturdiness is easier to handle. This a rather easy, but a very technical part.
- Decompose $\mathbb{R}^n \setminus A$ into Whitney cubes and define the extension in the vertices v by using suitable approximating similarities of f in sets of the type $A \cap B(v,r)$. Here the radius r must be carefully chosen in order to guarantee the thickness of this intersection.
- Triangulate the Whitney cubes and extend affinely to each simplex.
- The result will be a continuous map $F: \mathbb{R}^n \to \mathbb{R}^n$.
- The final step is based on showing that the assumptions for the following theorem from [3, 3.7] are satisfied if ε is small enough. Indeed, if a mapping has been extended in a suitable way using approximating similarities, it should not be surprising that it can be well approximated by similarities. However, the details of the proof are again quite technical.

Theorem 2.9 *Let $F: \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following condition for some $\varepsilon \leq 1/100$: For every ball $B = B(x,r)$ there is a similarity $S = S_{x,r}$ such that*

$$\|S \circ F - \mathrm{id}\|_B \leq \varepsilon r.$$

Then F is 50ε-power-quasisymmetric.

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