A Novel Single-Loop Decoupled Schoenflies-Motion Generator: Concept and Kinematics Analysis

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Abstract. Schoenflies-motion generators (SMGs) are 4-degrees-of-freedom (dof) manipulators whose end effector can perform translations along three independent directions and rotations around one fixed direction (Schoenflies motions). Such motions constitute the 4-dimensional (4-D) Schoenflies subgroup of the 6-D displacement group. The most known SMGs are the serial robots named SCARA. Pick-and-place tasks are typical industrial applications that can be accomplished by SMGs. Over the SCARA, lower-mobility parallel manipulators (PMs) have been proposed as SMGs. Here, a novel type of SMG with parallel architecture is presented together with its kinematics analysis. The proposed SMG has a single-loop not-overconstrained architecture, actuators on or near the base and decoupled kinematics.

Keywords: Parallel manipulators · Schoenflies subgroup · Position analysis · Singularity analysis · Decoupled kinematics

1 Introduction

A number of industrial applications (e.g., pick-and-place tasks) require rigid body’s translations along three independent directions together with rotations around one fixed direction. The displacement set of this type constitutes the 4-D Schoenflies subgroup of the 6-D displacement group [1–3] and the 4-dof manipulators whose end effector is constrained to perform Schoenflies displacements are called Schoenflies-motion generators (SMGs) [2]. The most known SMGs are the serial robots named SCARA. Parallel SMGs have been also proposed in the literature (see, for instance, [3–11]). Parallel architectures feature two rigid bodies, one fixed (base) and the other movable (platform), that are connected with one another by a number of kinematic chains (limbs). Most of the parallel SMGs have four limbs (e.g., [4–6, 8, 11]) and one actuator per limb located on the base. Nevertheless, two-limbed symmetric (i.e., single-loop) architectures with serial [3] or hybrid [7, 10] limbs have been proposed, too. Other parallel SMG are simply obtained by adding a double Cardan shaft (i.e., a limb of \(\text{RUPUR}^1\) type), which connects the base to the platform, in a translational parallel manipulator.

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1 Hereafter, R, P, U, S, and C stand for revolute pair, prismatic pair, universal joint, spherical pair, and cylindrical pair respectively. The underlining indicates the actuated joints and the sequence of joint types, which are encountered by moving from the base to the platform on a limb, is given with a string of capital letters.
Parallel SMGs are faster \[6\] than their serial counterparts due to the possibility, they have, of locating the actuators on the base. Nevertheless, reduced workspace and cumbersome multi-loop topologies with complex kinematics are their main drawbacks. In particular, parallel SMGs usually cannot make the platform perform a complete rotation and some tricks have to be devised to overcome this drawback \[12\]. So, reducing the limb number by keeping the actuators on or near the base is an appealing design choice. This can be done by adopting single-loop architectures with two actuators per limb \[3\].

Also, the possibility of decoupling position and orientation (decoupled kinematics) would allow simpler and more intuitive control strategies, and a not-overconstrained architecture would make it possible to avoid jamming without using small tolerances during manufacturing. Here, a novel parallel SMG of PRRS-RRC type (Fig. 1) is presented together with its kinematics analysis. The proposed SMG has a single-loop not-overconstrained architecture, actuators on or near the base and a simple and decoupled kinematics.

The paper is organized as follows. Section 2 presents the novel SMG. Then, Sect. 3 solves in closed form its direct and inverse position analyses. Eventually, Sect. 4 discusses the most critical issues of the proposed SMG, and Sect. 5 draws the conclusions.

Fig. 1. The PRRS-RRC Schoenflies-motion generator
2 The **PRRS-RRC** Shoenflies-Motion Generator

Figure 1 shows the SMG of **PRRS-RRC** type. With reference to Fig. 1, the platform is connected to the base by two limbs, one of **PRRS** type and the other of **RRC** type, which form a single-loop not-overconstrained spatial mechanism with 4 dof. The Cartesian reference $O_bx_by_z_b$ is fixed to base. The axes of the R and C pairs and the sliding direction of the P pair are all parallel to the $z_b$ axis.

In the **PRRS** limb (see Fig. 1), the P pair and the adjacent R pair are actuated. Point $B_2$ lies on the axis of the actuated R pair and is fixed to the slider of the P pair. The actuated-joint variable of the P pair is the signed distance, $d$, of $B_2$ from $O_b$. The plane parallel to the $x_by_b$ plane and passing through $B_2$ intersects the axis of the passive R pair at $D_2$, the axis of the C pair at $A_p$, and the axis parallel to the $z_b$ axis and passing through the center of the S pair at $A_2$. $a_3$ and $a_4$ are the lengths of the segments $B_2D_2$ and $D_2A_2$, respectively; whereas, $\theta_3$ and $\theta_4$ are the joint variables of the actuated R pair and of the passive R pair, respectively. The point $A_p$ is fixed to the platform.

In the **RRC** limb (see Fig. 1), the two R pairs are actuated. Their actuated-joint variables are $\theta_1$ and $\theta_2$. The $x_by_b$ plane intersects the axis of the R pair adjacent to the base at $B_1$, the axis of the other R pair at $D_1$, and the axis of the C pair at $A_1$. $a_0$, $a_1$ and $a_2$ are the lengths of the segments $O_bB_1$, $B_1D_1$ and $D_1A_1$, respectively.

Regarding the platform (see Fig. 1), $a_p$ is the constant distance of the center of the S pair from the axis of the C pair and is equal to the length of the segment $A_pA_2$. $O_p$ is the reference point of the platform, and $h$ is the length of the segment $A_pO_p$.

Hereafter, the coordinates of $O_p$ measured in $O_bx_by_z_b$ will be denoted $(x_p, y_p, z_p)^T$ and will locate the position of the platform; whereas, the angle, $\varphi$, between the segment $A_pA_2$ and a line parallel to $x_b$ and passing through $A_p$ (see Fig. 1) will be used to uniquely determine the platform orientation.

The **RRC** limb constrains the platform to perform Schoenflies motions with rotation axis parallel to the $z_b$ axis. As it will be clearer later, it controls the position of point $A_1$ on the $x_by_b$ plane through its two actuated R pairs; whereas, the **PRRS** limb controls the coordinate $z_p$ of point $O_p$ through the actuated P pair and, independently, the platform orientation through its actuated R pair.

3 Position Analysis

According to the above-introduced notations, the 4-tuple $q = (\theta_1, \theta_2, \theta_3, d)^T$ collects the coordinates of the joint space (i.e., all the actuated-joint variables); whereas, the 4-tuple $k = (x_p, y_p, z_p, \varphi)^T$ collects the coordinates of the operational space that uniquely define the platform pose.

The analysis of the **RRC** limb (Fig. 1) brings to write the following relationships

\[ x_p = a_0 + a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) \tag{1a} \]
\[ y_p = a_0 + a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2) \tag{1b} \]
Moreover, the analysis of the PRRS limb yields

\[\begin{align*}
x_p &= a_3 \cos \theta_3 + a_4 \cos(\theta_3 + \theta_4) - a_p \cos \varphi \\
y_p &= a_3 \sin \theta_3 + a_4 \sin(\theta_3 + \theta_4) - a_p \sin \varphi \\
z_p &= d - h
\end{align*}\]

(2a) (2b) (2c)

### 3.1 Direct Position Analysis

The direct position analysis (DPA) consists in calculating the platform poses (i.e., the values of \( \mathbf{k} \)) compatible with assigned values of the actuated-joint variables (i.e., with one value of \( \mathbf{q} \)). The closed-form solution of the DPA follows.

Since, in the DPA, the values of \( \theta_1, \theta_2 \) and \( d \) are assigned, Eqs. (1a), (1b), and (2c) straightforwardly give a unique value for \( x_p, y_p \), and \( z_p \) (i.e., for the platform position), respectively. Once \( x_p \) and \( y_p \) have been computed, the elimination of \( \cos(\theta_3 + \theta_4) \) and \( \sin(\theta_3 + \theta_4) \) from Eqs. (2a) and (2b), obtained by exploiting the trigonometric identity \( \cos^2 x + \sin^2 x = 1 \), yields

\[
(x_p + a_p \cos \varphi - a_3 \cos \theta_3)^2 + (y_p + a_p \sin \varphi - a_3 \sin \theta_3)^2 = a_4^2
\]

(3)

which can be transformed as follows by expanding and simplifying its left-hand side

\[
b_0 + b_1 \sin \varphi + b_2 \cos \varphi = 0
\]

(4)

where

\[
b_0 = x_p^2 + y_p^2 + a_3^2 - a_4^2 - 2a_3(x_p \cos \theta_3 + y_p \sin \theta_3),
\]

\[
b_1 = 2a_p(y_p - a_3 \sin \theta_3),
\]

and

\[
b_2 = 2a_p(x_p - a_3 \cos \theta_3).\]

Since, in the DPA, \( \theta_3 \) is known, Eq. (4) is a trigonometric equation that contains only one unknown: the angle \( \varphi \). The introduction of the trigonometric identities \( \cos \varphi = (1 - t^2)/(1 + t^2) \) and \( \sin \varphi = 2t/(1 + t^2) \) with \( t = \tan(\varphi/2) \) into Eq. (4) yields the following quadratic equation

\[
t^2(b_0 - b_2) + 2t b_1 + (b_0 + b_2) = 0
\]

(5)

whose two solutions are

\[
\varphi_{1,2} = 2\arctan\left(\pm\frac{b_1 \pm \sqrt{b_1^2 + b_2^2 - b_0^2}}{b_0 - b_2}\right)
\]

(6)

The two values of \( \varphi \) (i.e., the two platform orientations), Eq. (6) provides, depend only on the assigned value of the actuated-joint variable \( \theta_3 \). From a geometric point of view, they correspond to the two intersections of the two circumferences, one with center at \( D_2 \) and radius \( a_4 \) and the other with center at \( A_p \) and radius \( a_p \), that lie on the plane parallel to the \( x_b, y_b \) plane and passing through \( B_2 \).
The conclusions are: (i) the DPA can be solved by using simple explicit formulas, and (ii) the DPA has one solution for the platform position that depends only on the actuated-joint variables $\theta_1$, $\theta_2$ and $d$ and two solutions for the platform orientation that depend only on the remaining actuated-joint variable $\theta_3$ (i.e., position and orientation are decoupled).

### 3.2 Inverse Position Analysis

The inverse position analysis (IPA) consists in calculating the actuated-joint variables (i.e., the values of $q$) compatible with an assigned platform pose (i.e., with one value of $k$). The closed-form solution of the IPA follows.

Since, in the IPA, the values of $x_p$, $y_p$, and $z_p$ are assigned, the actuated-joint variable $d$ can be immediately computed from Eq. (2c) as follows: $d = z_p + h$; whereas, $\theta_1$ and $\theta_2$ can be computed by solving the system of two equations in two unknowns constituted by Eqs. (1a) and (1b). Indeed, the elimination of $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ from Eqs. (1a) and (1b), obtained by exploiting the trigonometric identity $\cos^2 x + \sin^2 x = 1$, yields

$$
(x_p - a_0 - a_1 \cos \theta_1)^2 + (y_p - a_0 - a_1 \sin \theta_1)^2 = a_2^2
$$

which can be transformed as follows by expanding and simplifying its left-hand side

$$
c_0 + c_1 \sin \theta_1 + c_2 \cos \theta_1 = 0
$$

where $c_0 = (x_p - a_0)^2 + (y_p - a_0)^2 + a_1^2 - a_2^2$, $c_1 = -2a_1(y_p - a_0)$, and $c_2 = -2a_1(x_p - a_0)$.

Then, the introduction of the trigonometric identities $\cos \theta_1 = (1 - f^2)/(1 + f^2)$ and $\sin \theta_1 = 2f/(1 + f^2)$ with $t = \tan (\theta_1/2)$ into Eq. (8) yields the following quadratic equation

$$
f^2(c_0 - c_2) + 2f c_1 + (c_0 + c_2) = 0
$$

whose two solutions are

$$
(\theta_1)_{1,2} = 2 \arctan \left( \frac{-c_1 \pm \sqrt{c_1^2 + c_2^2 - c_0^2}}{c_0 - c_2} \right)
$$

Once, the two values of $\theta_1$ have been computed through formula (10) the two corresponding values of $\theta_2$ can be computed from Eqs. (1a) and (1b) as follows:

$$
\theta_2 = \text{ATAN2}[(y_p - a_0 - a_1 \sin \theta_1), (x_p - a_0 - a_1 \cos \theta_1)] - \theta_1
$$

From a geometric point of view, the two computed solutions of $(\theta_1, \theta_2)$ correspond to the two intersections of the two circumferences of the $x_y$ plane one with center at $A_1$ and radius $a_2$ and the other with center at $B_1$ and radius $a_2$. 

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Eventually, since the angle $\phi$ is assigned in the IPA, Eq. (3) has the only unknown $\theta_3$. Such equation can be transformed as follows by expanding and simplifying its left-hand side

$$g_0 + g_1 \sin \theta_3 + g_2 \cos \theta_3 = 0$$  \hspace{1cm} (12)

where $g_0 = (x_p + a_p \cos \phi)^2 + (y_p + a_p \sin \phi)^2 + a_3^2 - a_4^2$, $g_1 = -2a_3(y_p + a_p \sin \phi)$, and $g_2 = -2a_3(x_p + a_p \cos \phi)$. After the introduction of the trigonometric identities $\cos \theta_3 = (1 - s^2)/(1 + s^2)$ and $\sin \theta_3 = 2s/(1 + s^2)$ with $s = \tan(\theta_3/2)$, Eq. (12) becomes the following quadratic equation in $s$

$$s^2(g_0 - g_2) + 2s g_1 + (g_0 + g_2) = 0$$  \hspace{1cm} (13)

whose two solutions are

$$\theta_{3,1,2} = 2 \arctan \left( \frac{-g_1 \pm \sqrt{g_1^2 + g_2^2 - g_0^2}}{g_0 - g_2} \right)$$  \hspace{1cm} (14)

From a geometric point of view, the two values of $\theta_3$ computed with formula (14) correspond to the two intersections of the two circumferences, one with center at $A_2$ and radius $a_4$ and the other with center at $B_2$ and radius $a_3$, that lie on the plane parallel to the $x_1y_2$ plane and passing through $B_2$.

The conclusion is that the IPA has only one solution for $d$ and, at most, four solutions for $(\theta_1, \theta_2, \theta_3)$, that is, at most four values of $q$ are compatible with one value of $k$.

## 4 Discussion

The singularity analysis is a central issue for parallel mechanisms. Singularities are mechanism configurations where the linear relationship between actuated-joint rates and platform twist (instantaneous input-output relationship (InI/O)) fails. In the studied SMG, configurations (constraint singularities) where the platform instantaneous motion may not be a Schoenflies motion are not present since the RRC limb has always connectivity four. The absence of constraint singularities makes it possible to deduce the InI/O by taking into account only $\dot{k} = (\dot{x}_p, \dot{y}_p, \dot{z}_p, \dot{\phi})^T$, instead of the whole platform twist, as instantaneous output. The time derivatives of Eqs. (1a), (1b), (2c), and (3) yield the following InI/O

$$\dot{x}_p = -[a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2)]\dot{\theta}_1 - a_2 \sin(\theta_1 + \theta_2)]\dot{\theta}_2$$  \hspace{1cm} (15a)

$$\dot{y}_p = [a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2)]\dot{\theta}_1 + a_2 \cos(\theta_1 + \theta_2)]\dot{\theta}_2$$  \hspace{1cm} (15b)
\[
\dot{z}_p = \dot{d} \tag{15c}
\]

\[
m_x \dot{x}_p + m_y \dot{y}_p + \phi a_p (m_y \cos \varphi - m_x \sin \varphi) = \dot{\theta}_3 a_3 (m_y \cos \theta_3 - m_x \sin \theta_3) \tag{15d}
\]

where \( m_x = x_p + a_p \cos \varphi - a_3 \cos \theta_3 \) and \( m_y = y_p + a_p \sin \varphi - a_3 \sin \theta_3 \).

The analysis of Eqs. (15a)–(15d) reveals that, if the actuated-joint rates are assigned, \( \dot{\Omega}_p = (\dot{x}_p, \dot{y}_p, \dot{z}_p)^T \) is always uniquely determined, but \( \dot{\phi} \) is not determined when the segments \( A_2D_2 \) and \( A_2A_p \) are aligned (i.e., a parallel singularity occurs). Also, it reveals that, if \( \dot{\mathbf{h}} \) is assigned, only two geometric conditions make one or more actuated-joint rates undetermined (i.e., a serial singularity occurs): (i) the 2-tuple \( (\dot{\theta}_1, \dot{\theta}_2) \) is not determined when the segments \( A_1D_1 \) and \( D_1B_1 \) are aligned, and (ii) \( \dot{\theta}_3 \) is not determined when the segments \( A_2D_2 \) and \( D_2B_2 \) are aligned.

If the actuated \( \mathbf{P} \) pair and the two actuated \( \mathbf{R} \) pairs of the \( \mathbf{RRC} \) limb are locked, the position of point \( A_p \) (Fig. 1) will be locked, too, and the segments \( B_2A_p, B_2D_2, D_2A_2 \), and \( A_pA_2 \) will behave like frame, input link, coupler, and follower, respectively, of a four-bar linkage with \( \theta_3 \) and \( \varphi \) as input and output variables, respectively. The singularities of this four-bar linkage correspond to the above-identified parallel singularity and serial singularity (ii). Also, if Grashof’s rule is satisfied and \( A_pA_2 \) is the shortest bar the angle \( \varphi \) can perform a complete rotation. In general, the angle \( \varphi \) can always perform a complete rotation if and only if \( (a_3 + a_4) \geq (a_p + \sqrt{x_p^2 + y_p^2}) \). Unfortunately, the parallel singularity condition is encountered two times during a complete rotation of the platform. Nevertheless, this parallel singularity can be safely crossed by suitably changing the length and/or the orientation of the frame \( B_2A_p \) with the \( \mathbf{RRC} \) limb.

Eventually, it is worth stressing that two coaxial motors put on the base can drive both the two actuated \( \mathbf{R} \) pairs of the \( \mathbf{RRC} \) limb since a toothed belt can transmit the motion to the second \( \mathbf{R} \) pair. So, the proposed SMG practically has three actuators on the base and only one near the base.

## 5 Conclusions

A novel parallel Schoenflies-motion generator of \( \text{PRRS-} \mathbf{RRC} \) type has been presented together with its kinematics analysis. The proposed \( \text{PRRS-} \mathbf{RRC} \) has a single-loop not-overconstrained architecture, actuators on or near the base and a simple and decoupled kinematics.

In particular, both the direct and the inverse position analyses are solved by using simple explicit formulas with a clear geometric meaning. Three actuators control the platform position and, independently, the remaining fourth controls the platform rotation. There is no constraint singularity, and only one parallel singularity condition safely crossable by using suitable path planning strategies for making the platform perform a complete rotation.

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2 It is worth reminding that serial singularities lie on (and identify) the workspace boundary.
Future works will address the dimensional synthesis and the dynamic behavior of the proposed manipulator.

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