Preface to the Second Edition

The main stimulus for preparing this new edition was the lively development of the subject since the publication of the first edition in 2011. I have tried to incorporate the new results as much as possible, although some of them could only be mentioned or sketched due to their complexity and space limitations. For the sake of consistency and completeness, I also added some material predating 2011 that was not included in the first edition due to a self-imposed limitation on the size. On the other hand, some inessential material has been removed, so neither edition is a subset of the other one. Needless to say, I eliminated the errors, gaps and inconsistencies which had been found.

Let me now describe the main changes in this edition.

Chapter 2, which summarizes the basic Stein manifold theory, has been slightly expanded. In particular, I added the statement and proof of the parametric version of the Cartan-Oka-Weil theorem in Sect. 2.8. Although these extensions follow easily from the classical theory, it seems impossible to find them in the literature. Since they are very important in Oka theory, I decided to include them here.

Chapter 3 is basically unchanged, except for Sect. 3.12 where some important recent works are mentioned.

In the theory of holomorphic automorphisms, discussed in Chap. 4, major new developments occurred in the study of the density property of Stein manifolds and affine algebraic manifolds, and of geometric structures on them; Sect. 4.10 has been rewritten accordingly. The class of complex manifolds called long $\mathbb{C}^n$'s is now somewhat better understood, and these new developments are presented in Sect. 4.21. There are several other improvements and new results in Chap. 4 which are too numerous to mention here. This beautiful and very useful subject would clearly deserve a book in its own right.

Chapters 5 and 6, which constitute the core of the book, received some topical improvements, minor rearrangements, and additions of newly discovered examples of Oka manifolds. The presentation of the relative Oka principle on 1-convex manifolds in Sect. 6.13 reflects new developments which complete the outline of proof given in the first edition. I added Sect. 6.14 where a relative Oka principle for sections of branched holomorphic maps is presented. This result was originally proved in 2003, but was not included in the first edition. In the meantime new applications have been found, two of which are included here. This result may have the potential of extending the Oka principle for Oka pairs of sheaves due to Forster and Ramspott; however, further investigations are needed.

Chapter 7 is new. Sections 7.1–7.2 contain a discussion of the known relationships between the Oka property and several other holomorphic flexibility properties (by which I mean anti-hyperbolicity properties) of complex manifolds. Section 7.3 brings a summary of what we know about which compact complex surfaces are Oka manifolds. There is an expanded version of Sect. 7.4 on Oka maps. In Sect. 7.5, which was contributed by Finnur Lárusson, the reader can find a homotopy theoretic point of view on Oka theory. The final Sect. 7.6 contains miscellaneous new results and a collection of open problems.

Chapters 8–10 roughly correspond to Chaps. 7–9 in the first edition. Sections 8.10–8.13 are new, although the material predates 2011. Section 9.1 has been expanded and moved here from the last chapter. This enabled me to simplify the topological aspects of proofs of some of the subsequent results. Section 9.7 offers a new and considerably simpler proof of the splitting lemma for biholomorphic maps close to the identity on Cartan pairs, also in complex spaces with singularities. Besides its original use in the construction of holomorphic submersions from Stein manifolds, many new applications of this gluing technique have been found, especially to the problem of exposing boundary points of domains. These results play a major role in the study of the so-called squeezing function and of the boundary behavior of invariant metrics. I have rewritten Sect. 9.8 in light of the new results concerning the existence of proper holomorphic immersions and embeddings of Stein manifolds into Stein manifolds with the (volume) density property. Also, there is a new application to the Hodge theory of q-complete manifolds. Section 9.11 contains several new recent results on the existence of proper holomorphic embeddings of open Riemann surfaces into the affine plane $\mathbb{C}^2$. In particular, it has been proved by Wold and the author than any circular planar domain (possibly infinitely connected) with at most finitely many punctures embeds properly holomorphically into $\mathbb{C}^2$.

The changes in Chap. 10 pertain mainly to the *soft Oka principle* in Sects. 10.9–10.11. A more complete presentation of this subject is now available in the book by Cieliebak and Eliashberg, *From Stein to Weinstein and back*, published in 2012. The improvements which concern this text are described at the relevant places, but the overall presentation and the proofs have not been changed since the interested reader can consult their book. These improvements mainly pertain to complex dimension 2 where there is an abundance of exotic Stein structures on surfaces of suitable topological type.

The Oka principle for sections of holomorphic maps over Stein spaces, presented in Chaps. 5 and 6, has reached a mature stage and has essentially been axiomatized. For stratified fibre bundles, considered in Chap. 5, the relevant condition implying the Oka principle is that the fibres satisfy the *Convex Approximation Property*, CAP, characterizing the class of *Oka manifolds* which became one of the central notions of the theory. For holomorphic submersions, the relevant condition is the *Homotopy Approximation Property*, HAP, for sections over small open sets in the base. The gist of the proofs in these two chapters, which follow ideas of Grauert, Henkin and Leiterer, and Gromov, is that these local conditions imply the Oka principle for global sections. This reduces the problem to finding geometric conditions implying CAP or HAP. The notions of ellipticity and subellipticity continue to play an important role, although some finer conditions such as holomorphic flexibility and the density property have become major new sources of examples. A possible next step would be to axiomatize the relevant geometric conditions on the source side; here we are talking of Cartan pairs and $\mathcal{C}$-strings in Stein spaces. This has the potential of extending the Oka principle way beyond its current scope. I leave this task for another day and another person.

I wish to thank the numerous colleagues who have contributed to this book by collaboration, discussions, or simply by pointing out the deficiencies and proposing improvements. First and foremost, I thank Finnur Lárusson for having contributed Sect. 7.5 and for his lively interest and collaboration on this subject, Frank Kutzschebauch for having explained to me the developments concerning the flexibility properties of affine algebraic and Stein manifolds and for numerous discussions of other topics, and the collaborators and colleagues whose ideas and communications contributed in an essential way to this text: Rafael Andrist, Luka Boc Thaler, Gregery Buzzard, Barbara Drinovec Drnovšek, Josip Globevnik, Jürgen Leiterer, Erik Løw, Takeo Ohsawa, Jasna Prezelj, Tyson Ritter, Jean-Pierre Rosay, Marko Slapar, Dror Varolin, Jörg Winkelmann, and Erlend F. Wold. I sincerely thank Peter Landweber who carefully read the draft and proposed numerous improvements.

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Finally, my sincere thanks to the staff of Springer-Verlag for their professional work in the technical preparation of the book.

Ljubljana Franc Forstnerič
May 1, 2017

Preface to the First Edition

This book is an attempt to present a coherent account of Oka theory, from the classical Oka-Grauert theory originating in the works of Kiyoshi Oka and Hans Grauert to the contemporary developments initiated by Mikhael Gromov.

At the core of Oka theory lies the heuristic *Oka principle*, a term coined by Jean-Pierre Serre in 1951: *Analytic problems on Stein manifolds admit analytic solutions if there are no topological obstructions.* The Cartan-Serre Theorems A and B are primary examples. The main exponent of the classical Oka-Grauert theory is the equivalence between topological and holomorphic classification of principal fiber bundles over Stein spaces. On the interface with affine algebraic geometry the Oka principle holds only rarely, while in projective geometry we have Serre's GAGA principle, the equivalence of analytic and algebraic coherent sheaves on compact projective algebraic varieties. In smooth geometry there is the analogous *homotopy principle* originating in the Smale-Hirsch homotopy classification of smooth immersions.

Modern Oka theory focuses on those properties of a complex manifold Y which ensure that any continuous map $X \to Y$ from a Stein source space X can be deformed to a holomorphic map; the same property is considered for sections of a holomorphic submersion $Y \to X$. By including the Runge approximation and the Cartan extension condition one obtains several ostensibly different Oka properties. Gromov's main result is that a geometric condition called ellipticity—the existence of a dominating holomorphic spray on Y—implies all forms of the Oka principle for maps or sections $X \to Y$. Subsequent research culminated in the result that all Oka properties of a complex manifold Y are equivalent to the following Runge approximation property:

A complex manifold Y is said to be an Oka manifold *if every holomorphic map $f\colon K \to Y$ from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n \to Y$.*

The related concept of an *Oka map* pertains to the Oka principle for lifting holomorphic maps from Stein sources. The class of Oka manifolds is dual to the class of Stein manifolds in a sense that can be made precise by means of abstract homotopy theory. Finnur Lárusson constructed a model category containing all complex manifolds in which Stein manifolds are cofibrant, Oka manifolds are fibrant, and Oka maps are fibrations. This means that

Stein manifolds are the natural sources of holomorphic maps, while Oka manifolds are the natural targets.

Oka manifolds seem to be few and special; in particular, no compact complex manifold of Kodaira general type is Oka. However, special and highly symmetric objects are often more interesting than average generic ones.

A few words about the content. Chapter 1 contains some preparatory material, and Chap. 2 is a brief survey of Stein space theory. In Chap. 3 we construct open Stein neighborhoods of certain types of sets in complex spaces that are used in Oka theory. Chapter 4 contains an exposition of the theory of holomorphic automorphisms of Euclidean spaces and of the density property, a subject closely intertwined with our main theme. In Chap. 5 we develop Oka theory for stratified fiber bundles with Oka fibers (this includes the classical Oka-Grauert theory), and in Chap. 6 we treat Oka-Gromov theory for stratified subelliptic submersions over Stein spaces. Chapters 7 and 8 contain applications ranging from classical to the recent ones. In Chap. 8 we present results on regular holomorphic maps of Stein manifolds; highlights include the optimal embedding theorems for Stein manifolds and Stein spaces, proper holomorphic embeddings of some bordered Riemann surfaces into $\mathbb{C}^2$, and the construction of noncritical holomorphic functions, submersions and foliations on Stein manifolds. In Chap. 9 we explore implications of Seiberg-Witten theory for the geometry of Stein surface, and we present the Eliashberg-Gompf construction of Stein structures on manifolds with suitable handlebody decomposition. A part of this story is the *Soft Oka principle*.

This book would not have existed without my collaboration with Jasna Prezelj who explained parts of Gromov's work on the Oka principle in her dissertation (University of Ljubljana, 2000). Josip Globevnik suggested that we look into this subject, while many years earlier Edgar Lee Stout proposed that I study the Oka-Grauert principle. My very special thanks go to the colleagues who read parts of the text and offered suggestions for improvements: Barbara Drinovec-Drnovšek, Frank Kutzschebauch, Finnur Lárusson, Takeo Ohsawa, Marko Slapar, and Erlend Fornæss Wold. I am grateful to Reinhold Remmert for his invitation to write a volume for the Ergebnisse series, and to the staff of Springer-Verlag for their professional help.

Finally, I thank Angela Gheorghiu for all those incomparably beautiful arias, and my family for their patience.

Ljubljana Franc Forstnerič
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