Chapter 2
Profinite Graphs

Unless otherwise specified, in this chapter \( C \) is a pseudovariety of finite groups, i.e., a nonempty class of finite groups closed under subgroups, quotients and finite direct products.

2.1 First Notions and Examples

A profinite graph is a profinite space \( \Gamma \) with a distinguished nonempty subset \( V(\Gamma) \), the vertex set of the graph \( \Gamma \), and two continuous maps 
\[
d_0, d_1 : \Gamma \to V(\Gamma)
\]
whose restrictions to \( V(\Gamma) \) are the identity map \( \text{id}_{V(\Gamma)} \) (to simplify the notation, we sometimes write \( d_i m \), rather than \( d_i (m) \) \( m \in \Gamma, i = 0, 1 \)). This implies that the distinguished subset \( V(\Gamma) \) is necessarily closed. The elements of \( V(\Gamma) \) are called the vertices of \( \Gamma \), the elements of \( E(\Gamma) = \Gamma - V(\Gamma) \) are the edges of \( \Gamma \), and \( d_0(e) \) and \( d_1(e) \) are the initial and terminal vertices of an edge \( e \), respectively (also called the origin and terminus of \( e \)). An edge \( e \) with \( d_0(e) = d_1(e) = v \) is called a loop or a loop based at \( v \). We refer to \( d_0 \) and \( d_1 \) as the incidence maps of the graph \( \Gamma \).

Observe that a profinite graph is also a graph in the usual sense, or, more precisely, an oriented graph (see Appendix A), if we dispense with the topology. The set of edges \( E(\Gamma) \) of a profinite graph \( \Gamma \) need not be a closed subset of \( \Gamma \). If \( E(\Gamma) \) is closed (and therefore compact), it is enough to check the continuity of \( d_0 \) and \( d_1 \) on \( V(\Gamma) \) and \( E(\Gamma) \) separately, since then \( V(\Gamma) \) and \( E(\Gamma) \) are disjoint and clopen.

Associated with each edge \( e \) of \( \Gamma \) we introduce symbols \( e^1 \) and \( e^{-1} \). We identify \( e^1 \) with \( e \). Define incidence maps for these symbols as follows: \( d_0(e^{-1}) = d_1(e) \) and \( d_1(e^{-1}) = d_0(e) \). Given vertices \( v \) and \( w \) of \( \Gamma \), a path \( p_{vw} \) from \( v \) to \( w \) is a finite sequence \( e_1^{\epsilon_1}, \ldots, e_m^{\epsilon_m} \), where \( m \geq 0, e_i \in E(\Gamma), \epsilon_i = \pm 1 \) \( i = 1, \ldots, m \) such that \( d_0(e_1^{\epsilon_1}) = v, d_1(e_m^{\epsilon_m}) = w \) and \( d_1(e_i^{\epsilon_i}) = d_0(e_{i+1}^{\epsilon_{i+1}}) \) for \( i = 1, \ldots, m - 1 \). Such a path is said to have length \( m \). Observe that a path is always meant to be finite. The underlying graph of the path \( p_{vw} \) consists of the edges \( e_1, \ldots, e_m \) and their
vertices $d_i(e_j)$ ($i = 0, 1; j = 1, \ldots, m$). The path $p_{vw}$ is called reduced if whenever $e_i = e_{i+1}$, then $\varepsilon_i = \varepsilon_{i+1}$, for all $i = 1, \ldots, m - 1$.

**Example 2.1.1**  (a) A finite abstract graph $\Gamma$ (see Appendix A) with the discrete topology is a profinite graph.

(b) Let $N = \{0, 1, 2, \ldots\}$ and $\tilde{N} = \{\tilde{n} \mid n \in N\}$ be copies of the set of natural numbers (with the discrete topology). Define

$$I = N \cup \tilde{N} \cup \{\infty\}$$

to be the one-point compactification of the space $N \cup \tilde{N}$. Recall that then in the topology of $I$ each set $\{n\}$ and $\{\tilde{n}\}$ is open ($n \in N$), and the basic open neighbourhoods of $\infty$ are the complements of finite subsets of $N \cup \tilde{N}$. Clearly $I$ is a profinite space. We make $I$ into a profinite graph by setting

$$V(I) = N \cup \{\infty\},$$

$$E(I) = \tilde{N},$$

$$d_0(\tilde{n}) = n, d_1(\tilde{n}) = n + 1, \text{ for } \tilde{n} \in E(I),$$

and $d_i(n) = n$, for $n \in V(I)$ ($i = 1, 2$).

Observe that in this case the subset of edges $E(I)$ is open, but not closed in $I$.

(c) Let $p$ be a prime number and let $Z_p$ be the additive group of the ring of $p$-adic integers. Define a graph

$$\Gamma = \Gamma(Z_p, \{1\})$$

with set of vertices $V = V(\Gamma) = Z_p$ and whose set of edges is $E = E(\Gamma) = \{(\alpha, 1) \mid \alpha \in Z_p\}$. Then $V(\Gamma)$ and $E(\Gamma)$ are profinite spaces. We define the topology of

$$\Gamma = V(\Gamma) \cup E(\Gamma)$$

to be the disjoint topology: a subset $A$ of $\Gamma$ is open if and only if $A \cap V$ is open in $V$ and $A \cap E$ is open in $E$. One easily sees that $\Gamma$ is a profinite space. Observe that the subset of edges $E = E(\Gamma)$ of $\Gamma$ is both open and closed (clopen) in the topology of $\Gamma$. The incidence maps are the continuous maps

$$d_i : \Gamma \rightarrow V \quad (i = 0, 1)$$

defined as $d_0(\alpha) = \alpha$, $d_0(\alpha, 1) = \alpha$ and $d_1(\alpha) = \alpha$, $d_1(\alpha, 1) = \alpha + 1$ ($\alpha \in Z_p$).

With these definitions $\Gamma$ becomes a profinite graph. [This is an instance of profinite graphs obtained from profinite groups in a standard manner, the so-called Cayley graphs: see Example 2.1.12.] The subgroup of integers $Z = \langle 1 \rangle$ is dense in $Z_p$ and the topology of $Z$ induced by the topology of $Z_p$ is the discrete topology. Let

$$\Gamma(Z, \{1\}) = \{\alpha \in V(\Gamma) \mid \alpha \in Z\} \cup \{(\alpha, 1) \mid \alpha \in Z\},$$

Then $\Gamma(Z, \{1\})$ is an abstract discrete graph

$$\cdots -2 -1 0 1 2 \cdots$$

which is dense in the profinite graph $\Gamma = \Gamma(Z_p, \{1\})$. 

2.1 First Notions and Examples

More generally, let $\beta$ be a fixed element of $\mathbb{Z}_p$, and define

$$\Gamma(\mathbb{Z} + \beta, \{1\}) = \{ \alpha \in V(\Gamma) \mid \alpha \in \mathbb{Z} + \beta \} \cup \{ (\alpha, 1) \in E(\Gamma) \mid \alpha \in \mathbb{Z} + \beta \}.$$

Then $\Gamma(\mathbb{Z} + \beta, \{1\})$ is an abstract discrete graph

\[ \cdots \quad (\beta - 2, 1) \quad (\beta - 1, 1) \quad \beta \quad (\beta, 1) \quad (\beta + 1, 1) \quad (\beta + 2, 1) \cdots \]

which is also dense in the profinite graph $\Gamma = \Gamma(\mathbb{Z}_p, \{1\})$. Note that $\Gamma(\mathbb{Z}_p, \{1\})$ is a disjoint union of uncountably many abstract discrete graphs of the form $\Gamma(\mathbb{Z} + \beta, \{1\})$:

$$\Gamma(\mathbb{Z}_p, \{1\}) = \bigcup_{\lambda \in \Lambda} \Gamma(\mathbb{Z} + \beta\lambda, \{1\}),$$

where $\{\beta\lambda \mid \lambda \in \Lambda\}$ is a complete set of representatives of the cosets of the subgroup $\mathbb{Z}$ in the group $\mathbb{Z}_p$.

Let $\Gamma$ and $\Delta$ be profinite graphs. A \textit{qmorphism} or a \textit{quasi-morphism} of profinite graphs or a map of graphs

$$\alpha : \Gamma \to \Delta$$

is a continuous map such that $d_j(\alpha(m)) = \alpha(d_j(m))$, for all $m \in \Gamma$ and $j = 0, 1$. If in addition $\alpha(e) \in E(\Delta)$ for every $e \in E(\Gamma)$, we say that $\alpha$ is a \textit{morphism}.

The composition of qmorphisms of profinite graphs is again a qmorphism, so that profinite graphs and their qmorphisms form a category. Similarly profinite graphs and their morphisms form a category. If $\alpha$ is a surjective (respectively, injective, bijective) qmorphism, we say that $\alpha$ is an \textit{epimorphism} (respectively, \textit{monomorphism}, \textit{isomorphism}). An isomorphism $\alpha : \Gamma \to \Gamma$ of the graph $\Gamma$ to itself is called an \textit{automorphism}. Note that a monomorphism of graphs sends edges to edges, and hence it is always a morphism. A nonempty closed subset $\Gamma$ of a profinite graph $\Delta$ is called a \textit{profinite subgraph} of $\Delta$ if whenever $m \in \Gamma$, then $d_j(m) \in \Gamma$ ($j = 0, 1$).

The equality $d_j(\alpha(m)) = \alpha(d_j(m))$ ($j = 0, 1; m \in \Gamma$) implies that a qmorphism of profinite graphs maps vertices to vertices. However, the next example shows that a qmorphism can map an edge to a vertex.

\textbf{Example 2.1.2 (Subgraph collapsing)} Let $\Delta$ be a profinite subgraph of a profinite graph $\Gamma$. Consider the natural continuous map $\alpha : \Gamma \to \Gamma/\Delta$ to the quotient space $\Gamma/\Delta$ with the quotient topology [the points of $\Gamma/\Delta$ are the equivalence classes of the relation $\sim$ on $\Gamma$ defined as follows: if $m, m' \in \Gamma$, then $m \sim m'$ if and only if either $m = m'$ or $m, m' \in \Delta$; if $m \in \Gamma$, then $\alpha(m)$ is the equivalence class of $m$; a subset $U$ of $\Gamma/\Delta$ is open if $\alpha^{-1}(U)$ is open in $\Gamma$]. Define a structure of profinite graph on the space $\Gamma/\Delta$ as follows: $V(\Gamma/\Delta) = \alpha(V(\Gamma))$, $d_0(\alpha(m)) = \alpha(d_0(m))$, $d_1(\alpha(m)) = \alpha(d_1(m))$, for all $m \in \Gamma$. Then clearly $\alpha$ is a qmorphism of graphs and $\Gamma/\Delta$ becomes a quotient graph of $\Gamma$. We shall say that $\Gamma/\Delta$ is obtained from $\Gamma$ by \textit{collapsing} $\Delta$ to a point. Observe that $\alpha$ maps any edge of $\Gamma$ which is in $\Delta$ to a vertex of $\Gamma/\Delta$. 
We note that if \( \alpha : \Gamma \to \Delta \) is an epimorphism of profinite graphs, then \( \Delta \) has the quotient topology (i.e., for \( A \subseteq \Delta \), one has that \( A \) is open in \( \Delta \) if and only if \( \alpha^{-1}(A) \) is open in \( \Gamma \)), since \( \Gamma \) and \( \Delta \) are compact Hausdorff spaces. We then say that \( \Delta \) is a quotient graph of \( \Gamma \) and \( \alpha \) is a quotient morphism of graphs.

If \( \Gamma \) is a profinite graph and \( \varphi : \Gamma \to Y \) is a continuous surjection onto a profinite space \( Y \), there is no assurance that there exists a profinite graph structure on \( Y \) so that \( \varphi \) is a qmorphism of graphs. The following construction provides necessary and sufficient conditions for this to happen.

**Construction 2.1.3** Let \( \Gamma \) be a profinite graph and let \( \varphi : \Gamma \to Y \) be a continuous surjection onto a profinite space \( Y \). Then we construct a quotient qmorphism of graphs

\[ \tilde{\varphi} : \Gamma \to \Gamma \varphi \]

with the following properties.

(a) There is a continuous surjection of topological spaces \( \psi : \Gamma \varphi \to Y \) such that the diagram

\[ \begin{array}{ccc}
\Gamma & \xrightarrow{\varphi} & Y \\
\downarrow{\tilde{\varphi}} & & \downarrow{\psi} \\
\Gamma \varphi & \xrightarrow{\psi} & Y 
\end{array} \]

commutes.

(b) If \( Y \) admits a profinite graph structure so that \( \varphi \) is a qmorphism, then \( \psi \) is an isomorphism of profinite graphs.

(c) Consequently, there exists a profinite graph structure on \( Y \) such that \( \varphi \) is a qmorphism of graphs if and only if whenever \( m, m' \in \Gamma \) with \( \varphi(m) = \varphi(m') \), then \( \varphi d_0(m) = \varphi d_0(m') \) and \( \varphi d_1(m) = \varphi d_1(m') \). If this is the case, then that structure is unique (isomorphic to \( \Gamma \varphi \)) and the incidence maps of \( Y \) are defined by \( d_i \varphi(m) = \varphi d_i(m) \) (\( m \in \Gamma, i = 0, 1 \)).

(d) If \( E(\Gamma) \) is a closed subset of \( \Gamma \) and \( \varphi(E(\Gamma)) \cap \varphi(V(\Gamma)) = \emptyset \), then \( \tilde{\varphi} \) is a morphism of profinite graphs and \( \psi(\varphi(E(\Gamma))) \cap \psi(\varphi(V(\Gamma))) = \emptyset \).

To construct \( \Gamma \varphi \), define a map

\[ \tilde{\varphi} : \Gamma \to Y \times Y \times Y \]

by

\[ \tilde{\varphi}(m) = (\varphi(m), \varphi d_0(m), \varphi d_1(m)) \quad (m \in \Gamma) \].

Let \( \Gamma \varphi = \tilde{\varphi}(\Gamma) \). Then \( \Gamma \varphi \) admits a unique graph structure such that \( \tilde{\varphi} : \Gamma \to \Gamma \varphi \) is a qmorphism of graphs, namely one is forced to define the incidence maps \( \tilde{d}_0 \) and \( \tilde{d}_1 \) of \( \Gamma \varphi \) by

\[ \tilde{d}_0(\varphi(m), \varphi d_0(m), \varphi d_1(m)) = (\varphi d_0(m), \varphi d_0(m), \varphi d_0(m)) \quad (m \in \Gamma) \]
and
\[ d_1(\varphi(m), \varphi d_0(m), \varphi d_1(m)) = (\varphi d_1(m), \varphi d_1(m), \varphi d_1(m)) \quad (m \in \Gamma) \]
(one easily checks that these are well defined, and that \( \tilde{d} \) is indeed a morphism of profinite graphs). Next note that there exists a unique map \( \psi_\varphi : \Gamma_\varphi \to Y \) such that 
\[ \psi_\varphi \tilde{d} = \varphi, \text{ namely, } \psi_\varphi(\varphi(m), \varphi d_0(m), \varphi d_1(m)) = \varphi(m). \]
If \( Y \) is a profinite graph and \( \varphi \) is a morphism of profinite graphs, then \( \psi_\varphi \) is an isomorphism of graphs because in this case the map \( \rho : Y \to \Gamma_\varphi \) given by \( \rho \varphi(m) = (\varphi(m), \varphi d_0(m), \varphi d_1(m)) \) is a well-defined morphism of graphs and it is inverse to \( \psi_\varphi \). This proves properties (a) and (b). Property (c) is clear. Property (d) is easily verified. \( \square \)

Before stating the following proposition we recall briefly the concept of an inverse limit in the category of graphs (see Sect. 1.1). Let \((I, \preceq)\) be a directed partially ordered set (a directed poset). An inverse system of profinite graphs \( \{ \Gamma_i, \varphi_{ij}, I \} \) over the directed poset \( I \) consists of a collection of profinite graphs \( \Gamma_i \) indexed by \( I \) and morphisms of profinite graphs \( \varphi_{ij} : \Gamma_i \to \Gamma_j \), whenever \( i \preceq j \), in such a way that \( \varphi_{ii} = \text{Id}_i \), for all \( i \in I \), and \( \varphi_{jk} \varphi_{ij} = \varphi_{ik} \), whenever \( i \succeq j \succeq k \). The inverse limit (or projective limit) of such a system \( \Gamma = \lim_{\leftarrow} \Gamma_i \) is the subset of \( \prod_{i \in I} \Gamma_i \) consisting of those tuples \( (m_i) \) with \( \varphi_{ij}(m_i) = m_j \), whenever \( i \succeq j \). Such an inverse limit is in a natural way a profinite graph whose space of vertices is
\[ V(\Gamma) = \lim_{\leftarrow} V(\Gamma_i). \]
Observe that the natural projections \( \varphi_i : \Gamma \to \Gamma_i \) are morphisms of profinite graphs. Note that if each \( \varphi_{ij} \) is a morphism, then so are the canonical projections \( \varphi_i \).

Let \( \Gamma \) be a profinite graph and consider the set \( \mathcal{R} \) of all open equivalence relations \( R \) on the set \( \Gamma \) (i.e., the equivalence classes \( xR \) are open for all \( x \in \Gamma \)). For \( R \in \mathcal{R} \), denote by \( \varphi_R : \Gamma \to \Gamma/R \) the corresponding quotient map as topological spaces. One defines a partial ordering \( \preceq \) on \( \mathcal{R} \) as follows: for \( R_1, R_2 \in \mathcal{R} \), we say that \( R_1 \succeq R_2 \) if there exists a map \( \varphi_{R_1,R_2} : \Gamma/R_1 \to \Gamma/R_2 \) such that the diagram
\[ \begin{array}{ccc} \Gamma & \xrightarrow{\varphi_{R_1}} & \Gamma/R_1 \\ \downarrow \varphi_{R_2} & & \downarrow \varphi_{R_1,R_2} \\ \Gamma/R_2 \end{array} \]
commutes. Then (cf. RZ, Theorem 1.1.2) \((\mathcal{R}, \preceq)\) is in fact a directed poset, \( \{ \Gamma/R, \varphi_{R_1,R_2} \} \) is an inverse system over \( \mathcal{R} \), and, as topological spaces, the collection
of quotient maps \( \{ \varphi_R \mid R \in \mathcal{R} \} \) induces a homeomorphism from \( \Gamma \) to \( \lim_{R \in \mathcal{R}} \Gamma / R \); in fact we identify these two spaces by means of this homeomorphism and write

\[
\Gamma = \lim_{R \in \mathcal{R}} \Gamma / R. \tag{2.1}
\]

Consider now the subset \( \mathcal{R}' \) of \( \mathcal{R} \) consisting of those \( R \in \mathcal{R} \) such that \( \Gamma / R \) admits a graph structure (which is unique according to part (c) of Construction 2.1.3) so that \( \varphi_R : \Gamma \to \Gamma / R \) is a qmorphism of profinite graphs. We check next that the poset \( (\mathcal{R}', \preceq) \) is directed. Indeed, let \( R_1, R_2 \in \mathcal{R}' \). Since \( \mathcal{R} \) is directed, there exists an \( R \in \mathcal{R} \) such that \( R \succeq R_1, R_2 \). Let \( \varphi_R : \Gamma \to \Gamma / R \) be the corresponding quotient map. Let \( \Gamma_{\varphi_R} \) and \( \tilde{\varphi}_R : \Gamma \to \Gamma_{\varphi_R} \) be as in Construction 2.1.3. Then \( \Gamma_{\varphi_R} = \Gamma / \tilde{R} \), where \( \tilde{R} \) is the equivalence relation on \( \Gamma \) whose equivalence classes are \( \{ \tilde{\varphi}_R^{-1}(x) \mid x \in \Gamma_{\varphi_R} \} \). Clearly \( \tilde{R} \in \mathcal{R}' \) and \( \tilde{R} \succeq R_1, R_2 \); hence \( \tilde{R} \succeq R_1, R_2 \), as needed.

Observe that if \( R_1, R_2 \in \mathcal{R}' \) and \( R_1 \succeq R_2 \), then the map \( \varphi_{R_1, R_2} : \Gamma / R_1 \to \Gamma / R_2 \) is in fact a qmorphism of finite graphs. Therefore the collection \( \{ \Gamma / R, \varphi_{R_1, R_2} \} \) of all finite quotient graphs of \( \Gamma \) is an inverse system of finite graphs and qmorphisms over the directed poset \( \mathcal{R}' \).

**Proposition 2.1.4** Let \( \Gamma \) be a profinite graph.

(a) \( \Gamma \) is the inverse limit of all its finite quotient graphs:

\[
\Gamma = \lim_{R \in \mathcal{R}'} \Gamma / R.
\]

Consequently

\[
V(\Gamma) = \lim_{R \in \mathcal{R}'} V(\Gamma / R).
\]

(b) If the subset \( E(\Gamma) \) of edges of \( \Gamma \) is closed, then a directed subposet \( \mathcal{R}'' \) of \( \mathcal{R}' \) can be chosen so that whenever \( R_1, R_2 \in \mathcal{R}'' \) with \( R_1 \succeq R_2 \), then \( \varphi_{R_1, R_2} : \Gamma / R_1 \to \Gamma / R_2 \) is a morphism of graphs and

\[
\Gamma = \lim_{R \in \mathcal{R}''} \Gamma / R.
\]

Consequently,

\[
V(\Gamma) = \lim_{R \in \mathcal{R}''} V(\Gamma / R) \quad \text{and} \quad E(\Gamma) = \lim_{R \in \mathcal{R}''} E(\Gamma / R).
\]

**Proof** (a) In view of (2.1) one simply has to show that \( \mathcal{R}' \) is cofinal in \( \mathcal{R} \), i.e., one has to show that whenever \( R \in \mathcal{R} \), there exists an \( R' \in \mathcal{R}' \) with \( R' \succeq R \). But this is clear from property (a) of Construction 2.1.3.

(b) Suppose that \( E(\Gamma) \) is closed. Then \( \Gamma = V(\Gamma) \cup E(\Gamma) \) and \( V(\Gamma) \) and \( E(\Gamma) \) are clopen subsets of \( \Gamma \). Let \( \tilde{\mathcal{R}} \) be the subset of \( \mathcal{R} \) consisting of those equivalence relations \( R \in \mathcal{R} \) whose equivalence classes \( xR \) are contained in either \( E(\Gamma) \) or \( V(\Gamma) \).
or \( V(\Gamma) \); this implies that if \( \varphi_R : \Gamma \to \Gamma/R \) is the canonical projection, then \( \varphi_R(V(\Gamma)) \cap \varphi_R(E(\Gamma)) = \emptyset \). Then one shows that \( \tilde{\mathcal{R}} \) is cofinal in \( \mathcal{R} \), so that

\[
\Gamma = \lim_{\leftarrow R \in \tilde{\mathcal{R}}} \Gamma/R.
\]

One can argue now as in part (a); we just indicate the main points: let \( \mathcal{R}'' \) be the subset of \( \tilde{\mathcal{R}} \) consisting of those equivalence relations \( R'' \) such that \( \Gamma/R'' \) has the structure of a graph in such a way that \( \varphi_{R''} : \Gamma \to \Gamma/R'' \) is a morphism of profinite graphs; note that \( \mathcal{R}'' \) is also a subset of \( \mathcal{R}' \); using property (d) of Construction 2.1.3 one shows that \( \mathcal{R}'' \) is cofinal in \( \tilde{\mathcal{R}} \), and hence the result easily follows as above. \( \square \)

**Lemma 2.1.5** Let \( \{\Gamma_i, \varphi_{ij}, I\} \) be an inverse system of profinite graphs and qmorphisms over a directed poset \( I \), and set

\[
\Gamma = \lim_{i \in I} \Gamma_i.
\]

(2.2)

Let \( \rho : \Gamma \to \Delta \) be a qmorphism into a finite graph \( \Delta \). Then there exists a \( k \in I \) such that \( \rho \) factors through \( \Gamma_k \), i.e., there exists a qmorphism \( \rho' : \Gamma_k \to \Delta \) such that \( \rho = \rho'\varphi_k \), where \( \varphi_k : \Gamma \to \Gamma_k \) is the projection.

**Proof** For \( i \in I \) denote by \( \mathcal{R}_i \) the set of all equivalence relations \( R \) of \( \Gamma_i \) such that the quotient \( \Gamma_i/R \) is a finite discrete graph and the natural projection \( \Gamma_i \to \Gamma_i/R \) is a qmorphism. Define an ordering on the set of pairs

\[
A = \{(i, R) \mid i \in I, R \in \mathcal{R}_i\}
\]

by setting \( (i, R_i) \succeq (j, R_j) \), if \( i \succeq j \) and \( (\varphi_{ij} \times \varphi_{ij})(R_i) \subseteq R_j \). Let us prove that \( (A, \succeq) \) is a directed poset. Fix \( i, j \in I \) and \( R_i \in \mathcal{R}_i, R_j \in \mathcal{R}_j \). Since \( I \) is a directed poset, there exists some \( k \in I \) with \( k \succeq i, j \). By Proposition 2.1.3, \( \Gamma_k \) is the inverse limit of all its finite quotient graphs; therefore there exists an \( R_k \in \mathcal{R}_k \) with \( (\varphi_{ki} \times \varphi_{kj})(R_k) \subseteq R_i \) and \( (\varphi_{kj} \times \varphi_{kj})(R_k) \subseteq R_j \), so that \( (k, R_k) \succeq (i, R_i), (j, R_j) \), as needed.

Now it is easy to see that

\[
\Gamma = \lim_{(i, R) \in A} \Gamma_i/R.
\]

Thus from now on we may assume that each \( \Gamma_i \) in the decomposition (2.2) is finite.

Assume first that each projection \( \varphi_i : \Gamma \to \Gamma_i \) is surjective. Let \( S \) be the equivalence relation on \( \Gamma \) whose equivalence classes are the clopen sets \( \rho^{-1}(m), m \in \Delta \); then \( \Gamma/S = \Delta \) and \( \rho \) is the natural projection \( \Gamma \to \Gamma/S \). Similarly, for \( i \in I \), let \( S_i \) be the equivalence relation on \( \Gamma \) whose equivalence classes are the clopen sets \( \varphi_i^{-1}(m), m \in \Gamma_i \), so that \( \varphi_i \) is the natural projection \( \Gamma \to \Gamma/S_i \). Since \( \Gamma = \lim_{i \in I} \Gamma_i \), we have that \( \bigcap_{i \in I} S_i \) is the trivial equivalence relation, i.e., \( \bigcap_{i \in I} S_i = D \), where \( D \) is the diagonal subset of \( \Gamma \times \Gamma \). Note that \( S \) and \( S_i (i \in I) \) are clopen subsets of \( \Gamma \times \Gamma \). Hence, it follows from the compactness of \( \Gamma \times \Gamma \) that there exists a finite subset \( F \) of \( I \) such that \( \bigcap_{j \in F} S_j \subseteq S \). Since the poset \( I \) is directed, there
exists a $k \in I$ with $S_k \subseteq \bigcap_{j \in F} S_j \subseteq S$. This means that there exists a quomorphism of graphs $\rho_k : \Gamma_k = \Gamma/S_k \to \Delta = \Gamma/S$ such that $\rho = \rho_k \varphi_k$.

Consider now a general $\varphi_i$. By the above, there exists some $k' \in I$ and a quomorphism of graphs $\rho_{k'} : \varphi_{k'}(\Gamma) \to \Delta$ such that $\rho = \rho_{k'} \varphi_{k'}$. Since $\Gamma_{k'}$ is finite, there exists a $k \geq k'$ such that $\varphi_{kk'}(\Gamma_k) \subseteq \varphi_{k'}(\Gamma)$. Then $\rho' = \rho_{k'} \varphi_{kk'}$ is the required quomorphism. □

An alternative proof of Lemma 2.1.5 above can be obtained along the lines of the proof of Lemma 1.1.16 in RZ.

A profinite graph $\Gamma$ is said to be connected if whenever $\varphi : \Gamma \to A$ is a quomorphism of profinite graphs onto a finite graph, then $A$ is connected as an abstract graph (see Sect. A.1 in Appendix A).

**Proposition 2.1.6**

(a) Every quotient graph of a connected profinite graph is connected.

(b) If

$$\Gamma = \lim_{\leftarrow} \Gamma_i$$

and each $\Gamma_i$ is a connected profinite graph, then $\Gamma$ is a connected profinite graph.

(c) Let $\Gamma$ be a connected profinite graph. If $|\Gamma| > 1$, then $\Gamma$ has at least one edge. Furthermore, if the set of edges $E(\Gamma)$ of $\Gamma$ is closed in $\Gamma$, then for any vertex $v \in V(\Gamma)$, there exists an edge $e \in E(\Gamma)$ such that either $v = d_0(e)$ or $v = d_1(e)$.

(d) Let $\Gamma$ be a profinite graph, and let $\Delta$ be a connected profinite subgraph of $\Gamma$. Consider the quotient graph $\Gamma/\Delta$ obtained by collapsing $\Delta$ to a point and let $\alpha : \Gamma \to \Gamma/\Delta$ be the natural projection. Then the inverse image $\Lambda = \alpha^{-1}(\Lambda)$ in $\Gamma$ of a connected profinite subgraph $\Lambda$ of $\Gamma/\Delta$ is a connected profinite subgraph.

**Proof** Part (a) is obvious. Let $A$ be a finite quotient graph of $\Gamma$. Then (see Lemma 2.1.5) there exists an $i \in I$ such that $A$ is also a quotient graph of $\Gamma_i$. It follows that $A$ is connected, proving (b).

To check the first assertion in (c) observe that by Proposition 2.1.4, $\Gamma$ has a finite quotient graph with at least two elements; since such a finite quotient graph is connected, it has at least one edge, and hence so does $\Gamma$. To check the second assertion in (c), write $\Gamma$ as an inverse limit $\Gamma = \lim_{\leftarrow} \Gamma_i$ of finite quotient graphs $\Gamma_i$ in such a way that

$$E(\Gamma) = \lim_{\leftarrow} E(\Gamma_i)$$

(see Proposition 2.1.4(b)). For $i \in I$, let $\varphi_i : \Gamma \to \Gamma_i$ denote the canonical projection, and if $i, j \in I$ with $i \geq j$, let $\varphi_{ij} : \Gamma_i \to \Gamma_j$ denote the canonical morphism. Put $v_i = \varphi_i(v)$ ($i \in I$). Since $\Gamma_i$ is a connected finite graph, the set $S_i =$
of edges of \( \Gamma_i \) starting or ending at \( v_i \) is nonempty; moreover, \( \varphi_{ij}(S_i) \subseteq S_j \). Hence the collection \( \{S_i\}_{i \in I} \) is an inverse system of nonempty finite sets. Thus

\[
\lim_{i \in I} S_i \neq \emptyset
\]

(see Sect. 1.1). Let \( e \in \lim_{i \in I} S_i \). Then \( e \) is an edge of \( \Gamma \) with either \( d_0(e) = v \) or \( d_1(e) = v \).

(d) This is clear if \( \Gamma \) is finite. Write \( \Gamma = \varprojlim_{i \in I} \Gamma_i \), where each \( \Gamma_i \) is a connected finite quotient graph of \( \Gamma \) (see Proposition 2.1.4(a)).

Let \( \Delta_i \) be the image of \( \Delta \) in \( \Gamma_i \) under the canonical projection. Then \( \Delta = \varprojlim_{i \in I} \Delta_i \) and \( \Gamma / \Delta = \varprojlim_{i \in I} \Gamma_i / \Delta_i \).

Let \( \Lambda_i \) be the image of \( \Lambda \) in \( \Gamma_i / \Delta_i \), and denote by \( \tilde{\Lambda}_i \) its inverse image in \( \Gamma_i \). Since \( \tilde{\Lambda} = \varprojlim_{i \in I} \tilde{\Lambda}_i \), \( \tilde{\Lambda} \) is connected according to part (b).

\[\Box\]

**Lemma 2.1.7**

(a) Let \( D \) be an abstract subgraph of a profinite graph \( \Gamma \). Then the topological closure \( \overline{D} \) of \( D \) in \( \Gamma \) is a profinite graph. If \( D \) is connected as an abstract graph (see Sect. A.1 in Appendix A), then \( \overline{D} \) is a connected profinite graph.

(b) Let \( \{\Delta_j \mid j \in J\} \) be a collection of connected profinite subgraphs of a profinite graph \( \Gamma \). If \( \bigcap_{j \in J} \Delta_j \neq \emptyset \), then \( \Delta = \bigcup_{j \in J} \Delta_j \) is connected.

**Proof** To prove (a), let \( m \in \overline{D} \). By the continuity of \( d_i \), \( d_i(m) \in \overline{V(D)} \) \( (i = 1, 2) \), so that \( \overline{D} \) is a (profinite) graph with \( V(\overline{D}) = \overline{V(D)} \). If \( \varphi : \overline{D} \to A \) is a qmorphism of profinite graphs onto a finite graph, then \( \varphi(\overline{D}) = \varphi(D) = A \) by continuity. Since \( D \) is a connected abstract graph, one easily checks that \( \varphi(D) \) is a finite connected graph; hence \( \overline{D} \) is a connected profinite graph. This proves (a).

For part (b) note that if \( \alpha : \Delta \to A \) is a qmorphism onto a finite graph \( A \), then \( \alpha(\Delta_j) \) is a connected finite subgraph of \( A \) \( (j \in J) \). Since \( A = \bigcup_{j \in J} \alpha(\Delta_j) \), and \( \bigcap_{j \in J} \alpha(\Delta_j) \neq \emptyset \), it follows that \( A \) is a connected abstract graph. \[\Box\]

**Example 2.1.8** (A connected profinite graph which is not connected as an abstract graph and with a vertex with no edge beginning or ending at it) Let \( I \) be the graph considered in Example 2.1.1(b): \( I = \mathbb{N} \cup \mathbb{N} \cup \{\infty\} \) is the one-point compactification of a disjoint union of two copies \( \mathbb{N} \) and \( \tilde{\mathbb{N}} = \{\tilde{n} \mid n \in \mathbb{N}\} \) of the natural numbers; \( V(I) = \mathbb{N} \cup \{\infty\} \), \( E(I) = \tilde{\mathbb{N}} \), \( d_0(\tilde{n}) = n, d_1(\tilde{n}) = n + 1 \) for \( \tilde{n} \in E(I) \), and \( d_i(n) = n \) for \( n \in V(I) \) \( (i = 1, 2) \).

\[\begin{array}{ccccccc}
0 & 1 & 2 & 3 & \ldots & \infty \\
\emptyset & 1 & \tilde{1} & \tilde{2} & 3 & \end{array}\]
Then $I$ is a connected profinite graph; to see this consider the connected finite graphs $I_n$

\begin{center}
\begin{tikzpicture}
\node [above] at (0,0) {$0$};
\node [above] at (1,0) {$1$};
\node [above] at (2,0) {$2$};
\node [above] at (3,0) {$\cdots$};
\node [above] at (4,0) {$n-1$};
\node [above] at (5,0) {$n$};
\node [below] at (0,0) {$0$};
\node [below] at (1,0) {$1$};
\node [below] at (2,0) {$2$};
\node [below] at (3,0) {$\cdots$};
\node [below] at (4,0) {$n-1$};
\node [below] at (5,0) {$n$};
\draw (0,0) -- (1,0);
\draw (1,0) -- (2,0);
\draw (2,0) -- (3,0);
\draw (3,0) -- (4,0);
\draw (4,0) -- (5,0);
\end{tikzpicture}
\end{center}

with vertices $V(I_n) = \{0, 1, 2, \ldots, n\}$ and edges $E(I_n) = \{\tilde{0}, \tilde{1}, \ldots, \tilde{n-1}\}$ such that $d_0(\tilde{i}) = i$, $d_1(\tilde{i}) = i + 1$ ($i = 0, \ldots, n - 1$) and $d_j(\tilde{i}) = i$ ($i = 0, \ldots, n$; $j = 0, 1$). If $n \leq m$, define $\varphi_{m,n} : I_m \rightarrow I_n$ to be the map of graphs that sends the segment $[0, n]$ identically to $[0, n]$, and the segment $[n, m]$ to the vertex $n$. Then $(I_n, \varphi_{m,n})$ is an inverse system of graphs, and

$$I = \lim_{\longrightarrow \in \mathbb{N}} I_n,$$

where $\infty = (n)_{n \in \mathbb{N}}$. Hence $I$ is a connected profinite graph. We observe that there is no edge $e$ of $I$ which has $\infty$ as one of its vertices; and so $I$ is not connected as an abstract graph.

**Lemma 2.1.9** Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a profinite graph which is the disjoint union of two open profinite subgraphs $\Gamma_1$ and $\Gamma_2$; then $\Gamma$ is not connected. In particular, a profinite graph that contains two different vertices and no edges is not connected.

**Proof** Collapse $\Gamma_1$ to a point $v_1$ and $\Gamma_2$ to a different point $v_2$ (see Example 2.1.2), to get a disconnected finite quotient graph $\tilde{\Gamma} = \{v_1\} \cup \{v_2\}$ consisting of two vertices and no edges. \(\square\)

A maximal connected profinite subgraph of a profinite graph $\Gamma$ is called a **connected profinite component** of $\Gamma$.

**Proposition 2.1.10** Let $\Gamma$ be a profinite graph.

(a) Let $m \in \Gamma$. Then there exists a unique connected profinite component of $\Gamma$ containing $m$, which we shall denote by $\Gamma^*(m)$.

(b) Any two connected profinite components of $\Gamma$ are either equal or disjoint.

(c) $\Gamma$ is the union of its connected profinite components.

**Proof** Part (c) follows from (a). Part (b) follows from (a) and Lemma 2.1.7(b). To prove (a) observe first that the result is obvious if $\Gamma$ is finite. By Proposition 2.1.4, $\Gamma$ can be represented as an inverse limit $\lim_{\longrightarrow \in I} \Gamma_i$ of finite quotient graphs. For $i \in I$, let $\varphi_i : \Gamma \rightarrow \Gamma_i$ denote the projection. Since the image of a connected profinite graph is connected, the graphs $\Gamma_i^*(\varphi_i(m))$ form an inverse system. It suffices to show that the profinite subgraph $\lim_{\longrightarrow \in I} \Gamma_i^*(\varphi_i(m))$ of $\Gamma$ is the connected profinite component of $\Gamma$ containing $m$. This profinite subgraph is connected by Proposition 2.1.6(b). If $\Gamma'$ is a connected profinite subgraph of $\Gamma$ containing $m$, then $\Gamma' = \lim_{\longrightarrow \in I} \varphi_i(\Gamma')$. Therefore $\varphi_i(\Gamma') \subseteq \Gamma_i^*(\varphi_i(m))$ for all $i \in I$. Hence
$\Gamma' \subseteq \lim_{\leftarrow i} \Gamma_i^*(\varphi_i(m))$; therefore $\lim_{\leftarrow i} \Gamma_i^*(\varphi_i(m))$ is maximal connected containing $m$, as desired. The uniqueness of connected profinite components containing $m$ follows from Lemma 2.1.7(b).

\[\square\]

Exercise 2.1.11

(a) Let $\Delta$ be a profinite graph. Define the space of connected profinite components of $\Delta$ as a quotient space $\Delta/\sim$, where $\sim$ is the equivalence relation defined as follows: $m_1 \sim m_2$ if and only if $\Delta^*(m_1) = \Delta^*(m_2)$. Prove that $\Delta/\sim$ is a profinite space. [Hint: write $\Delta$ as an inverse limit of finite quotient graphs.]

(b) Let $\Delta$ be a profinite subgraph of a profinite graph $\Gamma$. Define the operation of collapsing the connected profinite components of $\Delta$ to points as a natural mapping to the quotient space $\Gamma/\sim$, where $\sim$ is the equivalence relation defined as follows: $m_1 \sim m_2$ if $m_1 = m_2$, for $m_1, m_2 \in \Gamma - \Delta$, or $\Delta^*(m_1) = \Delta^*(m_2)$ for $m_1, m_2 \in \Delta$. Prove that $\Gamma/\sim$ is a profinite quotient graph of $\Gamma$.

Example 2.1.12 (The Cayley graph) Let $G$ be a profinite group (whose operation is denoted as multiplication and whose identity element is denoted by 1) and let $X$ be a closed subset of $G$. Put $\tilde{X} = X \cup \{1\}$. Define the Cayley graph $\Gamma(G, X)$ of $G$ with respect to the subset $X$ as follows:

$$\Gamma(G, X) = G \times \tilde{X},$$

where $G \times \tilde{X}$ has the product topology. Define the space of vertices of $\Gamma(G, X)$ to be $V(\Gamma(G, X)) = \{(g, 1) \mid g \in G\}$. We identify this space of vertices with $G$ by means of the homeomorphism $(g, 1) \mapsto g$ ($g \in G$).

Finally, the incidence maps

$$d_0, d_1 : \Gamma(G, X) = G \times \tilde{X} \longrightarrow V(\Gamma(G, X)) = G$$

are defined by

$$d_0(g, x) = g \quad \text{and} \quad d_1(g, x) = gx, \quad (g \in G, x \in X \cup \{1\}).$$

Clearly $d_0$ and $d_1$ are continuous and they are the identity map when restricted to $V(\Gamma(G, X)) = \{(g, 1) \mid g \in G\} = G$. Therefore the Cayley graph $\Gamma(G, X)$ is a profinite graph.

Note that the space of edges is $E(\Gamma(G, X)) = \Gamma(G, X) - V(\Gamma(G, X)) = G \times (X - \{1\})$:

$$g \xrightarrow{(g, x)} gx,$$

where $x \in X - \{1\}$. It is a closed (and hence clopen) subset of $\Gamma(G, X)$ if and only if 1 is an isolated point of $\tilde{X}$. Observe that if $1 \notin X$, then $V(\Gamma(G, X)) = G$ and $E(\Gamma(G, X)) = G \times X$, and in this case $E(\Gamma(G, X))$ is clopen. If $1 \in X$, then $\tilde{X} = X$. If 1 is in $X$ and it is an isolated point of $X$ (for example, if $X$ is finite), then $X - \{1\}$ is also a closed subspace and we have $\Gamma(G, X) = \Gamma(G, X - \{1\})$. Note that the Cayley graph $\Gamma(G, X)$ does not contain loops since the elements of the form $(g, 1)$ are vertices by definition.
Let \( \varphi : G \rightarrow H \) be a continuous homomorphism of profinite groups and let \( X \) be a closed subset of \( G \). Put \( Y = \varphi(X) \). Then \( \varphi \) induces a morphism of the corresponding Cayley graphs

\[
\tilde{\varphi} : \Gamma(G, X) \rightarrow \Gamma(H, Y).
\]

In particular, if \( U \) is an open normal subgroup of \( G \) and \( X_U = \varphi_U(X) \), where \( \varphi_U : G \rightarrow G/U \) is the canonical epimorphism, then \( \varphi_U \) induces a corresponding epimorphism of Cayley graphs \( \tilde{\varphi}_U : \Gamma(G, X) \rightarrow \Gamma(G/U, X_U) \). One easily checks that

\[
\Gamma(G, X) = \lim_{\leftarrow U \triangleleft_o G} \Gamma(G/U, X_U)
\]

is a decomposition of \( \Gamma(G, X) \) as an inverse limit of finite Cayley graphs.

**Example 2.1.13** (An infinite connected profinite graph all of whose proper connected profinite subgraphs are finite) Let \( \Gamma = \Gamma(\hat{\mathbb{Z}}, \{1\}) \) be the Cayley graph of the free profinite group \( \hat{\mathbb{Z}} \) of rank one with respect the subset \( \{1\} \). Then

\[
\Gamma = \lim_{n \geq 2} \Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\}),
\]

with canonical maps

\[
\varphi_{mn} : \Gamma(\mathbb{Z}/m\mathbb{Z}, \{1\}) \rightarrow \Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\}) \quad (n|m).
\]

Let

\[
\varphi_n : \Gamma \rightarrow \Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\})
\]

denote the projection \((n \in \mathbb{N})\). Assume that \( \Delta \) is a connected proper profinite subgraph of \( \Gamma \). Put \( \Delta_n = \varphi_n(\Gamma) \). Then \( \Delta_n \) is a connected subgraph of the finite graph \( \Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\}) \).

Since \( \Delta \neq \Gamma \), there exists some \( n_0 \in \mathbb{N} \) such that \( \Delta_{n_0} \neq \Gamma(\mathbb{Z}/n_0\mathbb{Z}, \{1\}) \). Observe that for every \( m \in \mathbb{N} \) with \( n_0|m \), the connected components of \( \varphi_{mn_0}^{-1}(\Delta_{n_0}) \) are isomorphic to \( \Delta_{n_0} \). Therefore, \(|\Delta_m| = |\Delta_{n_0}| \). Thus \( \Delta \) is finite.

It is easy to check that if \( \Delta \) is a proper connected subgraph of \( \Gamma \) with \( t + 1 \) vertices, then there exists a \( \gamma \in \hat{\mathbb{Z}} \) such that the vertices of \( \Delta \) are \( \gamma, \gamma + 1, \ldots, \gamma + t \) and with edges \((\gamma, 1), (\gamma + 1, 1), \ldots, (\gamma + t - 1, 1)\):

\[
\begin{array}{c c c c c}
\gamma & \gamma + 1 & \gamma + 2 & \ldots & \gamma + t - 1 & \gamma + t \\
(\gamma, 1) & (\gamma + 1, 1) & \ldots & (\gamma + t - 1, 1)
\end{array}
\]
2.1.14 Circuits. Let \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \), where \( \varepsilon_i = \pm 1 \) \((i = 1, \ldots, n)\) and \( n \geq 1 \) is a natural number. Define \( \text{Circ}_n(\varepsilon) \) to be a graph with \( n \) vertices (that we take to be the elements of \( \mathbb{Z}/n\mathbb{Z} \)) and \( n \) edges \( e_1, \ldots, e_n \) such that \( d_0(e_i) = i - 1 \) and \( d_1(e_i) = i \), if \( \varepsilon_i = 1 \), and \( d_0(e_i) = i \) and \( d_1(e_i) = i - 1 \), if \( \varepsilon_i = -1 \). We refer to a graph of the form \( \text{Circ}_n(\varepsilon) \) as a circuit of length \( n \) or as an \( n \)-circuit. A circuit of length 1 is a loop. Note that if \( n \geq 2 \) and \( \varepsilon = (1, \ldots, 1) \), then \( \text{Circ}_n(\varepsilon) = \Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\}) \).}

2.2 Groups Acting on Profinite Graphs

Let \( G \) be a profinite group and let \( \Gamma \) be a profinite graph. We say that the profinite group \( G \) acts on the profinite graph \( \Gamma \) on the left, or that \( \Gamma \) is a \( G \)-graph, if

(i) \( G \) acts continuously on the topological space \( \Gamma \) on the left, i.e., there is a continuous map \( G \times \Gamma \rightarrow \Gamma \), denoted \((g, m) \mapsto gm\), \( g \in G, m \in \Gamma \), such that \( (gh)m = g(hm) \) and \( 1m = m \), for all \( g, h \in G, m \in \Gamma \), where 1 is the identity element of \( G \); and

(ii) \( d_j(gm) = gd_j(m) \), for all \( g \in G, m \in \Gamma, j = 0, 1 \).

Observe that if \( G \) acts on \( \Gamma \), then for a fixed \( g \in G \), the map \( \rho_g : \Gamma \rightarrow \Gamma \) given by \( m \mapsto gm \) \((m \in \Gamma)\) is an automorphism of the graph \( \Gamma \). Hence (cf. RZ, Remark 5.6.1), \( G \) acts on a profinite graph \( \Gamma \) if and only if there exists a continuous homomorphism

\[ \rho : G \rightarrow \text{Aut}(\Gamma), \]

where \( \text{Aut}(\Gamma) \) is the group of automorphisms of \( \Gamma \) as a profinite graph, and where the topology on \( \text{Aut}(\Gamma) \) is induced by the compact-open topology. The kernel of the action of \( G \) on \( \Gamma \) is the kernel of \( \rho \), i.e., the closed normal subgroup of \( G \) consisting of all the elements \( g \in G \) such that \( gm = m \), for all \( m \in \Gamma \).

One defines actions on the right in a similar manner. We shall consider only left actions in this chapter.

Let \( G \) be a profinite group that acts continuously on two profinite graphs \( \Gamma \) and \( \Gamma' \). A morphism of graphs

\[ \varphi : \Gamma \rightarrow \Gamma' \]
is called a $G$-map of graphs if
\[ \varphi(gm) = g\varphi(m), \quad \text{for all } m \in \Gamma, g \in G. \]

Assume that a profinite group $G$ acts on a profinite graph $\Gamma$ and let $m \in \Gamma$. Define
\[ G_m = \{ g \in G \mid gm = m \} \]
to be the stabilizer (or $G$-stabilizer, if one needs to specify the group $G$) of the element $m$. It follows from the continuity of the action and the compactness of $G$ that $G_m$ is a closed subgroup of $G$. Clearly,
\[ G_m \leq G_{d_j(m)}, \quad \text{for every } m \in \Gamma, j = 0, 1. \]

If the stabilizer $G_m$ of every element $m \in \Gamma$ is trivial, i.e., $G_m = 1$, we say that $G$ acts freely on $\Gamma$. If $m \in \Gamma$, the $G$-orbit of $m$ is the closed subset $Gm = \{ gm \mid g \in G \}$.

If a profinite group $G$ acts on a profinite graph $\Gamma$, then $G$ acts on the profinite space $V(\Gamma)$ of vertices and $G$ acts on $E(\Gamma)$. The space
\[ G\backslash \Gamma = \{ Gm \mid m \in \Gamma \} \]
of $G$-orbits with the quotient topology is a profinite space which admits a natural profinite graph structure as follows:
\[ V(G\backslash \Gamma) = G\backslash V(\Gamma), \quad d_j(Gm) = Gd_j(m), \quad j = 0, 1. \]

We say that $G\backslash \Gamma$ is the quotient graph of $\Gamma$ under the action of $G$. The corresponding quotient map
\[ \Gamma \longrightarrow G\backslash \Gamma \]
is an epimorphism of profinite graphs given by $m \mapsto Gm$ ($m \in \Gamma, g \in G$). We observe that it sends edges to edges (it is a morphism).

If $N \trianglelefteq G$, there is an induced action of $G/N$ on $N\backslash \Gamma$ defined by
\[ (gN)(Nm) = N(gm), \quad g \in G, m \in \Gamma. \]

The following result is straightforward.

**Lemma 2.2.1** Let a profinite group $G$ act on a profinite graph $\Gamma$.

(a) Let $\mathcal{N}$ be a collection of closed normal subgroups of $G$ filtered from below (i.e., the intersection of any two groups in $\mathcal{N}$ contains a group in $\mathcal{N}$) and assume that
\[ G = \lim_{\mathcal{N} \in \mathcal{N}} G/N. \]

Then the collection of graphs $\{ N\backslash \Gamma \mid N \in \mathcal{N} \}$ is an inverse system in a natural way and
\[ \Gamma = \lim_{\mathcal{N} \in \mathcal{N}} N\backslash \Gamma. \]
(b) Let $N \triangleleft_c G$. For $m \in \Gamma$, denote by $m'$ the image of $m$ in $N \setminus \Gamma$. Consider the natural action of $G/N$ on $N \setminus \Gamma$ defined above. Then $(G/N)_m'$ is the image of $G_m$ under the natural epimorphism $G \to G/N$. In particular, if $G_m \leq N$, for all $m \in \Gamma$, then $G/N$ acts freely on $N \setminus \Gamma$.

Let $G$ be a profinite group. If $\{\Gamma_i, \varphi_{ij}, I\}$ is an inverse system of profinite $G$-graphs and $G$-maps over the directed poset $I$, then

$$
\Gamma = \lim_{\leftarrow \atop i \in I} \Gamma_i
$$

is in a natural way a profinite $G$-graph.

Next we show that every profinite $G$-graph admits a decomposition as an inverse limit of finite $G$-graphs.

**Proposition 2.2.2** Let a profinite group $G$ act on a profinite graph $\Gamma$.

(a) Then there exists a decomposition

$$
\Gamma = \lim_{\leftarrow \atop i \in I} \Gamma_i
$$

of $\Gamma$ as the inverse limit of a system of finite quotient $G$-graphs $\Gamma_i$ and $G$-maps $\varphi_{ij} : \Gamma_i \to \Gamma_j$ ($i \geq j$) over a directed poset $(I, \preceq)$.

(b) If $G$ is finite and acts freely on $\Gamma$, then the decomposition of part (a) can be chosen so that $G$ acts freely on each $\Gamma_i$.

**Proof** The proof follows the same pattern as the proof of Proposition 2.1.4; we only indicate the main steps and changes. We prove (a) and (b) at the same time.

Let $R$ be an open equivalence relation on $\Gamma$. Assume that $G$ acts continuously on the finite discrete space $\Gamma/R$ in such a way that the canonical projection $\varphi_R : \Gamma \to \Gamma/R$ is a $G$-map of $G$-spaces: this is equivalent to saying that whenever $m, m' \in \Gamma$ and $mR = m'R$, then $(gm)R = (gm')R$, for all $g \in G$ (we term such $R$ a $G$-invariant equivalence relation). Then (see Sect. 1.3) there exists a set $\mathcal{R}$ of $G$-invariant open equivalence relations on $\Gamma$ such that $(\mathcal{R}, \preceq)$ is a directed poset, $\{\Gamma/R, \varphi_{RR'}\}$ is an inverse system of finite $G$-spaces and $G$-maps over $\mathcal{R}$ and

$$
\Gamma = \lim_{\leftarrow \atop R \in \mathcal{R}} \Gamma/R
$$

as topological $G$-spaces. Moreover, if $G$ is finite and acts freely on $\Gamma$, one can modify the set $\mathcal{R}$ so that the action of $G$ on each $\Gamma/R$ is free and the decomposition (2.3) still holds.

Let $\mathcal{R}'$ be the subset of $\mathcal{R}$ consisting of those $R \in \mathcal{R}$ such that in addition $\Gamma/R$ has the structure of a $G$-graph and $\varphi_R : \Gamma \to \Gamma/R$ is a $G$-map of $G$-graphs.

Let $R \in \mathcal{R}$ and apply Construction 2.1.3 to get the maps $\overline{\varphi}_R : \Gamma \to \Gamma_{\varphi R}$ and $\psi_{\varphi R} : \Gamma_{\varphi R} \to \Gamma/R$. For $g \in G$ and $m \in \Gamma$, define

$$
g(\varphi(m), \varphi d_0(m), \varphi d_1(m)) = (g \varphi(m), g \varphi d_0(m), g \varphi d_1(m)).
$$
This makes $\Gamma_{\psi_R}$ into a $G$-graph and one checks that $\tilde{\psi}_R$ is a $G$-map of $G$-graphs and $\psi_{\psi_R}$ is a $G$-map of $G$-spaces. Let $\hat{R}$ be the open equivalence relation on $\Gamma$ whose equivalence classes are $\{\tilde{\psi}_R^{-1}(x) \mid x \in \Gamma_{\psi_R}\}$, so that $\Gamma_{\psi_R} = \Gamma / \hat{R}$. Therefore $\hat{R} \geq R$. From this one sees, as in the proof of Proposition 2.1.4, that $\mathcal{R}'$ is a directed poset that is cofinal in $\mathcal{R}$. Observe that if $G$ acts freely on $\Gamma / R$, then it acts freely on $\Gamma_{\psi_R}$. Hence both (a) and (b) follow from the decomposition (2.3) (see Sect. 1.1). 

We remark that part (b) of the above proposition can be sharpened in the following sense. When $G$ is infinite, it obviously cannot act freely on a finite graph; hence, if $G$ acts freely on $\Gamma$, it is not possible to obtain a $G$-decomposition of $\Gamma$ as in part (a) if in addition one requires that $G$ acts freely on each $\Gamma_i$. However, one can obtain a decomposition as in part (a) so that, for each $i$, a finite quotient $G_i$ of $G$ acts freely on $\Gamma_i$, and $G$ is the inverse limit of the $G_i$. We make this precise in Proposition 3.1.3. The following example shows how to do this in the case of Cayley graphs.

**Example 2.2.3 (The Cayley graph as a $G$-graph)** Let $G$ be a profinite group and let $X$ be a closed subset of $G$. Let $\Gamma(G, X)$ be the Cayley graph of $G$ with respect to $X$ (see Example 2.1.12). Define a left action of $G$ on $\Gamma(G, X)$ by setting

$$g' \cdot (g, x) = (g'g, x) \quad \forall x \in \tilde{X} = X \cup \{1\}, \ g', g \in G.$$

Clearly $gd_i(m) = d_i(gm)$, for all $g \in G$, $m \in \Gamma(G, X)$, $i = 0, 1$. Thus, $G$ acts (continuously and freely) on the Cayley graph $\Gamma(G, X)$.

Now, if $\mathcal{N}$ is the collection of all open normal subgroups of $G$, we have

$$\Gamma(G, X) = \lim_{\mathcal{N} \in \mathcal{N}} \Gamma(G/N, X_N),$$

where $X_N$ is the image of $X$ in $G/N$. Note that $G/N$ acts freely on $\Gamma(G/N, X_N)$.

The next lemma sometimes provides a useful way of checking whether certain $G$-graphs are connected.

**Lemma 2.2.4**

(a) Let $G = \langle X \rangle$ be an abstract group generated by a subset $X$. Assume that $G$ acts on an abstract graph $\Gamma$. Let $\Delta$ be a connected subgraph of $\Gamma$ such that $\Delta \cap x\Delta \neq \emptyset$, for all $x \in X$. Then

$$G\Delta = \bigcup_{g \in G} g\Delta$$

is a connected subgraph of $\Gamma$.

(b) Let $X$ be a closed subset of a profinite group $G$ that generates the group topologically, i.e., $G = \overline{\langle X \rangle}$. Assume that $G$ acts on a profinite graph $\Gamma$. Let $\Delta$ be a connected profinite subgraph of $\Gamma$ such that $\Delta \cap x\Delta \neq \emptyset$, for all $x \in X$. Then

$$G\Delta = \bigcup_{g \in G} g\Delta$$

is a connected profinite subgraph of $\Gamma$. 
(c) Let $G$ be a profinite group and let $X$ be a closed subset of $G$. The Cayley graph $\Gamma(G, X)$ is connected if and only if $G = \overline{(X)}$.

Proof (a) Put

$$Y = \{x^\varepsilon \mid \varepsilon = \pm 1, x \in X\},$$

and let $Y_n$ be the set of elements of $G$ that can be written as a product of not more than $n$ elements of $Y$ ($n = 0, 1, 2, \ldots$). Since $G\Delta = \bigcup_{n=0}^{\infty} Y_n\Delta$, and $Y_0 \subseteq Y_1 \subseteq \cdots$, it suffices to prove that $Y_n\Delta$ is a connected graph. We show this by induction on $n$. If $n = 0$, then $Y_0\Delta = \Delta$. Assume that $Y_n\Delta$ is connected. From our assumption that $x\Delta \cap \Delta \neq \emptyset$, we deduce that $x^{-1}\Delta \cap \Delta \neq \emptyset$, for all $x \in X$. Observe that if $w$ is a word in $Y$ of length $n + 1$, then $w = w'x^\varepsilon$, for some $w' \in Y_n$ and some $x \in X$; hence $w\Delta \cap w'\Delta \neq \emptyset$; and so, $w\Delta \cup Y_n\Delta$ is connected. It follows that $Y_{n+1}\Delta$ is connected.

(b) By Proposition 2.2.2 there exists a decomposition $\Gamma = \lim_{\leftarrow} \Gamma_i$, where all $\Gamma_i$ are finite quotient $G$-graphs of $\Gamma$. Hence it suffices to prove the result for $\Gamma$ finite.

In that case the kernel $K$ of the action of $G$ on $\Gamma$ is an open normal subgroup of $G$. Therefore, replacing $G$ by its quotient $G/K$ if necessary, we may assume that $G$ is finite; and then the result follows from part (a).

(c) Let $U$ be the collection of all open normal subgroups of $G$. Then

$$\Gamma(G, X) = \lim_{\leftarrow} \Gamma(G/U, X_U),$$

where $X_U$ is the image of $X$ on $G/U$ under the canonical map $G \to G/U$. Therefore we may assume that $G$ is finite, in which case the result follows from part (a): consider the connected subgraph $\Delta$ of $\Gamma(G, X)$ consisting of the vertices $1$ and $\{x \mid x \in X\}$ and the collection of edges $\{(1, x) \mid x \in X - \{1\}\}$; then $\Gamma(G, X) = G\Delta$.

\[\square\]

2.3 The Chain Complex of a Graph

We shall use the following notation and terminology. Given a pseudovariety of finite groups $\mathcal{C}$, we say that $R$ is a pro-$\mathcal{C}$ ring if it is an inverse limit of finite rings which are in $\mathcal{C}$ as abelian groups; if $\mathcal{C}$ is the class of all finite rings, we write profinite rather than pro-$\mathcal{C}$. Let $X$ be a profinite space and let $R$ be a pro-$\mathcal{C}$ ring. We denote by $[[RX]]$ the free profinite $R$-module on the space $X$. Similarly, $[[R(X, *)]]$ denotes the free profinite $R$-module on a pointed space $(X, *)$. The complete group algebra $[[RG]]$ is the inverse limit of the finite group algebras

$$[[RG]] = \lim_{\leftarrow} \left( (R/I)(G/U) \right),$$

where $I$ and $U$ range over the open ideals of $R$ and the open normal subgroups of $G$, respectively.

Let $G$ be a profinite group, and let $X$ be a profinite $G$-space. Then $[[RX]]$ naturally becomes a profinite $[[RG]]$-module. Similarly, if $(X, *)$ is a pointed profinite
Let $\Gamma$ be a profinite graph. Define

$$E^*(\Gamma) = \Gamma / V(\Gamma)$$

to be the quotient space of the space $\Gamma$ modulo the subspace of vertices $V(\Gamma)$. We think of $E^*(\Gamma)$ as a pointed space with the image of $V(\Gamma)$ as the distinguished point.

Let $R$ be a profinite ring and consider the free profinite $R$-modules $[[R(E^*(\Gamma), *)]]$ and $[[RV(\Gamma)]]$ on the pointed profinite space $(E^*(\Gamma), *)$ and on the profinite space $V(\Gamma)$, respectively. Denote by $C(\Gamma, R)$ the chain complex

$$0 \longrightarrow [[R(E^*(\Gamma), *)]] \xrightarrow{d} [[RV(\Gamma)]] \xrightarrow{\varepsilon} R \longrightarrow 0 \quad (2.4)$$

of free profinite $R$-modules and continuous $R$-homomorphisms $d$ and $\varepsilon$ determined by $\varepsilon(v) = 1$, for every $v \in V(\Gamma)$, $d(\tilde{e}) = d_1(e) - d_0(e)$, where $\tilde{e}$ is the image of an edge $e \in E(\Gamma)$ in the quotient space $E^*(\Gamma)$, and $d(*) = 0$. Obviously, $\text{Im}(d) \subseteq \text{Ker}(\varepsilon)$. If we need to emphasize the role of the ring $R$ we sometimes write $d_R$ for the map $d$.

Note that if $E(\Gamma)$ is closed in $\Gamma$, then $*$ is an isolated point of $E^*(\Gamma)$, and so $[[R(E^*(\Gamma), *)]] = [[RE(\Gamma)]]$; this is the case in many important examples.

The homology groups of $\Gamma$ are defined as the homology groups of the chain complex $C(\Gamma, R)$ in the usual way:

$$H_0(\Gamma, R) = \text{Ker}(\varepsilon)/\text{Im}(d), \quad H_1(\Gamma, R) = \text{Ker}(d).$$

A qmorphism

$$\alpha : \Gamma \longrightarrow \Delta$$

of profinite graphs naturally induces continuous maps

$$\alpha_V : V(\Gamma) \longrightarrow V(\Delta) \quad \text{and} \quad \alpha_{E^*} : (E^*(\Gamma), *) \longrightarrow (E^*(\Delta), *),$$

which in turn extend to continuous $R$-homomorphisms

$$\tilde{\alpha}_V : [[RV(\Gamma)]] \longrightarrow [[RV(\Delta)]] \quad \text{and} \quad \tilde{\alpha}_{E^*} : [[R(E^*(\Gamma), *)]] \longrightarrow [[R(E^*(\Delta), *)]].$$

Then the following diagram

$$\begin{array}{ccc}
0 & \longrightarrow & [[R(E^*(\Gamma), *)]] \\
\downarrow{\tilde{\alpha}_{E^*}} & & \downarrow{\tilde{\alpha}_V} \\
0 & \longrightarrow & [[R(E^*(\Delta), *)]]
\end{array}$$

$$\begin{array}{ccc}
& & \xrightarrow{\varepsilon} R \longrightarrow 0 \\
\xrightarrow{id_R} & & \downarrow{\varepsilon} \\
& & R \longrightarrow 0
\end{array}$$

commutes. In other words, the triple $\tilde{\alpha} = (\tilde{\alpha}_{E^*}, \tilde{\alpha}_V, id_R)$ is a morphism

$$\tilde{\alpha} : C(\Gamma, R) \longrightarrow C(\Delta, R)$$
of complexes. Therefore, if
\[ \Gamma = \lim_{i \in I} \Gamma_i \]
is an inverse limit of an inverse system of profinite graphs \( \Gamma_i \), the corresponding chain complexes \( C(\Gamma_i, R) \) form an inverse system and
\[ C(\Gamma, R) = \lim_{i \in I} C(\Gamma_i, R). \]

Furthermore, the homomorphism \( \tilde{\alpha} \) induces continuous homomorphisms of homology groups
\[ \alpha_0^* : H_0(\Gamma, R) \to H_0(\Delta, R) \quad \text{and} \quad \alpha_1^* : H_1(\Gamma, R) \to H_1(\Delta, R). \]
Of course, \( \alpha_1^* \) is just the restriction of \( \tilde{\alpha}_{E^*} \) to \( \text{Ker}(d) \). The statements in the following lemma are easily verified and we leave them to the reader.

**Lemma 2.3.1** Let \( R \) be a profinite ring.

(a) Let
\[ \alpha : \Gamma \to \Delta \]
be a morphism of profinite graphs. If \( \alpha \) is surjective, then
\[ \alpha_0^* : H_0(\Gamma, R) \to H_0(\Delta, R) \]
is surjective. If \( \alpha \) is injective, so is
\[ \alpha_1^* : H_1(\Gamma, R) \to H_1(\Delta, R). \]

(b) If \( \Gamma = \lim_{\leftarrow} \Gamma_i \) is the inverse limit of an inverse system of profinite graphs \( \Gamma_i \), then
\[ H_0(\Gamma, R) = \lim_{\leftarrow} H_0(\Gamma_i, R) \quad \text{and} \quad H_1(\Gamma, R) = \lim_{\leftarrow} H_1(\Gamma_i, R). \]

In the next proposition we prove that the connectivity of a profinite graph is equivalent to the triviality of its 0-homology group.

**Proposition 2.3.2** A profinite graph \( \Gamma \) is connected if and only if \( H_0(\Gamma, R) = 0 \), independently of the choice of the profinite ring \( R \).

**Proof** Write \( \Gamma \) as an inverse limit \( \Gamma = \lim_{i \in I} \Gamma_i \) of finite quotient graphs \( \Gamma_i \). By Proposition 2.1.6, \( \Gamma \) is a connected profinite graph if and only if each \( \Gamma_i \) is connected as an abstract graph. On the other hand, by Lemma 2.3.1, \( H_0(\Gamma, R) = 0 \) if and only if \( H_0(\Gamma_i, R) = 0 \), for each \( i \). Hence it suffices to prove the theorem for finite \( \Gamma \). In this case the sequence (2.4) becomes
\[ 0 \to [RE(\Gamma)] \xrightarrow{d} [RV(\Gamma)] \xrightarrow{\varepsilon} R \to 0. \]
where if $X$ is a set, $[RX]$ denotes the free $R$-module on the set $X$. Observe that $\varepsilon d = 0$, so that $\text{Im}(d) \leq \text{Ker}(\varepsilon)$.

Assume first that $\Gamma$ is connected. Let

$$\varepsilon \left( \sum_{i=1}^{t} n_i v_i \right) = \sum_{i=1}^{t} n_i = 0 \quad (v_1, \ldots, v_t \in V(\Gamma); n_1, \ldots, n_t \in R).$$

Fix $v_0 \in V(\Gamma)$. Then $\sum_{i=1}^{t} n_i v_i = \sum_{i=1}^{t} n_i (v_i - v_0)$; hence it suffices to check that for every pair of distinct vertices $v, w$ of $\Gamma$, there exists some $c \in [RE(\Gamma)]$ with $d(c) = w - v$. To verify this let $e^{e_1}, \ldots, e^{e_m}$ be a path from $v$ to $w$. Define $c = \sum_{i=1}^{s} e_i$, where we think of $e_i$ as an element of $R$. Then $d(c) = w - v$. Hence the sequence is exact at $[RV(\Gamma)]$, i.e., $H_0(\Gamma, R) = 0$.

Assume now that the sequence is exact at $[RV(\Gamma)]$. Let $v' \in V(\Gamma)$ and let $\Gamma'$ be the connected component of $v'$ in $\Gamma$. Suppose that $\Gamma' \neq \Gamma$, and let $\Gamma''$ be the complement of $\Gamma'$ in $\Gamma$; then $\Gamma''$ is a subgraph of $\Gamma$. Choose $v'' \in V(\Gamma'')$. Clearly $v' - v'' \in \text{Ker}(\varepsilon)$. Then there exists

$$\sum_{i=1}^{s} n_i e_i \in [RE(\Gamma)] \quad (e_i \in E(\Gamma), n_i \in R, i = 1, \ldots, s)$$

such that $d(\sum_{i=1}^{s} n_i e_i) = v' - v''$. We may assume that $v'$ is a vertex of $e_1$ and $e_1, \ldots, e_t \in \Gamma'$, while $e_{t+1}, \ldots, e_s \in \Gamma''$ and $v''$ is a vertex of $e_s$. Clearly

$$d([RE(\Gamma')]) \leq [RV(\Gamma')],$$
$$d([RE(\Gamma'')]) \leq [RV(\Gamma'')]$$

and

$$[RV(\Gamma)] = [RV(\Gamma')] \oplus [RV(\Gamma'')] .$$

Therefore $d(\sum_{i=1}^{t} n_i e_i) = v'$. However, $v' \notin \text{Ker}(\varepsilon)$, a contradiction. Thus $\Gamma = \Gamma'$, and $\Gamma$ is connected.

\[\square\]

### 2.4 $\pi$-Trees and $\mathcal{C}$-Trees

Let $\mathcal{C}$ be a pseudovariety of finite groups and consider the set of primes $\pi = \pi(\mathcal{C})$ involved in $\mathcal{C}$ (see Sect. 1.3). Let $\mathbb{Z}_{\mathcal{C}}$ denote the pro-$\mathcal{C}$ completion of the group of integers $\mathbb{Z}$. This is the free pro-$\mathcal{C}$ group of rank 1; it also has, in a natural way, a ring structure. One has

$$\mathbb{Z}_{\mathcal{C}} = \prod_{p \in \pi} \mathbb{Z}_p / p^{n_p} \mathbb{Z}_p ,$$

where

$$n_p = n_p(\mathcal{C}) = \sup \{ n \mid n \in \mathbb{N}, p^n \mid |C|, C \in \mathcal{C} \} .$$
If \( n_p = \infty \), then, by convention, we agree that \( p^\infty \mathbb{Z}_p = 0 \). Note that every abelian pro-\( C \) group is in a unique way a profinite \( \hat{\mathbb{Z}} \)-module.

A profinite graph \( \Gamma \) is said to be a \( C \)-tree if \( \Gamma \) is connected and \( H_1(\Gamma, \mathbb{Z}_\hat{\mathbb{C}}) = 0 \). Thus \( \Gamma \) is a \( C \)-tree if and only if the sequence \( C(\Gamma, \mathbb{Z}_\hat{\mathbb{C}}) \) (see Sect. 2.3)

\[
0 \longrightarrow \left[ [\mathbb{Z}_\hat{\mathbb{C}}(E^*(\Gamma), \mathbb{C})] \right] \xrightarrow{d} \left[ [\mathbb{Z}_\hat{\mathbb{C}}V(\Gamma)] \right] \xrightarrow{\varepsilon} \mathbb{Z}_\hat{\mathbb{C}} \longrightarrow 0 \tag{2.5}
\]

is exact. Note that if the set of edges \( E(\Gamma) \) of \( \Gamma \) is closed, then the sequence (2.5) becomes

\[
0 \longrightarrow \left[ [\mathbb{Z}_\hat{\mathbb{C}}E(\Gamma)] \right] \xrightarrow{d} \left[ [\mathbb{Z}_\hat{\mathbb{C}}V(\Gamma)] \right] \xrightarrow{\varepsilon} \mathbb{Z}_\hat{\mathbb{C}} \longrightarrow 0.
\]

**Lemma 2.4.1** Let \( C \) be a pseudovariety of finite groups. A profinite graph \( \Gamma \) is a \( C \)-tree if and only if the sequence \( C(\Gamma, \mathbb{F}_p) \)

\[
0 \longrightarrow \left[ [\mathbb{F}_pE^*(\Gamma), \mathbb{C})] \right] \xrightarrow{d} \left[ [\mathbb{F}_pV(\Gamma)] \right] \xrightarrow{\varepsilon} \mathbb{F}_p \longrightarrow 0
\]

is exact for every \( p \in \pi(C) \), where \( \mathbb{F}_p \) is the field with \( p \)-elements.

**Proof** First observe that a proabelian group is the direct product of its \( p \)-Sylow subgroups. So, for any profinite space \( X \),

\[
[\mathbb{Z}_\hat{\mathbb{C}}X] = \prod_{p \in \pi(C)} \left[ [(\mathbb{Z}_p/p^n_p\mathbb{Z}_p)X] \right].
\]

Therefore,

\[
C(\Gamma, \mathbb{Z}_\hat{\mathbb{C}}) = \prod_{p \in \pi(C)} C(\Gamma, \mathbb{Z}_p/p^n_p\mathbb{Z}_p),
\]

where \( n_p = n_p(C) \). Hence the sequence \( C(\Gamma, \mathbb{Z}_\hat{\mathbb{C}}) \) is exact if and only if the sequence \( C(\Gamma, \mathbb{Z}_p/p^n_p\mathbb{Z}_p) \) is exact for each \( p \in \pi(C) \). Therefore it suffices to prove that \( C(\Gamma, \mathbb{Z}_p/p^n_p\mathbb{Z}_p) \) is exact if and only if \( C(\Gamma, \mathbb{F}_p) \) is exact.

We observe that \( C(\Gamma, \mathbb{Z}_p/p^n_p\mathbb{Z}_p) \) and \( C(\Gamma, \mathbb{F}_p) \) are sequences of free abelian pro-\( p \) groups of exponent \( p^n_p \) and free abelian pro-\( p \) groups of exponent \( p \), respectively. Moreover, if \( X \) is a profinite space, \( [[\mathbb{F}_pX]] \) is the Frattini quotient

\[
[[((\mathbb{Z}_p/p^n_p\mathbb{Z}_p)X)]] / \Phi([[((\mathbb{Z}_p/p^n_p\mathbb{Z}_p)X)]])
\]

of \( [[((\mathbb{Z}_p/p^n_p\mathbb{Z}_p)X)]]) \); this is obvious if \( X \) is finite, and in general this can be deduced by a standard inverse limit argument.

Exactness of \( C(\Gamma, \mathbb{Z}_p/p^n_p\mathbb{Z}_p) \) at \( [[((\mathbb{Z}_p/p^n_p\mathbb{Z}_p)V(\Gamma))]] \) is equivalent to exactness of \( C(\Gamma, \mathbb{F}_p) \) at \( [[\mathbb{F}_p(V(\Gamma))]] \), because any of these statements is equivalent to \( \Gamma \) being connected, according to Proposition 2.3.2. Hence from now on we assume that \( \Gamma \) is connected as a profinite graph, and we must show that injectivity of the map \( d \) of \( C(\Gamma, \mathbb{Z}_p/p^n_p\mathbb{Z}_p) \) is equivalent to injectivity of the map \( d \) of \( C(\Gamma, \mathbb{F}_p) \).
To prove this we will also work with the chain complex $C(\Gamma, \mathbb{Z}_p)$. Consider the commutative diagram

$$
\begin{array}{ccc}
[\mathbb{Z}_p(E^*(\Gamma), *)] & \xrightarrow{d\mathbb{Z}_p} & d([\mathbb{Z}_p(E^*(\Gamma), *)]) \\
\downarrow & & \downarrow \\
[\mathbb{Z}_p/p^n\mathbb{Z}_p(E^*(\Gamma), *)] & \xrightarrow{d'} & d([\mathbb{Z}_p/p^n\mathbb{Z}_p(E^*(\Gamma), *)]) \\
\downarrow & & \downarrow \\
[\mathbb{F}_p(E^*(\Gamma), *)] & \xrightarrow{d\mathbb{F}_p} & d([\mathbb{F}_p(E^*(\Gamma), *)]) 
\end{array}
$$

where the vertical maps are the natural quotient maps, and the maps $d\mathbb{Z}_p$, $d'$ and $d\mathbb{F}_p$ denote the maps induced by the homomorphisms $d$ of $C(\Gamma, \mathbb{Z}_p)$, $C(\Gamma, \mathbb{Z}_p/p^n\mathbb{Z}_p)$ and $C(\Gamma, \mathbb{F}_p)$, respectively.

Since the sequence $C(\Gamma, \mathbb{Z}_p)$ is exact at $[\mathbb{Z}_pV(\Gamma)]$ and since $\mathbb{Z}_p$ is the free $\mathbb{Z}_p$-module of rank 1, the map $\varepsilon$ splits, and we have

$$
[\mathbb{Z}_pV(\Gamma)] = d([\mathbb{Z}_p(E^*(\Gamma), *)]) \oplus \mathbb{Z}_p.
$$

Similarly, we have

$$
[\mathbb{Z}_p/p^n\mathbb{Z}_pV(\Gamma)] = d([\mathbb{Z}_p/p^n\mathbb{Z}_p(E^*(\Gamma), *)]) \oplus \mathbb{Z}_p/p^n\mathbb{Z}_p
$$

and

$$
[\mathbb{F}_pV(\Gamma)] = d([\mathbb{F}_p(E^*(\Gamma), *)]) \oplus \mathbb{F}_p.
$$

From this it follows that the last line of the diagram is obtained from the first or second line by taking quotients modulo the subgroups of $p$-th powers (the Frattini subgroups); and the second line is obtained from the first by taking quotients modulo the subgroups of $p^n$-th powers. It follows that if $d\mathbb{Z}_p$ (respectively, $d'$) is an isomorphism, then so is $d\mathbb{F}_p$. Conversely, assume that $d\mathbb{F}_p$ is an isomorphism. Since $d([\mathbb{Z}_p(E^*(\Gamma), *)])$ is a subgroup of $[\mathbb{Z}_pV(\Gamma)]$, it is a torsion-free pro-$p$ group, and so a free abelian pro-$p$ group (cf. RZ, Theorem 4.3.3 and Example 3.3.8(c)). Therefore there exists a continuous homomorphism

$$
\alpha : d([\mathbb{Z}_p(E^*(\Gamma), *)]) \longrightarrow [\mathbb{Z}_p(E^*(\Gamma), *)]
$$

such that $d\mathbb{Z}_p\alpha$ is the identity map on $d([\mathbb{Z}_p(E^*(\Gamma), *)])$; therefore $\alpha$ is injective. On the other hand,

$$
\text{Ker}(d\mathbb{Z}_p) \leq \Phi([\mathbb{Z}_p(E^*(\Gamma), *)]) \quad \text{and} \quad (\text{Ker}(d\mathbb{Z}_p)) + \text{Im}(\alpha) = [\mathbb{Z}_p(E^*(\Gamma), *)],
$$

where $\Phi([\mathbb{Z}_p(E^*(\Gamma), *)])$ is the subgroup of $p$-th powers of $[\mathbb{Z}_p(E^*(\Gamma), *)]$, i.e., its Frattini subgroup. So $\text{Im}(\alpha) = [\mathbb{Z}_p(E^*(\Gamma), *)]$ (cf. RZ, Corollary 2.8.5). Therefore $\alpha$ is an isomorphism, and hence $d\mathbb{Z}_p$ is an isomorphism. Thus, $d'$ is also an isomorphism.
The above lemma shows that in fact the concept of a \(C\)-tree depends only on the primes involved in the pseudovariety \(C\). This suggests the following definition. Let \(\pi\) be a nonempty set of prime numbers, and denote by \(\mathbb{Z}_{\hat{\pi}}\) the profinite group (ring)

\[ \mathbb{Z}_{\hat{\pi}} = \prod_{p \in \pi} \mathbb{Z}_p. \]

We say that a profinite graph \(\Gamma\) is a \(\pi\)-tree if it is connected as a profinite graph and one has \(H_1(\Gamma, \mathbb{Z}_{\hat{\pi}}) = 0\). In other words, \(\Gamma\) is a \(\pi\)-tree if and only if the sequence

\[ 0 \rightarrow \left[ \mathbb{Z}_{\hat{\pi}}(E_*(\Gamma), *) \right] \xrightarrow{d} \left[ \mathbb{Z}_{\hat{\pi}} V(\Gamma) \right] \xrightarrow{\varepsilon} \mathbb{Z}_{\hat{\pi}} \rightarrow 0 \]  

is exact. If \(\pi = \{p\}\) consists of only one prime, we write \(p\)-tree rather than \(\{p\}\)-tree. When \(\pi\) is the set of all prime numbers, we normally use the term profinite tree rather than \(\pi\)-tree. The following proposition is an immediate consequence of Lemma 2.4.1.

**Proposition 2.4.2** Let \(C\) be a pseudovariety of finite groups and let \(\Gamma\) be a profinite graph. Let \(\pi = \pi(C)\). The following conditions are equivalent:

(a) \(\Gamma\) is a \(C\)-tree;

(b) \(\Gamma\) is a \(\pi\)-tree;

(c) let \(R\) be a quotient ring of \(\hat{\mathbb{Z}}\) such that the order \#\(R\) of \(R\) as a profinite group involves precisely the primes in the set \(\pi\). Then the sequence

\[ 0 \rightarrow \left[ R(E_*(\Gamma), *) \right] \xrightarrow{d} \left[ RV(\Gamma) \right] \xrightarrow{\varepsilon} R \rightarrow 0 \]  

is exact;

(d) for a given prime \(p\), let \(R_p\) denote one of the following rings: \(\mathbb{Z}_p\), \(\mathbb{F}_p\) or \(\mathbb{Z}_p/p^n\mathbb{Z}_p\), for some positive integer \(n\). Then, for every \(p \in \pi\), the sequence

\[ 0 \rightarrow \left[ R_p(E_*(\Gamma), *) \right] \xrightarrow{d} \left[ R_p V(\Gamma) \right] \xrightarrow{\varepsilon} R_p \rightarrow 0 \]  

is exact.

**Proposition 2.4.3** Let \(\pi\) be a nonempty set of prime numbers. Then the following statements hold.

(a) Every finite tree is a \(\pi\)-tree.

(b) Every connected profinite subgraph of a \(\pi\)-tree is a \(\pi\)-tree.

(c) If \(\Delta_1\) and \(\Delta_2\) are \(\pi\)-subtrees of a \(\pi\)-tree such that \(\Delta_1 \cap \Delta_2 \neq \emptyset\), then \(\Delta_1 \cup \Delta_2\) is a \(\pi\)-subtree.

(d) An inverse limit of \(\pi\)-trees is a \(\pi\)-tree. In particular, an inverse limit of finite trees is a \(\pi\)-tree.

(e) If \(\emptyset \neq \pi' \subseteq \pi\), then every \(\pi\)-tree is a \(\pi'\)-tree.

**Proof** Part (b) follows from Lemma 2.3.1(a). Part (c) follows from (b) and Lemma 2.1.7. The first statement in part (d) is a consequence of Lemma 2.3.1(b);
and the second then follows from (a). Part (e) is a consequence of the definition of a \( \pi \)-tree. To prove (a), let \( \Gamma \) be a finite tree. In this case the sequence (2.6) becomes

\[
0 \longrightarrow [\mathbb{Z}_p E(\Gamma)] \xrightarrow{d} [\mathbb{Z}_p V(\Gamma)] \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0.
\]

Since \( \Gamma \) is connected, this sequence is exact at \( [\mathbb{Z}_p V(\Gamma)] \) by Proposition 2.3.2. It remains to see that \( d \) is an injection. For this define a map

\[
\rho : V(\Gamma) \longrightarrow [\mathbb{Z}_p E(\Gamma)]
\]

as follows: fix a vertex \( v_0 \in V(\Gamma) \); since \( \Gamma \) is an abstract tree, for each vertex \( v \in V(\Gamma) \) there is a unique path \( e^{e_1}_1, \ldots, e^{e_t}_t \) from \( v_0 \) to \( v \) of minimal length; define

\[
\rho(v) = \varepsilon_1 e_1 + \cdots + \varepsilon_t e_t \quad (e_1, \ldots, e_t \in E(\Gamma); \varepsilon_i = \pm 1, i = 1, \ldots, t).
\]

Since \( [\mathbb{Z}_p V(\Gamma)] \) is a free \( \mathbb{Z}_p \)-module, this map extends to a \( \mathbb{Z}_p \)-homomorphism (also denoted \( \rho \)) \( \rho : [\mathbb{Z}_p V(\Gamma)] \to [\mathbb{Z}_p E(\Gamma)] \). Then \( \rho d \) is the identity map on \([\mathbb{Z}_p E(\Gamma)]\); thus \( d \) is an injection. \( \square \)

**Exercise 2.4.4** Let \( T \) be a \( \pi \)-tree.

(a) \( T \) does not contain circuits.
(b) If \( v, w \in V(T) \) and there exists a path \( p_{vw} \) from \( v \) to \( w \), then there is a unique reduced path from \( v \) to \( w \).

**Example 2.4.5** (A \( \pi \)-tree which is not an inverse limit of finite trees) It is not always possible to decompose a \( \pi \)-tree as an inverse limit of finite trees. For example, let \( p \) be a prime number. The Cayley graph \( \Gamma = \Gamma(\mathbb{Z}_p, \{1\}) \) is a \( p \)-tree (see Theorem 2.5.3 below). Let \( \tilde{\Gamma} \) be a finite quotient graph of \( \Gamma \). Then \( \tilde{\Gamma} \) is also a quotient graph of a graph of the form \( \Gamma(\mathbb{Z}/p^n\mathbb{Z}, \{1\}) \) (see Lemma 2.1.5), which is a circuit. Hence, if \( |\tilde{\Gamma}| \geq 2 \), then \( \tilde{\Gamma} \) is not a tree.

**Lemma 2.4.6** Let \( \Delta \) be a profinite subgraph of a profinite graph \( \Gamma \), and let \( R \) be a profinite ring. Then

(a) \( V(\Delta) \) is a closed subspace of \( V(\Gamma) \), and \( (E^*(\Delta), *) \) is naturally embedded in \( (E^*(\Gamma), *) \);  
(b) \( V(\Gamma/\Delta) \) is naturally homeomorphic with \( V(\Gamma)/V(\Delta) \), and \( E^*(\Gamma/\Delta, *) \) is naturally homeomorphic with \( (E^*(\Gamma)/E^*(\Delta), *) \), where, in this last case, the distinguished point \( * \) is the image of \( E^*(\Delta) \) in \( E^*(\Gamma)/E^*(\Delta) \);

(c) \[
\left[\left[ R(E^*(\Gamma/\Delta), *) \right] \right] \cong \left[\left[ R(E^*(\Gamma), *) \right] \right] / \left[\left[ R(E^*(\Delta), *) \right] \right].
\]

**Proof** Parts (a) and (b) are straightforward. To prove (c) consider the natural continuous map

\[
\iota : (E^*(\Gamma/\Delta), *) \longrightarrow \left[\left[ R(E^*(\Gamma), *) \right] \right] / \left[\left[ R(E^*(\Delta), *) \right] \right].
\]
We must show that \([R(E^*(\Gamma), \ast)]/[R(E^*(\Delta), \ast)]\) is the free profinite \(R\)-module on the space \((E^*(\Gamma/\Delta), \ast)\) with respect to the map \(\iota\) (see Sect. 1.7). Let \(\varphi : (E^*(\Gamma/\Delta), \ast) \to A\) be a continuous map of pointed spaces into a profinite \(R\)-module \(A\). Then \(\varphi\) induces a continuous map
\[
\varphi_1 : \left( E^*(\Gamma), \ast \right) \to A,
\]
and this in turn induces a continuous \(R\)-homomorphism
\[
\bar{\varphi}_1 : \left[ \left[ R\left( E^*(\Gamma), \ast \right) \right] \right] / \left[ \left[ R\left( E^*(\Delta), \ast \right) \right] \right] \to A
\]
such that \(\bar{\varphi}_1(\left[ R\left( E^*(\Delta), \ast \right) \right]) = 0\). Therefore \(\bar{\varphi}_1\) induces a continuous \(R\)-homomorphism
\[
\bar{\varphi} : \left[ \left[ R\left( E^*(\Gamma), \ast \right) \right] \right] / \left[ \left[ R\left( E^*(\Delta), \ast \right) \right] \right] \to A
\]
such that \(\bar{\varphi}\iota = \varphi\). The uniqueness of \(\bar{\varphi}\) is clear since \(\iota\left( E^*(\Gamma/\Delta), \ast \right)\) generates \(\left[ \left[ R\left( E^*(\Gamma), \ast \right) \right] \right] / \left[ \left[ R\left( E^*(\Delta), \ast \right) \right] \right]\).

\[\square\]

**Lemma 2.4.7** Let \(\Delta\) be a \(\pi\)-subtree of a connected profinite graph \(\Gamma\) and let
\[
\alpha : \Gamma \to \Gamma/\Delta
\]
be the corresponding canonical epimorphism of graphs. Then the induced homomorphism
\[
\alpha_1^* : H_1(\Gamma, \mathbb{Z}_p) \to H_1(\Gamma/\Delta, \mathbb{Z}_p)
\]
is an isomorphism. In particular, if \(\Gamma\) is a \(\pi\)-tree, then so is \(\Gamma/\Delta\).

**Proof** We may assume that \(\pi\) consists of just one prime \(p\). Let
\[
\beta : \Delta \to \Gamma
\]
be the natural embedding. Then \(\beta\) and \(\alpha\) induce a monomorphism \(\tilde{\beta} : C(\Delta, \mathbb{Z}_p) \to C(\Gamma, \mathbb{Z}_p)\) and an epimorphism \(\tilde{\alpha} : C(\Gamma, \mathbb{Z}_p) \to C(\Gamma/\Delta, \mathbb{Z}_p)\) of chain complexes, respectively, and the following diagram
commutes. Note that the first row is exact because $\Delta$ is a $p$-tree, the second row is exact because $\Gamma$ is connected.

By Lemma 2.4.6, $\text{Ker}(\tilde{\alpha}_E^*) = \tilde{\beta}_E^*(\{[Z_p(E^*(\Delta), *)]\})$, in other words, the first column of the diagram is an exact sequence. From this it easily follows that $\alpha_1^*$ is an injection. Indeed, let $a \in H_1(\Gamma, Z_p)$ be such that $\alpha_1^*(a) = 0$; i.e., $a \in \{[Z_p(E^*(\Gamma), *)]\}$ with $d^\Gamma(a) = 0$ and $\tilde{\alpha}_E^*(a) = 0$. Then there exists a $b \in \{[Z_p(E^*(\Delta), *)]\}$ such that $\tilde{\beta}_E^*(b) = a$. Now, since $d^\Delta$ and $\tilde{\beta}_V$ are injections, we deduce from the commutativity of the diagram that $b = 0$. Thus $a = 0$.

Next we observe that $\text{Ker}(\tilde{\alpha}_V) = \tilde{\beta}_V(\text{Ker}(\epsilon^\Delta));$ indeed, first we notice that this is straightforward if $V(\Gamma)$ is finite; in general we use an inverse limit argument.

Now we can easily deduce that $\alpha_1^*$ is a surjection: if $c \in \{[Z_p(E^*(\Gamma/\Delta), *)]\}$ and $d^{\Gamma/\Delta}(c) = 0$, choose $\tilde{c} \in \{[Z_p(E^*(\Gamma), *)]\}$ such that $\tilde{\alpha}_E^*(\tilde{c}) = c$; then $d^\Gamma(\tilde{c}) \in \text{Ker}(\tilde{\alpha}_V)$, and so there exists a $y \in \text{Ker}(\epsilon^\Delta)$ with $\tilde{\beta}_V(y) = d^\Gamma(\tilde{c})$; hence there exists a $y' \in \{[Z_p(E^*(\Delta), *)]\}$ with $d^\Delta(y') = y$; then $c' = \tilde{c} - \tilde{\beta}_E^*(y') \in \text{Ker}(d^\Gamma)$ and $\tilde{\alpha}_E^*(c') = c$, as needed. \hfill $\square$

**Lemma 2.4.8** Let $R$ be a profinite ring. Then the following statements hold.

(a) Let $\{X_i \mid i \in I\}$ be a collection of closed subspaces of a profinite space $Y$. Set $X = \bigcap_{i \in I} X_i$. Then

$$[[RX]] = \bigcap_i [[RX_i]].$$

(b) Let $\{(X_i, *) \mid i \in I\}$ be a collection of closed pointed subspaces of a profinite pointed space $(Y, *)$. Set $(X, *) = \bigcap_{i \in I} (X_i, *)$. Then

$$[[R(X, *)]] = \bigcap_i [[R(X_i, *)]].$$

(c) Let $Y$ and $Z$ be closed subspaces of the profinite pointed space $(X, *)$ such that $* \in Y$ and $* \notin Z$. Then there are natural isomorphisms

$$[[R(X, *)]]/[[RZ]] \cong [[R(X/Z, *)]] \quad \text{and} \quad [[R(X, *)]]/[[RY]] \cong [[R(X/Y, *)]].$$

**Proof** The proofs of (a) and (b) are similar. We only prove (a). Assume first that $I = \{1, 2\}$, i.e., $X = X_1 \cap X_2$. Write $Y$ as the inverse limit

$$Y = \lim_{j \in J} Y_j$$

of its finite quotient spaces. Denote by $\varphi_j : Y \rightarrow Y_j$ the projection ($j \in J$), and define $X_{1j} = \varphi_j(X_1)$ and $X_{2j} = \varphi_j(X_2)$.

Since $X_{1j}$ and $X_{2j}$ are finite, we have

$$[[RX_{1j} \cap X_{2j}]] = [[RX_{1j}]] \cap [[RX_{2j}]].$$
It is easy to verify that 
\[ X_1 \cap X_2 = \left( \lim_{j \in J} X_{1j} \right) \cap \left( \lim_{j \in J} X_{2j} \right) = \lim_{j \in J} (X_{1j} \cap X_{2j}) . \]

Hence 
\[
\left[ R(X_1 \cap X_2) \right] = \left[ R \left( \lim_{j \in J} (X_{1j} \cap X_{2j}) \right) \right] = \lim_{j \in J} \left[ R(X_{1j} \cap X_{2j}) \right] \\
= \lim_{j \in J} \left( \left[ R X_{1j} \right] \cap \left[ R X_{2j} \right] \right) = \left( \lim_{j \in J} \left[ R X_{1j} \right] \right) \cap \left( \lim_{j \in J} \left[ R X_{2j} \right] \right) \\
= \left[ R X_1 \right] \cap \left[ R X_2 \right]
\]
(for the second and fourth equalities see RZ, Proposition 5.2.2).

Assume now that \( I \) is any indexing set. By the case considered above we may assume that the collection \( \{ X_i \mid i \in I \} \) is filtered from below, i.e., that the intersection of any two sets in the collection contains a set in the collection. So we may think of this collection as an inverse system of sets and 
\[ X = \bigcap_{i \in I} X_i = \lim_{i \in I} X_i . \]

Also, using again the case above, the collection of profinite \( R \)-submodules \( \left[ R X_i \right] \) of \( \left[ R Y \right] \) is filtered from below. Therefore, 
\[
\left[ R X \right] = \left[ R \left( \lim_{i \in I} X_i \right) \right] = \lim_{i \in I} \left[ R X_i \right] = \bigcap_{i \in I} \left[ R X_i \right] .
\]

(c) We prove the second statement, the first being similar. The quotient map \((X, \ast) \to (X/Y, \ast)\) induces a continuous epimorphism of free profinite modules \( f : \left[ R(X, \ast) \right] \to \left[ R(X/Y, \ast) \right] \). Since \( f(\left[ R(Y, \ast) \right]) = 0 \), \( f \) induces an epimorphism 
\[ \rho : \left[ R(X, \ast) \right]/\left[ R(Y, \ast) \right] \to \left[ R(X/Y, \ast) \right] . \]

On the other hand, the natural map \((X/Y, \ast) \to \left[ R(X/Y, \ast) \right]/\left[ R(Y, \ast) \right] \) induces a continuous homomorphism 
\[ \psi : \left[ R(X/Y, \ast) \right] \to \left[ R(X, \ast) \right]/\left[ R(Y, \ast) \right] . \]

Finally, observe that the composition \( \psi \rho \) is the identity map on \( \left[ R(X, \ast) \right]/\left[ R(Y, \ast) \right] \). Thus \( \rho \) is an isomorphism. \( \square \)

**Proposition 2.4.9** Let \( \pi \) be a nonempty set of prime numbers. Suppose that \( \{ \Delta_i \mid i \in I \} \) is a family of \( \pi \)-subtrees of a \( \pi \)-tree \( T \), and let \( \Delta = \bigcap_{i \in I} \Delta_i \). Then \( \Delta \) is either empty or a \( \pi \)-tree.

**Proof** Assume that \( \Delta \neq \emptyset \). By Lemma 2.4.8 one has 
\[
\left[ \mathbb{Z}_\pi V(\Delta) \right] = \bigcap_{i \in I} \left[ \mathbb{Z}_\pi V(\Delta_i) \right]
\]
and
\[
\left[\mathbb{Z}_\hat{\pi}^* \left( E^* (\Delta), \ast \right) \right] = \bigcap_{i \in I} \left[\mathbb{Z}_\hat{\pi}^* \left( E^* (\Delta_i), \ast \right) \right].
\]

Consider the exact sequence
\[
0 \longrightarrow \left[\mathbb{Z}_\hat{\pi}^* \left( E^* (T), \ast \right) \right] \xrightarrow{d} \left[\mathbb{Z}_\hat{\pi}^* V(T) \right] \xrightarrow{\varepsilon} \mathbb{Z}_\hat{\pi}^* \longrightarrow 0
\]
associated with \( T \). Denote by \( \varepsilon^\Delta, \varepsilon^\Delta_i, d^\Delta, d^\Delta_i \) the restrictions of \( \varepsilon \) and \( d \) to \( \Delta \) and \( \Delta_i \), respectively. Then
\[
\text{Ker}(\varepsilon^\Delta) = \left[\mathbb{Z}_\hat{\pi}^* V(\Delta) \right] \cap \text{Ker}(\varepsilon) = \left( \bigcap_{i \in I} \left[\mathbb{Z}_\hat{\pi}^* V(\Delta_i) \right] \right) \cap \text{Ker}(\varepsilon) = \bigcap_{i \in I} \text{Ker}(\varepsilon^\Delta_i);
\]
moreover,
\[
\text{Im}(d^\Delta) = \bigcap_{i \in I} \text{Im}(d^\Delta_i)
\]
because \( d \) is injective. Since each \( \Delta_i \) is connected, we have \( \text{Ker}(\varepsilon^\Delta_i) = \text{Im}(d^\Delta_i) \), for every \( i \), by Proposition 2.3.2. It follows that \( \text{Im}(d^\Delta) = \text{Ker}(\varepsilon^\Delta) \). So, by Proposition 2.3.2, \( \Delta \) is connected, and therefore a \( \pi \)-tree according to Proposition 2.4.3(b).

It follows from Proposition 2.4.9 that given a nonempty subset \( W \) of a \( \pi \)-tree \( T \), there exists a smallest \( \pi \)-subtree \( [W] \) containing \( W \), namely the intersection of all \( \pi \)-subtrees containing \( W \). If \( W \) consists of two vertices \( v \) and \( w \), we use the notation \([v, w]\) rather than \([\{v, w\}]\) and call it the chain connecting \( v \) and \( w \). Observe that if \([v, w]\) is finite, then it is just the underlying graph of the unique reduced path from \( v \) to \( w \).

**Lemma 2.4.10** A profinite subgraph \( \Delta \) of a \( \pi \)-tree \( T \) is a \( \pi \)-tree if and only if \([v, w] \subseteq \Delta\), for all \( v, w \in V(\Delta) \).

**Proof** If \( \Delta \) is a \( \pi \)-tree, then by definition \([v, w] \subseteq \Delta\), for all \( v, w \in V(\Delta) \). Conversely, suppose \( \Delta \) is a profinite subgraph of \( T \) and that \([v, w] \subseteq \Delta\), for all \( v, w \in V(\Delta) \). To prove that \( \Delta \) is a \( \pi \)-tree, it suffices to show that \( \Delta \) is connected (see Proposition 2.4.3(b)). Write \( T \) as an inverse limit of finite quotient graphs,
\[
T = \operatorname{lim}_{i \in I} T_i,
\]
and let \( \varphi_i : T \rightarrow T_i \) denote the projection \((i \in I)\). It suffices to prove that \( \varphi_i(\Delta) \) is a connected graph for each \( i \in I \). Given vertices \( \bar{v} \) and \( \bar{w} \) of \( \varphi_i(\Delta) \), let \( v, w \in V(\Delta) \) with \( \varphi_i(v) = \bar{v} \) and \( \varphi_i(w) = \bar{w} \). Since \([v, w] \subseteq \Delta\) and \([v, w]\) is a \( \pi \)-tree, we have that \( \varphi_i([v, w]) \) is a connected subgraph of the finite graph \( \varphi_i(\Delta) \) containing \( \bar{v} \) and \( \bar{w} \). Therefore, \( \varphi_i(\Delta) \) is connected. \( \square \)
Example 2.4.11 (A $\pi$-tree that coincides with its infinite chains) Let $\Gamma = \Gamma(\hat{\mathbb{Z}}, 1)$ be the Cayley graph of the free profinite group $\hat{\mathbb{Z}}$ of rank 1 with respect to its subset $\{1\}$. This is a $\pi$-tree for any nonempty set of prime numbers $\pi$ (see Theorem 2.5.3 below and Proposition 2.4.3(e)). The proper $\pi$-subtrees of $\Gamma$ are precisely the proper connected profinite subgraphs of $\Gamma$, and these are precisely the finite $\pi$-subtrees (see Example 2.1.13). Therefore, if $v, w$ are vertices of $\Gamma$, then $[v, w] = \Gamma$, unless $[v, w]$ is finite, in which case $[v, w]$ has vertices $\gamma, \gamma + 1, \ldots, \gamma + t$, where $\gamma = v$ or $\gamma = w$ and $t$ is a natural number.

Let $G$ be a profinite group that acts on a $\pi$-tree $T$. A $\pi$-subtree $T'$ of $T$ is $G$-invariant if whenever $g \in G$ and $m \in T'$, one has $gm \in T'$; and such $T'$ is minimal if it does not contain any proper $G$-invariant $\pi$-subtrees. Minimal $G$-invariant $\pi$-subtrees are especially useful when they are unique. In the next proposition we begin the study of minimal $G$-invariant $\pi$-subtrees $T'$ of $T$. A more systematic study is carried out in Chap. 8.

Proposition 2.4.12 Let $G$ be a profinite group acting on a $\pi$-tree $T$. Then the following assertions hold.

(a) There exists a minimal $G$-invariant $\pi$-subtree $D$ of $T$.
(b) If $|D| > 1$, then $D$ is unique. In particular, if $|G| > 1$ and $G$ acts freely on $T$ or if $G$ is infinite and the stabilizer of some $m \in D$ is finite, then $D$ is the unique minimal $G$-invariant $\pi$-subtree of $T$.
(c) Assume that $D$ is unique. Let $N \triangleleft G$ be such that there exists a unique minimal $N$-invariant $\pi$-subtree $L$ of $T$. Then $L = D$.

Proof (a) Consider the collection $\mathcal{T}$ of all $G$-invariant $\pi$-subtrees of $T$ ordered by reverse inclusion. Since $T \in \mathcal{T}$, $\mathcal{T} \neq \emptyset$. Let $\{T_i\}_{i \in I}$ be a linearly ordered chain in $\mathcal{T}$. By the compactness of $T$, the set $\bigcap T_i$ is nonempty. Then, by Proposition 2.4.9, $\bigcap T_i$ is a $G$-invariant $\pi$-subtree. So $\{T_i\}_{i \in I}$ possesses an upper bound in $\mathcal{T}$. Therefore we can apply Zorn’s lemma to conclude that there exists a minimal $G$-invariant $\pi$-subtree.

(b) This will be proved after Corollary 4.1.9.

(c) Let $g \in G$; then $N$ acts on $gL$ and so $gL$ is minimal $N$-invariant; hence $gL = L$. This means that $G$ acts on $L$. Therefore $D \subseteq L$; but obviously $L \subseteq D$, since $N$ acts on $D$; thus $L = D$. $\square$

2.5 Cayley Graphs and $C$-Trees

A pseudovariety of finite groups $C_0$ is said to be closed under extensions with abelian kernel if whenever

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

is an exact sequence of finite groups with $A, H \in \mathcal{C}_0$ and $A$ is abelian, then $G \in \mathcal{C}_0$. By the Kaloujnine–Krassner theorem (cf. Kargapolov and Merzljakov 1979, Theorem 6.2.8) such an extension group $G$ can be embedded in the wreath product $A$ by $H$; it follows that to check that a pseudovariety of finite groups $\mathcal{C}$ is closed under extensions with abelian kernel, it suffices to verify that any semidirect product of an abelian group in $\mathcal{C}$ by a group in $\mathcal{C}$ is in $\mathcal{C}$.

Next we give an example showing that a pseudovariety which is closed under extensions with abelian kernel is not necessarily extension-closed.

Example 2.5.1 (A pseudovariety closed under extensions with abelian kernel that is not extension-closed) Let $\Delta = A_5$ be the alternating group of degree 5. This is the finite simple nonabelian group with smallest order. Let $\mathcal{C}(\Delta)$ be the collection of all the finite direct products of copies of $\Delta$. Observe that $\mathcal{C}(\Delta)$ is closed under homomorphic images (cf. RZ, Lemma 8.2.4). For a finite group $G$, denote by $S(G)$ its maximal solvable normal subgroup. Define $\mathcal{V}$ to be the set of all finite groups $G$ such that $G/S(G) \in \mathcal{C}(\Delta)$.

We shall show that $\mathcal{V}$ is a pseudovariety of finite groups that is closed under extensions with abelian kernel, but not extension-closed.

We claim first that $\mathcal{V}$ is a pseudovariety. Clearly $\mathcal{V}$ is closed under finite direct products; moreover, since $\mathcal{C}(\Delta)$ is closed under homomorphic images, so is $\mathcal{V}$. It remains to prove that $\mathcal{V}$ is closed under taking subgroups. Let $G \in \mathcal{V}$ and let $H$ be a proper subgroup of $G$. We use induction on the order of $G$ to show that $H \in \mathcal{V}$. If $G$ is solvable or $G \cong \Delta$, then $H$ is solvable and the result is clear. Observe that $H/S(H)$ is a quotient of $H/H \cap S(G)$. If $S(G) \neq 1$, the result follows from the induction hypothesis since $H/H \cap S(G) \leq G/S(G)$ and $|G/S(G)| < |G|$. Thus from now on we may assume that $G \in \mathcal{C}(\Delta)$, i.e., $G = \Delta_1 \times \cdots \times \Delta_n$ ($n \geq 2$), where each $\Delta_i$ is isomorphic to $\Delta$. Since $H$ is a proper subgroup of $G$, there is some $i$ such that $H_i = H \cap \Delta_i \neq \Delta_i$, $1 \leq i \leq n$. Then $H_i$ is solvable. So $H_i \leq S(H)$ and $S(H/H_i) = S(H)/H_i$. Now, since $H/H_i \leq G/\Delta_i \in \mathcal{V}$, we conclude from the induction hypothesis that

$$H/S(H) = (H/H_i)/S(H/H_i) \in \mathcal{C}(\Delta).$$

This proves the claim.

It follows easily from the definition that $\mathcal{V}$ is closed under extensions with abelian kernel. Let us show now that it is not extension-closed. For this consider the wreath product $R = \Delta \wr C$ of $\Delta$ with a group $C$ of order 2; this is a semidirect product of $B = \Delta \times \Delta$ by $C$; both of these groups are in $\mathcal{V}$; and the action of $C$ on $B$ permutes the two factors $\Delta$. Let $K \triangleleft R$ and assume that $K$ is solvable. We claim that $K = 1$. Note that $K \cap B = 1$, for otherwise $K$ must contain one of the copies of $\Delta$, contradicting the solvability of $K$. If $K \neq 1$, we have $R = B \times K = B \times C$, contradicting the definition of $R$. This proves the claim. Therefore $S(R) = 1$. Finally, observe that $R \notin \mathcal{C}(\Delta)$. Thus $R \notin \mathcal{V}$.
2.5 Cayley Graphs and C-Trees

If \((X, \ast)\) is a pointed profinite space, we denote by \(F = F_C(X, \ast)\) the free pro-C group on the pointed space \((X, \ast)\). The next two results establish conditions under which the Cayley graph of a free pro-C group with respect to one of its bases is a C-tree. We begin with a study of the augmentation ideal (see Sect. 1.10) of a free pro-C group.

**Lemma 2.5.2** Let \(C\) be a pseudovariety of finite groups. Then \(C\) is closed under extensions with abelian kernel if and only if for every pointed profinite space \((Y, \ast)\), the augmentation ideal \(((I F))\) of the complete group algebra \([Z\hat{\times}\!F]\) of the free pro-C group \(F = F_C(Y, \ast)\) is a free \([Z\hat{\times}\!F]\)-module with respect to the map \(\iota : (Y, \ast) \to ((I F))\) defined by \(\iota(y) = y - 1\) \((y \in Y)\).

**Proof** The augmentation ideal \(((I F))\) is topologically generated by the space \(Y - 1 = \{y - 1 \mid y \in Y\}\) as a \([Z\hat{\times}\!F]\)-module (see Sect. 1.10).

Assume first that \(C\) is closed under extensions with abelian kernel. We shall prove that \(((I F))\) satisfies the required universal property of a free \([Z\hat{\times}\!F]\)-module with respect to the map \(\iota\). We must prove that given a map of pointed spaces \(\psi : Y \to M\) to a profinite \([Z\hat{\times}\!F]\)-module \(M\), there exists a unique continuous \([Z\hat{\times}\!F]\)-module homomorphism \(\tilde{\psi} : ((I F)) \to M\) such that \(\tilde{\psi}\iota = \psi\).

\[
\begin{array}{ccc}
Y - 1 & \to & (I F) \\
\uparrow & \uparrow & \uparrow \\
Y & \to & M \\
\downarrow & \downarrow & \downarrow \\
\iota & \psi & \tilde{\psi}
\end{array}
\]

Observe that if such a \(\tilde{\psi}\) exists, then it is unique since \(\iota(Y)\) generates \(((I F))\) as a \([Z\hat{\times}\!F]\)-module.

We may assume that \(M\) is finite since \(M\) is an inverse limit of finite \([Z\hat{\times}\!F]\)-modules (see Sect. 1.7). Note that \(M \in C\) since \(M\) is automatically a \(Z\hat{\times}\!\)module and so an abelian pro-C group.

Since \(M\) is in particular an \(F\)-module, we may construct the corresponding semidirect product \(M \rtimes F\). We remark that \(M \rtimes F\) is a pro-C group since \(C\) is closed under extensions with abelian kernel. Since \(F\) is a free pro-C group on \((Y, \ast)\), there exists a unique continuous homomorphism

\[
\rho : F \longrightarrow M \rtimes F
\]

such that \(\rho(y) = (\psi(y), y)\) \((y \in Y)\).

Define now a map

\[
\delta : F \longrightarrow M
\]

by the equation \((\delta(f), f) = \rho(f)\), for all \(f \in F\). Then \(\delta\) is continuous and it is a derivation, that is,

\[
\delta(f_1f_2) = \delta(f_1) + f_1\delta(f_2), \quad \forall f_1, f_2 \in F
\]

(see Sect. 1.10). Now, (see 1.10.7 in Sect. 1.10), there exists an isomorphism

\[
\text{Der}(F, M) \cong \text{Hom}_{[Z\hat{\times}\!F]}((I F), M).
\]
and under this isomorphism $\delta$ corresponds to a $[[\hat{\mathbb{Z}}C F]]$-homomorphism
\[ \tilde{\psi} : ((IF)) \longrightarrow M \]
such that $\tilde{\psi}(f - 1) = \delta(f)$, for all $f \in F$. Then
\[ \tilde{\psi} \iota(y) = \tilde{\psi}(y - 1) = \delta(y) = \psi(y), \quad \forall y \in Y, \]
and thus $\tilde{\psi} \iota = \psi$.

Conversely, assume that $((IF))$ is a free $[[\hat{\mathbb{Z}}C F]]$-module on the pointed space $(Y, \ast)$ with respect to the map $\iota$, for every profinite pointed space $(Y, \ast)$, where $F = F(Y, \ast)$ denotes the free pro-$C$ group on the pointed profinite space $(Y, \ast)$. Let $A, H \in \mathcal{C}$, with $A$ abelian. Assume that $A$ is an $H$-module, and let $G = A \rtimes H$ be the corresponding semidirect product. To prove that $\mathcal{C}$ is closed under extensions with abelian kernel it suffices to show that $G \in \mathcal{C}$, as pointed out above.

Let $\{(a_y, h_y) \mid y \in Y\}$ be a generating set of $G = A \rtimes H$, with $a_y \in A, h_y \in H$, for all $y \in Y$, where $(Y, \ast)$ is a certain finite pointed indexing set and $a_\ast = 1, h_\ast = 1$. Then $H = \langle h_y \mid y \in Y \rangle$. Let $F = F_C(Y, \ast)$ be the free pro-$C$ group on the pointed space $(Y, \ast)$ and let
\[ \varphi : F \longrightarrow H \]
be the continuous epimorphism determined by $\varphi(y) = h_y (y \in Y)$. Then the action of $H$ on $A$ induces an action of $F$ on $A$ via $\varphi$:
\[ f \cdot a = \varphi(f)a, \quad (a \in A, f \in F). \]
Let $\tilde{G} = A \rtimes F$ be the corresponding semidirect product, and let
\[ \tilde{\varphi} : \tilde{G} = A \rtimes F \longrightarrow G = A \rtimes H \]
be the epimorphism induced by $\varphi$.

Since, by assumption, $((IF))$ is a free $[[\hat{\mathbb{Z}}C F]]$-module on $(Y, \ast)$ and $A$ is an $[[\hat{\mathbb{Z}}C F]]$-module, there exists a continuous $[[\hat{\mathbb{Z}}C F]]$-homomorphism
\[ \tilde{\psi} : ((IF)) \rightarrow A \]
such that $\tilde{\psi}(y - 1) = a_y (y \in Y)$. Define
\[ d : F \longrightarrow A \]
by $d(f) = \tilde{\psi}(f - 1) (f \in F)$. Then $d$ is a continuous derivation (see 1.10.7 in Sect. 1.10). Hence the map
\[ \rho : F \longrightarrow \tilde{G} = A \rtimes F \]
given by $\rho(f) = (d(f), f) (f \in F)$, is a continuous homomorphism (cf. RZ, Lemma 9.3.6). Define $\alpha : F \rightarrow G$ to be the composite $\alpha = \tilde{\varphi} \rho$. Observe that
\[ \alpha(y) = (a_y, h_y) \quad (y \in Y); \]
therefore $\alpha$ is an epimorphism, and thus $G \in \mathcal{C}$, as needed. □
Theorem 2.5.3 Let $C$ be a pseudovariety of finite groups. Then $C$ is closed under extensions with abelian kernel if and only if for every profinite pointed space $(Y, \ast)$, the Cayley graph $\Gamma = \Gamma(F, Y)$ of the free pro-$C$ group $F = F(Y, \ast)$ with respect to $Y$ is a $C$-tree.

Proof We think of $(Y, \ast)$ as being embedded in $F$; in particular $\ast$ is identified with $1$. Since $1 \in Y$, $\Gamma = \Gamma(F, Y) = F \times Y$ and $V(\Gamma) = F \times \{1\}$. Consider the sequence associated with the graph $\Gamma$ and $\hat{Z}_C$ as in Eq. (2.4) of Sect. 2.3:

$$0 \longrightarrow \left[\left[\hat{Z}_C((F \times Y)/(F \times \{1\}), \ast)\right]\right] \overset{d}{\longrightarrow} \left[\left[\hat{Z}_C F\right]\right] \overset{\varepsilon}{\longrightarrow} \hat{Z}_C \longrightarrow 0,$$

where $d(f, y) = fy - f$ ($y \in Y$) and $\varepsilon(f) = 1$ ($f \in F$). We have to prove that this sequence is exact for every $(Y, \ast)$ if and only if $C$ is closed under extensions with abelian kernel.

By Lemma 2.2.4, $\Gamma$ is a connected profinite graph since $F$ is topologically generated by $Y$. Therefore, by Proposition 2.3.2, the above sequence is exact at $\left[\left[\hat{Z}_C F\right]\right]$. It remains to prove that $d$ is a monomorphism. Now, $\text{Ker}(\varepsilon)$ is the augmentation ideal $((IF))$ of $\left[\left[\hat{Z}_C F\right]\right]$, which is generated as a topological $\left[\left[\hat{Z}_C F\right]\right]$-module by the subspace $\{y - 1 | y \in Y\}$ (see Sect. 1.10).

On the other hand, $\left[\left[\hat{Z}_C((F \times Y)/(F \times \{1\}), \ast)\right]\right]$ is a free $\left[\left[\hat{Z}_C F\right]\right]$-module on the quotient space $F\backslash((F \times Y)/(F \times \{1\}), \ast)$ (cf. RZ, Proposition 5.7.1). The space $F\backslash((F \times Y)/(F \times \{1\}), \ast)$ can be identified with the pointed space $\{(1, y) | y \in Y\}$, $\ast)$. Since $d(1, y) = 1 - y$ ($y \in Y$), to show that $d$ is a monomorphism is equivalent to showing that the augmentation ideal $((IF))$ is free on the subspace $\{(1 - y | y \in Y\}$, $\ast)$, as a profinite $\left[\left[\hat{Z}_C F\right]\right]$-module. But, according to Lemma 2.5.2, this is the case if and only if $C$ is closed under extensions with abelian kernel. □
Profinite Graphs and Groups
Ribes, L.
2017, XV, 471 p., Hardcover
ISBN: 978-3-319-61041-2