Chapter 2
Coordinate Systems and Transformations

2.1 Coordinate Systems

This chapter describes the coordinate systems used in depicting the position and orientation (pose) of the aerial robot and its manipulator arm(s) in relation to itself and its environment. A reference frame provides a relationship in pose between one coordinate system and another. Every reference frame can relate back to a universal coordinate system. All positions and orientations are referenced with respect to the universal coordinate system or with respect to another Cartesian coordinate system [3].

In Table 2.1 we give the list of variable nomenclature used throughout this chapter and the remainder of the book.

The angles of rotation about the center of mass of the aerial robot make up the overall attitude of the body. In order to track the changes of these attitude angles while the body is in motion, two coordinate systems are required. The body $\{B\}$ frame coordinate system is attached in the vicinity of the geometric center of the robot and typically aligned with the z-axis of the vehicle. The world $\{W\}$ frame coordinate system is fixed to the earth and is taken as an inertial coordinate system [5].

2.1.1 Global Coordinate System

The world inertial frame $\{W\}$ is fixed with the origin, L_W, at a known location on earth. The inertial frame follows an East-North-Up (ENU) convention with the z_W axis pointing up, y_W pointing east, and x_W pointing forward or north. The inertial frame is shown in Fig. 2.1.

The North-East-Down convention is often used in aviation systems. For the purposes of an aerial robot affixed with manipulators, an ENU convention will be used throughout the aircraft and manipulator coordinate frames. An ENU convention is used for a number of reasons. First, an aerial manipulator involves multiple links

M. Orsag et al., *Aerial Manipulation*, Advances in Industrial Control,
https://doi.org/10.1007/978-3-319-61022-1_2

Table 2.1 Table of variable nomenclature

Coordinate frames	
L	Origin of reference frame
W	World or inertial frame
B	Body or local frame
T	Tool or end-effector frame
Three-dimensional quantities	
x_i, y_i, z_i	Coordinate axes of frame i
Ψ, Θ, Φ	Orientation axes
ψ, θ, ϕ	Orientation (roll, pitch, yaw angle)
$\mathbf{p}_j^i$	Position of the i frame in the j frame
θ_j^i	Orientation of the i frame in the j frame
$\mathbf{R}_j^i$	Rotation of the i frame in the j frame
$\mathbf{T}_j^i$	Transformation of the i frame in the j frame
Joint space variables	n-dimensional vectors where n is the number of DOF
$\mathbf{q}$	Joint position
$\dot{\mathbf{q}}$	Joint velocity
$\dot{\mathbf{q}}$	Joint acceleration
τ	Joint torque
$\mathbf{J}$	Jacobian matrix
$\mathbf{w}$	6-DOF tool pose vector
Spatial variables	
$\mathbf{v}$	Velocity of a rigid body
$\mathbf{a}$	Acceleration of a rigid body
$\mathbf{m}$	Mass of a rigid body
$\mathbf{f}$	Force acting on a rigid body
$\mathbf{I}$	Inertia tensor of a rigid body
Control variables	
K_i	PID Control Gains: K_P Proportional, K_I Integral, K_D Derivative
Misc. variables	
$S_\theta, C_\theta, T_\theta$	$Sin(\theta), Cos(\theta), Tan(\theta)$
β	Aerodynamic drag
g	Scalar value of gravitational acceleration

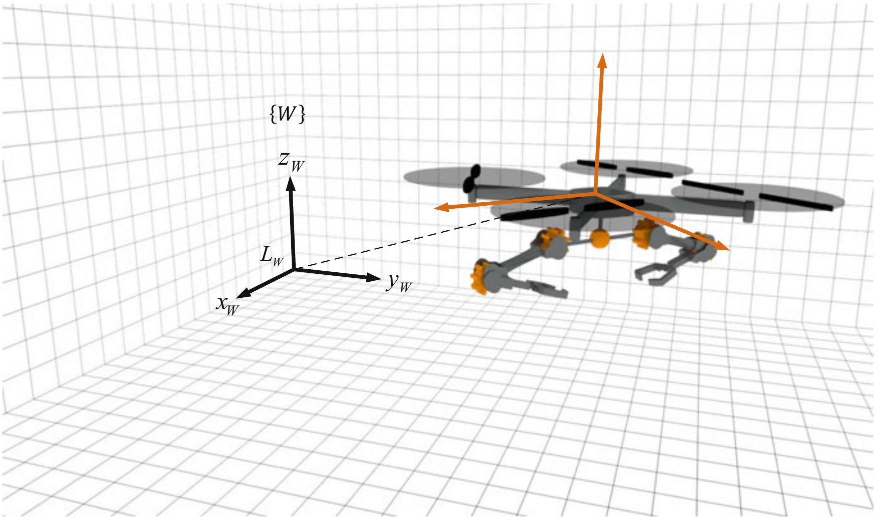

Fig. 2.1 Global (inertial) coordinate frame $\{W\}$ for the aerial robot. The origin is denoted as L_W. The frame is fixed, and all other frames refer back to the global frame. The frame is right-handed with the z-axis pointed upward

which are often described using Denavit–Hartenberg notation [4] (further detailed in Chap. 4). A right-handed coordinate system with the z-axis upward facilitates consistency, where every frame is the same. Ground robots are commonly labeled in this manner. Further, a right-handed coordinate system also allows for easier tracking of the tool-tip or end-effector frame.

2.1.2 Local Coordinate System

A generalized 6-degree of freedom coordinate system is utilized to represent the pose of the aerial robot. The vehicle or body reference frame $\{B\}$ is placed at the vehicle center of mass, geometric center, or at the top of the vehicle body. In almost all cases, the body frame z-axis is coincident with the z-axis of the vehicle as shown in Fig. 2.2. The axes of the local body frame align with those of the global world frame. For example, z_W and z_B both point up. A local coordinate system and body frame is needed when using fixed sensors that measure quantities relative to the vehicle's body such as inertial measurement and ranging sensors. Figure 2.2 also introduces a graphical representation of the local body frame which depicts the principal axes as unit vectors. Further, an arrow representing another vector is drawn from one frame to the next to indicate the change in position and how they are relative to each other. A local frame attaches to the rigid body to relate both position and orientation back to the previous frame.

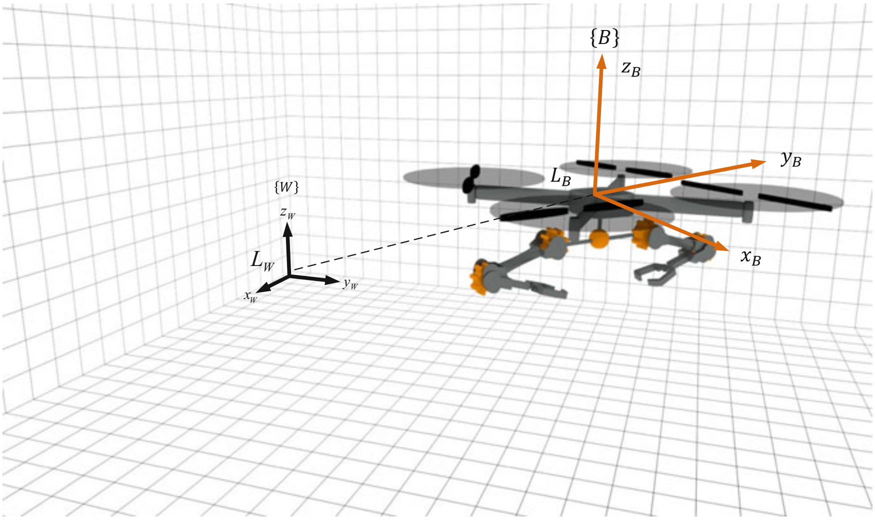

Fig. 2.2 Local (body) coordinate frame $\{B\}$ for the aerial robot where the z-axis points up. The origin is L_B

2.1.3 Coordinate System Representation

With the global and local coordinate systems established, the position and orientation of any point in space can be described using a frame which is the relationship between one coordinate system with respect to another. The frame contains four vectors: one for position and three vectors to describe the orientation or commonly known as a 3×3 rotation matrix. For example, the body frame $\{B\}$ is described by the position vector $\mathbf{p}$ with respect to the world frame $\{W\}$ with a frame origin of L. In this textbook, the trailing superscript is the frame being referenced and the trailing subscript is the frame with respect to. The position of the body frame with respect to the world or inertial reference frame can be expressed in standard form as:

$$\mathbf{p}_W^B = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.1}$$

where x, y, and z are the individual components of the vector. Figure 2.3 illustrates the description of the position of the body frame with respect to the world frame [2, 8].

In addition to position, the orientation of the point in space must also be described. The position of the body frame is described in points of a vector, whereas the orientation of the body must be described using the attached body frame $\{B\}$. The rotations around the axes are roll, pitch, and yaw for the x-axis, y-axis, and z-axis, respectively, and represented symbolically as angles of ψ, θ, and ϕ [10, 13]. The orientation of

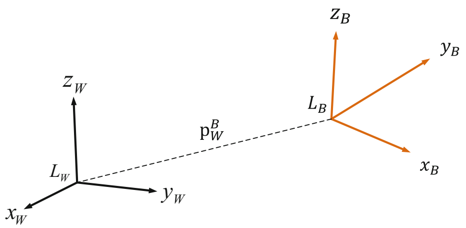

Fig. 2.3 Position of the body frame with respect to the world frame

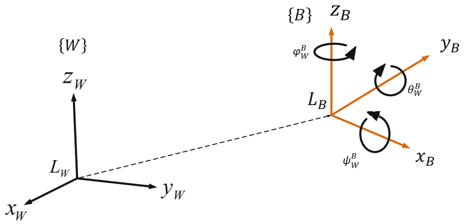

Fig. 2.4 Orientation of the body frame with respect to the world frame

the body frame with respect to the world or inertial reference frame can be expressed in standard form as:

$$\theta_W^B = \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix} \tag{2.2}$$

where the frames are again right-handed with the z-axis pointed upward as shown in Fig. 2.4.

These are not the Euler angles which will be described in the next section.

2.2 Coordinate Transformations

The previous section established the position, orientation, frame, and relationship between frames of a rigid body. It is now necessary to discuss transforming or mapping from one frame to the next.

Looking solely at translation and given that both frames have the same orientation, describing a point in space from one frame to the next is relatively easy. For example, we wish to describe the position, $\mathbf{p}_B^{sensor}$, of a fixed sensor (i.e., range finder) on the air vehicle in terms of the $\{W\}$ frame. The relationship of the body to world frame was given by $\mathbf{p}_W^B$, which is the origin of $\{B\}$ relative to $\{W\}$. The position vectors are defined by frames with the same orientation so the position of the sensor relative to the world is calculated by vector addition:

$$\mathbf{p}_W^{sensor} = \mathbf{p}_W^B + \mathbf{p}_B^{sensor} \tag{2.3}$$

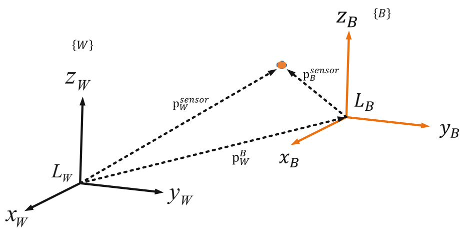

Fig. 2.5 Translation of a point (i.e., sensor) from one frame to the next with fixed orientation

and illustrated in Fig. 2.5. The position of the sensor is fixed, but the description of how it relates to the world is different than how it relates to the body.

Simple translation only involves vector addition between frames. In contrast, the description of orientation is far more complex as will be seen in the following sections.

2.2.1 Orientation Representation

To represent orientation via the rotation matrix $\mathbf{R}_j^i$, one must transform from the coordinate frame L_i toward the coordinate frame L_j. Unlike position, there exist a substantial amount of fundamentally different ways to represent the body orientation. This book, however, does not strive to provide an exhaustive summary, but rather points to more common representations, which are frequently used in robotics and aerial robotics likewise [6]. The 3×3 rotation matrix contains orthogonal unit vectors with unit magnitudes in their columns giving the relationship:

$$\mathbf{R}_j^i = [\mathbf{R}_i^j]^T = [\mathbf{R}_i^j]^{-1} \tag{2.4}$$

The components of the rotation matrix are the dot products of the basis vectors of the two relative frames [11]:

$$\mathbf{R}_j^i = \begin{bmatrix} \mathbf{x}_i \cdot \mathbf{x}_j & \mathbf{y}_i \cdot \mathbf{x}_j & \mathbf{z}_i \cdot \mathbf{x}_j \\ \mathbf{x}_i \cdot \mathbf{y}_j & \mathbf{y}_i \cdot \mathbf{y}_j & \mathbf{z}_i \cdot \mathbf{y}_j \\ \mathbf{x}_i \cdot \mathbf{z}_j & \mathbf{y}_i \cdot \mathbf{z}_j & \mathbf{z}_i \cdot \mathbf{z}_j \end{bmatrix} \tag{2.5}$$

which consists of nine elements. In most cases, there will be a vector offset in translation between the two frames and the frames will not have the same orientation. Thus, it is necessary to systematically describe both translation and rotation between frames.

2.2.2 *Euler Angles*

Although perhaps inferior to other conventions, Euler angles are still a predominant method of describing body orientation and shown in Fig. 2.6. Using only three angles of rotation that form a minimal representation (i.e., three parameters only), Euler angles suffer from singularities, or gimbal lock as they are usually referred to as shown in Fig. 2.7. However, Euler angles are commonly used and remarkably intuitive. This is exactly why they will be used throughout this book.

Unfortunately, there are many standard formulations and notations for Euler angles. Different authors may use different sets of rotation axes to define Euler angles, or different names for the same angles. Therefore, before discussing Euler angles, we first have to define the convention. For the description of aircraft and spacecraft motion, Ψ is the "roll" angle; Θ the "pitch" angle; and Φ the "yaw" angle. It is worth mentioning that in some literature, authors denote Φ as roll and Ψ as the yaw angle.

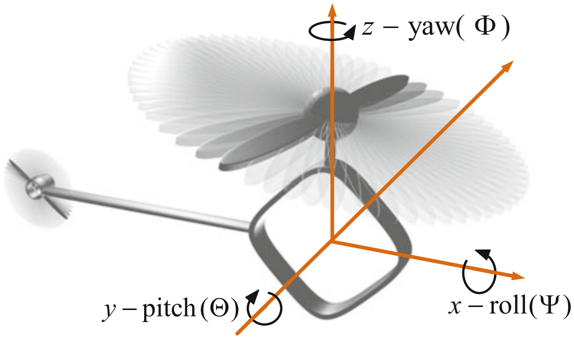

Fig. 2.6 Euler angle representation with three angles: roll Ψ, pitch Θ, and yaw Φ

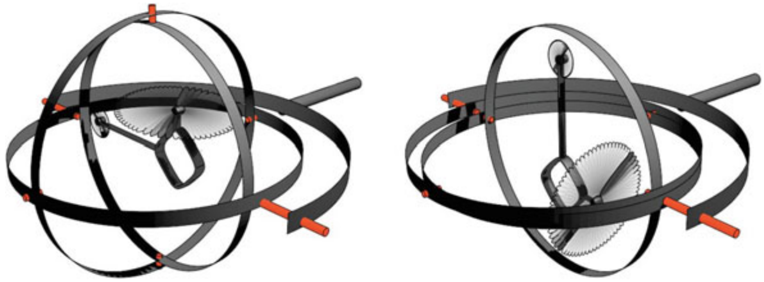

Fig. 2.7 The term gimbal lock was coined when the Euler angles were tightly connected to mechanical gyroscopes. Then, a gimbal lock would occur when the axes of two of the three gimbals in a gyroscope are driven into a parallel configuration, thus locking the system into rotation in a degenerate two-dimensional space. In this example, we rotate the UAV for 90° pitch, which locks the yaw and roll angle in a single degree of freedom rotation

2.2.3 Change of Frame

Rotation matrices allow the mapping of coordinates of the inertial frame into the body-fixed frame. To this end, we use Euler angles as previously mentioned to describe the orientation of the rigid body. In this analysis, we will use the ZYX Euler angles [7, 12]. Consider the elementary rotations in the following equations:

$$\mathbf{R}_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\psi & -S_\psi \\ 0 & S_\psi & C_\psi \end{bmatrix} \tag{2.6}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \tag{2.7}$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

The rotation matrix from the world frame $\{W\}$ to the body frame $\{B\}$ is the product of the three elementary rotations, which denote rotation around the z-axis followed by rotation around the y-axis and finally followed by rotation around the x-axis. Combined, the transformation from the body frame to the world frame is:

$$\mathbf{R}_W^B = \mathbf{R}_z(\psi) \cdot \mathbf{R}_y(\theta) \cdot \mathbf{R}_x(\phi) \tag{2.9}$$

or

$$\mathbf{R}_W^B = \begin{bmatrix} \mathrm{C}_\theta \mathrm{C}_\psi & \mathrm{S}_\phi \mathrm{S}_\theta \mathrm{C}_\psi - \mathrm{C}_\phi \mathrm{S}_\psi & \mathrm{C}_\phi \mathrm{S}_\theta \mathrm{C}_\psi + \mathrm{S}_\phi \mathrm{S}_\psi \\ \mathrm{C}_\theta \mathrm{S}_\psi & \mathrm{S}_\phi \mathrm{S}_\theta \mathrm{S}_\psi + \mathrm{C}_\phi \mathrm{C}_\psi & \mathrm{C}_\phi \mathrm{S}_\theta \mathrm{S}_\psi - \mathrm{S}_\phi \mathrm{C}_\psi \\ -\mathrm{S}_\theta & \mathrm{S}_\phi \mathrm{C}_\theta & \mathrm{C}_\phi \mathrm{C}_\theta \end{bmatrix} \tag{2.10}$$

where $\mathbf{R}_{zyx}(\psi, \theta, \phi) \in SO(3)$. The ZYX Euler angle rotation is a commonly used convention.

2.2.4 Translation and Rotation

Given the position and orientation of the body frame with respect to the world frame, the overall translation and rotation is succinctly represented by the 4×4 homogeneous transformation matrix. This matrix provides a unified method to represent translational and rotational displacements [1]. Taking Eq. 2.1 and rewriting it in a generalized vector form, we have:

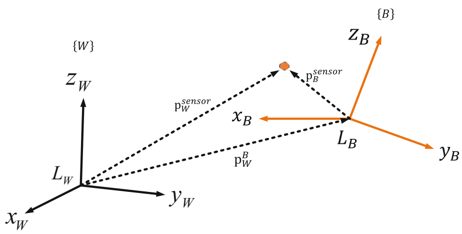

Fig. 2.8 Translation and rotation of a point (i.e., sensor) from one frame to the next with requiring an offset vector and rotation matrix

$$\mathbf{p}_j^i = \begin{bmatrix} \mathbf{x}_j^i \\ \mathbf{y}_j^i \\ \mathbf{z}_j^i \end{bmatrix} \tag{2.11}$$

where the origin of frame $\{i\}$ is not coincident with frame $\{j\}$ but has a vector translational offset. Further, frame $\{j\}$ is rotated with respect to frame $\{i\}$ which is described by the rotation matrix, $\mathbf{R}_j^i$. Using the example of a fixed point with a known position (i.e., sensor) on the $\{B\}$ frame, $\mathbf{p}_B^{sensor}$, we wish to calculate the position of the point relative to the $\{W\}$ frame, $\mathbf{p}_W^{sensor}$. The origin of frame $\{B\}$ is given by $\mathbf{p}_W^B$. Figure 2.8 illustrates the relationship between the frames requiring a translation and rotation.

To change the orientation of the sensor to match that of the world frame, the position vector is multiplied by the rotation matrix. Next, the translation offset is added to generate a description of the position and orientation of the sensor as shown here:

$$\mathbf{p}_W^{sensor} = \mathbf{R}_W^B \cdot \mathbf{p}_B^{sensor} + \mathbf{p}_W^B \tag{2.12}$$

Combining the rotation matrix and translation components, we have the shorthand generalized notation of the entire 4×4 homogeneous transformation matrix from the $\{i\}$ to $\{j\}$ frame.

$$\mathbf{T}_j^i = \begin{bmatrix} \mathbf{R}_j^i & p_j^i \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{2.13}$$

and the expanded version with the identity matrix as a placeholder for rotation:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.14}$$

For rotation, the 3×3 upper left elements represent rotation between one frame and then next frame. This rotation was previously described in Eq. 2.10.

The homogeneous transformation matrix creates a four-dimensional space $\mathbf{R}^4$. To recover the original physical three-dimensional vector, a 3×4 homogeneous conversion matrix can be utilized and defined as follows [9]:

$$\mathbf{H} = \frac{1}{\sigma}[I, 0] \tag{2.15}$$

where σ is the scaling factor and I is the identity matrix. For convenience, the scaling factor is typically set to $\sigma = 1$. As will be done in future chapters, four-dimensional homogeneous coordinates can be obtained from three-dimensional physical coordinates by adding a fourth component as seen in Eq. 2.15.

2.3 Motion Kinematics

Given coordinate systems, frames, position and orientation representation, and transforms, this section describes the motion of the rigid body both through linear and angular rates of change and rotational transformations of the moving rigid body.

2.3.1 Linear and Angular Velocities

The angular velocity components p, q, and r are the projection values on the body coordinate system of rotation angular velocity ω which denotes the rotation from the world coordinate system to the body coordinate system. Let us define the mathematical model of the body of the quadrotor. The vector $[x \quad y \quad z \quad \phi \quad \theta \quad \psi]^T$ contains the linear and angular position of the vehicle in the world frame, and the vector $[u \quad v \quad w \quad p \quad q \quad r]^T$ contains the linear and angular velocities in the body frame. Therefore, the two reference frames have the relationship:

$$\mathbf{v} = \mathbf{R} \cdot \mathbf{v_B} \tag{2.16}$$

$$\omega = \mathbf{T} \cdot \omega_\mathbf{B} \tag{2.17}$$

where $\mathbf{R}$ is defined in Eq. 2.10. $\mathbf{v}$ is the linear velocity vector, $\mathbf{v} = [\dot{x} \quad \dot{y} \quad \dot{z}]^T \in \mathbb{R}^3$, and the angular velocity vector is $\omega = [\dot{\phi} \quad \dot{\theta} \quad \dot{\psi}]^T \in \mathbb{R}^3$. $\mathbf{v_b} = [u \quad v \quad w]^T \in \mathbb{R}^3$ and $\omega_\mathbf{b} = [p \quad q \quad r]^T \in \mathbb{R}^3$. The angular transformation matrix, $\mathbf{T}$, is [9]:

$$\mathbf{T} = \begin{bmatrix} 1 & S_\phi T_\phi & C_\phi T_\phi \\ 0 & C_\phi & -S_\phi \\ 0 & \frac{S_\phi}{C_\phi} & \frac{C_\phi}{C_\phi} \end{bmatrix} \tag{2.18}$$

2.3.2 Rotational Transformations of a Moving Body

Now consider small rotations $\delta\phi, \delta\theta, \delta\psi$ from one frame to another $\mathbf{R}_W^B(\delta\phi, \delta\theta, \delta\psi)$ (2.10); using the small angle assumption to ignore the higher-order terms gives:

$$\delta\mathbf{R}_W^B \simeq \begin{bmatrix} C_{\delta\theta}C_{\delta\psi} & S_{\delta\phi}S_{\delta\theta}C_{\delta\psi} - C_{\delta\phi}S_{\delta\psi} & C_{\delta\phi}S_{\delta\theta}C_{\delta\psi} + S_{\delta\phi}S_{\delta\psi} \\ C_{\delta\theta}S_{\delta\psi} & S_{\delta\phi}S_{\delta\theta}S_{\delta\psi} + C_{\delta\phi}C_{\delta\psi} & C_{\delta\phi}S_{\delta\theta}S_{\delta\psi} - S_{\delta\phi}C_{\delta\psi} \\ -S_{\delta\theta} & S_{\delta\phi}C_{\delta\theta} & C_{\delta\phi}C_{\delta\theta} \end{bmatrix}$$
$$\simeq \begin{bmatrix} 1 & -\delta\psi & \delta\theta \\ \delta\psi & 1 & -\delta\phi \\ -\delta\theta & \delta\phi & 1 \end{bmatrix} = \mathbf{I}_{3\times 3} + \begin{bmatrix} 0 & -\delta\psi & \delta\theta \\ \delta\psi & 0 & -\delta\phi \\ -\delta\theta & \delta\phi & 0 \end{bmatrix}. \tag{2.19}$$

If we denote the small angle rotations $\delta\phi, \delta\theta, \delta\psi$, through the rotation speed vector $\boldsymbol{\Omega} \cdot \partial t = \left[\frac{\partial\psi}{\partial t}\ \frac{\partial\theta}{\partial t}\ \frac{\partial\phi}{\partial t}\right]^T \cdot \partial t$, we can write the skew-symmetric matrix using its more familiar vector representation of a rotation speed cross-product:

$$\partial t \begin{bmatrix} 0 & -\frac{\partial\psi}{\partial t} & \frac{\partial\theta}{\partial t} \\ \frac{\partial\psi}{\partial t} & 0 & -\frac{\partial\phi}{\partial t} \\ -\frac{\partial\theta}{\partial t} & \frac{\partial\phi}{\partial t} & 0 \end{bmatrix} = \partial t \begin{bmatrix} \frac{\partial\psi}{\partial t} \\ \frac{\partial\theta}{\partial t} \\ \frac{\partial\phi}{\partial t} \end{bmatrix} \times = \partial t \boldsymbol{\Omega} \times \tag{2.20}$$

This aforementioned rotation transformation can be applied to any vector in the moving/rotating body frame. For instance, let us imagine a person standing inside a helicopter that is rotating with angular speed Ω shown in Fig. 2.9. The vector connecting this person to the body frame of the helicopter is now $\mathbf{p}_B^P$. If the person remains still w.r.t. the body frame of the helicopter, then its position at a certain time instance t in the world frame is simply:

$$\mathbf{p}_W^P(t) = \mathbf{R}_W^B(\Psi, \Theta, \Phi)\mathbf{p}_B^P + \mathbf{p}_W^B. \tag{2.21}$$

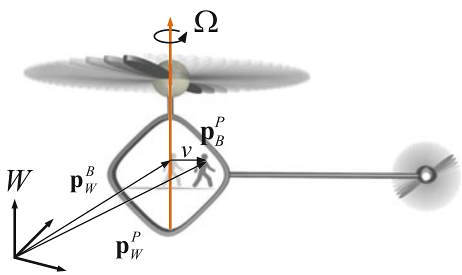

Fig. 2.9 Figure showing a person, moving with linear speed $\mathbf{v}$ inside a helicopter, rotating with angular speed $\boldsymbol{\Omega}$. The distance between the person and the body frame of the helicopter $\mathbf{p}_B^P$ changes in time, and this change is a linear combination of its angular rotation and linear speed

After infinitesimally small period of time, we can take a second reading only to find that the person is now repositioned at a different location in the world frame:

$$\begin{aligned}\mathbf{p}_W^P(t+\partial t) &= \partial\mathbf{R}_W^B(\partial\psi,\partial\theta,\partial\phi)\mathbf{R}_W^B(\psi,\theta,\phi)\mathbf{p}_B^P + \mathbf{p}_W^B \\ &= (\mathbf{I}_{3\times 3} + \partial t\boldsymbol{\Omega}\times)\,\mathbf{R}_W^B(\psi,\theta,\phi)\mathbf{p}_B^P + \mathbf{p}_W^B \end{aligned} \tag{2.22}$$

Comparing the two time instances, while letting the time interval reach infinitely small values, yields a derivation of the vector $\mathbf{p}_B^P$ in a world coordinate frame $\frac{\partial}{\partial t}^W$:

$$\frac{\partial}{\partial t}^W \left(\mathbf{p}_W^P\right) = \lim_{\partial t\to 0}\frac{\mathbf{p}_W^P(t+\partial t) - \mathbf{p}_W^P(t)}{\partial t} = \boldsymbol{\Omega}\times\mathbf{R}_W^B\,(\phi,\theta,\psi)\,\mathbf{p}_B^P \tag{2.23}$$

If we now imagine this person moving around the helicopter, the equations tend to become somewhat more complicated. If this person moves around, he or she changes the size of the vector $\left\|\mathbf{p}_B^P(t)\right\|$ to $\left\|\mathbf{p}_B^P(t+\partial t)\right\|$. It is important to note that for this example, the change in size of the vector occurs without the change of orientation in the body frame. Therefore, we can write:

$$\begin{aligned}\mathbf{p}_B^P(t+\partial t) &= \left\|\mathbf{p}_B^P(t+\partial t)\right\|\cdot\frac{\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} \\ &= \left(\left\|\mathbf{p}_B^P(t+\partial t)\right\| - \left\|\mathbf{p}_B^P(t)\right\|\right)\cdot\frac{\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} + \left\|\mathbf{p}_B^P(t)\right\|\frac{\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} \\ &= \underbrace{\left(\left\|\mathbf{p}_B^P(t+\partial t)\right\| - \left\|\mathbf{p}_B^P(t)\right\|\right)}_{\text{change in size contribution}}\cdot\frac{\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} + \mathbf{p}_B^P(t) \\ &= v\cdot\partial t\cdot\frac{\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} + \mathbf{p}_B^P(t). \end{aligned} \tag{2.24}$$

In previous equation, we used v to denote the linear speed of the factor, or the rate of change of its size. It is also important to note that this is a scalar, multiplied with the original direction of the vector $\frac{\mathbf{p}_B^P(t)}{\|\mathbf{p}_B^P(t)\|}$. Since we observe these changes in infinitesimally small time frames, it is safe to assume that both contributions, rotation and sizewise, can be viewed together as a linear combination of the two. In total, the new position of the person in the second time instance $\mathbf{p}_W^P(t+\partial t)$ w.r.t the world frame thus becomes

$$\begin{aligned}&\mathbf{p}_W^P(t+\partial t) = \partial\mathbf{R}_W^B(\partial\psi,\partial\theta,\partial\phi)\mathbf{R}_W^B(\psi,\theta,\phi)\mathbf{p}_B^P(t+\partial t) + \mathbf{p}_W^B \\ &= \left(\mathbf{I}_{3\times 3} + \partial t\boldsymbol{\Omega}\times\right)\mathbf{R}_W^B(\psi,\theta,\phi)\left(v\cdot\partial t\cdot\frac{\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} + \mathbf{p}_B^P(t)\right) + \mathbf{p}_W^B \\ &= v\cdot\partial t\cdot\frac{\mathbf{R}_W^B(\psi,\theta,\phi)\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} + \mathbf{R}_W^B(\psi,\theta,\phi)\mathbf{p}_B^P(t) + \partial t\boldsymbol{\Omega}\times\mathbf{R}_W^B(\psi,\theta,\phi)\mathbf{p}_B^P(t) + \mathbf{p}_W^B \end{aligned} \tag{2.25}$$

Repeating the steps applied to a stationary person moving inside the helicopter, we derive a time derivative of a vector in a moving frame observed in the inertial, world frame:

$$\frac{\partial}{\partial t}{}^{W}\left(\mathbf{p}_W^P\right) = \lim_{\partial t \to 0} \frac{\mathbf{p}_W^P(t+\partial t) - \mathbf{p}_W^P(t)}{\partial t}$$
$$= \boldsymbol{\Omega} \times \mathbf{R}_W^B(\phi, \theta, \psi)\,\mathbf{p}_B^P + v\frac{\mathbf{R}_W^B(\psi, \theta, \phi)\mathbf{p}_B^P(t)}{\left\|\mathbf{p}_B^P(t)\right\|} \tag{2.26}$$

only to find that as such, the derivation is a linear combination of its rotating and linear components. Careful reader should notice that multiplying the vector $\mathbf{p}_B^P(t)$ with $\mathbf{R}_W^B$ implies that it is transformed and expressed in the world frame at this point.

References

1. Bajd T, Mihelj M, Lenarčič J, Stanovnik A, Munih M (2010) Robotics. Intelligent systems, control and automation: science and engineering. Springer, Netherlands
2. Bottema O, Roth B (2012) Theoretical kinematics. Dover Books on Physics, Dover Publications, New York
3. Craig JJ (1986) Introduction to robotics: mechanics and control. Addison-Wesley, Reading (Mass.), Don Mills (Ont.), Wokingham, England
4. Denavit J, Hartenberg RS (1955) A kinematic notation for lower-pair mechanisms based on matrices. Trans ASME E J Appl Mech 22:215–221
5. Duffy J (1991) Kinematic geometry of mechanisms (k. h. hunt). SIAM Rev 33(4):678–679
6. Erdman AG, Sandor GN (1997) Mechanism design: analysis and synthesis. Number volume 1 in Prentice Hall international editions series. Prentice-Hall International
7. Lee D, Burg TC, Dawson DM, Shu D, Xian B, Tatlicioglu E (2009) Robust tracking control of an underactuated quadrotor aerial-robot based on a parametric uncertain model. In: 2009 IEEE international conference on systems, man and cybernetics, pp 3187–3192
8. Noble B, Daniel JW (1998) Applied linear algebra, 3rd edn. Prentice Hall, Upper Saddle River (Index included)
9. Schilling RJ (1990) Fundamentals of robotics: analysis and control. Prentice Hall, Englewood Cliffs
10. Shames IH (2006) Engineering mechanics statics and dynamics. Pearson Education
11. Siciliano B, Khatib O (2016) Springer handbook of robotics. Springer handbooks. Springer International Publishing
12. Siciliano B, Sciavicco L, Villani L, Oriolo G (2008) Robotics: modelling, planning and control, 1st edn. Springer Publishing Company, Incorporated
13. Symon KR (1971) Mechanics. Addison-Wesley world student series. Addison-Wesley Publishing Company

http://www.springer.com/978-3-319-61020-7

Aerial Manipulation
Orsag, M.; Korpela, C.; Oh, P.; Bogdan, S.
2018, XV, 235 p. 148 illus., 98 illus. in color. With online
files/update., Hardcover
ISBN: 978-3-319-61020-7