Chapter 2
Scalarization

2.1 General Idea

There were attempts to reduce multi-objective optimization problems to single-objective ones from the very beginning of their investigation [65, 99]. The reduction of a problem of multi-objective optimization to a single-objective optimization one normally is called scalarization. To find a discrete representation of the set of Pareto optimal solutions, a sequence of single-objective optimization problems should be solved, and they should hold the following theoretical properties:

- a solution of the single-objective optimization problem should provide an element of the set of Pareto optimal solutions,
- every element of the set of Pareto optimal solutions should be obtainable as a solution of a single-objective optimization problem constructed by the considered scalarization method.

In the case of nonlinear non-convex objective functions the single-objective problem, constructed by means of the scalarization, is non-convex as well, and therefore it can be solved exactly only in exceptional cases. Consequently, a found solution normally is not a true Pareto solution but a vector in some neighborhood of the Pareto set. To get a discrete representation (approximation) of the whole Pareto set, a sequence of auxiliary single-objective optimization problems should be solved. The quality of the approximation depends not only on the precision of a particular result of the single-objective optimization but also, and even stronger, on the sequence of parameters used in the construction of the auxiliary single-objective optimization problems. The methods of approximation of the whole Pareto set are discussed in the next chapter, while in this chapter we focus on the single-objective optimization problems which satisfy both theoretical requirements to the scalarization formulated above.

P.M. Pardalos et al., *Non-Convex Multi-Objective Optimization*, Springer Optimization and Its Applications 123, DOI 10.1007/978-3-319-61007-8_2

A classical scalarization method is the weighted sum method where the auxiliary single-objective function is defined by the formula

$$F(\mathbf{x}) = \sum_{i=1}^{m} w_i f_i(\mathbf{x}),\ w_i \geq 0,\ \sum_{i=1}^{m} w_i = 1, \tag{2.1}$$

where the weights w_i are parameters, and the selection of a vector $\mathbf{w} = (w_1, \ldots, w_m)^T$ gives the element of the Pareto front $\mathbf{f}(\mathbf{x}(\mathbf{w}))$

$$\mathbf{x}(\mathbf{w}) = \arg\min_{\mathbf{x}\in\mathbf{A}} F(\mathbf{x}). \tag{2.2}$$

The simplicity of the weighted sum scalarization is indeed an advantage of this method. However, its crucial disadvantage, in the case of non-convex objectives, is not guaranteed existence of $\mathbf{w}$ yielding an arbitrary element of the Pareto set. The latter fact is explained and illustrated in many textbooks, e.g., [135]. In the subsequent brief review of the scalarization methods, only the methods suitable for the computation of every Pareto optimal solution of non-convex problems are discussed.

An equivalent definition of proper efficiency was proposed in [234]. With the new definition of properness, a multi-objective problem is transformed to a more convenient one. Under determined conditions the original and the transformed problems have the same Pareto and properly efficient solutions. Therefore, this transformation may be employed for convexification and simplification in order to improve the computational efficiency of the solution of a given problem. Some existing results about the weighted sum method in the multi-objective optimization literature are generalized using the special case of the transformation scheme [234].

2.2 Tchebycheff Method

Different version of parametric scalar problems were proposed involving a reference point besides of weights of objectives. The weighted Tchebycheff method is one of the well-known methods of such a type, see, e.g., [42, 135]. Let $\mathbf{u} \in \mathbb{R}^m$ be a utopian point, i.e., $u_i < \min_{\mathbf{x}\in\mathbf{A}} f_i(\mathbf{x})$, and $\mathbf{w}$ be a vector of weights. The weighted Tchebycheff problem is of the form

$$\min_{x\in\mathbf{A}} \max_{i=1,\ldots,m} w_i\,(f_i(\mathbf{x}) - u_i). \tag{2.3}$$

Any Pareto optimal solution can be found by solving the weighted Tchebycheff problem with appropriate parameters, and with any parameters its solution corresponds to a weakly Pareto optimal solution of the original problem. Let us emphasize that the stated property of the weighted Tchebycheff problem holds for the non-convex problems.

The objective function of minimization problem (2.3) is nondifferentiable even in the case of smooth objectives of the original multi-objective problem. It is well known that the numerical solution of nondifferentiable problems is more complicated than the solution of smooth ones. However, the problem (2.3) with smooth $f_i(\cdot)$ can be reduced to the following equivalent differentiable form:

$$\min_{t \geq 0} t, \tag{2.4}$$
$$w_i\,(f_i(\mathbf{x}) - u_i) \leq t,\ i = 1, \ldots, m$$
$$\mathbf{x} \in \mathbf{A}.$$

The disadvantage of the Tchebycheff method is the presence of weakly Pareto optimal solutions among the found ones. The other disadvantage is the difficulty to solve the single-objective optimization problem (2.3) in case of non-convex objective functions. The latter problem, however, is inherent to the non-convex problems, whereas the weakly Pareto optimal solutions can be avoided by an adjustment of the formulation of the single-objective problem. For example, in [203, 204] it is suggested to augment the objective function in (2.3) by a small term proportional to the Manhattan distance between the decision and the utopian points

$$\min_{\mathbf{x} \in \mathbf{A}} \left(\left[\max_{i=1,\ldots,m} w_i\,(f_i(\mathbf{x}) - u_i) \right] + \rho \sum_{i=1}^{m} (f_i(\mathbf{x}) - u_i) \right), \tag{2.5}$$

where ρ is a small positive scalar.

A similar modification of the weighted Tchebycheff problem is proposed in [97]

$$\min_{\mathbf{x} \in \mathbf{A}} \left(\max_{i=1,\ldots,m} w_i \left[(f_i(\mathbf{x}) - u_i) + \rho \sum_{i=1}^{m} (f_i(\mathbf{x}) - u_i) \right] \right). \tag{2.6}$$

For both modified weighted Tchebycheff problems (2.5) and (2.6) it is true that they generate only strongly Pareto optimal solutions, and any such solution can be found by selecting the appropriate parameters [135]. The metrics used in (2.5) and (2.6) are indeed very similar and differ in the slightly different slopes of the hyper-planes which form the sphere in the considered metrics. Such a sphere can be imagined as a merger of spheres in L_1 and L_∞ metrics; we refer to [97] for details and further references.

The weakly Pareto optimal solutions can be excluded in the Tchebycheff metric-based approach also by the application of the lexicographic minimization of two objectives where the second objective is L_1 distance between the utopian and searched solutions:

$$\operatorname{lex}\min_{\mathbf{x} \in \mathbf{A}} \left[\left(\max_{i=1,\ldots,m} w_i\,(f_i(\mathbf{x}) - u_i) \right),\ \sum_{i=1}^{m} (f_i(\mathbf{x}) - u_i) \right]. \tag{2.7}$$

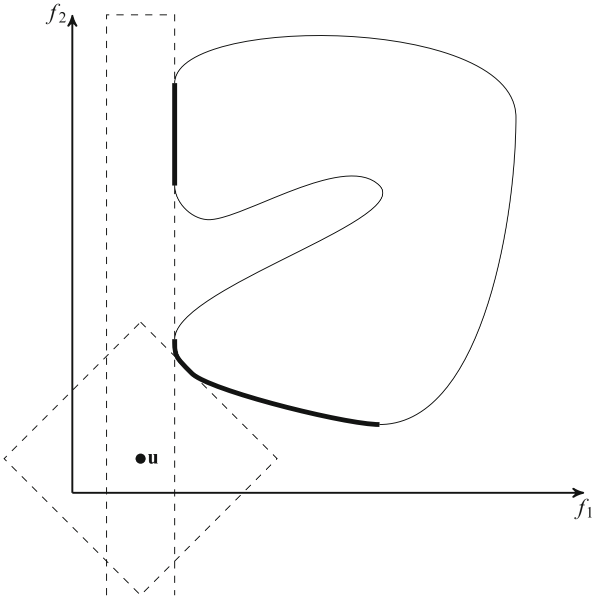

Fig. 2.1 Illustration of the lexicographic weighted Tchebycheff method

For illustration we refer to Figure 2.1.

As shown in [204] the solution of the lexicographic weighted Tchebycheff problem (2.7) is Pareto optimal. On the other hand, for a Pareto optimal decision $\tilde{\mathbf{x}}$ there exists a vector of weights $\mathbf{w}$, $w_i > 0, i = 1, \ldots, m$, $\sum_{i=1}^{m} w_i = 1$ such that $\tilde{\mathbf{x}}$ is a unique optimal decision for the problem (2.7).

The most important advantage of the Tchebycheff metric-based approach is the very weak assumptions ensuring the potential possibility to find every Pareto optimal solution. Various versions of the Tchebycheff methods are widely used for solving practical problems.

2.3 Achievement Scalarization Function

A method which is formally similar to the weighted Tchebycheff method is defined by the formula (2.3) with the only difference, namely instead of the utopian vector $\mathbf{u}$ any reference point $\mathbf{z}$ can be used

$$\min_{x \in \mathbf{A}} \max_{i=1,\ldots,m} w_i \, (f_i(\mathbf{x}) - z_i). \tag{2.8}$$

The solution $\mathbf{f}(\mathbf{x}_{\mathbf{w},\mathbf{z}})$, where $\mathbf{x}_{\mathbf{w},\mathbf{z}}$ is the minimizer in (2.8), is a weakly Pareto optimal solution independently of the feasibility or infeasibility of $\mathbf{z}$. The formula (2.8) defines a special case of the Achievement Scalarizing Function method proposed in [228].

2.4 *k*th-Objective Weighted-Constraint Problem

A scalarization method proposed in [26] is applicable not only to the problems with a disconnected Pareto front but also to the problems with a disconnected feasible set under the mild assumptions that the objective functions are continuous and bounded from below with a known lower bound; the latter assumption is reformulated as $f_i(\mathbf{x}) > 0$. This scalarization technique is named by its authors the "*k*th-objective weighted-constraint problem," since for each fixed k, the kth-objective is minimized, while the other weighted objective functions are incorporated as constraints:

$$\min_{\mathbf{x}\in\mathbf{A}} w_k f_k(\mathbf{x}), \tag{2.9}$$
$$w_i f_i(\mathbf{x}) \leq w_k f_k(\mathbf{x}),\ i = 1, \ldots, m,\ i \neq k,$$
$$w_i > 0,\ \sum_{i=1}^{m} w_i = 1.$$

As shown in [26], $\tilde{\mathbf{x}}$ is a weakly Pareto optimal decision of the original multi-objective optimization problem if and only if there exists some $\mathbf{w}$ such that $\tilde{\mathbf{x}}$ is an optimal decision of (2.9) for all $k = 1, \ldots, m$.

Despite different formulas in the problem statement (2.3) and (2.9), the weakly Pareto optimal solution defined by the m times repeated solution of (2.9) with different k can be obtained by the solution of (2.3) with the same weights. Since the objective functions in (2.9) are assumed positive, the utopian vector in (2.3) is assumed equal to zero.

Let $\tilde{\mathbf{x}}$ be an optimal decision for m problems (2.9) with different k. Then the following inclusion is true

$$\tilde{\mathbf{x}} \in \bigcap_{i=1}^{m} \mathbf{A_i} \tag{2.10}$$
$$\mathbf{A_i} = \{\mathbf{x} :\ \mathbf{x} \in \mathbf{A},\ w_i f_i(\mathbf{x}) \geq w_j f_j(\mathbf{x}),\ j = 1, \ldots, m,\ j \neq i\}.$$

Because of the equality $w_i f_i(\tilde{\mathbf{x}}) = w_j f_j(\tilde{\mathbf{x}})$, $j \neq i$, which follows from (2.10), and since

$$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathbf{A_i}} f_i(\mathbf{x}),\ i = 1, \ldots, m, \tag{2.11}$$

$\tilde{\mathbf{x}}$ is a minimizer of the minimization problem

$$\min_{\mathbf{x}\in\mathbf{A}} \max_{1\leq i\leq m} w_i f_i(\mathbf{x}), \tag{2.12}$$

which for $\mathbf{f}(\mathbf{x}) > \mathbf{0}$, $\mathbf{u} = \mathbf{0}$, is coincident with the optimal decision for (2.3).

2.5 Pascoletti and Serafini Scalarization

Pascoletti and Serafini scalarization [156] involves two vectors of parameters $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^m$, and is defined by the following minimization problem:

$$
\begin{aligned}
&\min_{t,\mathbf{x}} t, \\
&\mathbf{u} + t\mathbf{v} - \mathbf{f}(\mathbf{x}) \in \mathbb{R}^m_+, \\
&t \in \mathbb{R},\ \mathbf{x} \in \mathbf{A},
\end{aligned}
\tag{2.13}
$$

where $\mathbb{R}^m_+$ denotes the non-negative hyper-orthant. The formulation (2.13) is a special case of more general definition in [50, 51] where a nonempty closed pointed convex cone is used instead of $\mathbb{R}^m_+$. Assuming that the functions $f_i(\cdot)$ are continuously differentiable, and the feasible region $\mathbf{A}$ is compact, it can be proved that a weakly Pareto optimal solution of the multi-objective minimization problem

$$
\min_{\mathbf{x}\in\mathbf{A}} \mathbf{f}(\mathbf{x}), \tag{2.14}
$$

can be reduced to the solution of (2.13), i.e., the following statements are true:

- If $\tilde{\mathbf{x}}$ is a Pareto optimal decision vector of (2.14), then $(0, \tilde{\mathbf{x}})^T$ is a global minimizer of (2.13).
- Let $(\tilde{t}, \tilde{\mathbf{x}})^T$ be a global minimizer of (2.13), then $\tilde{\mathbf{x}}$ is a weakly Pareto optimal decision of (2.14).

http://www.springer.com/978-3-319-61005-4

Non-Convex Multi-Objective Optimization
Pardalos, P.; Žilinskas, A.; Žilinskas, J.
2017, XII, 192 p. 18 illus., 4 illus. in color., Hardcover
ISBN: 978-3-319-61005-4