Chapter 2
Continuous Observability of the Heat Equation Under a Single Mobile Point Sensor

2.1 Introduction

Let us consider the following Dirichlet problem:

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \tag{2.1}$$

$$t \in (0,\theta) \stackrel{\Delta}{=} T, \quad x \in \Omega \subset R^n, \quad Q = \Omega \times (0,\theta)\,, \;\; \Sigma = \partial\Omega \times (0,\theta),$$

$$u(x,t)|_\Sigma = 0, \qquad u(\cdot,0) = u_0(\cdot) \in L^2(\Omega).$$

We will assume that Ω is a bounded domain and that its boundary $\partial\Omega$ is of the class C^{2s_0+1}, where $2s_0 + 1 \geq [n/2] + 3$.

Next, we assume that the observation equation is given by (1.24):

$$y(t) = u(s(t),t), \quad t \in (\varepsilon,\theta) \stackrel{\Delta}{=} T_\varepsilon, \tag{2.2}$$

where ε is any number in $(0,\theta)$ and $s(t)$, $t \in T_\varepsilon$ describes the location of the sensor in Ω at time t. As we discussed it in Sect. 1.3 of Chap. 1, this type of observation requires a corresponding regularity of the solutions of (2.1), which is provided by the above assumption on $\partial\Omega$ (see Sect. 2.2 for details).

Methodology Our goal in this chapter is to investigate both the observability and $L^\infty(T_\varepsilon)$**-continuous observability** of system (2.1), (2.2). In the former we need to establish a 1–1 correspondence between solutions to (2.1) and their outputs (2.2). In the latter we need to prove the following estimate for solutions to (2.1):

$$\exists \gamma > 0 \;\text{ such that } \;\; \| \, u(s(\cdot),\cdot) \, \|_{L^\infty(\varepsilon,\theta)} \geq \; \gamma \, \| \, u(\cdot,\theta) \, \|_{L^2(\Omega)} \,. \tag{2.3}$$

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To achieve this goal, we will discuss, along the ideas of [39],[1] a rather general method to construct suitable trajectories for sensors that are able to ensure (2.3). This method is constructive and makes use of the representation of the solutions to (2.1) in the form of Fourier series expansions along the eigenfunctions as well as the separability of $C(\bar{\Omega})$. To some extent, it can be treated as an analogue of Galerkin's method coupled with the a priori estimate techniques when being applied in the framework of the observability theory.

The proposed methodology uses *a priori estimates of instantaneous type* for solutions of the system at hand. In particular, in this chapter we use the maximum principle for the heat equation. The latter allows us to extend the results of this chapter to a number of systems (including time-varying), admitting similar a priori estimates, when, instead of the eigenfunctions, an arbitrary appropriate basis—as in the Galerkin scheme—can be used.

We state the proposed method in the form of an abstract algorithm. Each iteration of the algorithm can be associated with some part of the observation time interval and provides $L^\infty(T_\varepsilon)$-continuous observability at time θ in the corresponding finite-dimensional subspace of $L^2(\Omega)$, spanned by the eigenfunctions of (2.1). Note that we deal here with the case when the space for outputs ($L^\infty(T_\varepsilon)$) is not Hilbert. Therefore, we need to construct a countable net (which can be specified in infinitely many ways) in the set of all possible pairs, namely: {solutions, and their respective observation outputs}. Then we establish the relationship between this net and appropriate countable sets of pairs $\{x^k, t_k\}_{k=1}^\infty$ that form the "skeletons" for required observation trajectories.

The main *existence* result of this chapter can be formulated as follows.

Theorem 2.1 *Given any* $\varepsilon \in (0, \theta), \theta > 0$*, the system (2.1)–(2.2) is* $L^\infty(\varepsilon, \theta)$*-continuously observable on* $(0, \theta)$ *for any curve*

$$s(\cdot) \stackrel{\Delta}{=} \hat{x}(\cdot),$$

constructed along Procedure A described in Sect. 2.3.

The remainder of this chapter is organized as follows. Section 2.2 deals with some preliminary results. In Sect. 2.3 we introduce a scheme for the construction of *continuous* observation curves that are able to ensure required observability at a final time in any finite-dimensional subspace (specified in advance), spanned by the eigenfunctions of (2.1). Procedure A then describes the general algorithm for the construction of observation curves. Theorem 2.1 is proven in Sect. 2.4. In the same section we discuss a number of important corollaries and consider the specific case of the one-dimensional heat equation. Section 2.6 deals with an application of the method to solve the discrete-time observability problem with a scanning sensor. For stationary sensors and an infinite time horizon a similar problem was studied

[1] This method was originally announced in a paper presented by the author at the IV International Conference on Control of Distributed Parameter Systems, held at Vorau, Austria, 1988.

in [31], where the authors made use of the results of Sakawa [92]. Section 2.7 deals with approximate controllability results related by duality with the main observability results of the chapter. We restrict ourselves here to the case when $n \leq 3$ and establish approximate controllability of the heat equation with a scanning point actuator in $H^{-1}(\Omega)$.

2.2 Auxiliary Results

Let us remind the reader that the general formula for the solutions to the mixed boundary problem (2.1) admits the following representation:

$$u(x,t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} < u_0(\cdot),\ \omega_i(\cdot) > \omega_i(x). \tag{2.4}$$

Here

$$< u_0(\cdot),\ \omega_i(\cdot) > = \int_{\Omega} u_0(x)\omega_i(x)dx$$

and $\{\lambda_i, \omega_i\}_{i=1}^{\infty}$ are the eigenelements solving the respective spectral problem, namely:

$$\Delta\omega_i(x) = -\lambda_i\omega_i(x), \quad \omega_i|_{\partial\Omega} = 0, \quad \|w_i\|_{L_2(\Omega)} = 1,$$
$$\lambda_{i+1} \geq \lambda_i > 0, \quad \lambda_i \to +\infty, \quad i \to +\infty.$$

It can be shown (e.g., [85]) that, if $\partial\Omega \in C^{2s}$ and $u_0(\cdot) \in H_D^{2s-1}(\Omega)$, then the generalized solution of (2.1) belongs to $H^{2s,s}(Q)$ and the following estimate is verified

$$\| u(\cdot,\cdot) \|_{H^{2s,s}(Q)} \leq \quad \text{const} \ \| u_0(\cdot) \|_{H^{2s-1}(\Omega)}, \tag{2.5}$$

where

$$H_D^r(\Omega) = \{\phi \mid \ \phi \in H^r(\Omega),\ \phi\ |_{\partial\Omega} = \ldots \ = \ \Delta^{[(r-1)/2]}\phi\ |_{\partial\Omega} = 0\}.$$

Furthermore, since the norm

$$\{v, v\}^{1/2} \ = \ (\sum_{k=1}^{\infty} \lambda_k^r v_k^2)^{1/2}, \quad v_k = \int_{\Omega} \ v(x)\ \omega_k(x)\ dx,$$

is equivalent to the standard one in $H_D^r(\Omega)$, due to the smoothing effect, this means that all the solutions of (2.1) are classical on $\bar{\Omega} \times [\varepsilon, \theta]$ (see, e.g., [85]).

It is well known [29, 85] that any solution to system (2.1) satisfies the *maximum principle,* namely:

$$\max_{x\in\bar{\Omega}} \mid u(x,t') \mid \geq \max_{x\in\bar{\Omega}} \mid u(x,t'') \mid . \quad 0 < t' \leq t'', \tag{2.6}$$

This principle is instrumental for the method presented in the next section.

2.3 $C[\varepsilon,\theta]$-Continuous Observability in Finite-Dimensional Subspaces

Let $L^2_{(k)}(\Omega)$ denote the finite-dimensional subspace of $L^2(\Omega)$ spanned by $\omega_i(\cdot), \quad i = 1, 2, \ldots, k$. Note that

$$u(\cdot,t) \in L^2_{(k)}(\Omega) \;\; \forall t \geq 0 \quad \text{if and only if} \quad u(x,0) \in L^2_{(k)}(\Omega). \tag{2.7}$$

In this section we intend to discuss the problem of continuous observability for system (2.1) in any of $L^2_{(k)}(\Omega), k = 1, \ldots$. We will show that we can find *continuous* observation curves capable of delivering this property in each case.

Let $\{x^k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ be any two sequences, respectively, of points in $\bar{\Omega}$ and of instants of time in T. We say that an observation curve $s(\cdot)$ has a *skeleton* $\{x^k, t_k\}_{k=1}^{K}$, where K can also be infinity, if it is continuous at all the instants $\{t_k\}_{k=1}^{K}$, and satisfies the following condition

$$\hat{x}(\cdot) = x^k, \quad k = 1, \ldots .$$

Lemma 2.1 (Continuous Observability in Finite-Dimensional Subspaces) *Select any* $\varepsilon \in (0,\theta), \theta > 0$ *and a positive integer* k*. There exist continuous observation curves for which the system (2.1), (2.2), (2.7) becomes* $C[\varepsilon,\theta]$*-continuously observable.*

Proof of Lemma 2.1 In order to specify a required curve

$$s(\cdot) \overset{\Delta}{=} \hat{x}(\cdot),$$

it suffices to determine an appropriate *finite* skeleton.

Denote by $Y_{\varepsilon k}$ the set of all the possible outputs (2.2), generated by the initial conditions from $L^2_{(k)}(\Omega)$ and taken on T_ε. Recall that (2.1), (2.2) is $C[\varepsilon,\theta]$-continuously observable in $L^2_{(k)}(\Omega)$ if and only if the mapping

$$\mathbf{P}: \; C[\varepsilon,\theta] \supset Y_{\varepsilon k} \;\to\; L^2_{(k)}(\Omega), \quad \mathbf{P}u(\hat{x}(\cdot),\cdot) = u(\cdot,\theta)$$

can be well defined and is bounded as follows:

$$\| \mathbf{P} \| = \sup\{\| \mathbf{P}u(\hat{x}(\cdot),\cdot) \| \mid u(\hat{x}(\cdot),\cdot) \in Y_{\varepsilon k}, \; \| u(\hat{x}(\cdot),\cdot) \|_{C([\varepsilon,\theta]} \leq 1\} \; < \; \infty.$$

Thus, we need to show that the pre-image of the set

$$\{u(\hat{x}(\cdot), \cdot) \mid u(\hat{x}(\cdot), \cdot) \in Y_{\varepsilon k}, \quad \| u(\hat{x}(\cdot), \cdot) \|_{C[\varepsilon,\theta]} \leq 1\}$$

for the mapping $u(\cdot, \theta) \rightarrow u(\hat{x}(\cdot), \cdot)$ is bounded in $L^2(\Omega)$.

Select any *monotone* sequence

$$\varepsilon = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots < \theta$$

and denote $\tau_k = (t_{k-1}, t_k), \quad k = 1, \ldots$.

Step 1: $k = 1$ Let $u(x, t)$ be generated by $u(\cdot, 0) \in L^2_{(1)}(\Omega)$. Then, due to (2.4),

$$u(x, t) = e^{-\lambda_1 t} u_{01} \omega_1(x), \quad u_{01} = \int_{\Omega} u(x, 0)\, \omega_1(x)\, dx.$$

Let $x^1_{(1)}$ be any solution of the following optimization problem:

$$| \omega_1(x) | \rightarrow \max, \qquad x \in \bar{\Omega},$$

that is,

$$| \omega_1(x^1_{(1)}) | = \max_{x \in \bar{\Omega}} | \omega_1(x) | . \tag{2.8}$$

Select any t^1_1 from $\tau_1 = (t_0, t_1)$ and consider an arbitrary continuous curve $\hat{x}(t), \ t \in T$ such that $s(t^1_1) = x^1_{(1)}$. Then, if $u(x, t)$ is such that

$$\| u(\hat{x}(\cdot), \cdot) \|_{C[\varepsilon,\theta]} = \max\{| u(\hat{x}(t), t) | \mid t \in [\varepsilon, \theta]\} \leq 1,$$

we have the following estimate:

$$| u_{01} | \leq e^{+\lambda_1 t^1_1} | \omega_1^{-1}(x^1_{(1)}) | . \tag{2.9}$$

Note that $| \omega_1^{-1}(x^1_{(1)}) | \neq 0$, because otherwise, due to (2.8), $\omega_1(x) \equiv 0$.

Combining (2.4) and (2.9) yield that

$$\| u(\cdot, \theta) \|_{L^2(\Omega)} \leq e^{+\lambda_1 (t^1_1 - \theta)} (\text{meas } \{\Omega\})^{1/2},$$

since

$$1 = \| \omega_1(\cdot) \|^2_{L^2(\Omega)} \leq \max \{| \omega_1(x) |^2 \mid x \in \bar{\Omega}\} \text{ meas } \{\Omega\}.$$

This gives us the conclusion of Lemma 2.1 for system (2.1), (2.2) in the subspace $L^2_{(1)}(\Omega)$ with $\gamma = e^{-\lambda_1 (t^1_1 - \theta)} (\text{meas } \{\Omega\})^{-1/2}$.

Step 2: The General Case Let

$$\Phi_k = \{v(\cdot) \mid \ \| v(\cdot) \|_{L^2(\Omega)} = 1, \quad v(\cdot) \in L^2_{(k)}(\Omega)\}.$$

Take any $\delta > 0$. Since Φ_k is a bounded finite-dimensional subset of $C(\bar{\Omega})$, we can find a finite δ-net for this set in this space. Denote it by Φ_k^δ,

$$\Phi_k^\delta = \{v_k^j(\cdot)\}_{j=1}^{J_k}, \quad v_k^j(\cdot) \in \Phi_k,$$

where J_k depends on δ. In other words, for any element $v(\cdot) \in \Phi_k$ there exists an integer $j = j_* \le J_k$ such that

$$\| v(\cdot) - v_k^{j_*}(\cdot) \|_{C(\bar{\Omega})} \le \delta.$$

The maximum principle (2.6) (applied for the set Φ_k) allows one to transform Φ_k^δ (due to finite dimension of Φ_k, (2.6) can be extended for $[0, \theta]$) into a δ-net $\Phi_k^\delta(\cdot)$ in the space $C(\bar{\Omega} \times [0, \theta])$,

$$\Phi_k^\delta(\cdot) = \{u_k^j(x, t), \ \ u_k^j(x, 0) = v_k^j(x)\}_{j=1}^{J_k}, \quad x \in \bar{\Omega}, \ \ t \in [0, \theta]$$

for the set of solutions $u(x, t)$ that are generated at instant $t = 0$ by initial conditions from Φ_k.

Next, consider any β in the interval $(0, 1)$ and select

$$\delta = \delta_k(\beta) \ = \ \beta \, \frac{1}{2} (\text{meas}\{\Omega\})^{-1/2} e^{-\lambda_k t_k}, \tag{2.10}$$

denoting accordingly $J_k = J_k(\beta)$.

Select any *monotone* sequence t_k^j , $j = 1, 2, \ldots J_k(\beta)$,

$$t_{k-1} < t_k^1 < t_k^2 < \ \ldots \ < t_k^{J_k(\beta)} < t_k$$

and introduce the following series of optimization problems for $j = 1, \ldots, J_k(\beta)$.

Problem (k, j): find $x_{(k)}^j$ satisfying

$$\max_{x \in \bar{\Omega}} \mid u_k^j(x, t_k^j) \mid \ = \ \mid u_k^j(x_{(k)}^j, t_k^j) \mid . \tag{2.11}$$

Note that Problems (k, j) can have multiple solutions. We can pick any of them.

Let $\hat{x}(\cdot)$ be an arbitrary continuous curve in $\bar{\Omega}$ with the skeleton $\{x_{(k)}^j, t_k^j\}_{j=1}^{J_k(\beta)}$. We show that it satisfies the requirements of Lemma 2.1.

Take any solution u of (2.1) such that $u(\cdot, t) \in L^2_{(k)}(\Omega), \ \forall t \in T$,

$$u(x, t) = \sum_{i=1}^{k} e^{-\lambda_i t} < u(\cdot, 0), \ \omega_i(\cdot) > \omega_i(x) \tag{2.12}$$

and assume that

$$\| u(\hat{x}(\cdot), \cdot) \|_{C[\varepsilon,\theta]} = \max\{| u(\hat{x}(t), t) | \mid t \in [\varepsilon, \theta]\} \leq 1. \tag{2.13}$$

This yields, in particular,

$$| u(\hat{x}(t_k^j), t_k^j) | \leq 1, \quad j = 1, \ldots, J_k(\beta). \tag{2.14}$$

Denote

$$\alpha = \| u(\cdot, 0) \|_{L^2(\Omega)} . \tag{2.15}$$

Without loss of generality we can assume that $\alpha \neq 0$.

Select an element $u_k^{j*}(\cdot, \cdot) \in \Phi_k^{\delta_k(\beta)}(\cdot)$ such that

$$| \alpha^{-1} u(x, t) - u_k^{j*}(x, t) | \leq \delta_k(\beta) \quad \text{for all } x \in \bar{\Omega} , \ t \in [0, \theta]. \tag{2.16}$$

In particular,

$$\| u(\cdot, t_k^{j*}) \|_{C(\bar{\Omega})} \leq \alpha \| u_k^{j*}(\cdot, t_k^{j*}) \|_{C(\bar{\Omega})} + \alpha \delta_k(\beta). \tag{2.17}$$

Next, making use of (2.11) and again (2.16), we obtain

$$\alpha \| u_k^{j*}(\cdot, t_k^{j*}) \|_{C(\bar{\Omega})} = | \alpha u_k^{j*}(x_{(k)}^{j*}, t_k^{j*}) | \leq | u(\hat{x}(t_k^{j*}), t_k^{j*}) | + \alpha \delta_k(\beta). \tag{2.18}$$

Combining (2.13), (2.17)–(2.18) leads us to

$$\| u(\cdot, t_k^{j*}) \|_{C(\bar{\Omega})} \leq 1 + 2\alpha \delta_k(\beta). \tag{2.19}$$

Note that, due to (2.12) and (2.15),

$$\begin{aligned} \alpha^2 e^{-2\lambda_k t_k^{j*}} &= e^{-2\lambda_k t_k^{j*}} \sum_{i=1}^{k} < u(\cdot, 0), \omega_i(\cdot) >^2 \leq \sum_{i=1}^{k} e^{-2\lambda_i t_k^{j*}} < u(\cdot, 0), \omega_i(\cdot) >^2 = \\ &= \int_{\Omega} u^2(x, t_k^{j*}) dx \leq \text{meas}\,\{\Omega\} \| u(\cdot, t_k^{j*}) \|_{C(\bar{\Omega})}^2 . \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), we derive that

$$\begin{aligned} &(\text{meas}\,\{\Omega\})^{-1/2} \Big(\sum_{i=1}^{k} e^{-2\lambda_i t_k^{j*}} < u(\cdot, 0), \ \omega_i(\cdot) >^2\Big)^{1/2} \\ &\quad - 2\delta_k(\beta) \Big(\sum_{i=1}^{k} < u(\cdot, 0), \ \omega_i(\cdot) >^2\Big)^{1/2} \leq 1. \end{aligned} \tag{2.21}$$

In turn, due to the inequality

$$e^{-\lambda_k t_k}(\sum_{i=1}^{k} < u(\cdot,0),\ \omega_i(\cdot) >^2)^{1/2} \ \le\ (\sum_{i=1}^{k} e^{-2\lambda_i t_k^{j*}} < u(\cdot,0),\ \omega_i(\cdot) >^2)^{1/2}$$

and (2.21), (2.10), we have:

$$\| u(\cdot,\theta) \|_{L^2(\Omega)} \ \le\ \| u(\cdot,t_k) \|_{L^2(\Omega)} \le$$

$$\le\ (\sum_{i=1}^{k} e^{-2\lambda_i t_k^{j*}} < u(\cdot,0),\ \omega_i(\cdot) >^2)^{1/2} \ \le\ (\text{meas}\ \{\Omega\})^{1/2}\frac{1}{1-\beta}. \tag{2.22}$$

The estimate (2.3), coupled with (2.13), implies the boundedness of $\mathbf{P}$ on $Y_{\varepsilon k} \subset C[\varepsilon,\theta]$. Respectively, this ensures the $C[\varepsilon,\theta]$-continuous observability of system (2.1), (2.2) in $L^2_{(k)}(\Omega)$ with

$$\gamma = (\text{meas}\ \{\Omega\})^{-1/2}(1-\beta) \ \le\ \| \mathbf{P} \|^{-1}, \tag{2.23}$$

so that

$$(\text{meas}\ \{\Omega\})^{-1/2}(1-\beta)\ \| u(\cdot,\theta) \|_{L^2(\Omega)} \le \max_{t\in[\varepsilon,\theta]} \mid u(\hat{x}(t),t) \mid .$$

This completes the proof of Lemma 2.1. ◇

We now describe an algorithm for constructing the observation curves that ensure the $C[\varepsilon,\theta]$-continuous (or exact) observability of (2.1), (2.2), (2.7), that is, in any $L^2_{(k)}(\Omega)$.

Procedure A: Constructions of Skeletons for Observation Curves Let $\varepsilon \in (0,\theta)$ and $\beta \in (0,1)$ be given.

1. Select an arbitrary monotone sequence $\{t_k\}_{k=1}^{\infty} \subset T_\varepsilon$. Denote

$$\lim_{k\to\infty} t_k = \hat{t} \le \theta.$$

2. Find $\delta_k = \delta_k(\beta), \quad k = 1,2\ldots$ according to (2.10).
3. Given positive integer k, find a $\delta_k(\beta)$-net $\Phi_k^{\delta_k(\beta)}(\cdot)$. This will give us the value of $J_k = J_k(\beta)$.
4. Selecting any monotone sequence $\{t_k^j\}_{j=1}^{J_k(\beta)}$ in $\tau_k = (t_{k-1},t_k)$, $t_0 = \varepsilon$ and, making use of (2.11), find a respective sequence of spatial points $\{x_{(k)}^j\}_{j=1}^{J_k(\beta)}$.
5. Repeat Steps 3 and 4 with $k+1$ instead of k.

Let $\varepsilon \in T$ and $\beta \in (0, 1)$ be given. We say that an observation curve $\hat{x}(t),\ t \in [0, \theta]$ is constructed along Procedure A if it has a skeleton

$$\{x^j_{(k)}, t^j_k\}, \quad k = 1, 2, \ldots, \quad j = 1, \ldots, J_k(\beta),$$

as in Procedure A and it is continuous in $[\varepsilon, \hat{t})$ (see Step 1 of Procedure A).

Lemma 2.1 yields the following assertion.

Corollary 2.1 (Observability Estimate) *Given* $\varepsilon \in (0, \theta), \theta > 0$ *and* $\beta \in (0, 1)$, *let* $\hat{x}(\cdot)$ *be any observation curve constructed along Procedure A. Then, for any* $k = 1, \ldots,$ *to ensure the estimate*

$$(\text{meas}\,\{\Omega\})^{-1/2}(1 - \beta) \parallel u(\cdot, \theta) \parallel_{L^2(\Omega)} \leq \parallel u(\hat{x}(\cdot), \cdot) \parallel_{L^\infty(T_\varepsilon)}$$

for the solutions to (2.1), (2.2), (2.7), it is sufficient to take into account the observations (2.2) over the time-interval $\tau_k = (t_{k-1}, t_k)$ *only.*

2.4 Proof of Theorem 2.1 and Some Corollaries

For any given $\varepsilon \in (0, \theta)$ and $\beta \in (0, 1)$ we intend to show that any curve $\hat{x}(\cdot)$ constructed along Procedure A satisfies the requirements of this theorem.

Let us recall that the estimate (2.3) is uniform over $k = 1, \ldots$. Denote by Y_ε the set of all the possible outputs (2.2) taken on T_ε. Observe that (2.1), (2.2) is $L^\infty(T_\varepsilon)$-continuously observable if and only if the mapping

$$\mathbf{P} :\ L^\infty(T_\varepsilon) \supset Y_\varepsilon \ \rightarrow\ L^2(\Omega), \quad \mathbf{P}u(\hat{x}(\cdot), \cdot) = u(\cdot, \theta)$$

exists and bounded (so that the domain of $\mathbf{P}$ may be extended to $\bar{Y}_\varepsilon$),

$$\parallel \mathbf{P} \parallel = \ \sup\{\parallel \mathbf{P}u(\hat{x}(\cdot), \cdot) \parallel \mid\ u(\hat{x}(\cdot), \cdot) \in Y_\varepsilon,\ \parallel u(\hat{x}(\cdot), \cdot) \parallel_{L^\infty(T_\varepsilon)} \leq 1\} \ < \ \infty.$$

To prove Theorem 2.1, we show that the pre-image of the set

$$\{u(\hat{x}(\cdot), \cdot) \mid u(\hat{x}(\cdot), \cdot) \in Y_\varepsilon,\ \parallel u(\hat{x}(\cdot), \cdot) \parallel_{L^\infty(T_\varepsilon)} \leq 1\}$$

for the mapping $u(\cdot, \theta) \ \rightarrow\ u(\hat{x}(\cdot), \cdot)$ is bounded in $L^2(\Omega)$.

Fix any positive μ. Let $u(x, t)$ be an arbitrary solution of the system (2.1) such that

$$\parallel u(\hat{x}(\cdot), \cdot) \parallel_{L^\infty(T_\varepsilon)} \leq\ 1. \tag{2.24}$$

Re-write the sum on the right in (2.4) as follows:

$$u(x,t) = u_N(x,t) + v_N(x,t),$$

where

$$u_N(x,t) = \sum_{i=1}^{N} e^{-\lambda_i t} < u(\cdot,0),\ \omega_i(\cdot) > \omega_i(x),$$

$$v_N(x,t) = \sum_{i=N+1}^{\infty} e^{-\lambda_i t} < u(\cdot,0),\ \omega_i(\cdot) > \omega_i(x),$$

$$\| \ v_N(\cdot,\theta) \ \|_{L^2(\Omega)} \leq \ \mu, \tag{2.25}$$

and

$$\| \ v_N(\cdot,\cdot) \ \|_{C(\bar{\Omega}\times[\varepsilon,\theta])} \leq \ \mu.$$

The latter and (2.24) imply that

$$| \ u_N(\hat{x}(t),t) \ | \leq \ 1+\mu \quad \forall t \in \tau_N,$$

where τ_N is defined in the step 4 of Procedure A. Applying estimate (2.3) with $k = N$ and with $1+\mu$ instead of 1 yields:

$$\| \ u_N(\cdot,\theta) \ \|_{L^2(\Omega)} \leq \ (\text{meas}\,\{\Omega\})^{1/2}\frac{1+\mu}{1-\beta}.$$

Finally, combining (2.25) and the last estimate, we arrive at

$$\| \ u(\cdot,\theta) \ \|_{L^2(\Omega)} \leq \ (\text{meas}\,\{\Omega\})^{1/2}\frac{1+\mu}{1-\beta} + \mu. \tag{2.26}$$

Thus, we have proved the boundedness of the mapping $\mathbf{P}$ defined on $Y_\varepsilon \subset L^\infty(T_\varepsilon)$. Recalling that $\mu > 0$ was selected arbitrarily, we obtain that

$$\| \ \mathbf{P} \ \| \leq \ (\text{meas}\,\{\Omega\})^{1/2}\frac{1+\mu}{1-\beta} + \mu, \quad \forall \mu > 0.$$

This allows us to conclude that the required inequality (2.3) holds with the same γ as in Lemma 2.1.

Corollary 2.2 *Given* $\varepsilon \in (0,\theta)$ *and* $\beta \in (0,1)$, *any observation curve* $\hat{x}(\cdot)$ *constructed along Procedure A makes the system (2.1), (2.2) be* $L^\infty(T_\varepsilon)$*-exactly observable with the constant* γ *from (2.23) in (2.3).*

Corollary 2.3 *Given* $\varepsilon \in (0, \theta)$ *and* $\beta \in (0, 1)$, *let* $\hat{x}(\cdot)$ *be constructed along Procedure A. Then*

$$\| u(\hat{x}(\cdot), \cdot) \|_{L^\infty(T_\varepsilon)} \geq \ (\text{meas } \{\Omega\})^{-1/2}(1-\beta) \ |< u(\cdot, 0), \omega_i(\cdot) >| \ e^{-\lambda_i \theta}, \quad i=1, \ldots \tag{2.27}$$

for any solution $u(x, t)$ *to the system (2.1).*

Proof of Corollary 2.3 follows from the representation (2.4) and the estimate (2.26) (or (2.3)).

Corollary 2.3 implies a "weaker" observability property in its turn.

Corollary 2.4 *Given* $\varepsilon \in (0, \theta)$ *and* $\beta \in (0, 1)$, *any curve* $\hat{x}(\cdot)$, *constructed along Procedure A, ensures the observability property of system (2.1), (2.2) on* (ε, θ).

Remark 2.1 (Countability) We stress that, to construct a curve according to Procedure A, one needs to deal with a countable number of pairs forming its skeleton.

Minimality in $L^\infty(T_\varepsilon)$ Consider a sequence of functions

$$\psi_i(t) = e^{-\lambda_i t}\omega_i(\hat{x}(t)) \ , \ t \in T, \ i = 1, 2, \ldots$$

and denote by $L^\infty_{(i)}(T_\varepsilon)$ the subspace of $L^\infty(T_\varepsilon)$ spanned by $\{P\psi_j(\cdot) \ , \ j = 1, \ldots ; j \neq i\}$. Let

$$d_i = \inf_{\psi(\cdot) \in L^\infty_{(i)}(T_\varepsilon)} \| \psi_i(\cdot) - \psi(\cdot) \|_{L^\infty(T_\varepsilon)}, \quad i = 1, 2, \ldots.$$

Estimate (2.27) implies the following result.

Corollary 2.5 (Minimality in $L^\infty(T_\varepsilon)$**)** *Let the function* $\hat{x}(\cdot)$ *be constructed as in Procedure A. Then*

$$d_i = \inf_{\psi(\cdot) \in L^\infty_{(i)}(T_\varepsilon)} \| \psi_i(\cdot) - \psi(\cdot) \|_{L^\infty(T_\varepsilon)} \geq \ (\text{meas } \Omega)^{-1/2}(1-\beta)e^{-\lambda_i \theta}, \ i = 1, 2, \ldots, \tag{2.28}$$

regardless of the spatial dimension of Ω.

Remark 2.2 (Minimality: Comparing to the Static Case) Estimates (2.28) may look unexpected when one compares them to the results of Theorem 7.1 from [81, 94] cited in Chap. 1 (see, in particular, (1.19) and (1.20)). However, in order to explain them, one can recall here that Corollary 2.3 was derived on the basis of the maximum principle for the heat equation, which is not affected by the multiplicity of eigenvalues and their growth. From this viewpoint, the techniques leading to (2.28) employ the eigenfunctions that are always distinct.

2.5 Explicit Observation Curves in 1-*D* Case

In this section we will discuss an example of geometrically explicit observation curves that can solve a continuous observability problem in a *robust* way.

Example 2.1 (Static Point Sensors) Consider first (for comparison) the following initial-boundary value problem:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t \in T = (0, \theta), \tag{2.29}$$

$$u(t,0) = u(t,1) = 0, \quad u(x,0) = u_0(x)$$

with a static point sensor located at $\bar{x}$:

$$y(t) = u(\bar{x}, t), \quad t \in T. \tag{2.30}$$

As we discussed in Chap. 1 the system (2.29), (2.30) is observable if and only if the point $\bar{x}$ is irrational. It can also be $L^2(0,\theta)$-continuously observable if $\bar{x}$ is an irrational number of special type (e.g., of *constant* type, see [10, 14, 17, 92]).

Example 2.2 (Mobile Point Sensors) Assume now that we have a mobile sensor that generates the following observation equation:

$$y(t) = u(\hat{x}(t), t), \quad t \in T. \tag{2.31}$$

and that the observation curves in (2.31) are of the form:

$$\hat{x}(t) = \begin{cases} 0 & \text{for } t \in (0, t_1), \\ x^*(t) & \text{for } t \in [t_1, t_2], \\ 0 & \text{for } t \in (t_2, \theta]. \end{cases} \tag{2.32}$$

Here, $t_1, t_2 \in (\varepsilon, \theta), t_1 < t_2, \varepsilon \in (0, \theta)$ and $x^*(t)$ is an arbitrary continuous function such that

$$x^*(t_1) = 0, \quad x^*(t_2) = 1. \tag{2.33}$$

Thus, the observation curve $\hat{x}(t)$ moves across the whole spatial domain $[0, 1]$ of system (2.29) during the time interval $[t_1, \ t_2]$.

The maximum principle, applied to the solutions of (2.29) in the domain D,

$$D = \{(x,t) \mid 0 \le x \le \hat{x}(t), \ \ t \in [t_1, t_2]\},$$

yields the estimate

$$\| u(\cdot, \theta) \|_{C[0,1]} \le \| u(\hat{x}(\cdot), \cdot) \|_{C[t_1,t_2]}, \tag{2.34}$$

which ensures the $C[\varepsilon, \theta]$-continuous observability at final time of system (2.29), (2.31).

Remark 2.3 (Robustness of Mobile Sensors in Example 2.2) An attractive property of mobile sensors in this example (in contrast to the static sensors in Example 2.1, which are ill-posed with respect to their positions; see also Chap. 1) is that the estimate (2.34) holds for any continuous curve $\hat{x}(\cdot)$ satisfying (2.32). This makes the $C[\varepsilon, \theta]$-continuous observability property of system (2.29), (2.31) *stable* with respect to the selection of the observation curves from this set.

2.6 Observability with Mobile Discrete-Time Point Sensors

Let us consider now the system (2.1) coupled with the discrete-time observations in place of (2.2):

$$y_i = u(x^i, t_i), \quad i = 1, 2, \ \ldots \ . \tag{2.35}$$

Here $\{y_i)\}_{i=1}^{\infty}$ are the observation data, taken, respectively, at points $\{x^i\}_{i=1}^{\infty} \subset \bar{\Omega}$ at instants $t_1 < t_2 < \ldots$. We assume that the outputs $\mathbf{y} = \{y_1, \ldots, y_i, \ldots\}$ lie in l^{∞}, equipped with the norm

$$\| \ \mathbf{y} \ \|_{l^{\infty}} = \sup_{i=1,\ldots} \ | \ y_i \ | \ .$$

Inequality (2.3) in the definition of l^{∞}-continuous observability, adjusted to system (2.1), (2.35), will take the following form:

$$\sup_{i=1,\ldots} \ | \ u(x^i, t_i) \ | \geq \ \gamma \ \| \ u(\cdot, \theta) \ \|_{L^2(\Omega)} \ . \tag{2.36}$$

Lemma 2.2 *Given $\varepsilon \in (0, \theta)$ and $\beta \in (0, 1)$, let the sequence $\{x^i, t_i\}_{i=1}^{\infty}$ in (2.35) be selected according to Procedure A. Then, system (2.1), (2.35) is l^{∞}-continuously observable at time $t = \theta$ and the estimate (2.36) holds with constant γ from (2.23).*

The proof of Lemma 2.2 immediately follows from the proof of Theorem 2.1.

The following assertion illustrates principal possibilities of the scheme developed in Sect. 2.3.

Theorem 2.2 *Let $\varepsilon \in (0, \theta), \theta >$ and $\beta \in (0, 1)$ be given. There exists a set of skeletons for (2.35) such that, for its every element, not only the skeleton itself, but also any its restriction to an arbitrary time-interval $(a, b) \subset T_{\varepsilon}$ regardless of the duration, makes the system (2.1), (2.35) be l^{∞}-continuously observable at final time $t = \theta$ with the same constant γ in (2.36).*

Proof Indeed, the scheme of proofs in Sects. 2.3 and 2.4 employ the observations taken only at a countable number of instants, namely, along the skeleton of the observation curve. In particular, these instants can be located in an arbitrary way in the interval T_{ε}.

Fix an arbitrary pair $\varepsilon \in T$ and $\beta \in (0,1)$. Let $\{t_i\}_{i=1}^{\infty}$ be an arbitrary set that is dense in T_ε and let $\{\delta_j\}_{j=1}^{\infty}$ be an arbitrary sequence of positive numbers such that

$$\lim_{j\to\infty} \delta_j = 0.$$

For each interval $(t_i - \delta_j, t_i) \bigcap T_\varepsilon$ select a sequence of pairs as described in Procedure A (with the same given ε and β):

$$\{x_{ij}^k, t_{ij}^k\}_{k=1}^{\infty}, \quad i, j = 1, \ldots. \tag{2.37}$$

The countability of the set of indices $i, j, k = 1, 2 \ \ldots$ allows us to select all the instants t_{ij}^k be all distinct. We can renumber the pairs in (2.37) in order to obtain the sequence of pairs, forming the skeleton required by Theorem 2.2.

2.7 Dual Approximate Controllability

We consider the following initial boundary value problem:

$$\frac{\partial z(x,t)}{\partial t} = \Delta z(x,t) + v(t)\delta(x - x^*(t)), \quad t \in T = (0,\theta), \quad x \in \Omega, \quad Q = \Omega \times T,$$
$$z(x,t)|_\Sigma = 0, \quad z(x,0) = 0, \tag{2.38}$$

where $v(\cdot) \in L^2(T)$ is control and $x^*(\cdot)$ is a given mobile control path, $x^*(t) \in \bar{\Omega}$ *a.e. in* T, and Ω satisfies the assumptions of Sect. 2.1. Let us also remind the reader (see Chap. 1) that system (2.38) is said to be approximately controllable in the Hilbert space H at time $t = \theta$ if the set of its attainable states $z(\cdot, \theta)$ at time $t = \theta$ is dense in H.

In this section we apply the observability results, obtained in Sects. 2.3 and 2.4, to derive the dual approximate controllability of (2.38). We restrict ourselves to the case when $n \le 3$, although Corollary 2.4 allows us to consider (in an appropriate space) an arbitrary space dimension.

Let us define the generalized solution of (6.1) by transposition (see [73, p. 186]) as a unique element of $L^2(Q)$ such that

$$\int_Q z(x,t)\left(-\frac{\partial \psi}{\partial t} + \Delta \psi\right) dxdt = \int_0^\theta \psi(x^*(t), t) v(t) dt \tag{2.39}$$

$$\forall \psi \in H^{2,1}(Q), \quad \psi \,|_\Sigma = 0, \quad \psi \,|_{t=\theta} = 0.$$

Indeed, for $n \leq 3$ any element ψ of $H^{2,1}(Q)$ is of Carathéodory type and, hence, $\psi(x^*(\cdot), \cdot) \in L^2(0, \theta)$. Furthermore the argument similar to that in [73, p. 202] gives

$$t \to z(\cdot, t) \quad \text{is a continuous function of } [0, \theta] \to H^{-1}(\Omega). \tag{2.40}$$

Theorem 2.3 (Approximate Controllability in $H^{-1}(\Omega)$) *Let $\varepsilon \in (0, \theta), \theta > 0$ and $\beta \in (0, 1)$ be given and $n \leq 3$. Let $x^*(\cdot)$ be any measurable curve such that $x^*(t) \equiv \hat{x}(\theta - t)$ for all $t \in (0, \theta - \varepsilon)$, where $\hat{x}(t)$, $t \in T_\varepsilon = (\varepsilon, \theta)$ is constructed along Procedure A. Then system (2.38) is approximately controllable in $H^{-1}(\Omega)$.*

Proof Fix any $\varepsilon \in (0, \theta)$ and $\beta \in (0, 1)$ and set up a system dual to (2.38) as follows:

$$\frac{\partial u(x, t)}{\partial t} = -\Delta u(x, t), \quad t \in T, \quad x \in \Omega, \tag{2.41}$$

$$u(x, t)|_\Sigma = 0, \quad u(x, \theta) = u_\theta(x), \quad u_\theta(\cdot) \in H_0^1(\Omega),$$

$$y(t) = u(x^*(t), t), \quad t \in T. \tag{2.42}$$

Note that system (2.41) is well posed in backward time and its solutions belong to $H^{2,1}(Q)$ (see (2.5)). The conclusion of Theorem 2.3 follows from the duality relations (see Chap. 1) and Corollary 2.4 (deduced from $L^\infty(T_\varepsilon)$-continuous observability), applied to the system (2.41), (2.42). Moreover, in order to prove this, one can consider only the set of controls of the following type:

$$v(\cdot) = \begin{cases} v^*(t) \in L^2(0, \theta - \varepsilon), & t \in (0, \theta - \varepsilon), \\ 0, & t \in (\theta - \varepsilon, \theta). \end{cases}$$

Indeed, in this case from (2.38) and (2.41)–(2.42) we obtain the identity

$$[z(\cdot, \theta), u_\theta(\cdot)] = \int_0^{\theta - \varepsilon} u(x^*(t), t) v^*(t) dt, \tag{2.43}$$

which is valid for $v^*(\cdot) \in L^2(0, \theta - \varepsilon)$, $u_\theta(\cdot) \in H_0^1(\Omega)$, where the symbol $[\cdot, \cdot]$ stands for the duality relation between $H_0^1(\Omega)$ and $(H_0^1(\Omega))' = H^{-1}(\Omega)$. Identity (2.43) implies the conclusion of Theorem 2.3. ◇

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