Chapter 2
Partial Orders and Pontryagin Duality

Abstract Partial orders, supernatural numbers, and Pontryagin duality, are discussed.

2.1 Partial Orders

Definition 2.1 (i) A partially ordered set is a set S with a relation $\prec$, and the properties:

(1) (reflexivity) $a \prec a$, for all $a \in S$;
(2) (antisymmetry) if $a \prec b$ and $b \prec a$, then $a = b$;
(3) (transitivity) if $a \prec b$ and $b \prec c$, then $a \prec c$.

(ii) A directed partially ordered set S, is a partially ordered set with the additional property that for $a, b \in S$, there exists $c \in S$ such that $a \prec c$ and $b \prec c$.

a, b are comparable if $a \prec b$ or $b \prec a$. A partially ordered set where any pair of elements is comparable, is a chain (total order).

Definition 2.2 Two partially ordered sets $(S, \prec)$ and $(S', \prec')$ are order isomorphic, if there is a bijective map f from S to S', and $f(a_1) \prec' f(a_2)$, if and only if $a_1 \prec a_2$.

Example 2.1

- the partial order 'subgroup' in a set of groups
- the partial order 'less or equal' in the set of natural numbers $\mathbb{N}$ (i.e., $a \prec b$ if $a \leq b$)
- the partial order 'divisibility' in the set of natural numbers $\mathbb{N}$ (i.e., $a \prec b$ if $a|b$)

For simplicity we use the same symbol $\prec$ for different partial orders, and its precise meaning is clear from the context.

Definition 2.3 An upper bound of a subset T of the partially ordered set S, is an element $a \in S$ such that $b \prec a$ for all $b \in T$. If the set of all upper bounds of T has a smallest element, it is called the supremum of T.

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An element $m \in S$ is called maximal, if there is no element $k \in S$ such that $m \prec k$. A partially ordered set might have many maximal elements, or it might have no maximal element.

Definition 2.4 A partially ordered set S, is called directed-complete partial order (dcpo) if one of the following two statements, which can be proved to be equivalent to each other [1–3], holds:

(1) Every directed subset of S has a supremum.
(2) Every chain in S has a supremum.

A chain which has a supremum, is called a complete chain.

Directed partially ordered sets which are not complete, can sometimes be enlarged into directed-complete partial orders, by adding extra elements.

Example 2.2 The set $\mathbb{N}$ of natural numbers, with divisibility as an order is a directed partially ordered set, but it is not a directed-complete partial order. For example the chain $p, p^2, p^3, \ldots$ where $p \in \Pi$, has no supremum. $\mathbb{N}$ has no maximal elements. Below we enlarge this set into the supernatural (Steinitz) numbers, which is a directed-complete partial order.

2.2 The Directed-Complete Partial Order of Supernatural (Steinitz) Numbers

The set $\mathbb{N}_S$ of supernatural (Steinitz) numbers [4, 5] is:

$$\mathbb{N}_S = \left\{ n = \prod p^{e_p} \mid p \in \Pi; \quad e_p \in \mathbb{Z}_0^+ \cup \{\infty\} \right\} \tag{2.1}$$

The index S indicates supernatural or Steinitz. Here:

- The exponents can take the value ∞.
- The product might contain an infinite number of prime numbers.

In this set only multiplication is well defined, and by definition

$$p^\infty p^e = p^\infty; \quad e \in \mathbb{Z}_0^+ \cup \{\infty\}. \tag{2.2}$$

$\mathbb{N}$ is a subset of $\mathbb{N}_S$. Indeed, if all $e_p \neq \infty$ and only a finite number of them are different from zero, the $\prod p^{e_p} \in \mathbb{N}$.

Definition 2.5

- Let (e_p) (where $p \in \Pi$ and $e_p \in \mathbb{Z}_0^+ \cup \{\infty\}$) be an infinite sequence of exponents. The $(e_p) \prec (e'_p)$ indicates that $e_p \leq e'_p$ for all p. By definition all numbers in $\mathbb{Z}_0^+$ are smaller than ∞.
- $n = \prod p^{e_p}$ is a divisor of $n' = \prod p^{e'_p}$, if $(e_p) \prec (e'_p)$. We denote this as $n|n'$ or as $n \prec n'$.

- $\mathscr{E}$ is the element of $\mathbb{N}_S$, corresponding to the sequence where all $e_p = 1$:

$$\mathscr{E} = \prod_{p \in \Pi} p \tag{2.3}$$

- Υ is the element of $\mathbb{N}_S$, corresponding to the sequence where all $e_p = \infty$:

$$\Upsilon = \prod_{p \in \Pi} p^{\infty} \tag{2.4}$$

Every element of $\mathbb{N}_S$ is a divisor of Υ.

The set $\mathbb{N}_S$ ordered by divisibility (as defined above) is a directed-complete partial order, with Υ as supremum. Examples of complete chains in $\mathbb{N}_S$, are

$$\begin{aligned}
&p, p^2, ..., p^{\infty}; \quad p \in \Pi \\
&p_1 \prec p_1^2 \prec ... \prec p_1^{\infty} \prec p_1^{\infty} p_2 \prec p_1^{\infty} p_2^2 \prec ... \prec p_1^{\infty} p_2^{\infty} \\
&2 \prec 2 \cdot 3 \prec 2 \cdot 3 \cdot 5 \prec ... \prec \mathscr{E} \\
&2^{\infty} \prec 2^{\infty} 3^{\infty} \prec 2^{\infty} 3^{\infty} 5^{\infty} \prec ... \prec \Upsilon
\end{aligned} \tag{2.5}$$

The suprema in these chains are p^{∞}, $p_1^{\infty} p_2^{\infty}$, $\mathscr{E}$ and Υ, correspondingly. They are examples of the elements that have been added into $\mathbb{N}$, in order to make it the directed-complete partial order $\mathbb{N}_S$.

We use the notation $\mathbb{N}_S(p)$ for the complete chain

$$\mathbb{N}_S(p) = \{p, p^2, ..., p^{\infty}\}. \tag{2.6}$$

2.3 Pontryagin Duality

Let G be an Abelian group and $\widetilde{G}$ its Pontryagin dual group, i.e. the group of its characters (we use the notation χ for characters). For locally compact Abelian groups, the Pontryagin duality theorem states that

$$\widetilde{\widetilde{G}} \cong G. \tag{2.7}$$

Let $\mathfrak{G}$ be a set of groups, and $\widetilde{\mathfrak{G}}$ the set of their Pontryagin dual groups. The partial order subgroup in $\mathfrak{G}$, endows a partial order in $\widetilde{\mathfrak{G}}$, where $\widetilde{A} \prec \widetilde{G}$ if $A \prec G$.

Definition 2.6 Let A be a subgroup of G (we denote this as $A \prec G$). The annihilator $\mathrm{Ann}_{\widetilde{G}}(A)$ of A, is the subgroup of $\widetilde{G}$:

$$\mathrm{Ann}_{\widetilde{G}}(A) = \{b \in \widetilde{G} \mid \chi_b(a) = 1, \forall a \in A\} \tag{2.8}$$

Table 2.1 The groups G relevant to this monograph, together with their Pontryagin dual groups $\widetilde{G}$, and the corresponding quantum system

G	$\widetilde{G}$	$\Sigma(G, \widetilde{G})$
$\mathbb{Z}(d)$	$\mathbb{Z}(d)$	$\Sigma[\mathbb{Z}(d)]$
$GF(p^e)$	$GF(p^e)$	$\Sigma[GF(p^e)]$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$	$\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$
$\widehat{\mathbb{Z}}$	$\mathbb{Q}/\mathbb{Z}$	$\Sigma[\widehat{\mathbb{Z}}, (\mathbb{Q}/\mathbb{Z})]$

The following proposition gives the Pontryagin dual group $\widetilde{A}$ of a subgroup A of a group G, and we present it without proof (e.g., [6]).

Proposition 2.1 *If $A \prec G$, then the Pontryagin dual group of A is isomorphic to $\widetilde{G}/\mathrm{Ann}_{\widetilde{G}}(A)$:*

$$\widetilde{A} \cong \widetilde{G}/\mathrm{Ann}_{\widetilde{G}}(A). \tag{2.9}$$

In quantum mechanics G can be used as the group of 'positions', and its Pontryagin dual $\widetilde{G}$ as the group of 'momenta'. We denote such a quantum system as $\Sigma(G, \widetilde{G})$. For some groups $G \cong \widetilde{G}$, and then we use the simpler notation $\Sigma(G)$ for the corresponding quantum system.

Definition 2.7 $\Sigma(A, \widetilde{A})$ is a subsystem of $\Sigma(G, \widetilde{G})$ if $A \prec G$ (in which case the $\widetilde{A}$ is related to $\widetilde{G}$ as in Eq. (2.9)). We denote this as $\Sigma(A, \widetilde{A}) \prec \Sigma(G, \widetilde{G})$.

The groups G relevant to this monograph, together with their Pontryagin dual groups $\widetilde{G}$, and the corresponding quantum system, are shown in Table 2.1.

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