Chapter 2
Some Algebraic and Combinatorial Non-existence Results

In this chapter we intend to explain some easily obtained non-existence theorems for orthogonal designs. Many of these results will be generalized in later chapters, but we feel that these simpler special cases will give the reader an idea as to how the subject developed and what sorts of propositions might be expected.

2.1 Weighing Matrices

To help in the development, and because of independent interest, we make a new definition.

Definition 2.1. A weighing matrix of weight k and order n is an $n \times n$ $\{0,1,-1\}$ matrix A such that $AA^\top = kI_n$. (Note: $A^\top A = AA^\top = kI_n$).

Such matrices have already appeared naturally as the "coefficient" matrices of an orthogonal design. (See Raghavarao [163] or [164] for why these are called weighing matrices and why there is interest in them by statisticians. See also J. Wallis [232].) We shall refer to such a matrix as a $W(n,k)$.

Hadamard [95] showed that $H(n) = W(n,n)$ only exist if $n = 1, 2$, or $\equiv 0 \pmod 4$. It is an easy exercise to show:

Proposition 2.1. *In order that a $W(n,n)$ exist, $n = 1, 2$ or $4|n$.*

The proof uses, in an essential way, the fact that entries in an Hadamard matrix are $\{\pm 1\}$, and the statement would be false without that, since $(3I_9)(3I_9)^\top = 9I_9$, for example.

Now, when n is odd, $\rho(n) = 1$, and an orthogonal design on one variable is nothing more than a weighing matrix.

We shall next attempt to find some necessary conditions on the type of an orthogonal design in order n. We shall only consider a few special cases here: namely, n odd and $n = 2b$, b odd. We shall come back to the general problem later.

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2.2 Odd Order

We have already seen that $\rho(n) = 1$, and we need only consider orthogonal designs on one variable, that is, weighing matrices.

Proposition 2.1 already tells us something: If n is odd and a $W(n,k)$ exists, then $k \neq n$.

Proposition 2.2. *If X is a $W(n,k)$, n odd, then $k = a^2$ for some $a \in \mathbb{Z}$.*

Proof. More generally, if X is a matrix of odd order n with rational entries and $XX^\top = qI_n$, then $q = r^2$ with $r \in \mathbb{Q}$; for $\det(X)^2 = q^2$ and since q^n is a square and n is odd, q is already a square. The proposition follows from the observation that if an integer is the square of a rational number, it is the square of an integer. $\square$

This proposition by itself does not to begin to tell the whole story in odd order, as the following example shows:

Example 2.1. There is no $W(5,4)$.

The property of being a weighing matrix is unaffected by row (or column) permutations. Multiplications of a row (or column) by -1 also does not affect the property of being a weighing matrix. Thus, there is no loss in generality if we assume a $W(5,4)$ has first row $[1\,1\,1\,1\,0]$. The inner product of rows 1 and 2 of our matrix is zero, and so there are an even number of non-zero entries under the 1's of the first row. There must, then, be a zero in the second row, last column. This then allows only three non-zero entries in the last column; a contradiction.

This example can be generalized.

Proposition 2.3. *If n is odd, then a necessary condition that a $W(n,k)$ exists is that $(n-k)^2 - (n-k) \geq n-1$.*

Proof. (We are indebted to P. Eades for this proof, which is much more illuminating than the proof we gave in Geramita-Geramita-Wallis [77].)

We start with a preliminary observation: If M is an $n \times n$ $\{0,1\}$ matrix with exactly k non-zero entries in each row and column, and if we number the rows of M by $r_1, \ldots, r_n$, then, for any $1 \leq j \leq n$,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} r_i \cdot r_j = k^2 - k \tag{2.1}$$

To see this let J be the $n \times n$ matrix of ones. Then $MJ = kJ = M^\top J$, and hence $MM^\top J = k^2 J$, and so

$$\sum_{i=1}^{n} r_i \cdot r_j = k^2$$

Since $r_j \cdot r_j = k$, equation (2.1) is then clear.

To see the relevance of this observation, suppose n is odd and W is a $W(n,k)$, and set $X = W * W$. Then X is a $\{0,1\}$-matrix with exactly k non-zero entries in every row and column. An additional fact about X is that the inner product of any two of its rows is even.

Let $Y = J - X$, and note that Y has exactly $n-k$ non-zero entries in every row and column and, since n is odd, the inner product of any pair of rows of Y is odd and so ≥ 1.

Applying our preliminary observation to Y, we obtain $(n-k)^2 - (n-k) \geq n-1$, as was to be shown. □

Remark 2.1. We conclude that $W(5,4)$, $W(11,9)$, $W(19,16)$, $W(29,25)$, $W(41,36)$, and so on, do not exist. This remark does not exclude $W(7,4)$, $W(13,9)$, $W(31,25), \ldots, W(111,100)$, among others.

This proposition is far from the last word on non-existence, as the following example shows:

Example 2.2. There is no $W(9,4)$. (This proof is due to J. Verner.)(Note that here we do have $(n-k)^2 - (n-k) = 5^2 - 5 = 20 > 8 = n-1$).

We now consider $W(2t+1,t)$ for integers $t \geq 2$. We see that

$$(n-k)^2 - (n-k) = (t+1)^2 - (t+1) = t^2 + t \ \geq 2t = n-1$$

but using Verner's method we have

Lemma 2.1. *There does not exist a $W = W(2t+1,t)$ for $t > 2$ (although a $W(2t+1,t)$ satisfies the known necessary conditions for t, a square).*

Proof. We use 'e' to denote an even number of elements and 'o' to denote an odd number of elements. We notice that orthogonality requires that an even number of non-zero elements overlap in any pair of rows. We use 'x' to denote a non-zero element.

We permute the rows and columns of W until the first row has its first k elements $+1$. We permute the columns so the second row has non-zero elements first. Similarly and other row can be permuted so its first elements are non-zero.

Let k be odd. Thus diagrammatically any three rows are

$$\begin{array}{cccc|cccc}
\multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{o}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{e}} \\
1 \ldots\ldots & \ldots\ldots & \ldots\ldots & 1 & 0 \ldots\ldots & \ldots\ldots & \ldots\ldots & 0 \\
x \ldots\ldots & x & 0 \ldots\ldots & 0 & x \ldots\ldots & x & 0 \ldots\ldots & 0 \\
\underbrace{x \ldots x}_{e} & \underbrace{0 \ldots 0}_{e} & \underbrace{x \ldots x}_{e} & \underbrace{0 \ldots 0}_{o} & \underbrace{x \ldots x}_{e} & \underbrace{0 \ldots 0}_{e} & \underbrace{x \ldots x}_{o} & \underbrace{0 \ldots 0}_{o}
\end{array}$$

Thus it is always possible to force the last column to contain a zero. Again the last column is all zeros which contradicts the definition of a weighing matrix.

Problem 2.1 (Research Problem). What other restrictions can be found for the non-existence of $W(n,k)$ k odd?

Let k be even. Thus diagrammatically any three rows are

$$
\begin{array}{l}
\overbrace{1\ \ldots\ldots\ldots\ldots\ldots\ldots\ 1}^{e}\ \overbrace{0\ \ldots\ldots\ldots\ldots\ldots\ldots\ 0}^{o} \\
x\ \ldots\ldots\ x\ 0\ \ldots\ldots\ 0\ \ x\ \ldots\ldots\ x\ 0\ \ldots\ldots\ 0 \\
\underbrace{x\ldots x}_{e}\ \underbrace{0\ldots 0}_{e}\ \underbrace{x\ldots x}_{e}\ \underbrace{0\ldots 0}_{e}\ \underbrace{x\ldots x}_{e}\ \underbrace{0\ldots 0}_{e}\ \underbrace{x\ldots x}_{e}\ \underbrace{0\ldots 0}_{o}
\end{array}
$$

Thus it is always possible to force the last column to contain a zero. Thus the last column is all zeros which contradicts the definition of a weighing matrix. □

The "boundary" values of proposition 2.3 are of special interest, that is, when $n=(n-k)^2-(n-k)+1$. An inspection of the proof shows that if A is a $W(n,k)$ for such an n and k and if we set $B=J-A*A$ (where J is the $n\times n$ matrix of 1's), then $BB^\top=(n-k-1)I_n+J$; that is B is the incidence matrix of a projective plane of order $n-k-1$ (see Hall [97]).

Thus, for example, the existence of $W(111,100)$ would imply the existence of a projective plane of order 10. Lam showed the projective plane of order 10 does not exist [141]. We shall not go into projective planes here but will sum up this discussion by stating:

Proposition 2.4. *A $W(m^2+m+1,m^2)$ exists only if a projective plane of order m exists.*

It is worth mentioning now, and we shall prove in Section 4.4, that for those m where it is currently known that a projective plane of order m exists, then it is also known that a $W(m^2+m+1,m^2)$ also exists. It is hard to believe that the additional structure in a weighing matrix will not make its existence more difficult than the existence of a projective plane, yet there is no evidence to the contrary. Clearly the existence problem for $W(m^2+m+1,m^2)$ merits considerable attention.

There is little more we can do to the non-existence problem for weighing matrices of odd order, and so we shall leave that subject now. This last tantalising connection between weighing matrices of odd order and projective planes has made us wonder what other combinatorial structures may be related to the non-boundary values of Proposition 2.3. This area is wide open for further study.

We have seen that $\rho(n)=2$ in this case, and so our investigation of orthogonal designs in these orders must centre about one-variable designs (that is, weighing matrices) and two-variable designs.

For our first theorem in these orders, we shall need a classical theorem about quadratic forms. The theorem can be stated for matrices without any reference to quadratic forms, however, and since we intend to come back to quadratic forms later, we shall for now just state the theorem in its unmotivated form.

Definition 2.2. Let R be any commutative ring with identity, and let A, B be two $n\times n$ symmetric matrices with entries in R. We say that A and B are *congruent* if there is an invertible matrix P, with entries in R, such that $PAP^\top = B$.

Notation: If A is an $n\times n$ matrix over the ring R and B is an $m\times m$ matrix over R, then $A\oplus B$ is the $(n+m)\times(n+m)$ matrix $\left[\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right]$ where 0 stands for the appropriate-sized matrix of zeros.

Theorem 2.1 (Witt Cancellation Theorem). *Let F be a field of characteristics $\neq 2$, and let A and B be symmetric $n\times n$ matrices over F. Let X be any symmetric matrix over F. If $A\oplus X$ is congruent to $B\oplus X$, then A is congruent to B.*

Proof. See Lam [142]. □

We now apply this to orthogonal designs.

Theorem 2.2 (Raghavarao-van Lint-Seidel). *Let $n\equiv 2 \pmod 4$, and let A be a rational matrix of order n with $AA^\top = kI_n$, $k\in\mathbb{Q}$. Then $k = q_1^2+q_2^2$, $q_1, q_2\in\mathbb{Q}$.*

Proof. (This theorem was first provided by Raghavarao in [163] and another proof later given by van Lint-Seidel in [150]. The proof we give here is different from both and is based on a suggestion of H. Ryser [171].)

It is a well known theorem of Lagrange that every rational number is the sum of four squares of rational numbers, so let $k = k_1^2+k_2^2+k_3^2+k_4^2$.

From the matrix

$$M = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ -k_2 & k_1 & -k_4 & k_3 \\ -k_3 & k_4 & k_1 & -k_2 \\ -k_4 & -k_3 & k_2 & k_1 \end{bmatrix}$$

It is easy to check that

$$MM^\top = kI_4. \tag{2.2}$$

The hypothesis of the theorem asserts that

$$AA^\top = kI_n. \tag{2.3}$$

Thus, from (2.2) we obtain that I_4 is congruent to kI_4 over $\mathbb{Q}$ and from (2.3) that I_n is congruent to kI_n over $\mathbb{Q}$.

By Witt's Cancellation Theorem, applied $\frac{n-2}{4}$ times, and the fact that the $n\equiv 2 \pmod 4$, we obtain I_2 is congruent to kI_2 over $\mathbb{Q}$; that is, there is a 2×2 rational matrix B such that $BB^\top = kI_2$. From this it is obvious that k is a sum of two squares in $\mathbb{Q}$. □

Corollary 2.1. *Let* $n \equiv 2 \pmod 4$*. If* X *is*

(a) a $W(n,k)$ *or*
(b) an orthogonal design of type (s_1, s_2) *in order* n*,*

then each of k*,* s_ℓ, s_2*,* s_1+s_2 *is a sum of two squares in* $\mathbb{Z}$*.*

Proof. (a) From the theorem we obtain that k is a sum of two squares of rational numbers. It is a famous theorem of Fermat (see, for example, Samuel [172]) that if n is an integer and $n = c^2d$ d square-free $(c,d \in \mathbb{Z})$, then n is a sum of two squares of integers if and only if whenever p is a prime and $p|d$, then $p = 2$ or $p = 1 \pmod 4$. It follows easily from this that if an integer is the sum of two squares of rational numbers, then it is the sum of two squares of integers.

(b) If X is $OD(n; s_1, s_2)$, then $X = A_1x_1 + A_2x_2$. Then A_i is a $W(n, s_i)$, and setting $x_1 = x_2 = 1$ we find $A_1 + A_2$ is a $W(n, s_1 + s_2)$. The result now follows from (a). □

Proposition 2.5. *Let* $n \equiv 2 \pmod 4$*, and let* A *be a rational matrix of order* n *satisfying:*

(i) $A = -A^\top$*;*
(ii) $AA^\top = kI_n$*.*

Then $k = r^2, r \in \mathbb{Q}$*.*

Proof. Since $AA^\top = kI_n$, we have $\det = (k^n)^{1/2}$. For $n \equiv 2 \pmod 4$, $\frac{n}{2} = s$ is odd. Now, since A is skew symmetric, $\det = q^2$ where $q =$ Pfaffian of A (see Artin [12]). Thus, $q^2 = k^s$, and since s is odd, $k = r^2$ for some $r \equiv \mathbb{Q}$. □

Corollary 2.2. *If* $n \equiv 2 \pmod 4$ *and* X *is an* $OD(n; s_1, s_2)$*, then* s_1s_2 *is a square in* $\mathbb{Z}$*.*

Proof. Let

$$X = A_1x_1 + A_2x_2\,;$$

then $A_iA_i^\top = s_iI_n$ and $A_1A_2^\top + A_2A_1^\top = 0$.
Let

$$B_1 = \frac{1}{s_1}A_1^\top A_1\,, \qquad B_2 = \frac{1}{s_1}A_1^\top A_2\,.$$

Then $B_1 = I_n$ and $B_1B_2^\top + B_2B_1^\top = 0$; that is, $B_2 = -B_2^\top$. Also, $B_2B_2^\top = \frac{s_2}{s_1}I_n$.

Thus, by the proposition, $\frac{s_2}{s_1}$ is a square in $\mathbb{Q}$; but then so is $s_1^2(\frac{s_2}{s_1}) = s_1s_2$. However, if an integer is the square of a rational number, it is the square of an integer. □

So far, all the conditions that we have found necessary for the existence of an orthogonal design in order $n \equiv 2 \pmod 4$ have not depended on the fact that the matrices we want should have entries $\{0, 1, -1\}$. In fact, we have proven half of the following:

Theorem 2.3. *Let* $n \equiv 2 \pmod 4$. *A necessary and sufficient condition that there exist two rational matrices* A *and* B *of order* n *such that* $AA^\top = q_1 I_n$, $BB^\top = q_2 I_n$, *and* $AB^\top + BA^\top = 0$ *is that* q_1, q_2 *each be a sum of two squares in* $\mathbb{Q}$ *and that* $q_1 q_2$ *be a square in* $\mathbb{Q}$.

Proof. We have already seen the necessity of three conditions, and so it remains only to show that they are sufficient.

Write $q_1 = r_1^2 u$, $q_2 = r_2^2 v$ where r_1, $r_2 \in \mathbb{Q}$ u, $v \in \mathbb{Z}$, and u and v are square-free. Since $q_1 q_2$ is a square, we obtain $u = v$. Since q_1 and q_2 are each a sum of two squares, we find that $u = v = s^2 + t^2$. Now $q_1 = (r_1 s)^2 + (r_1 t)^2$ and $q_2 = (r_2 s)^2 + (r_2 t)^2$.

Let

$$A_1 = \begin{bmatrix} r_1 s & r_1 t \\ -r_1 t & r_1 s \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -r_2 t & r_2 s \\ -r_2 s & -r_2 t \end{bmatrix};$$

then if $n = 2m$, m odd, one easily checks that $A = A_1 \otimes I_m$, $B = B_1 \otimes I_m$ are the required matrices. □

We can get one additional fact from knowing that in an orthogonal design the coefficient matrices are special.

2.3 Algebraic Problem

Proposition 2.6. *If* $n \equiv 2 \pmod 4$, $(n > 2)$ *and* X *is an* $OD(n; s_1, \ldots, s_\ell)$ *then* $s_1 + s_2 < n$.

Proof. If $s_1 + s_2 = n$, then by setting the variables in the design equal to one, we would obtain an Hadamard matrix, that is, a $W(n,n)$. This contradicts Proposition 2.1. □

This last proposition says, for example, that there is no $OD(10;1,9)$, although the existence of such an orthogonal design would not contradict Corollary 2.1 or Corollary 2.2.

2.4 Orthogonal Designs' Algebraic Problem

It has already become apparent that two separate kinds of theorem are being proved. If we glance back at the statement of Proposition 2.1, we see that the existence of an $OD(n; s_1, \ldots, s_\ell)$ depended on finding a collection of ℓ matrices, $A_1, \ldots, A_\ell$, satisfying two rather different types of conditions, which we shall label combinatorial and algebraic; namely,

$$\left\{ \begin{array}{c} \text{combinatorial} \\ \text{conditions} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{ll} \text{(0)} & \text{the } A_i \text{ are } \{0, 1, -1\} \text{ matrices, } 1 \le i \le \ell \\ \text{(i)} & A_i * A_j = 0,\ 1 \le i \ne \ell; \end{array} \right.$$

$$\left\{\begin{array}{c}\text{algebraic}\\ \text{conditions}\end{array}\right\} \leftrightarrow \left\{\begin{array}{ll}\text{(ii)} & A_iA_i^\top = s_iI_n, 1 \le i \le \ell\\ \text{(iii)} & A_iA_j^\top + A_jA_i^\top = 0, 1 \le i \ne j \le \ell.\end{array}\right.$$

When we looked at an orthogonal design in odd order (that is, a weighing matrix), the algebraic conditions yielded the simple statement that the weight had to be a square integer, while the combinatorial condition, in this case only (0), turned out to be the more mystifying and to have the deeper significance (re: the connection with finite projective planes).

In orders $n \equiv 2 \pmod 4$ the algebraic conditions are somewhat more substantial, and the only general combinatorial fact we know to date arose from the simple result that (apart from order 2) a Hadamard matrix can only exist in orders divisible by 4. In Geramita-Verner [82], by a rather tedious and un-instructive argument, it is shown that there is no $OD(18;1,16)$. This fact is not covered by any general theorem and appears to us to be but the tip of an iceberg, indicating what promises to be a rich source of possible combinatorial relations. The entire question of what combinatorial facts prohibit the existence of orthogonal designs in these orders $\equiv 2 \pmod 4$ is virtually untouched.

Theorem 2.3, however, opens up the possibility that the algebraic part of the problem may be tractable. This turns out to be the case and will occupy much of our efforts.

We state the algebraic problem after a definition.

Definition 2.3. A rational family of order n and type $[s_1,\dots,s_\ell]$, where the s_i are positive rational numbers, is a collection of ℓ rational matrices of order $n, A_1,\dots,A_\ell$, satisfying:

(a) $A_iA_i^\top = s_iI_n$, $1 \le i \le \ell$;
(b) $A_iA_j^\top + A_jA_i^\top = 0$, $1 \le i \ne j \le \ell$.

Algebraic Problem: Find necessary and sufficient conditions on n and $s_1,\dots,s_\ell$ in order that there exists a rational family of order n and type $(s_1,\dots,s_\ell)$.

Clearly, an orthogonal design gives rise to a rational family, but the converse is obviously not true. Nonetheless, if one wants to know if there is an $OD(n;s_1,\dots,s_\ell)$ and one has proved there can be no rational family in order n of type $[s_1,\dots,s_\ell]$, then ones knows there can be no orthogonal design of that order and type.

We may, in fact, rephrase many of the results so far proved for orthogonal designs in terms of rational families. For example, we have proved:

Proposition 2.7. *A rational family in order n cannot consist of more than $\rho(n)$ members; furthermore, there are rational families in order n consisting of ℓ members for every $\ell \le \rho(n)$.*

Proof. Examine the discussion after Proposition 1.1 and the proof of Theorem 1.1. □

We have also shown:

Proposition 2.8. *A necessary and sufficient condition that there exists a rational family of*

(a) type $[s]$ *in order* n*, if* n *is odd, is that* s *be a square in* $\mathbb{Q}$.
(b) type $[s]$ *in order* n*, if* $n \equiv 2 \pmod 4$*, is that* s *be a sum of two squares in* $\mathbb{Q}$;
(c) type $[s_1, s_2]$ *in order* n*, if* $n \equiv 2 \pmod 4$*, is that* $s_1 s_2$ *each be a sum of two squares in* $\mathbb{Q}$ *and* $s_1 s_2$ *be a square in* $\mathbb{Q}$.

We shall pursue this algebraic problem in greater depth in the next chapter. For now we shall concentrate on trying to find other combinatorial facts about orthogonal designs.

2.5 Geramita-Verner Theorem Consequences

The major combinatorial result so far found is motivated by the following theorem.

Theorem 2.4 (Delsarte-Goethals-Seidel [39]). *Let* A *be a* $W(n, n-1)$ *with the rows reordered so that the diagonal consists of zeros.*

(a) If $n \equiv 2 \pmod 4$*, then multiplication by* -1 *of rows (or columns) of* A *as necessary yields a matrix* $\bar{A}$ *with* $\bar{A} = \bar{A}^\top$.
(b) If $n \equiv 0 \pmod 4$*, then multiplication by* -1 *of rows (or columns) of* A *as necessary yields a matrix* $\bar{A}$ *with* $\bar{A} = -\bar{A}^\top$.

Proof. See Delsarte-Goethals-Seidel [39]. □

This result may be generalized to orthogonal designs as follows.

Theorem 2.5 (A. Geramita-J. Verner [82]). *Let* X *be* $OD(n; s_1, \ldots, s_\ell)$ *with* $\sum_{i=1}^{\ell} s_i = n-1$.

(a) If $n \equiv 2 \pmod 4$*, there is an* $OD(n; s_1, \ldots, s_\ell)$ *where* $\bar{X}$ *has zero-diagonal and* $\bar{X} = \bar{X}^\top$.
(b) If $n \equiv 0 \pmod 4$*, there is an* $OD(n; s_1, \ldots, s_\ell)$ *where* $\bar{X}$ *has zero-diagonal and* $\bar{X} = -\bar{X}^\top$.

Proof. If necessary, reorder the rows (or columns) so that the orthogonal design X has 0-diagonal. In this form if x_1 (say) occurs in position (i, j), $i < j$, then position (j,i) contains $\pm x_1$.

For suppose not, and assume, without loss of generality, that position (j,i) contains $\pm x_2$. Consider the various incidences between the i-th and j-th rows.

Count all occurrences of $\binom{\pm x_1}{\pm x_1}$, and assume there are t_1 of these; similarly, assume there are a total of t_2 occurrences of $\binom{\pm x_1}{\pm x_2}$ and $\binom{\pm x_2}{\pm x_1}$ and a total of t_3 occurrences of $\binom{\pm x_1}{\pm x_k}$ and $\binom{\pm x_k}{\pm x_1}$, $k \neq 1$, 2.

Since rows i and j are orthogonal it follows that each of t_1, t_2 and t_3 must be even.

Observe also that these incidences account for all but one of the x_ℓ's in rows i and j, namely, that occurring as $\binom{x_1}{0}$.

Thus, $2t_1 + t_2 + t_3 = 2s_\ell - 1$, and this is a contradiction. □

Now suppose $n \equiv 2 \pmod 4$, and multiply rows and columns of the orthogonal design by -1, as necessary, so that each variable in the first row and column appears with coefficient $= +1$. Call the resulting matrix $\bar{X}$, and replace every variable in it by $+1$ to obtain the $W(n, n-1)$,

$$\begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & \ddots & * & \\ \vdots & * & \ddots & \\ 1 & & & 0 \end{bmatrix}$$

By Theorem 2.4 part (a), multiplication of appropriate rows and columns of this matrix by -1 will make it symmetric. However, as the first row and column are already symmetric, it follows that the entire matrix is symmetric. Hence, $\bar{X} = \bar{X}^\top$, as was to be shown.

For $n \equiv 0 \pmod 4$, multiply the rows and columns of X so that each variable in the first row appears with coefficient $= +1$ and each variable in the first column appears with coefficient $= -1$. Call the resulting matrix $\bar{X}$, and set each variable $= +1$. The argument above, with Theorem 2.4 part (b) implies $\bar{X} = -\bar{X}^\top$.

Corollary 2.3. *Let $n \equiv 0 \pmod 4$. There is an $OD(n; s_1, \dots, s_\ell)$ with $\sum_{i=1}^{\ell} s_i = n-1$ if and only if there is an $OD(n; 1, s_1, \dots, s_\ell)$ with $1 + \sum_{i=1}^{\ell} s_i = n$.*

Proof. The sufficiency is evident.

To establish the necessity, one observes that in view of Theorem 2.5 part(b) if there is an orthogonal design of the type described, then there is one $\bar{X}$ where $\bar{X} = -\bar{X}^\top$ on the variables $x_1, x_2, \dots, x_\ell$. It is then easily verified that $Y = yI + \bar{X}$ is an $OD(n; 1, s_1, s_2, \dots, s_\ell)$. □

Corollary 2.4. *If $n \neq 1, 2, 4, 8$, then there is a $\rho(n)$-tuple, $(s_1, \dots, s_{\rho(n)})$ with $s_i > 0$ and $\sum_{i=1}^{\rho(n)} s_i \leq n$ which is not the type of an orthogonal design of order n.*

Proof. If n is odd, $n > 1$, there is no $W(n, 2)$ since 2 is not a square.

If $n \equiv 2 \pmod 4, n > 2$, there is no orthogonal design of type (1,2) in order n, by Corollary 2.2.

So let $n \equiv 0 \pmod 4, n \neq 4, 8$. In this case $n - \rho(n) > 0$. So we may consider the $\rho(n)$-tuple $(1, 1, \ldots, n - \rho(n))$. The sum of the entries in this tuple is $n - 1$, and so by Corollary 2.3 there is an orthogonal design on $\rho(n) + 1$ variables. This contradicts Theorem 1.3 part (a). $\square$

The full strength of Corollary 2.4 will best be realised after the algebraic question of orthogonal designs is dealt with in greater depth. We come back to this theorem again at the end of the next chapter.

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