1996

1. Let $a, b, c \in \mathbb{C}$. Find $\limsup\limits_{n\to\infty} |a^n + b^n + c^n|^{1/n}$.

2. A function $f \in C([1, +\infty))$ is such that for every $x \geq 1$ there exists a limit

$$\lim_{A\to\infty} \int_A^{Ax} f(u)du =: \varphi(x),$$

$\varphi(2) = 1$, and moreover the function φ is continuous at point $x = 1$. Find $\varphi(x)$.

3. A function $f \in C([0, +\infty))$ is such that

$$f(x) \int_0^x f^2(u)du \to 1, \text{ as } x \to +\infty.$$

Prove that

$$f(x) \sim \left(\frac{1}{3x}\right)^{1/3}, \text{ as } x \to +\infty.$$

4. Find

$$\sup_{\lambda} \left(\frac{\sum\limits_{k=0}^{n-1} (x_{k+1} - x_k) \sin 2\pi x_k}{\sum\limits_{k=0}^{n-1} (x_{k+1} - x_k)^2} \right),$$

where the supremum is taken over all possible partitions of $[0, 1]$ of the form $\lambda = \{0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1\}$, $n \geq 1$.

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5. Find general form of a function $f(z)$, which is analytic on the upper half-plane except the point $z = i$, and satisfies the following conditions:
◇ the point $z = i$ is a simple pole of $f(z)$;
◇ the function $f(z)$ is continuous and real-valued on the real axis;
◇ $\lim\limits_{\substack{z\to\infty \\ \mathrm{Im} z\geq 0}} f(z) = A$ $(A \in \mathbb{R})$.

6. Let $\mathscr{D}$ be a bounded connected domain with boundary $\partial\mathscr{D}$, and $f(z)$, $F(z)$ be functions analytic on $\overline{\mathscr{D}}$. It is known that $F(z) \neq 0$ and $\mathrm{Im}\frac{f(z)}{F(z)} \neq 0$ for every $z \in \partial\mathscr{D}$. Prove that the functions $F(z)$ and $F(z) + f(z)$ have equal number of zeros in $\mathscr{D}$.

7. A linear operator A on a finite-dimensional space satisfies

$$A^{1996} + A^{998} + 1996I = 0.$$

Prove that A has an eigenbasis. Here I is the unit operator.

8. Let $A_1, A_2, \dots, A_{n+1}$ be $n \times n$ matrices. Prove that there exist numbers a_1, a_2, $\dots$, a_{n+1} (not all of them equal 0) such that a matrix

$$a_1A_1 + \ldots + a_{n+1}A_{n+1}$$

is singular.

9. The trace of a matrix A equals 0. Prove that A can be decomposed into a finite sum of matrices, such that the square of each of them equals to zero matrix.

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Taras Shevchenko National University of Kyiv
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