Preface

When I am asked what sort of mathematics I do, I say "High School algebra, but taken somewhat further". This is the study and application of q-series, technically known as basic hypergeometric series. The theory of q-series has its roots in the work of Euler, Cauchy, Abel and, in particular, Jacobi. It was given great impetus by Ramanujan, and has been contributed to by many others, too numerous to list here.

We learn in High School that the geometric series

$$1 + x + x^2 + \cdots = \sum_{n \geq 0} x^n$$

sums to

$$\frac{1}{1 - x}$$

provided that $|x| < 1$, that is $-1 < x < 1$.

At some stage in the 20th Century, it became popular to use the symbol q instead of x (hence "q-series"), so

$$1 + q + q^2 + \cdots = \frac{1}{1 - q}$$

provided $|q| < 1$. But it is noteworthy that Ramanujan used the symbol x (and did not use $\sum$ notation!).

We shall see that in the hands of such luminaries as Euler, Jacobi and Ramanujan, the simple fact stated above became a fundamental tool in the development of q-series and its application to problems in the theory of numbers.

In 1621, Bachet made the claim that every number (non-negative integer) is the sum of four squares, but he had no proof. This problem remained open through the next 150 years, defeated Euler and was finally settled by Lagrange in 1770.

In 1828, Jacobi wrote a monumentally important paper, in which he proved the four-squares theorem (and much more) as a consequence of an identity which became known as the triple product identity. This book starts with a proof of Jacobi's triple product identity, and within the first two chapters (30 pages), we will have proved Lagrange's four-squares theorem, a problem that challenged mankind for a century and a half! Thus we see that the theory of q-series is a powerful tool, and this explains the title, for which I am indebted to my wife Terri.

Before I go into more detail of what you will find in my book, let me tell you more about it.

The book truly is elementary, and should be accessible to a committed undergraduate or potential postgraduate, or to an interested layperson who is prepared to do a little hard (but not abstract) algebra. There are occasions also when we use complex numbers (roots of unity) and calculus (differentiation).

The book did not come about as a textbook for any course, but it ought to be possible to use at least parts of it as a text. There are plenty of exercises for the reader, some of them explicit, others to fill gaps in the exposition. The anonymous reviewer has prevailed upon me to include more exercises than I might have done, and accordingly, Chapters 14 and 26 are essentially all exercises.

The level of sophistication rises gradually through the book, with the occasional hard bit. The reader is invited to jump over anything too hard and proceed.

The subject matter of the book is essentially a selection of topics I have worked on over the years, as well as topics others have worked on that I wish I had. I hope that I have made my gratitude and appreciation sufficiently clear when I have used the work of others. In writing the book, I have in most cases reworked the material in my published and unpublished papers in order for the material to be more accessible, and, where I have found possible, arranged or explained more clearly.

One of my chief aims is to bring out the extreme beauty of some of the results. A mathematical formula exhibits beauty if it contains an element of surprise. This surprise might just be the simplicity of the formula. Perhaps the supreme example of this is the formula described by G.H. Hardy as most typical of Ramanujan, which we meet in Chapter 5.

As will become clear, the principal influence on my life's work has been Ramanujan. I have had contact with him now for more than 50 years, and yet I understand only a small part of his output. But there are many others who have had an influence on me, and it would be remiss of me not to acknowledge them. It is difficult to list the people who have influenced me in order of importance, so I am bound to offend someone. I wish particularly to acknowledge George E. Andrews, who was my Ph.D. supervisor, and gave me my career, Leonard Carlitz, who published my first paper, Dick Askey, who encouraged me to learn the subject, George Szekeres, Frank Garvan, Shaun Cooper, James Sellers, David Hunt, G.H. Hardy, Bruce Berndt, Heng Huat Chan, Krishna Alladi, Hershel Farkas and Elizabeth Loew. I would also like to acknowledge the many people with whom I have discussed matters over the years, and all my co-authors.

Now let me describe the material of the book in some detail. In introductory Chapter 1, you will find not only a proof of the triple product identity, but also the series expansion of Euler's product,

$$E(q) = (1-q)(1-q^2)(1-q^3)\ \cdots\ = \prod_{n\geq 1}(1-q^n),$$

as well as Jacobi's series expansion of the cube of Euler's product, and in Chapter 2 I prove the four-squares theorem and Gauss's two-squares theorem.

Next come 11 chapters on matters associated with (unrestricted) partitions and Ramanujan's partition congruences.

Euler was apparently the first to study $p(n)$, the number of partitions of n. If n is a positive integer, one can ask for the number of ways in which n can be partitioned, that is written as a sum of one or more positive integers. For example, the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. Thus, the number of partitions of 4 is 5 (note that 4 is itself a partition of 4 and that $1+3$ is considered the same partition as $3+1$), and we write

$$p(4) = 5.$$

Euler knew that

$$1 + \sum_{n\geq 1} p(n)q^n = \frac{1}{E(q)},$$

and that $p(n)$ can be calculated recursively (that is the value of $p(n)$ can be found from the values of $p(k)$ for $k<n$.)

Ramanujan was convinced that there was a closed formula for $p(n)$, and he and Hardy found such a formula. In checking their formula, Ramanujan discovered amazing, totally unexpected, arithmetic properties of $p(n)$. The simplest of these properties is that if N is 4 more than a multiple of 5, then $p(N)$ is a multiple of 5. This can be written

$$p(5n+4) \equiv 0 \pmod{5}. \tag{P1}$$

Ramanujan also discovered that

$$p(7n+5) \equiv 0 \pmod{7} \tag{P2}$$

and that

$$p(11n+6) \equiv 0 \pmod{11}. \tag{P3}$$

In this book, you will find a handful of proofs of each of these. Indeed, one of the high points (for me) of the book is my recent simple proof of (P3) to be found in Chapter 3.

The congruence (P1) is an immediate consequence of the identity I alluded to earlier, which has become known as “Ramanujan’s most beautiful identity”,

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6}. \tag{P4}$$

We will see proofs of this in Chapters 5 and 6.

In these chapters on partitions, you will meet some marvellous identities, including the quintuple product identity and Winquist’s identity, the Atkin–Swinnerton-Dyer congruences, Ramanujan’s 5-dissection of Euler’s product and his factorisation of that identity, Ramanujan’s 7-analogue of his most beautiful identity, as well as the Andrews–Garvan crank.

I make it one of my goals in these chapters to obtain as much as I can with as little as I can. Thus, for example I prove all of (P1)–(P4), as well as Ramanujan’s partition congruences for powers of 5 and 7, making use only of the expansions of Euler’s product and its cube.

Before I go on to detail the rest of the contents of this book, let me say something about a technique I use that runs as a unifying thread through the book.

Quite often we have under consideration a sum of the form

$$S = \sum_{k_1,\cdots,k_p=-\infty}^{\infty} (-1)^L q^Q,$$

where L is linear and Q is quadratic in the variables $k_1, \cdots, k_p$.

We separate S into a number of sums according to some property of Q, for example the residue of Q modulo m.

We then make use of linear transformations of the variables to write the separate sums in such a way that they can be summed by one of the several sum-to-product identities that we have established. This versatile, and as we shall see, powerful, technique of using linear transformations I learned from studying the papers of Leonard Carlitz. Apparently, the technique has its origin in the work of Jacobi. I illustrate this technique in §1.10.

In Chapter 14 we study partitions where even parts come in two colours, and see that, as discovered by Hei-Chi Chan, congruences hold modulo powers of 3, analogous to Ramanujan’s partition congruences modulo powers of 5, 7 and 11.

The celebrated Rogers–Ramanujan identities and the Rogers–Ramanujan continued fraction feature in the next three chapters. We give Ramanujan’s proof of the Rogers–Ramanujan identities, and observe the phenomenon, discovered by George Szekeres, that when the Rogers–Ramanujan continued fraction (or its reciprocal) is expanded as a series, the sign of the coefficients is periodic with period 5. We also complete the proof, begun in Chapter 9, of an identity sent by Ramanujan in his first letter to Hardy, which Hardy says “defeated me completely”.

In Chapter 17, we briefly mention a famous set of forty identities of Ramanujan and make the observation that we find proofs of five of them in this book (two in this chapter).

In Chapter 18, we examine a continued fraction usually called the Ramanujan–Göllnitz-Gordon continued fraction which exhibits properties similar to the Rogers–Ramanujan continued fraction.

Jacobi discovered an identity which he described as *aequatio identica satis abstrusa"* ("a rather abstruse identity"), and which has gained importance in string and superstring theory. In Chapter 19 we give two simple proofs of this identity, and formulate it in two striking ways.

Chapter 20 is devoted to proving two modular identities, both of which have beautiful combinatorial interpretations as partition theorems. During the course of the book, we will give no fewer than five proofs of one of these, described by Venkatachaliengar as "a beautiful identity which looks extremely difficult to prove".

In 1917 or 1918, Ramanujan sent Hardy a letter from the nursing home Fitzroy House, in which he stated a striking identity. Chapter 21 is devoted to a proof of this identity, and this leads in the next chapter to a thorough investigation of the cubic theta function analogues introduced by Jon and Peter Borwein and Frank Garvan. These functions and the many relations between them prove to be very useful in our subsequent work.

The next seven chapters are devoted to results giving the number of representations of a number as a sum of certain combinations of figurate numbers, triangles, squares, pentagonal and octagonal numbers, extending the idea of two squares and so on. They culminate in Chapter 29, devoted to proving just three of a set of 298 such results discovered in a computer-assisted search by Ray Melham. In the course of this work we meet Lambert series, bilateral Lambert series, and the Jordan–Kronecker identity, useful in summing bilateral Lambert series.

In the next chapter, we return to the problem of four squares, and consider the problem of partitions, as opposed to representations. We find the generating function for the number of partitions of a number into four squares, $p_{4\square}(n)$, and spend considerable time and effort proving certain relations and Ramanujan-type identities satisfied by it. We also find a formula for $p_{4\square}(n)$ in terms of divisor functions, thus answering a quest of D.H. Lehmer.

In Chapter 31 we use the techniques introduced in the previous chapter to study partitions into four distinct squares of equal parity, a topic inspired by a conjecture made by Bill Gosper at the Ramanujan Centenary Conference in 1987, that if a number is the sum of four distinct odd squares then it is the sum of four distinct even squares. We prove the conjecture correct, and do much more.

In the next nine chapters, we consider various different partition-related functions, including partitions with odd parts distinct, partitions with even parts distinct, overpartitions (that is, partitions into two colours, the parts of one colour being distinct), bipartitions with odd parts distinct, overcubic partitions and generalised Frobenius partitions. In each case we prove certain identities, relations or congruences of the type we have seen in the context of unrestricted partitions.

On the way, we devote a chapter to gathering together many results involving Ramanujan's functions $\phi(q)$ and $\psi(q)$, which subsequently prove indispensable.

We also describe a parametrisation developed by Alaca, Alaca and Williams which has been put to good use in the proofs of certain partition-related results.

We finish with three chapters proving certain identities due to Ramanujan and others, and providing factorisations of these identities, and a final chapter on Ramanujan's tau function.

It is my hope that you will get inspiration from this book to investigate problems in q-series and their applications to partitions and other topics.

You might also read more widely. I suggest Bruce Berndt's "Number Theory in the Spirit of Ramanujan", Hei-Chi Chan's "Invitation to q-series", Shaun Cooper's "Ramanujan's Theta Functions", as well as Bruce Berndt's monumental five volume work "Ramanujan's Notebooks" and the four, soon to be five, volume work by Andrews and Berndt "Ramanujan's Lost Notebook". Of course, there are many other sources, including Ramanujan's Collected Papers, Hardy's book "Ramanujan Twelve lectures on subjects suggested by his life and work", the Notebooks of Ramanujan and "The Lost Notebook and Other Unpublished Papers". And then there are the many papers being produced on the subjects covered in this book and many other related areas.

There is one more thing I have to add, and that is Frank Garvan's computer package "qseries" has been invaluable to me, for which I am very grateful, and which I cannot recommend too highly.

In closing, I hope you enjoy dipping into my book and that you get a thrill from some of the remarkable and beautiful formulas.

Sydney, Australia Michael D. Hirschhorn

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