Chapter 2
Jacobi's Two-Squares and Four-Squares Theorems

2.1 Introduction

In 1640, Fermat stated that a prime p is the sum of two squares if and only if $p \equiv +1 \pmod 4$, and this was eventually proved by Euler in 1747. In 1801, Gauss showed that the number n is the sum of two squares if and only if the squarefree part of n has no divisor congruent to $-1 \pmod 4$. In 1829, Jacobi proved a result giving the number of representations of n as the sum of two squares, of which Gauss's result is, as we will see, a corollary.

Jacobi's two-squares theorem: The number of representations of $n \geq 1$ as the sum of two squares is given by

$$r\{\square + \square\}(n) = 4\left(d_{1,4}(n) - d_{3,4}(n)\right), \tag{2.1.1}$$

where $d_{r,m}(n)$ denotes the number of divisors d of n with $d \equiv r \pmod m$.

We will give two proofs of Jacobi's two-squares theorem, one directly from JTP, the other from (1.7.1). The second may be new.

In 1621 Bachet conjectured that every number is the sum of four squares, and the proof was completed by Lagrange in 1770. In 1829, Jacobi gave a formula for the number of representations of the number n as the sum of four squares, of which Lagrange's theorem is an immediate corollary.

Jacobi's four-squares theorem: The number of representations of $n \geq 1$ as the sum of four squares is given by

$$r\{\square + \square + \square + \square\}(n) = 8 \sum_{\substack{d \mid n \\ d \equiv 1,\, 2 \text{ or } 3 \pmod 4}} d. \tag{2.1.2}$$

We will give a proof of Jacobi's four-squares theorem from (1.7.1).

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2.2 Our First Proof of Jacobi's Two-Squares Theorem

We start by noting that (1.7.4) can be written

$$(a-a^{-1})(a^{-2}q;q)_\infty(a^2q;q)_\infty(q;q)_\infty=\sum_{-\infty}^{\infty}(-1)^k a^{2k+1}q^{(k^2+k)/2}. \tag{2.2.1}$$

Now consider k even, $k=2l$ and k odd, $k=2l+1$ in the right side of (2.2.1) and we find

$$\begin{aligned}(a-a^{-1})(a^{-2}q;q)_\infty(a^2q;q)_\infty(q;q)_\infty&=\sum_{-\infty}^{\infty}a^{4l+1}q^{2l^2+l}-\sum_{-\infty}^{\infty}a^{4l-1}q^{2l^2-l}\\ =a(-a^{-4}q;q^4)_\infty(-a^4q^3;q^4)_\infty(q^4;q^4)_\infty&-a^{-1}(-a^{-4}q^3;q^4)_\infty(-a^4q;q^4)_\infty(q^4;q^4)_\infty.\end{aligned} \tag{2.2.2}$$

If we differentiate (2.2.2) with respect to a using the product rule in the form

$$\left(\prod_k u_k\right)'=\left(\prod_k u_k\right)\sum_k\frac{u_k'}{u_k}$$

then multiply by a, we find (this is fairly tricky!)

$$\begin{aligned}&(a+a^{-1})(a^{-2}q;q)_\infty(a^2q;q)_\infty(q;q)_\infty\\ &\quad+2(a-a^{-1})(a^{-2}q;q)_\infty(a^2q;q)_\infty(q^2;q^2)_\infty\sum_{k\ge1}\left(\frac{a^{-2}q^k}{1-a^{-2}q^k}-\frac{a^2q^k}{1-a^2q^k}\right)\\ &\quad=a(a^{-4}q;q^4)_\infty(-a^4q^3;q^4)_\infty(q^4;q^4)_\infty\\ &\qquad\times\left(1-4\sum_{k\ge1}\left(\frac{a^{-4}q^{4k-3}}{1+a^{-4}q^{4k-3}}-\frac{a^4q^{4k-1}}{1+a^4q^{4k-1}}\right)\right)\\ &\quad+a^{-1}(a^{-4}q^3;q^4)_\infty(a^4q;q^4)_\infty(q^4;q^4)_\infty\\ &\qquad\times\left(1+4\sum_{k\ge1}\left(\frac{a^{-4}q^{4k-1}}{1+a^{-4}q^{4k-1}}-\frac{a^4q^{4k-3}}{1+a^4q^{4k-3}}\right)\right).\end{aligned} \tag{2.2.3}$$

If in (2.2.3) we set $a=1$ and divide by 2, we obtain

$$(q;q)_\infty^3=(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty\left(1-4\sum_{k\ge1}\left(\frac{q^{4k-3}}{1+q^{4k-3}}-\frac{q^{4k-1}}{1+q^{4k-1}}\right)\right). \tag{2.2.4}$$

If we invoke (1.4.9), (2.2.4) becomes

$$(q;q)_\infty^3 = \psi(q)\left(1 - 4\sum_{k\geq 1}\left(\frac{q^{4k-3}}{1+q^{4k-3}} - \frac{q^{4k-1}}{1+q^{4k-1}}\right)\right). \tag{2.2.5}$$

If we divide (2.2.5) by $\psi(q)$ and make use of (1.5.11), we obtain

$$\phi(-q)^2 = 1 - 4\sum_{k\geq 1}\left(\frac{q^{4k-3}}{1+q^{4k-3}} - \frac{q^{4k-1}}{1+q^{4k-1}}\right). \tag{2.2.6}$$

If in (2.2.6) we replace q by $-q$,

$$\phi(q)^2 = 1 + 4\sum_{k\geq 1}\left(\frac{q^{4k-3}}{1-q^{4k-3}} - \frac{q^{4k-1}}{1-q^{4k-1}}\right). \tag{2.2.7}$$

If we use the fact that $\dfrac{x}{1-x} = \sum_{l\geq 1} x^l$ for $|x| < 1$, we can write

$$\begin{aligned}\left(\sum_{-\infty}^{\infty} q^{k^2}\right)^2 &= 1 + 4\sum_{k,l\geq 1}\left(q^{(4k-3)l} - q^{(4k-1)l}\right)\\ &= 1 + 4\left(\sum_{d\equiv +1 \pmod 4,\ l\geq 1} q^{dl} - \sum_{d\equiv -1 \pmod 4,\ l\geq 1} q^{dl}\right).\end{aligned} \tag{2.2.8}$$

If $n \geq 1$, and we compare coefficients of q^n on both sides of (2.2.8), we obtain Jacobi's two-squares theorem (2.1.1).

Now suppose the prime factorisation of $n \geq 1$ is

$$n = 2^\alpha \prod_i p_i^{\alpha_i} \prod_j q_j^{\beta_j}, \tag{2.2.9}$$

where the $p_i \equiv +1 \pmod 4$ and $q_j \equiv -1 \pmod 4$ are distinct primes, $\alpha \geq 0$ and $\alpha_i \geq 1,\ \ \beta_j \geq 1$. We observe that if $P \neq 0$,

$$\begin{aligned}&\sum_{\substack{d|n\\ d\equiv +1 \pmod 4}} d^P - \sum_{\substack{d|n\\ d\equiv -1 \pmod 4}} d^P\\ &\qquad = \prod_i (1 + p_i^P + \cdots + p_i^{P\alpha_i}) \prod_j (1 - q_j^P + - \cdots + (-1)^{\beta_j} q_j^{P\beta_j})\\ &\qquad = \prod_i \frac{1 - p_i^{(\alpha_i+1)P}}{1 - p_i^P} \prod_j \frac{1 + (-1)^{\beta_j} q_j^{(\beta_j+1)P}}{1 + q_j^P}\end{aligned} \tag{2.2.10}$$

If we let $P \to 0$ in (2.2.10), we find that

$$d_{1,4}(n) - d_{3,4}(n) = \prod_i (\alpha_i + 1) \prod_j \frac{1 + (-1)^{\beta_j}}{2}. \tag{2.2.11}$$

It follows from (2.2.1) and (2.2.11) that for $n \geq 1$,

$$r\{\square + \square\}(n) = 4 \prod_i (\alpha_i + 1) \prod_j \frac{1 + (-1)^{\beta_j}}{2}. \tag{2.2.12}$$

Gauss's result is now clear. The number n is the sum of two squares if and only if all the β_j are even, that is, if and only if the squarefree part of n is divisible by no prime congruent to $-1 \pmod 4$, or, equivalently, has no divisor (prime or otherwise) congruent to $-1 \pmod 4$.

Before moving on to give our second proof of Jacobi's two-squares theorem, it is worth pointing out that the number of expressions, which we will denote by $p\{\square + \square\}(n)$, of $n \geq 1$ as the sum of two squares,

$$n = a^2 + b^2$$

with

$$a \geq b \geq 0$$

is given by

$$p\{\square + \square\}(n) = \begin{cases} 0 & \text{if any } \beta_j \text{ is odd} \\ \dfrac{1}{2} \displaystyle\prod_i (\alpha_i + 1) & \text{if all } \beta_j \text{ are even and any } \alpha_j \text{ is odd} \\ \dfrac{1}{2} \displaystyle\prod_i (\alpha_i + 1) + \dfrac{1}{2} & \text{if all } \beta_j \text{ and all } \alpha_i \text{ are even,} \\ & \text{that is, } n \text{ is a square or twice a square.} \end{cases} \tag{2.2.13}$$

2.3 Our Second Proof of Jacobi's Two-Squares Theorem

We start with (1.7.1),

$$(q;q)_\infty^3 = \sum_{k \geq 0} (-1)^k (2k+1) q^{(k^2+k)/2}.$$

Remarkably, (1.7.1) can be written

$$(q;q)_\infty^3 = \sum_{-\infty}^{\infty}(4k+1)q^{2k^2+k}. \tag{2.3.1}$$

Exercise: Verify (2.3.1).

This leads automatically to what follows.

$$\begin{aligned}
(q;q)_\infty^3 &= \left[\frac{d}{da}\left(\sum_{-\infty}^{\infty} a^{4k+1}q^{2k^2+k}\right)\right]_{a=1} \\
&= \left[\frac{d}{da}\left(a(-a^{-4}q;q^4)_\infty(-a^4q^3;q^4)_\infty(q^4;q^4)_\infty\right)\right]_{a=1} \\
&= \Bigg[(-a^{-4}q;q^4)_\infty(-a^4q^3;q^4)_\infty(q^4;q^4)_\infty \\
&\qquad \times \left(1-4\sum_{k\geq 1}\left(\frac{a^{-4}q^{4k-3}}{1+a^{-4}q^{4k-3}} - \frac{a^4q^{4k-1}}{1+a^4q^{4k-1}}\right)\right)\Bigg]_{a=1} \\
&= (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty\left(1-4\sum_{k\geq 1}\left(\frac{q^{4k-3}}{1+q^{4k-3}} - \frac{q^{4k-1}}{1+q^{4k-1}}\right)\right).
\end{aligned} \tag{2.3.2}$$

We note that (2.3.2) is identical with (2.2.4), so we can complete the proof of Jacobi's two-squares theorem as in §2.2.

2.4 A Proof of Jacobi's Four-Squares Theorem

We start with (1.7.1),

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{(k^2+k)/2}.$$

We can write this

$$(q;q)_\infty^3 = \frac{1}{2}\sum_{-\infty}^{\infty}(-1)^k(2k+1)q^{(k^2+k)/2}. \tag{2.4.1}$$

Exercise: Verify (2.4.1).

If we square (2.4.1), we find

$$(q;q)_\infty^6 = \frac{1}{4}\sum_{k,l=-\infty}^{\infty}(-1)^{k+l}(2k+1)(2l+1)q^{(k^2+k+l^2+l)/2}. \tag{2.4.2}$$

We now split the sum in two, according as $k+l$ is even or odd, and obtain

$$\begin{aligned}(q;q)_\infty^6 = \frac{1}{4}\Bigg(&\sum_{k\equiv l \pmod 2}(2k+1)(2l+1)q^{(k^2+k+l^2+l)/2}\\ &-\sum_{k\not\equiv l \pmod 2}(2k+1)(2l+1)q^{(k^2+k+l^2+l)/1}\Bigg).\end{aligned} \tag{2.4.3}$$

In the first sum, let $r=\frac{1}{2}(k+l),\ s=\frac{1}{2}(k-l), k=r+s,\ l=r-s$, and in the second let $r=\frac{1}{2}(k-l-1),\ s=\frac{1}{2}(k+l+1), k=r+s,\ l=s-r-1$, and we find

$$\begin{aligned}(q;q)_\infty^6 &= \frac{1}{4}\Bigg(\sum_{r,s=-\infty}^{\infty}(2r+2s+1)(2r-2s+1)q^{((r+s)^2+(r+s)+(r-s)^2+(r-s))/2}\\ &\quad-\sum_{r,s=-\infty}^{\infty}(2r+2s+1)(2s-2r-1)q^{((r+s)^2+(r+s)+(s-r-1)^2+(s-r-1))/2}\Bigg)\\ &= \frac{1}{4}\left(\sum_{r,s=-\infty}^{\infty}((2r+1)^2-(2s)^2)q^{r^2+r+s^2}-\sum_{r,s=-\infty}^{\infty}((2s)^2-(2r+1)^2)q^{r^2+r+s^2}\right)\\ &= \frac{1}{2}\sum_{r,s=-\infty}^{\infty}((2r+1)^2-(2s)^2)q^{r^2+r+s^2}\\ &= \frac{1}{2}\left(\sum_{-\infty}^{\infty}q^{s^2}\sum_{r=-\infty}^{\infty}(2r+1)^2q^{r^2+r}-\sum_{r=-\infty}^{\infty}q^{r^2+r}\sum_{-\infty}^{\infty}(2s)^2q^{s^2}\right)\\ &= (-q;q^2)_\infty^2(q^2;q^2)_\infty\left(1+4q\frac{d}{dq}\right)\left((-q^2;q^2)_\infty^2(q^2;q^2)_\infty\right)\\ &\quad-(-q^2;q^2)_\infty^2(q^2;q^2)_\infty\cdot 4\frac{d}{dq}\left((-q;q^2)_\infty^2(q^2;q^2)_\infty\right)\end{aligned}$$

$$= (-q;q^2)_\infty^2(-q^2;q^2)_\infty^2(q^2;q^2)_\infty^2 \left(1+4\left(2\sum_{k\geq 1}\frac{2kq^{2k}}{1+q^{2k}} - \sum_{k\geq 1}\frac{2kq^{2k}}{1-q^{2k}}\right)\right.$$
$$\left. -4\left(2\sum_{k\geq 1}\frac{(2k-1)q^{2k-1}}{1+q^{2k-1}} - \sum_{k\geq 1}\frac{2kq^{2k}}{1-q^{2k}}\right)\right)$$
$$= (-q;q)_\infty^4(q;q)_\infty^2\left(1-8\sum_{k\geq 1}\left(\frac{(2k-1)q^{2k-1}}{1+q^{2k-1}} - \frac{2kq^{2k}}{1+q^{2k}}\right)\right). \tag{2.4.4}$$

If we divide (2.4.4) by $(-q;q)_\infty^4(q;q)_\infty^2$ and use (1.5.8), we obtain

$$\phi(-q)^4 = 1-8\sum_{k\geq 1}\left(\frac{(2k-1)q^{2k-1}}{1+q^{2k-1}} - \frac{2kq^{2k}}{1+q^{2k}}\right). \tag{2.4.5}$$

If in (2.4.5) we put $-q$ for q, we find

$$\begin{aligned}
\phi(q)^4 &= 1+8\sum_{k\geq 1}\left(\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}} + \frac{2kq^{2k}}{1+q^{2k}}\right) \\
&= 1+8\sum_{k\geq 1}\left(\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}} + \frac{2kq^{2k}}{1-q^{2k}}\right) - 8\sum_{k\geq 1}\left(\frac{2kq^{2k}}{1-q^{2k}} - \frac{2kq^{2k}}{1+q^{2k}}\right) \\
&= 1+8\sum_{k\geq 1}\frac{kq^k}{1-q^k} - 8\sum_{k\geq 1}\frac{4kq^{4k}}{1-q^{4k}} \\
&= 1+8\sum_{\substack{k\geq 1\\ k\equiv 1,\ 2 \text{ or } 3 \pmod 4}}\frac{kq^k}{1-q^k} \\
&= 1+8\sum_{\substack{k,l\geq 1\\ k\equiv 1,\ 2 \text{ or } 3 \pmod 4}} kq^{kl}.
\end{aligned} \tag{2.4.6}$$

If $n \geq 1$ and we compare the coefficients of q^n on both sides of (2.4.6), we obtain Jacobi's four-squares theorem (2.1.2).

Endnotes.

§2.2 This proof of Jacobi's two-squares theorem appears in Hirschhorn (1985) [65].

§2.3 This proof of Jacobi's two-squares theorem has not been published.

§2.4 An earlier version of this proof of Jacobi's four-squares theorem appeared in Hirschhorn (1987) [66]. A somewhat clumsy proof, starting from JTP, appeared earlier, in Hirschhorn (1982) [64].

§2.4 It is not possible to give a formula for $p\{\square+\square+\square+\square\}(n)$, the number of expressions of n, $n = a^2+b^2+c^2+d^2$ with $a \geq b \geq c \geq d \geq 0$, comparable in simplicity to that for $p\{\square+\square\}(n)$ in §2.2, but see Chapter 30.

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