Chapter 2
Mathematical Modeling of Linear Dynamical Quantum Systems

Abstract This chapter provides a review of the mathematical theory of linear quantum systems, which is based on the Hudson–Parthasarathy quantum stochastic calculus as a mathematical tool for describing Markov open quantum systems interacting with external propagating quantum fields. A precise definition of linear quantum systems is given as well as quantum stochastic differential equations representing their linear equation of motion in the Heisenberg picture. The important notion of physical realizability for linear quantum stochastic differential equations is introduced, and necessary and sufficient conditions for physical realizability reviewed. Complete parameterizations for linear quantum systems are given, and transfer functions defined. Also, the special class of completely passive linear quantum systems is introduced and the notion of stability for linear quantum systems is developed.

Our aim here is to give a brief exposition of the key ideas behind quantum stochastic calculus, as required for the purposes of this monograph. For a more detailed exposition, we refer the reader to two excellent and comprehensive texts on the subject, K.R. Parthasarathy's *An Introduction to Quantum Stochastic Calculus* [1] and P.-A. Meyer's *Quantum Probability for Probabilists* [2]. We also mention the remarkably well-written tutorial paper [3]. Once the basis for quantum stochastic calculus has been laid out, we proceed to use it to formulate linear dynamical quantum systems and their stochastic dynamics.

H.I. Nurdin and N. Yamamoto, *Linear Dynamical Quantum Systems*,
Communications and Control Engineering, DOI 10.1007/978-3-319-55201-9_2

2.1 Quantum Stochastic Calculus

2.1.1 The Boson Fock Space, Exponential Vectors, and Fundamental Processes on the Fock Space

The linear quantum systems that are the main subject of this monograph consist of linearly coupled quantum harmonic oscillators that are in turn coupled to one or more distinct freely propagating optical fields. The freely propagating fields are bosons and each field can contain an indefinite number of bosons. A bosonic field can be mathematically described by a special type of Hilbert space known as a (*symmetric*) *Fock space*. Let $\mathcal{H}$ denote a complex Hilbert space with complex inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ linear in the second slot and antilinear in the first, referred to as the one-particle Hilbert space. Then the symmetric Fock space over $\mathcal{H}$, denoted by $\Gamma_s(\mathcal{H})$, is defined as

$$\Gamma_s(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{j=1}^{\infty} \mathcal{H}^{\otimes_s j},$$

where $\mathcal{H}^{\otimes_s j} = \underbrace{\mathcal{H} \otimes_s \mathcal{H} \otimes_s \cdots \otimes_s \mathcal{H}}_{j-\text{times}}$, and $\otimes_s$ denotes the symmetric tensor product. For any j elements $f_1, f_2, \ldots, f_j \in \mathcal{H}$, the symmetric tensor product of these elements is given by $f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_j = \frac{1}{j!} \sum_{\pi \in \mathcal{P}_j} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes_s f_{\pi(j)}$, where $\otimes$ is the ordinary tensor product on Hilbert spaces, and $\mathcal{P}_j$ denotes the set of all permutation maps π of $\{1, 2, \ldots, j\}$ to itself (there are $j!$ such maps). The Hilbert space $\mathcal{H}^{\otimes_s j}$ is referred to as the j-particle subspace of $\Gamma_s(\mathcal{H})$. Thus, the symmetric Fock space is the infinite direct sum of finite particle Hilbert spaces $\mathcal{H}^{\otimes_s j}$ (including the 0-particle space corresponding to $\mathbb{C}$), representing the fact that a bosonic field can contain an indefinite number of bosons. The symmetric nature of the Fock space, captured by the symmetric tensor product space on each finite particle subspace, reflects the fact that bosons are a class of indistinguishable particles (the other being fermions) with a wavefunction which is symmetric with respect to an interchange of any pair of its arguments (fermions, on the other hand, have wavefunctions that are antisymmetric with respect to an interchange of any pair of its arguments). Every vector $\psi \in \Gamma_s(\mathcal{H})$ can be expressed as an infinite-dimensional vector

$$\psi = (\psi_0, \psi_1, \psi_2, \ldots, \psi_j, \ldots),$$

with $\psi_0 \in \mathbb{C}$ and $\psi_j \in \mathcal{H}^{\otimes_s j}$ for $j \geq 1$. For any two elements $\psi, \phi \in \Gamma_s(\mathcal{H})$, we have the inner product

$$\langle \psi, \phi \rangle = \psi_0^* \phi_0 + \sum_{j=1}^{\infty} \langle \psi_j, \phi_j \rangle_{\otimes_s j},$$

with $\langle \cdot, \cdot \rangle_{\otimes_s j}$ denoting the inner product on $\mathcal{H}^{\otimes_s j}$. This inner product is defined via the identity

$$\langle u_1 \otimes_s \cdots \otimes_s u_n, v_1 \otimes_s \cdots \otimes_s v_n \rangle_{\otimes_s n} = \text{Perm}\left([\langle u_i, v_j \rangle_{\mathcal{H}}]_{i,j=1,2,\ldots,n} \right),$$

where Perm$(\cdot)$ denotes the permanent of a square matrix.

An important class of vectors in $\Gamma_s(\mathcal{H})$ is the class of exponential or coherent vectors $e(f)$ that is parametrized by $f \in \mathcal{H}$. It is defined as

$$e(f) = \left(1, f, \frac{1}{2!}f^{\otimes_s 2}, \ldots, \frac{1}{k!}f^{\otimes_s k}, \ldots\right),$$

(note that $f^{\otimes_s j} = f^{\otimes j}$) with inner product

$$\langle e(g), e(f)\rangle = \exp(\langle g, f\rangle_{\mathcal{H}}).$$

In particular, we have the norm

$$\|e(f)\| = \sqrt{\exp(\|f\|^2_{\mathcal{H}})},$$

where $\|\cdot\|_{\mathcal{H}}$ is the norm on $\mathcal{H}$, $\|f\|_{\mathcal{H}} = \sqrt{\langle f, f\rangle_{\mathcal{H}}}$.

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_r$ be Hilbert spaces. The symmetric Fock space has the property that $\Gamma_s(\oplus_{j=1}^r \mathcal{H}_j) \cong \otimes_{j=1}^r \Gamma_s(\mathcal{H}_j)$ for any integer $r \geq 1$, where $\cong$ denotes that the two spaces are unitarily equivalent. Therefore, $\Gamma_s(\oplus_{j=1}^r \mathcal{H}_j)$ can be identified with $\otimes_{j=1}^r \Gamma_s(\mathcal{H}_j)$. This identification can be made via the correspondence $e((f_1, f_2, \ldots, f_r)) \leftrightarrow e(f_1) \otimes e(f_2) \otimes \cdots \otimes e(f_r)$, for any $f_j \in \mathcal{H}_j$.

In this monograph, we will be working exclusively with a boson Fock space over $\mathcal{H} = L^2(\mathbb{R}_+; \mathbb{C}^m)$, with $\mathbb{R}_+$ denoting the set of nonnegative real numbers, which we denote throughout the rest of the monograph as $\mathcal{F}_m$, i.e., $\mathcal{F}_m = \Gamma_s(L^2(\mathbb{R}_+; \mathbb{C}^m))$. Let $\mathcal{F}_m(\mathcal{I}) = \Gamma_s(L^2(\mathcal{I}; \mathbb{C}^m))$ for any (Lebesque) measurable set $\mathcal{I} \subset \mathbb{R}_+$. If $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r$ are disjoint subsets of $\mathbb{R}_+$, then $\mathcal{F}_m(\mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_r) = \otimes_{j=1}^r \mathcal{F}_m(\mathcal{I}_j)$. In the following, we will frequently use the shorthand notation $t] \equiv [0, t], t) \equiv [0, t)$, $[t \equiv [t, \infty)$, and $(t \equiv (t, \infty)$.

2.1.2 Adapted Processes and Quantum Stochastic Integrals

Let $\mathcal{E}$ denote the complex linear space spanned by the exponential vectors on the Fock space $\mathcal{F}_m$, $\mathcal{E} = \text{span}\{e(f) \mid f \in L^2(\mathbb{R}_+; \mathbb{C}^m)\}$. For any $g \in L^2(\mathbb{R}_+; \mathbb{C}^m)$ and bounded self-adjoint operator $\Pi : L^2(\mathbb{R}_+; \mathbb{C}^m) \to L^2(\mathbb{R}_+; \mathbb{C}^m)$, define the operators $\mathcal{A}(f)$, $\mathcal{A}^*(f)$, and $\Lambda(\Pi)$ with domain $\mathcal{E}$ via their action on the coherent vectors:

$$\begin{aligned}
\mathcal{A}(g)e(f) &= \left(\int_0^\infty g(s)^* f(s)ds\right) e(f),\\
\mathcal{A}^*(g)e(f) &= \frac{d}{dt} e(f + tg)\Big|_{t=0},\\
\Lambda(\Pi)e(f) &= \mathcal{A}^*(\Pi f)e(f).
\end{aligned}$$

In particular, we have the relation $\langle \mathcal{A}^*(f)e(h), e(g)\rangle = \langle e(h), \mathcal{A}(f)e(g)\rangle$ and the commutation relations

$$
\begin{aligned}
[\mathcal{A}(f), \mathcal{A}(g)] &= [\mathcal{A}^*(f), \mathcal{A}^*(g)] = 0, \\
[\mathcal{A}(f), \mathcal{A}^*(g)] &= \int_0^\infty f(s)^* g(s) ds, \\
[\Lambda(\Pi_1), \Lambda(\Pi_2)] &= \Lambda([\Pi_1, \Pi_2]), \\
[\mathcal{A}(f), \Lambda(\Pi)] &= \mathcal{A}(\Pi^* f), \\
[\mathcal{A}^*(f), \Lambda(\Pi)] &= -\mathcal{A}^*(\Pi f).
\end{aligned}
$$

Also, $\mathcal{A}^*(f)$ is the adjoint of $\mathcal{A}(f)$ with domain $\mathrm{Dom}(A^*(f)) \supset \mathcal{E}$, and $\Lambda(\Pi)$ is self-adjoint on a domain $\mathrm{Dom}(\Lambda(\Pi)) \supset \mathcal{E}$.

Let e_i be a standard basis vector in $\mathbb{C}^m$ (as a column vector) that is 0 everywhere except at the i-th element which is 1, and let $1_{\mathcal{I}}$ be the indicator function on the set $\mathcal{I}$. Define the so-called fundamental processes $\mathcal{A}_j(t)$, $\mathcal{A}_j^*(t)$, and $\Lambda_{jk}(t)$ acting on $\mathcal{E}$ as

$$
\begin{aligned}
\mathcal{A}_j(t) &= \mathcal{A}(e_j 1_{t]}), \\
\mathcal{A}_j^*(t) &= \mathcal{A}^*(e_j 1_{t]}), \\
\Lambda_{jk}(t) &= \Lambda(e_j e_k^\top 1_{t]}),
\end{aligned}
$$

with the indices j and k ranging from 1 until m. The subscript j for $\mathcal{A}_j(t)$ and $\mathcal{A}_j^*(t)$ indicates that they are field *annihilation and creation processes* on the j-th bosonic field, respectively, while the subscript jk on $\Lambda_{jk}(t)$ indicates that it is a photon exchange operator from the k-th boson field to the j-th field. We may then *identify* (and we shall do so for the rest of the monograph without further comment) $\mathcal{A}(t)$ and $\mathcal{A}^*(t)$ as vectors of operators on $\mathcal{F}_m$:

$$
\begin{aligned}
\mathcal{A}(t) &= [\, \mathcal{A}_1(t)\ \mathcal{A}_2(t)\ \ldots\ \mathcal{A}_m(t)\,]^\top, \\
\mathcal{A}^*(t) &= [\, \mathcal{A}_1^*(t)\ \mathcal{A}_2^*(t)\ \ldots\ \mathcal{A}_m^*(t)\,]^\top,
\end{aligned}
$$

and identify $\Lambda(t)$ with the matrix of operators,

$$
\Lambda(t) = [\Lambda_{jk}(t)]_{j,k=1,2,\ldots,m}.
$$

From the properties of the creation process, we will freely make the identification $\mathcal{A}_j^*(t) = \mathcal{A}_j(t)^*$, and $\mathcal{A}^*(t) = \mathcal{A}(t)^*$ on $\mathcal{E}$. We also note the fact that $\Lambda(t) = \Lambda(t)^*$ on $\mathcal{E}$.

The process $\Lambda(t)$ is known as the *counting* or *gauge process*. Let $\mathfrak{h}$ denote the underlying Hilbert space of a quantum system that is coupled to bosonic fields on the Fock space $\mathcal{F}_m$. A process $\{X(t),\ t \geq 0\}$ defined on $\mathfrak{h} \otimes \mathcal{F}_m$ is said to be adapted if $X(t) \left(\psi(t) \otimes e(f 1_{[t})\right) = \phi(t) \otimes e(f 1_{[t})$ for any $f \in L^2(\mathbb{R}_+; \mathbb{C}^m)$, $\psi(t) \in \mathfrak{h} \otimes \mathcal{F}_m(t])$, and some $\phi(t) \in \mathfrak{h} \otimes \mathcal{F}_m(t])$. That is, an adapted process acts trivially (i.e., as the identity operator) on the future factor $\mathcal{F}_m([t)$ of $\mathcal{F}_m$.

Observe that $(\mathcal{A}(t) - \mathcal{A}(s))e(f) = e(f 1_{s]}) \otimes \mathcal{A}(1_{[s,t]}) e(f 1_{[s,t]}) \otimes e(f 1_{[t})$, and similarly, $(\mathcal{A}^*(t) - \mathcal{A}^*(s))e(f) = e(f 1_{s]}) \otimes \mathcal{A}^*(1_{[s,t]}) e(f 1_{[s,t]}) \otimes e(f 1_{[t})$, and

$(\Lambda(t) - \Lambda(s))e(f) = e(f 1_{s]}) \otimes \Lambda(1_{[s,t]})e(f 1_{[s,t]}) \otimes e(f 1_{[t})$. With these properties, one can define quantum stochastic integrals with respect to adapted processes:

$$\int_0^t (X_1(s)d\mathcal{A}(s) + X_2(s)d\mathcal{A}(s) + X_3(s)d\Lambda(s))$$
$$= \int_0^t (X_1(s) \otimes d\mathcal{A}(s) + X_2(s) \otimes d\mathcal{A}(s) + X_3(s) \otimes d\Lambda(s)), \qquad (2.1)$$

where X_1, X_2, X_3 are adapted processes, and the quantum stochastic integral is defined on $\mathfrak{h} \otimes \mathcal{F}_m$. The quantum stochastic integral then also defines an adapted process. These quantum stochastic integrals can be constructed in a fashion that is similar to the construction of classical Itō stochastic integrals; see [1, 2, 4] for details. Also, note the important property that any adapted process $X(t)$ commutes with $\mathcal{A}_j(1_{[\tau_1,\tau_2)})$, $\mathcal{A}_j^*(1_{[\tau_1,\tau_2)})$, and $\Lambda_{jk}(1_{[\tau_1,\tau_2)})$ for any j, k and any $t \leq \tau_1 < \tau_2 \leq \infty$.

From a physical point of view, the processes $\mathcal{A}_j(t)$, $\mathcal{A}_j^*(t)$, and $\Lambda_{jk}(t)$ arise as integrated version of idealized quantum white noise processes $\xi_j(t), j = 1, 2, \ldots, m$, satisfying the singular commutation relations $[\xi_j(t), \xi_k(s)^*] = \delta_{jk}\delta(t - s)$ [5, 6], introduced earlier in Chap. 1. That is, we may formally write $\mathcal{A}_j(t) = \int_0^t \xi_j(s)ds$, $\mathcal{A}_j^*(t) = \int_0^t \xi_j(s)^*ds$, and $\Lambda_{jk}(t) = \int_0^t \xi_j(s)^*\xi_k(s)ds$. Due to the singular nature of the quantum white noise processes, it is mathematically simpler to work with the more regular integrated processes $\mathcal{A}_j(t)$, $\mathcal{A}_j^*(t)$, and $\Lambda_{jk}(t)$ as these can be rigorously and explicitly constructed as processes on the Fock space $\mathcal{F}_m$, as we have already briefly elaborated upon. However, from the perspective of describing the physics involved in the interaction between the system and the bosonic environment, the white noise picture is more meaningful, and we shall often employ this in the discussion that ensues. That is to say that the more fundamental processes from the underlying physics are the quantum white noise processes rather than the mathematically more convenient "derived" integrated processes. It should be noted that quantum white noise processes arise as a consequence of making a Markov assumption regarding the interaction between the system and the bosonic environment, and it is only within this approximation that the quantum white noise processes can be given a sensible physical interpretation. The modeling of a bosonic environment as a quantum white noise process without the Markov approximation can be problematic and gives rise to physical inconsistencies, see, e.g., [7] for a discussion.

2.1.3 The Quantum Itō Table in Vacuum and the Quantum Itō Rule

The fundamental processes $\mathcal{A}_j(t)$, $\mathcal{A}_j^*(t)$, and $\Lambda_{jk}(t)$ are quantum stochastic processes on the Fock space $\mathcal{F}_m$. The processes $\mathcal{A}_j(t) + \mathcal{A}_j^*(t)$ and $-\imath\mathcal{A}_j(t) + \imath\mathcal{A}_j^*(t)$ are non-commuting processes that are each isomorphic to a classical standard Wiener processes, that is, each can be viewed as Fock space representations of the standard

Wiener process. On the other hand, $\Lambda_{jj}(t)$ is isomorphic to a classical Poisson process, that is, the latter is a realization of the former on the Fock space. As a quantum mechanical system, the bosonic fields have a quantum state that determines their statistics under measurement. We now assume that the bosonic fields are in the vacuum state $|\Omega\rangle = e(0)$. This is a state in which the fields do not contain any photons. In this state, we have that $\mathcal{A}(f)e(0) = 0$ for all $f \in L^2(\mathbb{R}_+; \mathbb{C}^m)$, and the forward-pointing differentials $d\mathcal{A}_j(t) = \mathcal{A}_j(t+dt) - \mathcal{A}_j(t)$, $d\mathcal{A}_j^*(t) = \mathcal{A}_j^*(t+dt) - \mathcal{A}_j^*(t)$, and $d\Lambda_{jk}(t)$ satisfy the quantum Itō product rule, as a quantum adaptation of the classical Itō product rules: $d\mathcal{A}_j(t)d\mathcal{A}_k^*(t) = \delta_{jk}dt$; $d\mathcal{A}_j^*(t)d\mathcal{A}_k(t) = d\mathcal{A}_j(t)d\mathcal{A}_k(t) = d\mathcal{A}_j^*(t)d\mathcal{A}_k^*(t) = 0$, and

$$\begin{aligned} &d\Lambda_{jk}(t)d\Lambda_{j'k'}(t) = \delta_{kj'}d\Lambda_{jk'}(t),\ d\mathcal{A}_j(t)d\Lambda_{kl}(t) = \delta_{jk}d\mathcal{A}_l(t), \\ &\quad d\Lambda_{jk}d\mathcal{A}_l^*(t) = \delta_{kl}d\mathcal{A}_j^*(t). \end{aligned}$$

Let $X(t)$ and $Y(t)$ be two adapted processes on $\mathfrak{h} \otimes \mathcal{F}_m$ that can be expressed as quantum stochastic integrals with respect to the fundamental processes $\mathcal{A}_j(t)$, $\mathcal{A}_j^*(t)$, and $\Lambda_{jk}(t)$, as in (2.1). The quantum Itō product rule holds for the forward-pointing differential of the product process $X(t)Y(t)$:

$$d(X(t)Y(t)) = (dX(t))Y(t) + X(t)dY(t) + dX(t)dY(t).$$

Note the third term that serves as a quantum analogue of the second-order correction term in the classical Itō stochastic calculus. Since X and Y are quantum stochastic integrals with respect to the fundamental processes, the correction term can be calculated using (2.1) together with the quantum Itō product rule for the fundamental processes given above.

2.1.4 The Hudson–Parthasarathy Quantum Stochastic Differential Equation

Using the quantum stochastic integrals and the quantum Itō rules, one can define adapted processes as solutions of quantum stochastic differential equations (QSDEs). A particularly important class of QSDEs that describe the physical scenario of Markov open quantum systems coupled to a bosonic environment is the Hudson–Parthasarathy QSDE of the form:

$$\begin{aligned} dU(t) = \Big(&\mathrm{Tr}\big((S-I)^\top d\Lambda(t)\big) + d\mathcal{A}(t)^* L - L^* S d\mathcal{A}(t) \\ &-(\imath H + 1/2 L^* L)dt\Big) U(t), \end{aligned} \tag{2.2}$$

with initial condition $U(0) = I$. Here, H is the Hamiltonian of the system and is a self-adjoint operator on the system Hilbert space $\mathfrak{h}$, $L = [\,L_1\ L_2\ \ldots\ L_m\,]^\top$ is a vector of operators L_j on $\mathfrak{h}$, and $S \in \mathbb{C}^{m\times m} \otimes \mathfrak{h}$ is a unitary operator (i.e., $S^*S = SS^* = I$), with entries S_{jk} being operators on $\mathfrak{h}$, called a *scattering matrix*. They are the operator-valued "coefficients" or parameters of the QSDE. The form of the equation is such that the solution $U(t)$ of the QSDE is a unitary adapted process. That is, $U(t)$ is an adapted process and is unitary for each t: $U(t)^*U(t) = U(t)U(t)^* = I$.[1]

Following [9], we denote a quantum system G, whose dynamics are governed by the unitary solution of a Hudson–Parthasarahy QSDE (2.2) with parameters S, L, and H, by the shorthand $G = (S, L, H)$.

We will now elaborate on properties of the Hudson–Parthasarathy QSDE that will be crucial for the (measurement-based) feedback control theory of open quantum systems. Let $\theta_t : L^2(\mathbb{R}; \mathbb{C}^m) \to L^2(\mathbb{R}; \mathbb{C}^m)$ denote the one-particle left shift operator, defined by $\theta_t f(\cdot) = f(\cdot + t)\ \forall t \in \mathbb{R}_+$. Let Θ_t be the so-called second quantization of the one-particle left shift operator, defined as an operator on $\mathcal{F}_{m-} = \Gamma_s(L^2(\mathbb{R}; \mathbb{C}^m))$ by its action on the exponential vectors as $\Theta_t e(f) = e(\theta_t f)$. It is straightforward to verify that $\Theta_t^* e(f) = e(f(\cdot - t))$ and Θ_t can be uniquely extended to a unitary operator on $\mathcal{F}_{m-}$, $\Theta_t^*\Theta_t = \Theta_t\Theta_t^* = I$. The solution $U(t)$ of a Hudson–Parthasarathy QSDE, with $U(t)$ extended from $\mathfrak{h} \otimes F_m$ to $\mathfrak{h} \otimes F_{m-}$ as $I \otimes U(t)$, has the property that it is a left cocycle with respect to Θ_t, meaning that the following holds for all $0 \le s \le t$,

$$U(t) = \Theta_s^* U(t-s)\Theta_s U(s).$$

We note that $V(t,s) = \Theta_s^* U(t-s)\Theta_s$ acts non-trivially only on the portion $\mathfrak{h} \otimes \mathcal{F}_m([s,t])$ of $\mathfrak{h} \otimes \mathcal{F}_m$. Note that $U(t) = V(t,0)$ and the cocycle property can be equivalently expressed as $V(t,s) = V(t,\tau)V(\tau,s)$ for all $0 \le s \le \tau \le t$. Indeed, we have that

$$\begin{aligned} V(t,\tau)V(\tau,s) &= \Theta_\tau^* U(t-\tau)\Theta_\tau\Theta_s^* U(\tau - s)\Theta_s \\ &= \Theta_\tau^* U(t-\tau)\Theta_{\tau-s} U(\tau - s)\Theta_s \\ &= \Theta_s^*\Theta_{\tau-s}^* U(t-\tau)\Theta_{\tau-s} U(\tau - s)\Theta_s \\ &= \Theta_s^* V(t-s, \tau - s) U(\tau - s)\Theta_s \\ &= \Theta_s^* U(t-s)\Theta_s \\ &= V(t,s), \end{aligned}$$

where in the second and third lines we have used the fact that $\Theta_\tau\Theta_s^* = \Theta_{\tau-s}$. A consequence of this property is that $\tilde{U}(t) = \Theta_t U(t)$ defines a strongly continuous one-parameter semigroup of unitary operators on $\mathfrak{h} \otimes \mathcal{F}_{m-}$, $\tilde{U}(t)\tilde{U}(s) = \tilde{U}(t+s)$ for all $s, t \ge 0$ with $\tilde{U}(0) = I$. Therefore there exists a densely defined essentially self-adjoint operator K such that $\tilde{U}(t) = e^{-\imath Kt}$. The operator K is the Stone generator

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[1] Technically, the existence of a unique solution of the Hudson–Parthasarathy QSDE that is unitary is guaranteed whenever the coefficients S, L, H are all bounded operators. When they are unbounded then additional technical assumptions need to be assumed, see, e.g., [8].

of the strongly continuous semigroup $\{\tilde{U}(t),\ 0 \leq s \leq t\}$. The difficult and long-standing problem of characterizing this generator was finally resolved by Gregoratti [10, 11] building on the work of Chebotarev [12].

Let us now turn to the Heisenberg picture of quantum mechanics, where states are fixed and operators evolve in time. This will be the natural setting for studying quantum filtering and control problems that will be considered later in the monograph. Let $j_t(\cdot) = U(t)^* \cdot U(t)$ and let $Z(t)$ denote any adapted process on the boson Fock space $\mathcal{F}_m$. Then by the cocycle property of $V(t, s)$ we find that, for any $t \geq s \geq 0$,

$$\begin{aligned} j_t(Z(s)) &= U(t)^* Z(s) U(t) \\ &= U(s)^* \Theta_s^* U(t-s)^* \Theta_s Z(s) \Theta_s^* U(t-s) \Theta_s U(s) \\ &= U(s)^* Z(s) U(s) \\ &= j_s(Z(s)). \end{aligned}$$

Using this property, it follows that for any (bounded linear) system operator X on $\mathfrak{h}$ and $0 \leq s \leq t$,

$$\begin{aligned} j_t(X) j_s(Z(s)) &= j_t(X) j_t(Z(s)) \\ &= U(t)^* X Z(s) U(t) \\ &= U(t)^* Z(s) X U(t) \\ &= j_t(Z(s)) j_t(X) \\ &= j_s(Z(s)) j_t(X). \end{aligned}$$

Therefore, $[j_t(X), j_s(Z(s))] = 0$ for any system operator X and any adapted process $Z(t)$ on $\mathcal{F}_m$. In other words, $\{j_t(X), j_s(Z(s)),\ 0 \leq s \leq t\}$ forms a commutative family of operators, a property of input–output Markov open quantum systems known as the *non-demolition* property. Now, let $Z(t)$ be an adapted process on $\mathcal{F}_m$ with the additional property that $[Z(t), Z(s)] = 0$ for all $s, t \geq 0$. For example, $Z(t) = e^{-\imath\phi}\mathcal{A}(t) + e^{\imath\phi}\mathcal{A}(t)^*$ ($\phi \in \mathbb{R}$) and $Z(t) = \Lambda(t)$, all have these properties. It follows that

$$\begin{aligned} j_t(Z(t)) j_s(Z(s)) &= j_t(Z(t)) j_t(Z(s)) \\ &= U(t)^* Z(t) Z(s) U(t) \\ &= U(t)^* Z(s) Z(t) U(t) \\ &= j_t(Z(s)) j_t(Z(t)) \\ &= j_s(Z(s)) j_t(Z(t)). \end{aligned}$$

Hence, when $Z(t)$ has the stipulated properties, $[j_t(Z(t)), j_s(Z(s))] = 0$ for all $s, t \geq 0$ and $\{j_t(Z(t)), j_s(Z(s)),\ 0 \leq s \leq t\}$ forms a commutative family of operators. This is a property of input–output Markov open quantum systems known as the *self non-demolition* property.

Now, let $Z(t)$ be an observable for each t, $Z(t) = Z(t)^*$, the self non-demolition property means that $\mathscr{Z}_t = \{j_s(Z(s)),\ \ 0 \le s \le t\}$ describes a continuous output process of the quantum fields that may be continuously monitored/measured (depending on what $Z(t)$ actually is), to yield a real-valued measurement record. When $Z(t)$ is self non-demolition, the non-demolition property means that information about $j_t(X)$ can be inferred from an observation record of $\mathscr{Z}_t$. This inference process is known as quantum filtering, the quantum analogue of classical nonlinear filtering theory [13], and will be treated in Chap. 4.

2.2 Linear Dynamical Quantum Systems: Joint Unitary Evolution of Oscillators and Boson Fields

We now begin to introduce and specialize to the class of linear quantum stochastic systems as the central theme of this monograph. This section describes the joint evolution of a set of coupled oscillators that are also coupled to external bosonic fields, via a certain form of interaction, which results in linear dynamics of the oscillators' position and momentum operators in the Heisenberg picture. The linear dynamics is a defining feature of linear quantum systems.

Let there be n independent quantum harmonic oscillators. The j-th quantum harmonic oscillator has position and momentum operators q_j and p_j acting on elements of the underlying Hilbert space $L^2(\mathbb{R};\mathbb{C})$, the space of all square integrable complex-valued function on $\mathbb{R}$. These operators satisfy the canonical commutation relations (CCR)[2]:

$$[q_j, p_k] = 2\imath\delta_{jk}\,,\ [q_j, q_k] = 0\,,\ [p_j, p_k] = 0. \tag{2.3}$$

The position and momentum operators of the n oscillators can be collected in a single column vector of operators x defined by $x = (q_1, p_1, q_2, p_2, \ldots, q_n, p_n)^\top$. We can then express the CCR more compactly as,

$$[x, x^\top] = xx^\top - (xx^\top)^\top = 2\imath\mathbb{J}_n,$$

with $\mathbb{J}_n = I_n \otimes \mathbb{J}$, with

$$\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Also note in passing that $xx^\top \neq (xx^\top)^\top$ since some elements of x do not commute with one another. The composite system of n quantum harmonic oscillators has a *quadratic Hamiltonian* H given by $H = (1/2)x^\top Rx$, with R a real symmetric $2n \times 2n$

[2] We take the common convention that units are taken such that $\hbar = 1$. Also, we will take (2.3) as the default CCR for the position and momentum operators of multiple distinct oscillators. However, it is easy to adapt the results of this chapter for a different definition of these operators that satisfy a different set of commutation relations; see Remark 2.2.

matrix. The quantum harmonic oscillators are also coupled to m distinct external quantum bosonic fields. They are coupled to the k-th quantum field via a singular interaction of the form Hamiltonian $H_k = \imath(L_k\xi_k^*(t) - L_k^*\xi_k(t))$ [5, 6], where $L_k = K_k x$ (with $K_k \in \mathbb{C}^{1\times 2n}$) is a linear coupling operator describing the coupling of the position and momentum operators to $\xi_k(t)$. Here $\xi_k(t)$ is a quantum white noise process as discussed in Sect. 2.1.2. We can collect the coupling operators $L_1, L_2, \ldots, L_m$ together in one *linear coupling vector* $L = [L_1\ L_2\ \ldots\ L_m]^\top = Kx$, with $K = [K_1^\top\ K_2^\top\ \ldots\ K_m^\top]^\top$. The *joint* evolution of the oscillators and quantum fields is then given by a unitary adapted process $U(t)$ satisfying the Hudson–Parthasarathy QSDE (2.2), with S being a fixed unitary matrix in $\mathbb{C}^{m\times m}$, i.e., the entries of S in this case are complex numbers rather than operators on $\mathfrak{h}$.

Using the shorthand notation introduced earlier, a linear quantum stochastic system with parameters $S, L = Kx, H = (1/2)x^\top Rx$ as described above can be expressed as $G = (S, Kx, (1/2)x^\top Rx)$. In a later section, we will slightly generalize the notion of a linear quantum system by allowing so-called Bogoliubov transformations to replace the scattering matrix S.

2.3 Equations of Motion: Real Quadrature Form and Complex Mode Form

Using the quantum Itō rule and the quantum Itō products, and exploiting the canonical commutation relations between the operators in x, the *Heisenberg evolution*

$$x(t) = U(t)^* x U(t) = \begin{bmatrix} U(t)^* q_1 U(t) \\ U(t)^* p_1 U(t) \\ U(t)^* q_2 U(t) \\ U(t)^* p_2 U(t) \\ \vdots \\ U(t)^* q_n U(t) \\ U(t)^* p_n U(t) \end{bmatrix}$$

of the vector x can be obtained [14, 15]. This is given by the QSDE,

$$\begin{aligned} dx(t) &= d(U(t)^* x U(t)) \\ &= A_o x(t)dt + B_o \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; x(0) = x, \\ d\mathcal{Y}(t) &= d(U(t)^* \mathcal{A}(t) U(t)) \\ &= C_o x(t)dt + D_o d\mathcal{A}(t), \end{aligned} \tag{2.4}$$

with

$$\begin{aligned}
A_o &= 2\mathbb{J}_n(R + \Im\{K^*K\}), \\
B_o &= 2\imath\mathbb{J}_n[\,-K^\dagger S \;\; K^\top S^\#\,], \\
C_o &= K, \\
D_o &= S.
\end{aligned}$$

Here,

$$\mathcal{Y}(t) = (\mathcal{Y}_1(t), \ldots, \mathcal{Y}_m(t))^\top = U(t)^*\mathcal{A}(t)U(t)$$

is a vector of *output fields* that is produced by the interaction of the quantum harmonic oscillators and the incoming quantum fields $\mathcal{A}(t)$. Note that the Heisenberg picture dynamics of $x(t)$ is linear, and $\mathcal{Y}(t)$ has a component which is a linear combination of elements of $x(t)$. The index n in the above will be referred to as the number of *degrees of freedom* or simply the *degree* of the linear quantum stochastic system.

2.3.1 Real Quadrature Form

In certain circumstances, it is convenient to write the dynamics (2.4) in the so-called (real) quadrature form as in [15–17]:

$$\begin{aligned}
dx(t) &= Ax(t)dt + Bdw(t);\; x(0) = x, \\
dy(t) &= Cx(t)dt + Ddw(t),
\end{aligned} \tag{2.5}$$

with

$$\begin{aligned}
w(t) &= 2(\Re\{\mathcal{A}_1(t)\}, \Im\{\mathcal{A}_1(t)\}, \ldots, \Re\{\mathcal{A}_m(t)\}, \Im\{\mathcal{A}_m(t)\})^\top, \\
y(t) &= 2(\Re\{\mathcal{Y}_1(t)\}, \Im\{\mathcal{Y}_1(t)\}, \ldots, \Re\{\mathcal{Y}_m(t)\}, \Im\{\mathcal{Y}_m(t)\})^\top.
\end{aligned}$$

Here,

$$A = 2\mathbb{J}_n(R + \Im\{K^*K\}), \tag{2.6}$$

$$B = 2\imath\mathbb{J}_n[\,-K^* \;\; K^\top\,]\mathrm{diag}(S, S^\#)\Gamma_m, \tag{2.7}$$

$$C = P_m^\top \begin{bmatrix} K + K^\# \\ -\imath K + \imath K^\# \end{bmatrix}, \tag{2.8}$$

$$D = \Gamma_m^{-1}\mathrm{diag}(S, S^\#)\Gamma_m, \tag{2.9}$$

where P_m denotes a $2m \times 2m$ *permutation matrix* acting as

$$P_m[\, a_1 \ a_2 \ \ldots \ a_{2m-1} \ a_{2m} \,]^\top \\ = [\, a_1 \ a_3 \ \ldots \ a_{2m-1} \ a_2 \ a_4 \ \ldots \ a_{2m} \,]^\top,$$

and

$$\Gamma_m = P_m \left(I_m \otimes \frac{1}{2} \begin{bmatrix} 1 & \imath \\ 1 & -\imath \end{bmatrix} \right).$$

Note that the matrices A, B, C, D in the quadrature form are all real and are in a one to one correspondence with the matrices A_o, B_o, C_o, D_o, and the quantum noise vector $w(t)$ satisfies the Itō relationship $dw(t)dw(t)^\top = (I + \imath \mathbb{J}_m)dt$.

Some remarks are in order about the nature of the quadrature quantum noise w. From the self non-demolition property we have that $[w_j(s), w_j(t)] = 0$ (w_j denotes the j-th component of w) for all $s, t \geq 0$ and all j. Indeed,

$$\begin{aligned} &[w_j(s), w_j(t)] \\ &\quad = [\mathcal{A}_j(t) + \mathcal{A}_j^*(t), \mathcal{A}_j(s) + \mathcal{A}_j^*(s)] \\ &\quad = [\mathcal{A}_j(t), \mathcal{A}_j(s)] + [\mathcal{A}_j(t), \mathcal{A}_j^*(s)] + [\mathcal{A}_j^*(t), \mathcal{A}_j(s)] + [\mathcal{A}_j^*(t), \mathcal{A}_j^*(s)] \\ &\quad = \min(t, s) - \min(t, s) \\ &\quad = 0. \end{aligned}$$

When the state is the vacuum state, w_j is isomorphic to a standard Wiener process, in the sense that it is a realization of a standard Wiener process as an operator-valued process on a Fock space. This will be demonstrated in Sect. 2.7.2 when Gaussian states of a bosonic field are discussed. However, for any $j = 1, 2, \ldots, m$, $w_{2j-1}(t)$ and $w_{2j}(t)$ do not commute for any t, meaning that they cannot be realized jointly on a common classical probability space. On the other hand, we do have that $[w_{2j-1}(s), w_k(t)] = 0$ for all $k \neq 2j$ and all $s, t \geq 0$, so that the collection of processes $\{w_{2j-1},\ w_k,\ k \neq 2j\}$ is equivalent to a collection of independent standard Wiener processes that can be jointly defined on a common classical probability space. The non-commutativity among components of $w(t)$ is a crucial difference from the usual vector of classical independent Wiener processes.

2.3.2 Complex Mode Form

In other situations, it may be more convenient to represent the dynamics of a linear quantum stochastic system in the (complex) mode form. In this form, instead of writing down the Heisenberg evolution of amplitude-phase quadrature pairs $(q_1, p_1, q_2, p_2, \ldots, q_n, p_n)^\top$, we write down the Heisenberg evolution of the annihilation–creation pairs $(a_1, a_1^*, a_2, a_2^*, \ldots, a_n, a_n^*)^\top$, where $a_j = \frac{1}{2}(q_j + \imath p_j)$ and

$a_j^* = \frac{1}{2}(q_j - \imath p_j)$. In terms of these operators, the quadratic Hamiltonian H and coupling operator L can be expressed as [18]

$$H = a^* \Omega_- a + \frac{1}{2} a^* \Omega_+ a^\# + \frac{1}{2} a^\top \Omega_+^\# a,$$

where $\Omega_+ = [\omega_{jk}^+]$ and $\Omega_- = [\omega_{jk}^-]$ are complex $n \times n$ matrices possessing the symmetries $\Omega_-^* = \Omega_-$ and $\Omega_+^\top = \Omega_+$, and

$$L = C_- a + C_+ a^\#,$$

where $C_\pm \in \mathbb{C}^{m \times n}$. Using these expression for H and L one can write down the QSDE for the Heisenberg evolution of a and $a^\#$. Let

$$\begin{aligned} a(t) &= U(t)^* a U(t) \\ &= (U(t)^* a_1 U(t), U(t)^* a_2 U(t), \ldots, U(t)^* a_n U(t))^\top, \end{aligned}$$

and introduce the *doubled-up matrix*

$$\Delta(E_-, E_+) \triangleq \begin{bmatrix} E_- & E_+ \\ E_+^\# & E_-^\# \end{bmatrix},$$

for any complex matrices $E_\pm$ of the same dimension. We also introduce the notation

$$\mathbb{K}_m \triangleq \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \tag{2.10}$$

and define the $\cdot^\flat$ involution operator that acts on a $2k \times 2l$ matrix X as

$$X^\flat = \mathbb{K}_l X^* \mathbb{K}_k. \tag{2.11}$$

Then the QSDE for $(a(t)^\top, a(t)^*)^\top$ and the output $\mathcal{Y}(t)$ can be expressed in the doubled-up form [18]

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} &= \tilde{A} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + \tilde{B} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}, \\ \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix} &= \tilde{C} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + \tilde{D} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} A_\mp &= -\frac{1}{2}(C_-^* C_\mp - C_+^\top C_\pm^\# - \imath \Omega_\mp), \\ \Omega_- &= -\frac{1}{2\imath}(A_- - A_-^*), \end{aligned}$$

$$\Omega_+ = -\frac{1}{2\imath}(A_+ + A_+^\top),$$
$$\tilde{A} = \Delta(A_-, A_+), \tag{2.13}$$
$$\tilde{B} = -\Delta(C_-, C_+)^\flat \Delta(S, 0), \tag{2.14}$$
$$\tilde{C} = \Delta(C_-, C_+), \tag{2.15}$$
$$\tilde{D} = \Delta(S, 0). \tag{2.16}$$

Note that it is necessary to adopt a doubled-up form since the evolutions of $a(t)$ and $a(t)^\#$ are, in general, coupled. The following example illustrates the description of a system in mode form.

Example 2.1 Consider the nondegenerate optical parametric amplifier (NOPA) from Sect. 1.5.2. We consider a lossless version of this device with $\kappa = 0$. Therefore, the device is only coupled to two fields collected in the vector $\mathcal{A}(t)$, and has parameters $\Omega_- = 0_{2\times 2}$, $\Omega_+ = \begin{bmatrix} 0 & \imath\epsilon/2 \\ \imath\epsilon/2 & 0 \end{bmatrix}$, $C_- = \begin{bmatrix} \sqrt{\gamma} & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}$, $C_+ = 0_{2\times 2}$, and $S = I_2$. The doubled-up form of the evolution of the cavity modes in the vector $a = [\, a_1 \; a_2 \,]^\top$ is

$$\begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} = \begin{bmatrix} -\gamma/2 & 0 & 0 & \epsilon/2 \\ 0 & -\gamma/2 & \epsilon/2 & 0 \\ 0 & \epsilon/2 & -\gamma/2 & 0 \\ \epsilon/2 & 0 & 0 & -\gamma/2 \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt$$
$$- \begin{bmatrix} \sqrt{\gamma} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma} & 0 \\ 0 & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma} \end{bmatrix} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix},$$
$$\begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix} = \sqrt{\gamma} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}.$$

In the physics literature, it is more common for the quantum stochastic dynamics to be written in the heuristic form of a quantum Langevin equation (introduced earlier in Chap. 1) driven by a quantum white noise process $\xi(t)$, where $\mathcal{A}(t) = \int_0^t \xi(s) ds$. For the mode form of the equation of motion, this quantum Langevin equation takes the form

$$\begin{bmatrix} \dot{a}(t) \\ \dot{a}(t)^\# \end{bmatrix} = \tilde{A} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} + \tilde{B} \begin{bmatrix} \xi(t) \\ \xi(t)^\# \end{bmatrix},$$
$$\begin{bmatrix} \eta(t) \\ \eta(t)^\# \end{bmatrix} = \tilde{C} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} + \tilde{D} \begin{bmatrix} \xi(t) \\ \xi(t)^\# \end{bmatrix},$$

where η is the output process that propagates from the system after the incoming white noise ξ interacts with the latter. The process η satisfies the same time commutation relations as ξ.

Remark 2.1 Note that for notational expediency, in the monograph we will often not explicitly write the time dependence in the QSDEs describing the evolution of the system.

2.3.3 Transfer Function of Linear Dynamical Quantum Systems

Whether one is working with a linear quantum system in the quadrature form (2.5) or the mode form (2.12), as with classical linear systems, one can define a transfer function. We start with a system $G = (A, B, C, D)$ in the quadrature form (2.5). Following [18], we can define the Laplace transform of an adapted quantum stochastic process such as $w(t)$ as $w[s] = \int_0^\infty e^{-st} dw(t)$. The "particular" solution of (2.5), that does not depend on the initial condition $x(0) = x$, is given by

$$y_p(t) = \int_0^t Ce^{A(t-s)} B dw(s) + Dw(t).$$

Now, taking Laplace transforms of both side of the equality gives

$$y_p[s] = (C(sI - A)^{-1}B + D)w[s],$$

and from this we can define the transfer function $\Xi_G(s)$ from $w[s]$ to $y_p[s]$ of G just as for a classical linear system,

$$\Xi_G[s] = C(sI - A)^{-1}B + D.$$

Similarly, if we are working in the mode form (2.12) we define $\mathcal{A}[s] = \int_0^\infty e^{-s\tau} d\mathcal{A}(\tau)$, $\mathcal{Y}[s] = \int_0^\infty e^{-s\tau} d\mathcal{Y}(\tau)$, $\xi[s] = \int_0^\infty e^{-s\tau} \xi(\tau) d\tau$, $\eta[s] = \int_0^\infty e^{-s\tau} \eta(\tau) d\tau$. Since $\mathcal{A}(t) = \int_0^t \xi(\tau) d\tau$ and $\mathcal{Y}(t) = \int_0^t \eta(\tau) d\tau$, note that $\mathcal{A}[s] = \xi[s]$ and $\mathcal{Y}[s] = \eta[s]$. Moreover, $\mathcal{A}^\#[s] = \mathcal{A}[s^*]^\#$ and analogous identities hold for $\mathcal{Y}[s]$, $\xi[s]$ and $\eta[s]$. The transfer function $\tilde{\Xi}_G$ from $(\mathcal{A}[s], \mathcal{A}^*[s])^\top$ to $(\mathcal{Y}[s], \mathcal{Y}^*[s])^\top$ or, equivalently, from $(\xi[s], \xi^*[s])^\top$ to $(\eta[s], \eta^*[s])^\top$, for G in the mode form $G = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is given by

$$\tilde{\Xi}_G[s] = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}.$$

The two transfer functions Ξ_G and $\tilde{\Xi}_G$ for the quadrature and mode forms of G, respectively, are related by the identity

$$\Xi_G[s] = \Gamma_m^{-1} \tilde{\Xi}_G[s] \Gamma_m,$$

where Γ_m is as defined before.

In analogy with the classical case, the transfer function is important in analyzing the input–output behavior of linear quantum systems.

2.4 Inclusion of Idealized Static Transformations on Bosonic Fields: The Bogoliubov Transformation

When $H = 0$ and $L = 0$, a linear unitary transformation of the input field $\mathcal{A}(t)$ is realized,

$$\mathcal{Y}(t) = S\mathcal{A}(t).$$

Such a transformation preserves the differential commutation relations, that is,

$$\left[d\begin{bmatrix}\mathcal{Y}(t)\\ \mathcal{Y}(t)^{\#}\end{bmatrix}, d\begin{bmatrix}\mathcal{Y}(t)\\ \mathcal{Y}(t)^{\#}\end{bmatrix}^{*}\right] = \left[d\begin{bmatrix}\mathcal{A}(t)\\ \mathcal{A}(t)^{\#}\end{bmatrix}, d\begin{bmatrix}\mathcal{A}(t)\\ \mathcal{A}(t)^{\#}\end{bmatrix}^{*}\right] = \mathbb{K}_m dt, \tag{2.17}$$

where for a (column) vector of operators v and w,

$$[v, w^*] = vw^* - (w^{\#}v^{\top})^{\top}.$$

However, a unitary S is not the only matrix that can result in preservation of the differential commutation relations. More generally, one may consider a so-called *Bogoliubov transformation* implemented by a *Bogoliubov matrix* $W = \Delta(W_+, W_-)$ of the form

$$\begin{bmatrix}\mathcal{Y}'(t)\\ \mathcal{Y}'(t)^{\#}\end{bmatrix} = W\begin{bmatrix}\mathcal{A}(t)\\ \mathcal{A}(t)^{\#}\end{bmatrix}, \tag{2.18}$$

such that $\mathcal{Y}'(t)$ satisfies the same differential commutation relation as $\mathcal{A}(t)$. This is guaranteed by the definition of a Bogoliubov matrix, that will now be given.

Definition 2.1 Let $\cdot^{\flat}$ denote the involution defined via (2.11). A complex $2m \times 2m$ matrix W is a Bogoliubov matrix if it is of a doubled-up form $W = \Delta(W_+, W_-)$ for some complex $m \times m$ matrices W_+, W_-, and W is $\flat$-unitary,

$$W^{\flat}W = WW^{\flat} = I_{2m}.$$

Notice that, by definition, $W^{\flat} = W^{-1}$ and W reduces to a unitary matrix when $W_- = 0$. Linear transformations of the form (2.18) preserve the differential commutation relation, so that the output field $\mathcal{Y}'(t)$ is a valid output field. However, when $W_- \neq 0$ it cannot be modeled within the Hudson–Parthasarathy QSDE framework. That is, there does *not* exist a unitary solution $U(t)$ of the QSDE such that $\mathcal{Y}'(t)$ in (2.18) can be written as $\mathcal{Y}'(t) = U(t)^*\mathcal{A}(t)U(t)$.

From a physical point of view, when $W_- \neq 0$ the "pathology" discussed above can be understood from the fact that transformations such as (2.18) represent the action of an idealized infinite-bandwidth squeezing device acting on an incoming boson field $\mathcal{A}(t)$. Squeezing here means that the device reduces the quantum fluctuations on some quadratures of the incoming field $\mathcal{A}(t)$ while increasing them in other quadratures. We will shortly illustrate this point by demonstrating how a Bogoliubov transformation arises as the infinite-bandwidth limit of a degenerate parameteric amplifier. Such an idealized device is not physical since it produces uniform squeezing across the continuum of frequencies contained in $\mathcal{A}(t)$, a process that would require an infinite amount of energy. From a mathematical physics perspective, the linear transformation (2.18) represents a transformation between two fields $\mathcal{A}(t)$ and $\mathcal{Y}'(t)$. The former arises from a Fock space representation of field operators satisfying the canonical commutation relations $[\mathcal{A}(f), \mathcal{A}^*(g)] = \int_0^\infty f(s)^* g(s) ds$ with respect to vacuum state of the field, while the latter arises from a Fock space representation with respect to another state induced by W (e.g., a squeezed field state as postulated in [6, Chap. 6]); see, e.g., [20] for a more in-depth treatment of these representations in the case of squeezed field states. When $W_- \neq 0$ the two representations are not unitarily equivalent, since the transformation involved violates the necessary and sufficient conditions of the Shale's Theorem [1, Theorem 22.11].

Although the transformation (2.18) with W_- represents an idealized non-physical scenario, it is convenient to include in the modeling repertoire as it allows some simplification in describing the system. That is, it allows high-bandwidth dynamic linear devices with fast dynamics to be approximated as static linear transformations without any dynamics. This would be appropriate in instances where the fast device is connected to other linear devices with appreciably lower bandwidth (slower dynamics). We illustrate this in the next example.

Example 2.2 Recall the degenerate parametric amplifier (DPA) introduced in Sect. 1.5.3. Under the assumption of losslessness ($\kappa = 0$), the DPA is modeled as a single oscillator coupled to a single field with $S = I$, $\Omega_- = 0$, $\Omega_+ = \imath\epsilon$, $\epsilon > 0$, $C_- = \sqrt{\gamma}$, and $C_+ = 0$. The matrix $\tilde{A}$ is Hurwitz (all its eigenvalues are in the left half plane) and the system is stable (in the sense that the oscillator's mean photon number is bounded at all times) if we take $\epsilon < \gamma$ as we shall do from now on. The mode form transfer function of the system is

$$\tilde{\Xi}_{\mathrm{DPA}}[s] = \frac{1}{P(s)} \begin{bmatrix} s^2 - (\gamma^2 + \epsilon^2)/4 & -\epsilon\gamma/2 \\ -\epsilon\gamma/2 & s^2 - (\gamma^2 + \epsilon^2)/4 \end{bmatrix},$$

where $P(s) = (s + \gamma/2)^2 - \epsilon^2/4$. The zeros of P are the poles of the transfer function, namely $s = \pm\epsilon/2 - \gamma/2$. In the frequency domain, the output white noise field is

$$\eta[s] = \frac{1}{P[s]} \left(s^2 - (\gamma^2 + \epsilon^2)/4\right) \xi[s] - \frac{1}{2P(s)} \epsilon\gamma\xi^*[s],$$

where $\xi^*[s] = \xi[s^*]^*$. In terms of quadratures $\eta^q = \eta + \eta^*$ and $\eta^p = (\eta - \eta^*)/\imath$, we find that

$$\eta^q[s] = \Xi^q_{\mathrm{DPA}}[s]\eta^q[s], \quad \eta^p[s] = \Xi^p_{\mathrm{DPA}}[s]\eta^p[s],$$

where

$$\Xi^q_{\mathrm{DPA}}[s] = \frac{s-(\gamma+\epsilon)/2}{s+(\gamma-\epsilon)/2} = \frac{1}{\Xi^p_{\mathrm{DPA}}[s]}.$$

In this example we consider the idealized case where $\gamma, \epsilon \to \infty$ (approximated in practice by suitably large values of these parameters) while maintaining the ratio $\frac{\epsilon}{\gamma}$ to be constant; see [6, 10.2.1.g] for a more complete discussion. Rescaling $\gamma = k\gamma_0$ and $\epsilon = k\epsilon_0$ is equivalent to the substitution of γ by γ_0 and ϵ by ϵ_0 and the rescaling of s as $\frac{s}{k}$:

$$\tilde{\Xi}_{\mathrm{DPA}}[s; \gamma = k\gamma_0, \epsilon = k\epsilon_0] = \tilde{\Xi}_{\mathrm{DPA}}[s/k; \gamma_0, \epsilon_0].$$

We now take the limit $k \to \infty$ in which the cavity has an instantaneous response. Thus the cavity's internal dynamics are essentially eliminated. This yields the output $\eta[s]$ given by

$$\eta[s] = -\cosh(r_0)\,\eta[s] - \sinh(r_0)\,\eta^*[s],$$

where

$$r_0 = \ln\frac{\gamma_0+\epsilon_0}{\gamma_0-\epsilon_0},$$

as a Bogoliubov transformation of the input. In this limit the DPA can thus be viewed as a static device that outputs a squeezed white noise field from a vacuum white noise source. The transfer function becomes a constant Bogoliubov matrix that is independent of frequency. That is,

$$\begin{aligned}\tilde{\Xi}_{\mathrm{DPA\,static}}[s] &= \lim_{k\to\infty} \tilde{\Xi}_{\mathrm{DPA}}[s/k; \gamma_0, \epsilon_0]\\ &= -\Delta(\cosh r_0, \sinh r_0), \forall s \in \mathbb{C},\end{aligned}$$

and the quadrature transfer functions are the constant functions

$$\Xi^q_{\mathrm{DPA\,static}}[s] = -e^{r_0},\ \Xi^p_{\mathrm{DPA\,static}}[s] = -e^{-r_0}.$$

Therefore, we see that in the high bandwidth regime, the DPA can be approximated by a Bogoliubov transformation implemented by the Bogoliubov matrix

$$W = -\begin{bmatrix}\cosh(r_0) & \sinh(r_0)\\ \sinh(r_0) & \cosh(r_0)\end{bmatrix}.$$

When a linear quantum system (2.12) with $\tilde{D} = I$ is driven by vacuum noise that has passed through a device modelled as a static Bogoliubov transformation $W = \Delta(W_-, W_+)$, as shown in part (a) of Fig. 2.1, then the equation of motion for the system is again of the form (2.12) but now with the matrix $\tilde{D} = \Delta(S, 0)$ being replaced by $\tilde{D} = W$ in (2.13)–(2.16). In particular, we have that

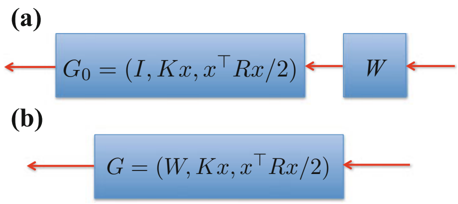

Fig. 2.1 **a** A static device performing a linear transformation W driving a linear quantum system G_0 with scattering matrix $S = I$, and **b** Equivalent representation of the cascaded system in part **a** as a single linear quantum system $G = (W, L, H)$

$$\tilde{A} = -\frac{1}{2}\tilde{C}^\flat \tilde{C} - \imath \tilde{\Omega}, \tag{2.19}$$

$$\tilde{B} = -\tilde{C}^\flat \tilde{D}, \tag{2.20}$$

$$\tilde{C} = \Delta(C_-, C_+), \tag{2.21}$$

$$\tilde{D} = W, \tag{2.22}$$

with

$$\tilde{\Omega} = -\imath \Delta(\imath \Omega_-, \imath \Omega_+).$$

Note the property that $\tilde{\Omega}^\flat = \tilde{\Omega}$. We denote systems of the type depicted in Fig. 2.1 part (a) with the shorthand notation $G = (W, L, H)$, as shown in part (b) of the figure.

2.4.1 Completely Passive Linear Dynamical Quantum Systems

This section will introduce a special class of linear quantum systems that shall be referred to as *completely passive* linear quantum stochastic systems, for reasons that will be explained below. This class will appear in several contexts in the monograph and are of interest in applications such linear optical quantum memories. Further discussions relating to this class of systems can be found in, e.g., [21–23] and in Chap. 3 of this monograph.

For $k = 1, \ldots, n$ let $a_k = \frac{q_k + \imath p_k}{2}$ be the annihilation operator for mode k and define $a = (a_1, \ldots, a_n)^\top$. The vector a satisfies the CCR

$$\begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a^* & a^\top \end{bmatrix} - \left(\begin{bmatrix} a^\# \\ a \end{bmatrix} \begin{bmatrix} a^\top & a^* \end{bmatrix} \right)^\top$$
$$= \mathrm{diag}(I_n, -I_n),$$

and note that,

$$\begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix} x,$$

where

$$\Sigma = \begin{bmatrix} \frac{1}{2} & \imath\frac{1}{2} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2} & \imath\frac{1}{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \frac{1}{2} & \imath\frac{1}{2} \end{bmatrix}.$$

Moreover, we also have $\begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix}^{-1} = 2\left[\, \Sigma^* \;\; \Sigma^\top \,\right]$ and, from the relation $\begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix} 2$ $[\, \Sigma^* \;\; \Sigma^\top \,] = I$, the identities,

$$\Sigma\Sigma^* = I/2 = \Sigma^\#\Sigma^\top, \;\; \Sigma\Sigma^\top = 0 = \Sigma^\#\Sigma^*. \tag{2.23}$$

Therefore,

$$x = \begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix}^{-1} \begin{bmatrix} a \\ a^\# \end{bmatrix} = 2\left[\, \Sigma^* \;\; \Sigma^\top \,\right] \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

We say that a linear quantum system $G = (S, Kx, (1/2)x^\top Rx)$ with n degrees of freedom is completely passive if $H = (1/2)x^\top Rx = (1/2)a^*\tilde{R}a + c$ and $L = Kx = \tilde{K}a$ for some real constant c, some complex $n \times n$ *Hermitian* matrix $\tilde{R}$, and some complex $m \times n$ matrix $\tilde{K}$. The terminology is motivated by physical considerations. Notice that the Hamiltonian H contains no terms of the form $c_1 a_j^2$, $c_2 a_k^{*2}$, $c_3 a_j a_k$ and $c_4 a_j^* a_k^*$ and, likewise, the coupling operator L contains no terms of the form $c_5 a_k^*$ (where $c_1, \dots, c_5$ denote arbitrary complex constants), with the indices j and k running over $1, \dots, n$. These terms correspond to interactions that require an external source of quanta (e.g., an external pump beam) to implement. Therefore, they cannot be realized using only passive optical components like phase shifters, beam splitters and mirrors, devices that will be discussed in more detail in Chap. 3. This is the physical motivation for referring to these class of systems as completely passive.

In a completely passive system, the Heisenberg picture evolution of the system annihilation operator $a(t)$ is not coupled to $a(t)^\#$ or to $\mathcal{A}(t)^\#$. In the complex mode form, the evolution of $a(t)$ is given by

$$\begin{aligned} da(t) &= (\imath\tilde{R} - (1/2)\tilde{K}^*\tilde{K})a(t)dt + \tilde{K}^* S d\mathcal{A}(t), \\ d\mathcal{Y}(t) &= \tilde{K}a(t)dt + Sd\mathcal{A}(t). \end{aligned}$$

The transfer function $\Xi[s]$ from $\mathcal{A}[s]$ to $\mathcal{Y}[s]$ given by,

$$\Xi[s] = \left(\tilde{K}\left(sI - (\imath\tilde{R} - (1/2)\tilde{K}^*\tilde{K})\right)^{-1}\tilde{K}^* + I\right)S,$$

has the special property that $\Xi[\imath\omega]$ is unitary for all for all ω for which $\imath\omega$ is not an imaginary eigenvalue of $\imath\tilde{R} - (1/2)\tilde{K}^*\tilde{K}$ [21]. It is easy to see that $\imath\tilde{R} - (1/2)\tilde{K}^*\tilde{K}$ cannot have any eigenvalues in the (open) right half plane, with a positive real part. Moreover, if this matrix has all its eigenvalues in the (open) left half plane (all eigenvalues have negative real part), meaning that the system is stable in a sense that will be discussed in Sect. 2.6, then $\Xi[s]$ is in fact lossless bounded real [22]. This means that $\Xi[s]$ is analytic for all s with $\Re\{s\} > 0$, $\Xi[\imath\omega]$ is unitary for all $\omega \in \mathbb{R}$ and, moreover, $\Xi[s]$ satisfies,

$$\Xi[s]^*\Xi[s] \leq I,$$

for all s with $\Re\{s\} > 0$.

2.5 Physical Realizability Conditions and Parameterizations for Linear Dynamical Quantum Systems

2.5.1 Physical Realizability Conditions for Linear QSDEs

We have now seen that the dynamics of linear quantum stochastic systems in the Heisenberg picture leads to an equation of motion in the form of linear QSDEs, i.e., as given by Eq. (2.5) in the quadrature form and (2.12) in the mode form. These equations can be viewed as a quantum analogue of the equations that describe classical linear stochastic systems. However, unlike classical linear stochastic systems, the matrices A, B, C, D in (2.5) or the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ in (2.12) cannot be arbitrary, but are constrained by quantum mechanics so that these equations represent the evolution of a valid (open) quantum system. When the linear QSDE represents a valid quantum system, it is said that the QSDE or the system represented by the QSDE is physically realizable. To avoid unnecessary repetition, a formal definition will be given below for systems in the quadrature form. This definition can be directly adapted to systems in the mode form due to the one-to-one relationship between the two forms.

Definition 2.2 (*Physical realizability*) A system of linear quantum stochastic differential equations of the form (2.5) is said to be physically realizable if there exists a quadratic Hamiltonian $H = (1/2)x^\top Rx$, a coupling operator $L = Kx$, and a Bogoliubov matrix W such that

$$A = 2\mathbb{J}_n(R + \Im\{K^*K\}), \tag{2.24}$$

$$B = 2\imath\mathbb{J}_n[\,-K^* \;\; K^\top\,]W\Gamma_m, \tag{2.25}$$

$$C = P_m^\top \begin{bmatrix} K + K^\# \\ -\imath K + \imath K^\# \end{bmatrix}, \tag{2.26}$$

$$D = \Gamma_m^{-1}W\Gamma_m. \tag{2.27}$$

Essentially, the definition states that if a system of QSDEs (2.5) is physically realizable then there is a linear quantum system $G = (W, L, H)$ for which the Heisenberg evolution of x is given by (2.5). G is, of course, by definition, a physical system. We have the following result [16, 24, 25]

Theorem 2.1 *A system of linear quantum stochastic differential equations of the form (2.5) is physically realizable if and only if*

$$A\mathbb{J}_n + \mathbb{J}_n A^\top + B\mathbb{J}_m B^\top = 0, \tag{2.28}$$

$$\mathbb{J}_n C^\top + B\mathbb{J}_m D^\top = 0, \tag{2.29}$$

$$D\mathbb{J}_m D^\top = \mathbb{J}_m. \tag{2.30}$$

Proof Note that the condition (2.30) is equivalent to the statement that D is a symplectic matrix (hence it is invertible with D^{-1} also symplectic). Without loss of generality, we may reduce the proof to the case with $D = I$ (i.e., $W = I$) for the following reasons. Let $dy_o = D^{-1}Cx(t)dt + dw(t)$, then $y(t) = Dy_{\mathrm{o}}(t)$. Let $C' = D^{-1}C$ and consider the linear quantum stochastic system,

$$dx(t) = Ax(t)dt + Bdw(t),$$
$$dy_{\mathrm{o}}(t) = C'x(t)dt + dw(t).$$

We will show that this system is physically realizable if and only if (2.28)–(2.30) is satisfied with C replaced with C' and $D = I$. From this it easily follows that the original system (2.5) is physically realizable if and only if (2.28)–(2.30) holds. This is because y is a symplectic transformation of y_{o}, so if the system with output y is physically realizable so is the system with output y_{o}, and vice-versa. Thus for the remainder of the proof we identify C with C' and D with I.

To this end, let us first show that physical realizability of the system $G = (A, B, C, I)$ implies (2.28)–(2.30). Indeed, physical realizability implies that the canonical commutation relation for $x(t)$,

$$x(t)x(t)^\top - (x(t)x(t)^\top)^\top = 2\imath\mathbb{J}_n,$$

and the differential commutation relation for $y(t)$,

$$dy(t)dy(t)^\top - (dy(t)dy(t)^\top)^\top = 2\imath\mathbb{J}_n dt,$$

must hold for all times $t \geq 0$. This follows from the fact that the joint evolution of the system and the bosonic fields is unitary (given by the solution of a Hudson–Parthasarathy QSDE). Since $x(0) = x$ satisfies $xx^\top - (xx^\top)^\top = 2\imath\mathbb{J}_n$, for the commutation relation to hold for all $x(t)$, $t \geq 0$, we must have

$$d(x(t)x(t)^\top - (x(t)x(t)^\top)^\top) = 0.$$

The condition (2.28) follows from this by substituting the expression $dx(t) = Ax(t)dt + Bdw(t)$ and using the quantum Itō product rule, and the Itō table to explicitly compute the differential. By the same procedure and using the expression $dy(t) = Cx(t)dt + dw(t)$, we obtain (2.29). The details of the calculations can be found in [16, 26].

We now consider the converse implication, that (2.28) and (2.29) imply that there exist $R = R^\top \in \mathbb{R}^{2n\times 2n}$ and $K \in \mathbb{C}^{m\times 2n}$ such that (2.24)–(2.27) holds. Indeed, define $R = \frac{1}{4}(-\mathbb{J}_n A + A^\top \mathbb{J}_n)$ and $K = -\frac{\imath}{2}[\, 0_{m\times m} \; I_{m\times m}\,]\Gamma_m^{-1} B^\top \mathbb{J}_n$; see [16, 26] for details of the origin of these expressions. Using these definitions and (2.28), we can directly calculate that $\mathbb{J}_n(R + \Im\{K^* K\}) = \mathbb{J}_n R + \mathbb{J}_n \Im\{K^* K\} = A$ and $2\imath \mathbb{J}_n[\, -K^* \; K^\top \,]\Gamma_n = B$. Using the last expression for B and (2.29), we then also get that $C = P_m^\top \begin{bmatrix} K + K^\# \\ -\imath K + \imath K^\# \end{bmatrix}$. This completes the proof. □

It should be noted from the above theorem that, remarkably, physical realizability is equivalent to preservation of canonical commutation relations for $x(t)$ and $y(t)$. The notion of physical realizability and the physical realizability constraints on the systems matrices were first introduced in [16, 26] for the quadrature form.

2.5.2 Parameterization of Linear Dynamical Quantum Systems

From the exposition in the preceding sections, we see that there are two sets of parameters that can be used to parameterize linear quantum stochastic systems with the same number of inputs and outputs. The first is the set of three parameters (W, K, R), with W a $2n \times 2n$ Bogoliubov matrix, K an $m \times 2n$ complex matrix, and R a real symmetric $2n \times 2n$ matrix, that describe the physical parameters of the system including the Hamiltonian H and the linear coupling operator L. The other parameterization is via the system matrices (A, B, C, D) that appear in the QSDE for the Heisenberg evolution (2.5) of the system.

The two parameterizations (W, K, R) and (A, B, C, D) are equivalent, and can be used interchangeably, according to which one may be more convenient for the purpose at hand. This is because of a bijective correspondence between the two parameterizations: to any given (W, K, R) parametrization there corresponds a unique (A, B, C, D) parameterization, and vice-versa. The (A, B, C, D) matrices for a given set of (W, K, R) matrices are given by (2.6)–(2.9) (recall that the class of symplectic matrices have a bijective correspondence with the class of Bogoliubov matrices). On the other hand, the (W, K, R) parameterization can be obtained from a given set of (A, B, C, D) matrices following the proof of Theorem 2.1.

2.5.3 Linear Dynamical Quantum Systems with Less Outputs Than Inputs

In general, not all the outputs of a system may be of interest or can be observed or utilized. In many situations, one may only be interested in certain pairs of the output field quadratures in $y(t)$ [16]. In the most general scenario, one can consider $y(t)$ having an even dimension $2n_y < 2m$ and D is a $2n_y \times 2m$ matrix satisfying $D\mathbb{J}_m D^\top = \mathbb{J}_{n_y}$. That is, we can consider outputs of the form:

$$dy(t) = Cx(t)dt + Ddw(t), \tag{2.31}$$

with $C \in \mathbb{R}^{2n_y \times 2n}$, $D \in \mathbb{R}^{2n_y \times 2m}$ with $n_y < m$. Generalizing the notion discussed in Sect. 2.5.1 (for the case where there are as many outputs as there are inputs), a linear quantum system with output (2.31) is physically realizable if and only if there exist matrices $C' \in \mathbb{R}^{2(m-n_y)\times 2n}$ and $D' \in \mathbb{R}^{2(m-n_y)\times 2m}$ such that the system

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t);\ x(0) = x, \\ dy'(t) &= \begin{bmatrix} C \\ C' \end{bmatrix} x(t)dt + \begin{bmatrix} D \\ D' \end{bmatrix} dw(t), \end{aligned} \tag{2.32}$$

is physically realizable with the same number of inputs and outputs. Therefore the matrices A, B, $[\, C^\top \ (C')^\top \,]^\top$, and $[\, D^\top \ (D')^\top \,]^\top$ satisfy the constraints (2.28)–(2.30) when C and D in (2.29) and (2.30) are replaced by $[\, C^\top \ (C')^\top \,]^\top$ and $[\, D^\top \ (D')^\top \,]^\top$, respectively. We can easily obtain the following necessary and sufficient condition for physical realizability of linear quantum systems with less outputs than inputs see, e.g., [25].

Theorem 2.2 *A linear quantum stochastic system with less outputs than inputs is physically realizable if and only if*

$$A\mathbb{J}_n + \mathbb{J}_n A^\top + B\mathbb{J}_m B^\top = 0, \tag{2.33}$$

$$\mathbb{J}_n C^\top + B\mathbb{J}_m D^\top = 0, \tag{2.34}$$

$$D\mathbb{J}_m D^\top = \mathbb{J}_{n_y}. \tag{2.35}$$

Proof The necessity of (2.33)–(2.35) is immediate from the definition of physically realizable systems with less outputs than inputs given above and the physical realizability conditions for systems with the same number of input and outputs. For the sufficiency, first note that for D satisfying (2.35), it follows from a construction used in the proof of [27, Lemma 6] that a matrix $D' \in \mathbb{R}^{2(m-n_y)\times 2m}$ can be constructed such that the matrix $\tilde{D} = [\, D^\top \ (D')^\top \,]^\top$ is symplectic. Now, define $C' = D'\mathbb{J}_m B^\top \mathbb{J}_n$ and $\tilde{C} = [\, C^\top \ (C')^\top \,]^\top$. Consider now a system $\tilde{G}$ with an equal number of inputs and outputs and system matrices $(A, B, \tilde{C}, \tilde{D})$. From the physical realizability conditions (2.33)–(2.35) and the definition of C' and $\tilde{C}$, it follows that $\tilde{G}$ satisfies (2.28)–(2.30) and is therefore physically realizable. It now follows from definition that the original system with output $y(t)$, of smaller dimension than $w(t)$, is physically realizable. This completes the proof. □

Thus the physical realizability constraints for systems with the same number of outputs as inputs, or less inputs than outputs, have essentially the same form, so the following corollary is immediate.

Corollary 2.1 *An arbitrary linear quantum stochastic system is physically realizable if and only if*

$$A\mathbb{J}_n + \mathbb{J}_n A^\top + B\mathbb{J}_m B^\top = 0, \tag{2.36}$$
$$\mathbb{J}_n C^\top + B\mathbb{J}_m D^\top = 0, \tag{2.37}$$
$$D\mathbb{J}_m D^\top = \mathbb{J}_{n_y}. \tag{2.38}$$

Remark 2.2 All the results relating to linear quantum systems in this chapter depart from the assumption that the position and momentum operators satisfy the default CCR given by (2.3). However, these operators may be defined differently and satisfy a different CCR. For instance, one may wish to take as the position and momentum pair for the j-th oscillator $q_j' = q_j/\sqrt{2}$ and $p_j' = p_j/\sqrt{2}$ satisfying $[q_j', p_j'] = \imath$, and defining $x' = (q_1', p_1', q_2', p_2', \ldots, q_n', p_n')^\top$ as the canonical internal operator that will satisfy the CCR $[x', (x')^\top] = \imath\mathbb{J}_n$. More generally, one may take $x' = \mathsf{T}x$ satisfying $[x', (x')^\top] = 2\imath\mathsf{T}\mathbb{J}_n\mathsf{T}^\top$, with an invertible real $2n \times 2n$ matrix T that would typically be diagonal. All of the structural results for linear quantum systems that have been obtained can be adapted to this different choice of CCR by making the substitutions (given here for the real quadrature form):

$$\mathsf{T}A\mathsf{T}^{-1} \to A,\ \mathsf{T}B \to B,\ C\mathsf{T}^{-1} \to C,$$
$$\mathsf{T}\mathbb{J}_n\mathsf{T}^\top \to \mathbb{J}_n,\ \mathsf{T}^{-\top}R\mathsf{T}^{-1} \to R,\ K\mathsf{T}^{-1} \to K,$$

while the system matrix D and scattering matrix S remain the same.

2.6 Stability of Linear Quantum Systems

As with classical linear systems, one can define a notion of stability for linear quantum systems. We will do this for systems in quadrature form but analogous notions and results also hold for systems in complex mode form. Consider a linear quantum system in quadrature form driven by fields that are all in the vacuum state. Then quantum expectation of $\langle x(t)\rangle$ is given by the ODE

$$\dot{\langle x(t)\rangle} = A\langle x(t)\rangle.$$

The system is said to be:

- *Asymptotically stable* or simply *stable* if $|\langle x(t)\rangle| \to 0$ as $t \to \infty$ for any initial state of the system.
- *Marginally stable* if $|\langle x(t)\rangle|$ does not go to 0 as $t \to \infty$, but remains bounded at all times $t \geq 0$ for any initial state of the system.
- *Unstable* if there exists some initial state of the system such that $|\langle x(t)\rangle| \to \infty$ as $t \to \infty$.

From standard linear state-space theory, we immediately have necessary and sufficient conditions for a system to be (asymptotically) stable, marginally stable and unstable. These are:

- A system is stable if and only A is Hurwitz. That is, all its eigenvalues have a negative real part.
- A system is marginally stable if and only if A has one or more eigenvalues on the imaginary axis with geometric multiplicity equal to its algebraic multiplicity while all other remaining eigenvalues have negative real part.
- A system is unstable if and only if A has at least one eigenvalue with positive real part or one eigenvalue on the imaginary axis with geometric multiplicity less than its algebraic multiplicity.

One can associate a natural energy functional to a linear quantum system. This energy functional $\mathcal{E}(x(t))$ is defined by

$$\mathcal{E}(x(t)) = \sum_{j=1}^{n} \langle q_j(t)^2 + p_j(t)^2 \rangle = \mathrm{Tr}(\langle P(t) \rangle),$$

where $P(t) = (1/2)\left(x(t)x(t)^\top + (x(t)x(t)^\top)^\top\right)$. Note that $P(t) = P(t)^\top$ and $\langle P(t)\rangle \geq 0$. Suppose that the system has some initial energy but not receiving any energy from the external input fields (all are in the vacuum state). Then $P(t)$ satisfies the Lyapunov differential equation:

$$\dot{P}(t) = AP(t) + P(t)A^\top + BB^\top,$$

with initial condition $P(0) = (1/2)\langle xx^\top + (xx^\top)^\top \rangle$. Notice that when the system is stable, A is Hurwitz and $\lim_{t\to\infty} P(t) = P_\infty$ exists, is unique, and given by the unique solution to the algebraic Lyapunov equation,

$$AP_\infty + P_\infty A^\top + BB^\top = 0.$$

Thus, when the system is stable the system's energy in the long time settles to a steady-state value. This is a necessity in applications as unbounded energy growth leads to the system being damaged or destroyed, so ensuring stability is an important goal in the control of linear quantum systems.

2.7 Gaussian States

The oscillators and the bosonic fields to which they can be coupled are defined on an infinite-dimensional Hilbert space. For the oscillator this is the Hilbert space $\otimes_{j=1}^{n} L^2(\mathbb{R}; \mathbb{C}) \equiv L^2(\mathbb{R}^n; \mathbb{C})$, where n is the number of oscillators, while for the bosonic fields it is the Fock space $\mathcal{F}_m = \Gamma_s(L^2(\mathbb{R}_+; \mathbb{C}^m))$, where m is the multiplicity space of the field, i.e., the number of distinct bosonic fields. As such, on these fields there exist observables that have a continuous spectra which are the quantum analogue of real-valued random variables taking values in $\mathbb{R}$. Moreover, one can define states on the oscillator or field that are the quantum analogue of the Gaussian states in classical probability theory.

Gaussian states are defined differently for single-mode oscillators (a collection of oscillators with a single well-defined frequency) and bosonic fields, since the latter supports an infinite number of modes (in frequency). In this section, we provide a definition of Gaussian quantum states, beginning with the case of a collection of single-mode oscillators and then proceeding to define Gaussian states for bosonic fields. The treatment here closely follows [18, 19].

Gaussian states have a special status with respect to linear quantum systems. Due to their linear dynamics, linear quantum systems have the property that they preserve Gaussian states. This means that if the oscillators are initialized in a Gaussian state and input fields are in a Gaussian state, then the joint state of the oscillators and output fields will be in Gaussian state at all times in the system's evolution. This will be discussed in more detail in Sect. 4.3.2 when quantum Kalman filters for linear quantum systems are introduced.

2.7.1 *Gaussian State of a Collection of Single-Mode Oscillators*

Consider a collection of n independent harmonic oscillators on the collective Hilbert space $L^2(\mathbb{R}^n; \mathbb{C}) = L^2(\mathbb{R}; \mathbb{C})^{\otimes n}$, with annihilation operators $a_1, a_2, \ldots, a_n$. As before, we let $a = [\, a_1 \; a_2 \; \ldots \; a_n \,]^\top$, and introduce $\mathcal{S}_n$ to denote the set of all annihilation operators for n independent oscillators, so $a \in \mathcal{S}(n)$. A quantum state on $L^2(\mathbb{R}^n; \mathbb{C})$ is said to be Gaussian if

$$\left\langle \exp\left(\imath \left\{ [\, u^* \; u^\top \,] \begin{bmatrix} a \\ a^\# \end{bmatrix} \right\}\right) \right\rangle = \exp\left\{ -1/2 [\, u^* \; u^\top \,] F \begin{bmatrix} u \\ u^\# \end{bmatrix} + \imath [\, u^* \; u^\top \,] \begin{bmatrix} \langle a \rangle \\ \langle a^\# \rangle \end{bmatrix} \right\},$$

for all $u \in \mathbb{C}^n$ and for a positive semidefinite Hermitian matrix $F \geq 0$ given by

$$F = \left\langle \begin{bmatrix} a - \langle a \rangle \\ (a - \langle a \rangle)^{\#} \end{bmatrix} \begin{bmatrix} (a - \langle a \rangle)^{*} & (a - \langle a \rangle)^{\top} \end{bmatrix} \right\rangle = \begin{bmatrix} I + N^{\top} & M \\ M^{*} & N \end{bmatrix} \tag{2.39}$$

with

$$N_{jk} = \langle (a_j - \langle a_j \rangle)^{*}(a_k - \langle a_k \rangle) \rangle, \quad M_{jk} = \big\langle (a_j - \langle a_j \rangle)(a_k - \langle a_k \rangle) \big\rangle . \tag{2.40}$$

We have the properties that $N = N^{*}$ and $M = M^{\top}$, and $F \geq 0$ implies that $N \geq 0$. F is referred to as the "covariance matrix" of the Gaussian state. Note that the definition of a Gaussian state given above is based on a quantum generalization of the characteristic function of a classical Gaussian probability distribution, which uniquely identifies the distribution.

Let us now consider zero-mean states satisfying $\langle a_j \rangle = 0$ for all j. The joint vacuum state of the n oscillators is a special state corresponding to $N = 0, M = 0$, with $F = F_{\mathrm{vac}}$ given by

$$F_{\mathrm{vac}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.41}$$

Here the subscript vac stands for "vacuum."

For a fixed $N \geq 0$, M is constrained by the condition that $F \geq 0$. When $n = 1$, N and M are scalars and the condition $F \geq 0$ is equivalent to $N \geq 0$ with $|M|^2 \leq N\,(N + 1)$. More generally, we can perform the diagonalization $V^{*}NV = \mathrm{diag}\,(N_1, \ldots, N_n)$ for some unitary V, in which case new modes $a' = Va$ can be defined. Then N_j can be interpreted as the average number of quanta in the mode a'_j. Note that, in general, one cannot simultaneously diagonalize N and M.

2.7.1.1 Generalized Araki–Woods Representation

Given a zero-mean Gaussian state characterized by the covariance matrix F in (2.39), modes having that state can be constructed via canonical transformations of vacuum modes. That is, we will show that for any Gaussian state for which a has covariance F given by (2.39), there exists a $2n \times 4n$ matrix $\tilde{S}_0$ such that

$$\begin{bmatrix} a \\ a^{\#} \end{bmatrix} = \tilde{S}_0 \begin{bmatrix} a_0 \\ a_0^{\#} \end{bmatrix} \tag{2.42}$$

where $a_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(n + n)$ has vacuum statistics, and

$$\tilde{S}_0 \tilde{S}_0^{\flat} = I. \tag{2.43}$$

Indeed, we will construct $\tilde{S}_0 = \Delta(E_-^0, E_+^0)$ for some $n \times 2n$ matrices E_-^0, E_+^0. This can be viewed as a generalization of a construction by Araki and Woods [28] for non-squeezed thermal states, see [20, 29].

2.7.1.2 Construction of Araki–Woods Vacuum Representation

Step 1: Diagonalize N. Determine a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that $V^*NV = \text{diag}\,(N_1, \ldots, N_n)$, with the eigenvalues ordered such that $N_1 \geq \cdots \geq N_n \geq 0$. We can restrict our attention to the case with N diagonalized in this manner.
Step 2: Ignore zero eigenvalues. Take the first n_+ eigen values to be strictly positive, with the remaining $n_0 = n - n_+$ being zero. With respect to the eigen decomposition $\mathbb{C}^n = \mathbb{C}^{n_+} \oplus \mathbb{C}^{n_0}$, decompose F as

$$F = \begin{bmatrix} I + N_{++} & 0 & M_{++} & M_{+0} \\ 0 & I & M_{0+} & M_{00} \\ M_{++}^\top & M_{+0}^\top & N_{++} & 0 \\ M_{0+}^\top & M_{00}^\top & 0 & 0 \end{bmatrix}.$$

However, if a positive semidefinite matrix has a zero on a diagonal then every entry on the corresponding row and column must vanish[3] so that in fact,

$$F \equiv \begin{bmatrix} I + N_{++} & 0 & M_{++} & 0 \\ 0 & I & 0 & 0 \\ M_{++}^\top & 0 & N_{++} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can thus restrict our attention to the case where N is diagonal and positive definite (thus invertible).
Step 3: Explicit Construction. We begin by noting the constraint $I + N \geq MN^{-1}M^*$, which follows from

$$0 \leq \begin{bmatrix} I & -MN^{-1} \\ 0 & 0 \end{bmatrix} F \begin{bmatrix} I & -MN^{-1} \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} I + N - MN^{-1}M^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, introduce the matrices X, Y, Z defined by:

$$X = \sqrt{I + N - MN^{-1}M^*},$$
$$Y = \sqrt{N} = \text{diag}\left(\sqrt{N_1}, \ldots, \sqrt{N_n}\right),$$

[3]To see why this is the case, consider a complex $k \times k$ matrix $E \geq 0$ with $E_{11} = 0$. Taking $u\,(t) = (tx_1, x_2, \ldots, x_k)^\top$ we have that $0 \leq u\,(t)^* Eu\,(t) = 2t\text{Re}\{\sum_{j>1} x_1^* E_{1j} x_j + \sum_{j,k>1} x_j^* E_{jk} x_k\}$. However, for this inequality to hold for all real t it must be that $\Re\{\sum_{j>1} x_1^* E_{1j} x_j\} = 0$. Now, replacing t with $\imath t$ shows that $\Im\{\sum_{j>1} x_1^* E_{1j} x_j\}$ also vanishes. As this is true for any x_j, it must be the case that $E_{1j} = E_{j1}^* = 0$ for all $j > 1$.

$$Z = MY^{-1}.$$

Note that $Y = Y^\top$ and from $Z = MY^{-1}$ we have that $YZ^\top = M^\top = M = ZY = ZY^\top$. The matrices satisfy

$$XX^* - YY^* + ZZ^* = I \text{ and } YZ^\top = ZY^\top. \tag{2.44}$$

Now take b_1 and b_2 to be any two distinct modes in $\mathcal{S}(n)$ (hence they are commuting). Fix the state to be the joint vacuum state for these two modes. Then we have the representation

$$a = Xb_1 + Yb_2^\# + Zb_2. \tag{2.45}$$

Indeed, it can be straightforwardly verified that

$$\begin{aligned}\langle a^\# a^\top \rangle &= Y^2 = N, \\ \langle aa^\top \rangle &= ZY^\top = ZY = M.\end{aligned}$$

We have thus constructed the representation (2.42) with $\tilde{S}_0 = \Delta(E_-^0, E_+^0)$ and

$$E_-^0 = \left[X \; Z \right], \;\; E_+^0 = \left[0 \; Y \right].$$

Property (2.43) follows from (2.44).

2.7.2 *Gaussian States of the Field and Their Fock Space Representation*

Consider m continuous-mode bosonic fields indexed by $j = 1, 2, \ldots, m$ with annihilation field operators $b_j(t)$ satisfying the field commutation relations $[b_j(t), b_k(t')^*] = \delta_{jk}\delta(t - t')$ and $[b_j(t), b_k(t')] = 0$. Let us introduce the shorthand notation,

$$\breve{b} = \begin{bmatrix} b \\ b^\# \end{bmatrix}.$$

A Gaussian state of a continuous-mode bosonic field is state with characteristic function of the form

$$\begin{aligned}&\left\langle \exp\left(\imath \int_0^\infty \breve{u}(t)^* \breve{b}(t)\, dt \right) \right\rangle \\ &\quad = \exp\left(-1/2 \int_0^\infty \breve{u}(t)^* F \breve{u}(t)\, dt + \imath \int_0^\infty \breve{u}(t)^* \langle \breve{b}(t) \rangle dt \right),\end{aligned}$$

for any $u \in L^2(\mathbb{R}_+; \mathbb{C}^m)$. Here, as in the single-mode case, F is a $2m \times 2m$ Hermitian matrix that is again of the form (2.39) with the entries of N and M specified by the correlation functions

$$\left\langle \left(b_j(t) - \langle b_j(t)\rangle\right)^* \left(b_k(t') - \langle b_k(t')\rangle\right)\right\rangle = N_{jk}\ \delta(t-t'),$$
$$\left\langle \left(b_j(t) - \langle b_j(t)\rangle\right) \left(b_k(t') - \langle b_k(t')\rangle\right)\right\rangle = M_{jk}\ \delta(t-t').$$

That is,

$$\left\langle \left(\breve{b}(t) - \langle \breve{b}(t)\rangle\right) \left(\breve{b}(t') - \langle \breve{b}(t')\rangle\right)^*\right\rangle \equiv F\delta\left(t-t'\right). \qquad (2.46)$$

Following the treatment in [18], we now consider a zero-mean Gaussian state $\omega_{N,M}(\cdot) = \langle\cdot\rangle$ of n continuous-mode bosonic fields with $\langle \breve{b}(t)\rangle = 0$ for $t \geq 0$. An important special case is a joint vacuum Gaussian state, when $N = 0$ and $M = 0$. This is the state when the fields are all empty (contain no photons). The vacuum state for the field is completely specified by the characteristic function,

$$\left\langle \exp\left(\imath \int_0^\infty \breve{u}\,(t)^* \,\breve{b}\,(t)\,dt\right)\right\rangle_{\text{vac}} = \exp\left(-1/2 \int_0^\infty \breve{u}\,(t)^* \,F_{\text{vac}}\breve{u}\,(t)\,dt\right),$$

for any $u \in L^2(\mathbb{R}_+; \mathbb{C}^m)$. For concreteness, let us use this definition to show that any component $w_j(t)$ of the quadrature noise $w(t)$ is a standard Wiener process under the vacuum state, as claimed in Sect. 2.3.2. First, from its definition we immediately have that (i) $w_j(0)^2 = 0$ and $\langle w_j(t)\rangle = 0$. Now, take $u_\lambda = \lambda e_j 1_{[t_1,t_2]}$ with $\lambda \in \mathbb{R}$ and $0 \leq t_1 \leq t_2$ (recall that e_j is a unit column vector with all entries 0 except for a 1 as the j-th component). From the definition of a vacuum state given above,

$$\begin{aligned}
&\left\langle \exp\left(\imath\lambda(w_j(t_2) - w_j(t_1))\right)\right\rangle_{\text{vac}} \\
&= \left\langle \exp\left(\imath \int_{t_1}^{t_2} dw_j(t)\right)\right\rangle_{\text{vac}} \\
&= \left\langle \exp\left(\imath \int_0^\infty u_\lambda(t)^\top (b(t) + b(t)^\#)dt\right)\right\rangle_{\text{vac}} \\
&= \left\langle \exp\left(-1/2 \int_0^\infty \|u_\lambda(t)\|^2 dt\right)\right\rangle_{\text{vac}} \\
&= \exp\left(-1/2\lambda^2(t_2 - t_1)\right), \ \forall \lambda \in \mathbb{R}.
\end{aligned}$$

Therefore, (ii) the increment $w_j(t_2) - w_j(t_1)$ has the characteristic function of a zero-mean Gaussian random variable with variance $t_2 - t_1$. Moreover, by adaptedness we have that (iii) $\langle (w(t_4) - w(t_3))(w(t_2) - w(t_1))\rangle = \langle (w(t_4) - w(t_3))\rangle\langle (w(t_2) - w(t_1))\rangle = 0$ for all $t_1 < t_2 < t_3 < t_4$, so that increments at disjoint time intervals are uncorrelated. Since $[w_j(t), w_i(s)] = 0$ for all $s, t \geq 0$, from (i)–(iii) we conclude that $w_j(t)$ has precisely the statistics of a standard Wiener process and is therefore a Fock space realization of the latter.

For field operators $b_1(t), b_2(t), \ldots, b_m(t)$ in a zero-mean jointly Gaussian state, we define $\mathcal{B}_j(f) = \int_0^\infty f(s)^* b_j(s)ds$ for any $f \in L^2(\mathbb{R}_+; \mathbb{C})$ and its adjoint process $\mathcal{B}_j^*(f) = \int_0^\infty f(s)b_j(s)^*ds$ as they are well-defined and more regular mathematical objects, and can be manipulated using the Hudson–Partharathy quantum stochastic calculus. We introduce these notations to distinguish them from the earlier notations $\mathcal{A}_j(f)$ and $\mathcal{A}_j^*(f)$ that were assigned for the special case of a vacuum field state, if field j is in the vacuum state then one can simply set $\mathcal{B}_j(f)$ to be $\mathcal{A}_j(f)$. The operators $\mathcal{B}_j(f)$ and $\mathcal{B}_j^*(f)$ satisfy the same commutation relations as $\mathcal{A}(f)$ and $\mathcal{A}^*(f)$, $[\mathcal{B}_j(f), \mathcal{B}_k^*(g)] = \delta_{jk}\int_0^\infty f(s)^*g(s)ds$, and concrete realizations of $\mathcal{B}(f) = (\mathcal{B}_1(f), \mathcal{B}_2(f), \ldots, \mathcal{B}_m(f))^\top$ and $\mathcal{B}(f)^\# = (\mathcal{B}_1^*(f), \mathcal{B}_2^*(f), \ldots, \mathcal{B}_m^*(f))^\top$ on a suitable Hilbert space are dependent on the state of the field. However, for arbitrary Gaussian states one can relate the associated realizations of $\mathcal{B}(f)$ and $\mathcal{B}^*(f)$ to the vacuum state representation of these operators, via the generalized Araki–Woods representation from the previous section. In particular, any operator $\mathcal{B}(f)$ associated with a zero-mean Gaussian state $\omega_{N,M}$ has a Fock space representation of the form

$$\mathcal{B}(f) = X\mathcal{A}_1(f) + Y\mathcal{A}_2(f)^\# + Z\mathcal{A}_2(f), \tag{2.47}$$

with complex $m \times m$ matrices X, Y, and Z as constructed in the previous section, determined by the values of the parameters N and M of $\omega_{N,M}$, and where $\mathcal{A}_1(f)$ and $\mathcal{A}_2(f)$ are two independent vacuum annihilation processes that can each be realized on a distinct copy of the Fock space $\mathcal{F}_m$. Note that by construction X, Y, Z guarantee that the commutation relations $[\mathcal{B}_j(f), \mathcal{B}_k^*(g)] = \delta_{jk}\int_0^t f(s)^*g(s)ds$ hold for any $f, g \in L^2(\mathbb{R}_+; \mathbb{C})$.

The Itō table for a jointly Gaussian state of the fields can be directly constructed by exploiting the generalized Araki–Woods representation (2.47) and the vacuum Itō table. Recall that $\mathcal{A}_1(f)$ and $\mathcal{A}_2(f)$ in (2.47) are vacuum representations on distinct copies of the Fock space $\mathcal{F}_m$. The extended Itō table for the integrated operators $\mathcal{B}_j(t) = \mathcal{B}_j(1_{[0,t]})$ and $\mathcal{B}_j^*(t) = \mathcal{B}_j^*(1_{[0,t]})$ when the field is in an arbitrary Gaussian state is then

$$\begin{array}{c|ll} \times & d\mathcal{B}_k^* & d\mathcal{B}_k \\ \hline d\mathcal{B}_j & (\delta_{jk} + N_{kj})dt & M_{jk}dt \\ d\mathcal{B}_j^* & M_{kj}^*dt & N_{jk}dt. \end{array} \tag{2.48}$$

Note that in general Gaussian states, the counting process Λ need not be defined. We can also define a QSDE of the Hudson–Parthasarathy type but in which the vacuum field operators $\mathcal{A}$ and $\mathcal{A}^\#$ are replaced by field operators $\mathcal{B}$ and $\mathcal{B}^\#$ corresponding to a non-vacuum zero-mean jointly Gaussian state of the field. This yields the QSDE (without the counting process Λ),

$$dU(t) = \big(-(\imath H + {}^1\!/\!_2(L^*(I+N^\top)L + L^\top N L^\# - L^* M L^\# - L^\top M^\# L))dt \\ + d\mathcal{B}(t)^* L - L^* d\mathcal{B}(t)\big)U(t) \quad (2.49)$$

with initial condition $U(0) = I$. As with the vacuum case, after interaction with the system, $\mathcal{B}$ is transformed to $\mathcal{B}^{\text{out}}$ according to $\mathcal{B}^{\text{out}}(t) = U(t)^*\mathcal{B}(t)U(t)$. Using (2.47), the QSDE can be expressed in terms of the vacuum operator $\mathcal{A}(t) = [\,\mathcal{A}_1(t)^\top \; \mathcal{A}_2(t)^\top\,]^\top$,

$$dU(t) = (-(\imath H + L_{N,M}^* L_{N,M}/2)dt + d\mathcal{A}(t)^* L_{N,M} - L_{N,M}^* d\mathcal{A}(t))U(t), \quad (2.50)$$

with

$$L_{N,M} = \begin{bmatrix} X^* L \\ Z^* L - Y^\top L^\# \end{bmatrix}.$$

Example 2.3 An example of a non-vacuum zero-mean Gaussian state of a single bosonic field is a *squeezed vacuum* field state where the parameters $M \neq 0$ and $N \neq 0$ satisfy the identity $|M|^2 = N(N+1)$. Such a field can be approximately produced in the laboratory using the DPA that was treated in Example 2.2. In the limit discussed in that example, where the DPA becomes idealized with an infinite bandwidth, its output is an ideal squeezed Gaussian field state satisfying the quantum Itō rule with

$$N = \sinh^2 r_0 = \frac{4\kappa_0\epsilon_0}{(\kappa_0^2 - \epsilon_0^2)^2},$$
$$M = \cosh r_0 \sinh r_0 = \frac{2\kappa_0\epsilon_0(\kappa_0^2 - \epsilon_0^2)}{(\kappa_0^2 - \epsilon_0^2)^2}.$$

It can be easily verified that the parameters M and N above satisfy $|M|^2 = N(N+1)$. In practice, treating the output of the DPA as a squeezed bosonic field will be valid as long as it is driving a quantum system with sufficiently slower dynamics (sufficiently lower bandwidth) than the DPA itself.

2.7.3 Coherent States

The Gaussian states that we have discussed so far have a common feature, they all have zero mean. We shall now discuss an important class of nonzero-mean Gaussian states that model the output beam from a laser and can facilitate quantum feedback control. These are the coherent states, and we will now give describe the coherent states of single-mode oscillators and bosonic fields.

2.7.4 Coherent States of a Single-Mode Oscillator

A coherent state of a single-mode oscillator is a pure state of the oscillator that is indexed by a complex number α and denoted by $|\alpha\rangle$. The coherent state $|\alpha\rangle$ is a normalized eigenvector of the oscillator's annihilation operator corresponding to the eigenvalue α, $a|\alpha\rangle = \alpha|\alpha\rangle$. In a coherent state, the annihilation operator has mean $\langle a\rangle = \langle\alpha|a|\alpha\rangle = \alpha$ and mean photon number of $\langle\alpha|a^*a|\alpha\rangle = |\alpha|^2$, while the complex "covariance" matrix F defined in (2.39) corresponds to that of a vacuum state, with $N = M = 0$. Therefore, the vacuum state of an oscillator is a particular coherent state with $\alpha = 0$.

A coherent state with $\alpha \neq 0$ can be generated form the vacuum state $|0\rangle$ by applying a unitary displacement operator $D(\alpha) = \exp(\alpha a^* - \alpha^* a)$, $|\alpha\rangle = D(\alpha)|0\rangle$. We have that $D(\alpha)^* a D(\alpha) = \alpha + a$, and we notice the duality between the Schrödinger picture state transformation $|0\rangle \overset{D(\alpha)}{\mapsto} |\alpha\rangle$ and Heisenberg picture operator transformation $b \overset{D(\alpha)^*(\cdot)D(\alpha)}{\mapsto} b + \alpha I$, $\langle\alpha|a|\alpha\rangle = \langle 0|a + \alpha|0\rangle$ for all $\alpha \in \mathbb{C}$. Therefore, α is also referred to as the amplitude of the coherent state.

2.7.5 Coherent States of a Bosonic Field

Coherent states of a bosonic field are directly related to the exponential vectors and can be interpreted in analogous way to the coherent states of single-mode oscillator. The coherent state of a single bosonic field is a pure state of the field that is indexed by a function $f \in L^2(\mathbb{R}_+;\mathbb{C})$ and denoted by $|f\rangle$. It is simply a normalized version of the exponential vector $e(f)$,

$$|f\rangle = \frac{e(f)}{\|e(f)\|} = e(f)\exp\left(-\|f\|^2/2\right).$$

It follows that that $A(g)|f\rangle = \langle g, f\rangle|f\rangle$, thus $|f\rangle$ is an eigenvector of $A(g)$ for any $g \in L^2(\mathbb{R}_+;\mathbb{C})$ corresponding to the eigenvalue $\langle g, f\rangle$. More formally, in terms of the annihilation field operator $b(t)$ (satisfying $[b(t), b(t')^*] = \delta(t-t')$ and $[b(t), b(t')] = 0$), we have that $b(t)|f\rangle = f(t)|f\rangle$ for all $t \geq 0$, mirroring the property of coherent states of a single-mode oscillator.

The analogue of the displacement operator $D(\alpha)$ for bosonic fields is the Weyl operator $W(g)$ defined through its action on the exponential vectors,

$$W(g)e(f) = \exp(-\imath\langle g, f\rangle - \|f\|^2/2 - \|g\|^2/2)e(f+g).$$

From its definition, $W(f)$ is a unitary operator on $\Gamma_s(L^2(\mathbb{R}_+,\mathbb{C}))$ and we have that $W(f)|\Omega\rangle = |f\rangle$, where $|\Omega\rangle = e(0)$ is the vacuum state of the bosonic filed. Therefore, any coherent state can be generated from $e(0)$ via $W(f)$. In the bosonic case, we also have the duality between the Schroedinger state transformation $|\Omega\rangle \overset{W(g)}{\mapsto} |f\rangle$, with the Heisenberg picture transformation

$$\mathcal{A}(g) \overset{W(g)^*(\cdot)W(g)}{\mapsto} W(g)^*\mathcal{A}(f)W(g),\ \langle f|\mathcal{A}(g)|f\rangle = \langle 0|W(g)^*\mathcal{A}(f)W(g)|0\rangle.$$

Note that $W(f) = W(f_{t]}) \otimes W(f_{[t})$. The Weyl operator $W(f_{t]})$ with time index t is an adapted process and satisfies the QSDE,

$$dW(f_{t]}) = \left(-1/2|f(t)|^2 dt + f(t)d\mathcal{A}(t)^* - f(t)^* d\mathcal{A}(t)\right) W(f_{t]}),\ \ W(f_{0]}) = I.$$

Thus we have that $W(f) = \lim_{t\to\infty} W(f_{t]})$. The time-indexed Weyl operator is realized in the laboratory by an electro-optic modulator (EOM) to produce the state $W(f_{t]})|\Omega\rangle$. In the Heisenberg picture, the EOM implements the displacement $b(t) \mapsto b(t) + f(t)I$, where $b(t)$ is a vacuum bosonic annihilation operator. Coherent states of a single-mode oscillator can be created in the steady-state by driving a stable linear quantum system with a coherent bosonic field. In the laboratory, this would be implemented by driving the inputs of the system with EOMs.

The treatment above can be easily extended to multiple quantum fields that are each in a coherent state. If there are m fields each in a coherent state with amplitudes $f_1, f_2, \ldots, f_m$ then we replace the Hilbert space $L^2(\mathbb{R}_+; \mathbb{C})$ with $L^2(\mathbb{R}_+; \mathbb{C}^m)$, replace f with the vector $f = (f_1, f_2, \ldots, f_m)^\top$, and generalize the Weyl operator $W(f_{t]})$ according to the solution of the QSDE,

$$dW(f_{t]}) = \left(-1/2\|f(t)\|^2 dt + d\mathcal{A}(t)^* f(t) - f(t)^* d\mathcal{A}(t)\right) W(f_{t]}),\ \ W(f_{0]}) = I.$$

Here $\mathcal{A}(t)$ is now a vector of m annihilation processes, one for each field as before.

Since coherent fields can be generated from the vacuum, a system driven by coherent fields can be analyzed by referring it back to the vacuum. Suppose the system $G = (S, L, H)$ is driven by coherent fields with amplitude vector f. Then we can take all fields to be in the vacuum state and consider the joint unitary evolution of the system and vacuum fields given by $U_f(t) = U(t)W(f_{t]})$, where $U(t)$ is the unitary evolution of G given by the Hudson–Parthasarathy QSDE. By applying the quantum Itō stochastic calculus, it is easy to show that $U_f(t)$ satisfies the QSDE,

$$\begin{aligned} dU_f(t) = \Big(&- \big(\imath(H + \Im\{L^* Sf(t)\}) + 1/2(L + f(t)I)^*(L + f(t)I)\big)dt \\ &+ d\mathcal{A}(t)^*(L + f(t)I) - (L + f(t)I)^* S d\mathcal{A}(t) \\ &+ \mathrm{Tr}\big((S - I)d\Lambda(t)^\top\big)\Big)\, U_f(t), \end{aligned}$$

with initial condition $U_f(0) = I$. The vacuum driven system is thus $G_{\mathrm{vac}} = (S, L + fI, H + \Im\{L^* Sf\})$. Let $j_t(\cdot) = U(t)^* \cdot U(t)$ as in Sect. 2.1.4 and also let $j_t^f(\cdot) = U_f(t)^* \cdot U_f(t)$. The equivalence between G and the vacuum referred system G_{vac} is in the sense that

$$\langle f \lfloor j_t(X) \lfloor f \rangle = \langle \Omega \lfloor j_t^f(X) | \Omega \rangle, \forall X \in B(\mathfrak{h}) \text{ and } \forall t \geq 0.$$

That is, for all $t \geq 0$, the quantum expectation of $j_t(X)$ under the coherent field $\lfloor f \rangle$ coincides with the quantum expectation of $j_t^f(X)$ under the vacuum state, for any bounded operator X on $\mathfrak{h}$.

Finally, we remark that the QSDEs for the Weyl operator and G_{vac} can have time-dependent parameters, $S(t)$, $L(t)$ and $H(t)$. For instance, in the case of G_{vac}, we have $L(t) = L + f(t)I$, and $H(t) = H + \Im\{L^* S f(t)\}$, while S is not time-dependent. This is a slightly more general model than the one presented in Sect. 2.1.4. However, this extension is valid and defines a unitary solution of the QSDE when $S(t)$, $L(t)$, and $H(t)$ are adapted processes that are bounded for each t, $H(t)$ self-adjoint and $S(t)$ unitary for all $t \geq 0$ [30].

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Linear Dynamical Quantum Systems
Analysis, Synthesis, and Control
Nurdin, H.I.; Yamamoto, N.
2017, XV, 262 p. 68 illus., Hardcover
ISBN: 978-3-319-55199-9