Chapter 2
Theoretical Foundations for Flows Involving Vorticity

Abstract This chapter introduces the fluid mechanics foundations that are relevant for this book. The fluid mechanics equations are given for inertial and non-inertial frames. The chapter presents vorticity kinematics and dynamics and the main theorems involving vorticity. The equations presented are necessary to the development of vorticity-based methods, both analytical and numerical. Some classical results of vortices in viscous and inviscid fluid are provided due to their relevance for the validation of numerical vortex menates and 3D axisymmetric flows are developed in details. They are conveniently used for the study of rotor. Two-dimensional potential flows and conformal mapping solutions are introduced. They are relevant for the implementation and validation of vortex methods and the derivation of Prandtl's tip-loss factor. A Matlab code to compute the Karman-Trefftz map is provided.

Further developments are found e.g. in the books of Lamb [29], Batchelor [4], and Saffman [43]. Useful relations involving tensors, operators, differential calculus, integration theorems and field formalism are found in Appendix C. In particular, Appendix C provides the definition of the operators grad div, curl and their notations using the "Del" operator ∇. Both the "Del" and literal notations are used in this book.

2.1 Fluid Mechanics Equations in Inertial and Non-inertial Frames

2.1.1 *Physical Quantities*

The following notations are adopted: $\underline{u}$ is the fluid velocity [m/s], ρ is the fluid density [kg/m^3], S is the entropy of the fluid [J/K], T is the temperature of the fluid [K], $p = p(\rho, T)$ is the static pressure of the fluid [kg/m/s^2], and $e = e(\rho, S) = e(\rho, T)$ is the internal energy of the fluid [m^2/s^2]. The enthalpy h, the total energy e_t and total enthalpy h_t are defined as:

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$$h = e + \frac{p}{\rho}, \qquad e_t = e + \frac{u^2}{2}, \qquad h_t = h + \frac{u^2}{2} \qquad [\mathrm{m^2/s^2}] \tag{2.1}$$

The *total pressure* p_t is defined as the sum of the static pressure p and the dynamic pressure $\frac{1}{2}\rho u^2$:

$$p_t = p + \frac{1}{2}\rho u^2 \tag{2.2}$$

The *vorticity* is defined as the rotational of the velocity:

$$\underline{\omega} \triangleq \mathrm{curl}\ \underline{u} \qquad [\mathrm{s^{-1}}] \tag{2.3}$$

The *dilatation* is defined as the divergence of the velocity:

$$\Theta \triangleq \mathrm{div}\, \underline{u} \qquad [\mathrm{s^{-1}}] \tag{2.4}$$

2.1.2 Conservation Laws

Introduction The main fluid-mechanics conservation laws are recalled here in integral and local forms. The conservation laws of fluid mechanics are obtained by consideration of a *material volume*, which is a volume consisting of the same fluid particles throughout time. For a material volume D_m, the conservations laws are stated as follows: the change in time of the mass of D_m is zero in a flow without sink or source of mass (conservation of mass); the change in time of the linear momentum of D_m is equal to the forces applied within the volume and at its boundary (Newton's law); the change in time of the angular momentum of D_m is equal to the moments applied within the volume and at its boundary (Newton's law); the change in time of the total energy of D_m is equal to the power of the external forces and the external heat (first law of thermodynamics). The conservation laws apply on the total values of the mass, momentum and energy in the material volume and these total values are obtained by integration of the quantities over the volume. Reynolds transport theorem (RTT) is used to express the time derivative of these integrals in terms of volume and surface integral. The equations of the conservation laws obtained are then referred to as *integral forms*. Using the divergence theorem, it is possible to express the equations under one volume integral. The equations are then said to be in *conservative forms*. The expressions below the integrals are then isolated and these are referred to as the *local forms* or *differential forms* of the conservation laws.

Material derivative The *material derivative* of a fluid property is defined as the time rate of change of this property as experienced by a fluid particle as it moves through the flow field. Writing $f(\underline{x}, t)$ the Eulerian tensorial field of any order corresponding to this property, the material derivative is defined as:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(\underline{x},t) = \frac{\partial f}{\partial t}(\underline{x},t) + [\nabla f(\underline{x},t)] \cdot \underline{u}(\underline{x},t) \tag{2.5}$$

The material derivative is a particular case of the *total derivative* where $\dot{\underline{x}} \equiv \underline{u}$. A thorough treatment requires the distinction between Eulerian and Lagrangian fields and coordinates. The material derivative is seen to consist of the sum of the *local derivative*, which corresponds to the variation of the quantity at a fixed point, and the *convective derivative* which is related to the fact that the particle experiences different values of the vector field f as it convects.

Conservative derivative The *conservative derivative* of a tensorial field f of any order is defined as:

$$\frac{\mathscr{D}f}{\mathscr{D}t}(\underline{x},t) \triangleq \frac{\partial f}{\partial t}(\underline{x},t) + \nabla\cdot \big(f(\underline{x},t) \otimes \underline{u}(\underline{x},t)\big) \tag{2.6}$$

where $\otimes$ is the tensor product. For a scalar field λ, the tensor product can be removed and the conservative derivative is $\frac{\mathscr{D}\lambda}{\mathscr{D}t} = \frac{\partial\lambda}{\partial t} = \nabla\cdot(\lambda\underline{u})$. The material and conservative derivatives are related using $\nabla\cdot(f \otimes \underline{u}) = f\nabla\cdot\underline{u} + [\nabla f]\cdot\underline{u}$ as:

$$\frac{\mathscr{D}f}{\mathscr{D}t} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} + f\nabla\cdot\underline{u}, \qquad \frac{\mathscr{D}(\lambda f)}{\mathscr{D}t} \equiv \lambda\frac{\mathrm{d}f}{\mathrm{d}t} + f\frac{\mathscr{D}\lambda}{\mathscr{D}t} \tag{2.7}$$

The conservation of mass writes $\frac{\mathscr{D}\rho}{\mathscr{D}t} = 0$ and hence:

$$\frac{\mathscr{D}(\rho f)}{\mathscr{D}t} = \rho\frac{\mathrm{d}f}{\mathrm{d}t} \tag{2.8}$$

Further, the material and convective derivatives are equal for incompressible flows (i.e. when $\nabla\cdot\underline{u} = 0$).

Reynolds transport theorem A general form of Reynolds transport theorem (RTT) is presented below. A fluid quantity f is studied in a *geometrical volume* $\Omega(t)$. The system is illustrated in Fig. 2.1. The surface marking the boundary of the volume, $\partial\Omega(t)$, moves with the local velocity $\underline{V}_b(\underline{x},t)$. This velocity may differ from the fluid velocity. For a *fixed volume*, the boundary velocity is $\underline{V}_b \equiv \underline{0}$. For a *material volume*, the boundary velocity is the fluid velocity $\underline{V}_b(\underline{x},t) \equiv \underline{u}(\underline{x},t)$. The geometrical volume may contain a discontinuity surface Σ moving at the local velocity $\underline{V}_\Sigma(\underline{x},t)$.

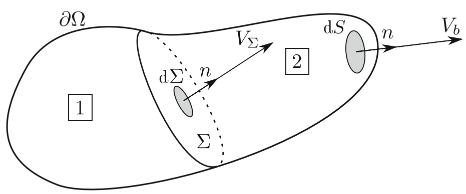

Fig. 2.1 Sketch of a domain Ω with a surface of discontinuity Σ moving at velocity $\underline{V}_\Sigma$ and a boundary $\partial\Omega$ moving at the speed $\underline{V}_b$

Examples of discontinuity surfaces are: shock waves in supersonic flows, vorticity sheets in shear flows, surfaces between immiscible fluids, actuator disks, etc. The time derivative of the integral quantity of the vector field $f(\underline{x}, t)$ in Ω is given by the generalized Reynolds transport theorem (RTT) as:

$$\left.\frac{\delta}{\delta t}\right|_{\underline{V}_b}\left[\int_{\Omega(t)} f \mathrm{d}v\right] = \int_{\Omega(t)} \frac{\partial f}{\partial t}\mathrm{d}v + \int_{\partial\Omega(t)} f\, \underline{V}_b \cdot \underline{\mathrm{d}S} - \int_{\Sigma(t)} [\![f]\!] \underline{V}_\Sigma \cdot \underline{\mathrm{d}S} \tag{2.9}$$

$$= \int_{\Omega(t)} \left[\frac{\partial f}{\partial t} + \nabla \cdot (f \otimes \underline{V}_b)\right] \mathrm{d}v + \int_{\Sigma(t)} \left[\!\left[f \otimes (\underline{V}_b - \underline{V}_\Sigma) \right]\!\right] \cdot \underline{\mathrm{d}S} \tag{2.10}$$

where $\underline{\mathrm{d}S} = \underline{n}\mathrm{d}S$ is pointing outward of the domain and the notation $[\![f]\!] = f_2 - f_1$ is used to represent the difference of the values of the quantity f on both sides of the discontinuity surface (see Fig. 2.1). The generalized divergence theorem (DT) from Eq. C.42 was used to go from Eq. 2.9 to 2.10. The generalized RTT and DT are obtained by expressing the theorems in the domains 1 and 2, and doing the sum of the results. The integral on the discontinuity surface appears naturally. The notation $\left.\frac{\delta}{\delta t}\right|_{\underline{V}_b}$ is introduced to emphasize that this derivative is not the material derivative and the domain considered is not a material domain. Yet, if the boundary of Ω is moving with the local fluid velocity $(\underline{u}(\underline{x}, t) \cdot \underline{n})\underline{n}$ on each of its point then the volume is a material domain and the time rate of change of the integral quantity followed in its motion is the material derivative, viz.: $\left.\frac{\delta}{\delta t}\right|_{\underline{u}} \equiv \frac{\mathrm{d}}{\mathrm{d}t}$. According to Eq. 2.9 then, the material derivative of an integral over a material domain D_m is:

$$\left.\frac{\delta}{\delta t}\right|_{\underline{u}}\left[\int_{D_m} f \mathrm{d}v\right] \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{D_m} f \mathrm{d}v\right] = \int_{D_m} \frac{\partial f}{\partial t}\mathrm{d}v + \int_{\partial D_m} f\, \underline{u} \cdot \underline{\mathrm{d}S} - \int_{\Sigma(t)} [\![f]\!] \underline{V}_\Sigma \cdot \underline{\mathrm{d}S} \tag{2.11}$$

$$= \int_{D_m} \frac{\mathscr{D} f}{\mathscr{D} t}\mathrm{d}v + \int_{\Sigma(t)} \left[\!\left[f \otimes (\underline{u} - \underline{V}_\Sigma) \right]\!\right] \cdot \underline{\mathrm{d}S} \tag{2.12}$$

where the conservative derivative is defined in Eq. 2.6. For a fixed volume, $\left.\frac{\delta}{\delta t}\right|_0$ implies a derivative with respect to t for $\underline{x}$ fixed, which is the definition of the partial derivative $\frac{\partial}{\partial t}$. The time variation of the quantity within a fixed volume is then:

$$\left.\frac{\delta}{\delta t}\right|_0\left[\int_{\Omega} f \mathrm{d}v\right] \equiv \frac{\partial}{\partial t}\left[\int_{\Omega} f \mathrm{d}v\right] = \int_{\Omega} \frac{\partial f}{\partial t}\mathrm{d}v - \int_{\Sigma(t)} [\![f]\!] \underline{V}_\Sigma \cdot \underline{\mathrm{d}S} \tag{2.13}$$

By considering the instantaneous "fixed" volume Ω that matches with the material volume D_m at a given time t, the material derivative of the integral $I = \int_{D_m} f \mathrm{d}v$ (Eq. 2.11) is seen to be the sum of two contributions:

$$\left.\frac{\delta I}{\delta t}\right|_{\underline{u}} \equiv \frac{\mathrm{d}I}{\mathrm{d}t} = \left.\frac{\delta I}{\delta t}\right|_0 + \int_{\partial D_m} f\, \underline{u} \cdot \underline{\mathrm{d}S} \tag{2.14}$$

The first term of the right hand side (RHS) corresponds to the variation of I as function of time if the volume was fixed. The second term corresponds to the variation of the integral due to the convection of the domain D_m. Equation 2.9 can also be applied to two identical volumes $\Omega(t)$ that matches at time t with two different boundary velocities $\underline{V}_b$ and $\underline{V}_{b,2}$. A typical choice is to take $\underline{V}_{b,2} = (\underline{u} \cdot \underline{n})\underline{n}$. The difference of Eqs. 2.11 and 2.9 applied for the volumes matching at t provides the material derivative of a volume integral delimited by an arbitrary surface moving with an arbitrary velocity $\underline{V}_b$ as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega(t)} f \mathrm{d}v\right] = \left.\frac{\delta}{\delta t}\right|_{\underline{V}_b}\left[\int_{\Omega(t)} f \mathrm{d}v\right] + \int_{\partial\Omega(t)} f\,(\underline{u} - \underline{V}_b) \cdot \underline{\mathrm{d}S} \tag{2.15}$$

The above is applied for instance to obtain the material derivative of a fixed volume integral (using $\underline{V}_b = 0$ and Eq. 2.13):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega} f \mathrm{d}v\right] = \int_{\Omega} \frac{\partial f}{\partial t}\mathrm{d}v + \int_{\partial\Omega} f\, \underline{u} \cdot \underline{\mathrm{d}S} - \int_{\Sigma(t)} [\![f]\!] \underline{V}_{\Sigma} \cdot \underline{\mathrm{d}S} \tag{2.16}$$

$$= \int_{\Omega} \frac{\mathscr{D} f}{\mathscr{D} t}\mathrm{d}v + \int_{\Sigma(t)} [\![f \otimes (\underline{u} - \underline{V}_{\Sigma})]\!] \cdot \underline{\mathrm{d}S} \tag{2.17}$$

Most of the conservation laws involve an integral of a quantity ρf. The application of Reynolds transport theorem for such quantity can be simplified once the equation of conservation of mass is proved (in the next paragraphs). Equation 2.8 follows directly from the conservation of mass and Eq. 2.12 leads then to the following result for the integral of a quantity ρf over a material volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{D_m} (\rho f) \mathrm{d}v\right] = \int_{D_m} \rho \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}v + \int_{\Sigma(t)} [\![\rho f \otimes (\underline{u} - \underline{V}_{\Sigma})]\!] \cdot \underline{\mathrm{d}S} \tag{2.18}$$

The same formula holds for a fixed volume according to Eq. 2.17.

Integral forms The conservations laws stated in the introduction are extended to any geometrical volume $\Omega(t)$ moving with the velocity $\underline{V}_b$, with or without surface of discontinuity, as long as Eq. 2.15 is used. The mass, linear momentum, angular momentum and total enthalpy in the volume Ω are obtained by integration of the quantities ρ, $\rho\underline{u}$, $\underline{r} \times \rho\underline{u}$ and ρe_t respectively. Their conservation laws writes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho \mathrm{d}v = 0 \tag{2.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho \underline{u} \mathrm{d}v = \int_{\Omega(t)} \rho \underline{F} \mathrm{d}v + \int_{\partial\Omega(t)} \underline{\underline{\sigma}} \cdot \underline{n} \mathrm{d}S \tag{2.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \underline{r} \times \rho \underline{u} \mathrm{d}v = \int_{\Omega(t)} \underline{r} \times \rho \underline{F} \mathrm{d}v + \int_{\partial\Omega(t)} \underline{r} \times (\underline{\underline{\sigma}} \cdot \underline{n}) \mathrm{d}S \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho e_t \mathrm{d}v = \int_{\Omega(t)} (\rho \underline{F} \cdot \underline{u}) \mathrm{d}v + \int_{\partial\Omega(t)} (\underline{u} \cdot \underline{\underline{\sigma}} - \underline{q_s}) \cdot \underline{n} \mathrm{d}S \tag{2.22}$$

In the above, $\underline{\underline{\sigma}}$ [N/m^2] is Cauchy's *stress tensor* accounting for surface forces. $\underline{F}$ are forces per mass [N/kg], linked to volume forces with $\underline{f} = \rho \underline{F}$ [N/m^3]. Examples of volume forces are: the gravity, inertial forces in non-inertial frames (Euler force, centrifugal force, Coriolis force), the Lorentz force due to a magnetic field, etc. and $\underline{q}_s$ is the surface *heat flux* [W/m^2] (see e.g. Eq. 2.90). Writing $\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{1}}} + \underline{\underline{\tau}}$ (see Eq. 2.85), the total enthalpy $h_t = e + \frac{p}{\rho} + \frac{u^2}{2}$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho h_t \mathrm{d}v = \int_{\Omega(t)} (\rho \underline{F} \cdot \underline{u}) \mathrm{d}v + \int_{\partial\Omega(t)} (\underline{u} \cdot \underline{\underline{\tau}} - \underline{q_s}) \cdot \underline{n} \mathrm{d}S + \int_{\Omega(t)} \frac{\partial p}{\partial t} \mathrm{d}v - \int_{\Sigma(t)} [\![p]\!] (\underline{V}_\Sigma \cdot \underline{n}) \mathrm{d}S \tag{2.23}$$

The conservation of *entropy* S for a volume without discontinuity surface is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho S \mathrm{d}v = \int_{\Omega(t)} \left[\frac{\underline{\underline{\tau}} : \underline{\underline{\mathscr{D}}}}{T} - \frac{\underline{q}_s \cdot \mathrm{grad}\, T}{T^2} \right] \mathrm{d}v - \int_{\partial\Omega(t)} \frac{\underline{q}_s}{T} \cdot \underline{n} \mathrm{d}S \tag{2.24}$$

where $\underline{\underline{\mathscr{D}}}$ is the deformation tensor (see Sect. 2.2.1, $\underline{\underline{\mathscr{D}}} = \frac{1}{2}(\mathrm{grad}\, \underline{u} + {}^t\, \mathrm{grad}\, \underline{u})$), and $\underline{\underline{\tau}} : \underline{\underline{\mathscr{D}}} = \sum_i \sum_j \tau_{ij} \mathscr{D}_{ji}$ is the result of the double tensorial contraction. The fundamental inequality of thermodynamics (second law) is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho S \mathrm{d}v \geq - \int_{\partial\Omega(t)} \frac{\underline{q}_s}{T} \cdot \underline{n} \mathrm{d}S \tag{2.25}$$

Local/differential forms The *local forms* of the conservation laws are obtained by transforming all the surface integrals over $\partial\Omega$ in Eqs. 2.19–2.22 into volume integrals using the generalized divergence theorem given in Eq. C.42. The local forms are then:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\mathrm{div}\,\underline{u} = 0, \qquad [\![\rho(\underline{u} - \underline{V}_\Sigma)]\!]\cdot\underline{n} = 0 \tag{2.26}$$

$$\rho\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \rho\underline{F} + \mathrm{div}\,\underline{\underline{\sigma}}, \qquad [\![\ \underline{\underline{\sigma}}\ ]\!]\cdot\underline{n} + \rho[\![\underline{u}]\!](\underline{u} - \underline{V}_\Sigma)\cdot\underline{n} = 0 \tag{2.27}$$

$${}^t\underline{\underline{\sigma}} = \underline{\underline{\sigma}} \qquad \text{(none)} \tag{2.28}$$

$$\rho\frac{\mathrm{d}e_t}{\mathrm{d}t} = \rho\underline{F}\cdot\underline{u} + \mathrm{div}(\underline{\underline{\sigma}}\cdot\underline{u} - \underline{q}_s), \qquad \left[\!\left[\underline{u}\cdot(\underline{\underline{\sigma}}\cdot\underline{n} + \underline{q}_s)\right]\!\right] + \rho[\![e_t]\!](\underline{u} - \underline{V}_\Sigma)\cdot\underline{n} = 0 \tag{2.29}$$

For each conservation law, the equation on the left side provides the "Jump-condition" that apply across the surface of discontinuity Σ if such surface is present (see Fig. 2.1). The local forms of the conservation of enthalpy and entropy with $\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{1}}} + \underline{\underline{\tau}}$ (see Eq. 2.85) are:

$$\rho\frac{\mathrm{d}h_t}{\mathrm{d}t} = \rho\underline{F}\cdot\underline{u} + \mathrm{div}(\underline{\underline{\tau}}\cdot\underline{u} - \underline{q}_s) + \frac{\partial p}{\partial t} \tag{2.30}$$

$$\rho\frac{\mathrm{d}S}{\mathrm{d}t} = -\,\mathrm{div}\,\frac{\underline{q}_s}{T} + \frac{\underline{\underline{\tau}} : \underline{\underline{\mathscr{D}}}}{T} - \frac{\underline{q}_s\cdot\mathrm{grad}\,T}{T^2} \tag{2.31}$$

The fundamental inequality of thermodynamics is:

$$\rho\frac{\mathrm{d}S}{\mathrm{d}t} \geq -\,\mathrm{div}\,\frac{\underline{q}_s}{T}, \qquad \rho[\![S]\!](\underline{u} - \underline{V}_\Sigma)\cdot\underline{n} \geq -\left[\!\left[\frac{\underline{q}_s}{T}\right]\!\right] \tag{2.32}$$

The conservation of kinetic energy is obtained by taking the scalar product of the momentum equation with $\underline{u}$:

$$\rho\frac{\mathrm{d}(u^2/2)}{\mathrm{d}t} = \rho\underline{F}\cdot\underline{u} + \underline{u}\cdot\mathrm{div}\,\underline{\underline{\sigma}} \tag{2.33}$$

2.1.3 Fluid-Mechanic Equations in a Non-inertial Frame

The fluid mechanics equations in a non-inertial frame are derived in this section. The non-inertial frame follows an arbitrary motion. A relevant application of these equations for rotor is the case of a frame rotating with a constant velocity.

Notations: Frame of references, basis and vectors coordinates Two frame of references (R) and (R') are considered where the first one is assumed to be inertial and the second one has an arbitrary motion with respect to (w.r.t.) the frame (R). The arbitrary motion includes spatial rotation and translation. Non-relativistic velocities are assumed and the time is assumed to be the same in both frames. The origins of the reference frames are the points O and O' and their systems of axes are identified with the orthonormal bases (e) and (e'). Cartesian coordinates related to the bases (e) and (e') are adopted in both reference frames to simplify the notations and derivations. The final results that will be obtained are yet independent of the coordinates system adopted in each frame. The reference frame (R') rotates around O' with the

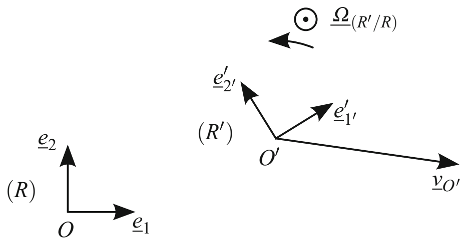

Fig. 2.2 Notations used for the inertial frame (R) and the non-inertial frame $(R)'$. In this sketch the rotation of the frame (R') is assumed to be along $\underline{e}'_{3'}$ for simplicity

instantaneous rotation vector $\underline{\Omega}_{(R'/R)}(t)$ with respect to (R) and translates with the instantaneous velocity $\underline{v}_{O'}(t)$. The Cartesian basis of (R) consists of the vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$. A sketch representing the notations adopted is shown in Fig. 2.2. A vector $\underline{V}$ is expressed in this coordinate system using the coordinates V_i as[1]:

$$\underline{V} = \underline{V}\big|_{(e)} = V_1\,\underline{e}_1 + V_2\,\underline{e}_2 + V_3\,\underline{e}_3 = V_i\,\underline{e}_i \tag{2.34}$$

The *Einstein summation* convention was used in the last equality. The notation $V|_{(e)}$ means that the vector is expressed in the basis (e), which is convenient when matricial products are involved. *Primed indices* will now be introduced as a convenient way to distinguish the coordinates and basis vectors of (R'). The Cartesian basis of (R') consists of the vectors $\underline{e}'_{1'}$, $\underline{e}'_{2'}$, $\underline{e}'_{3'}$ where primed indices have been used. The coordinates of a vector $\underline{V}$ in this coordinate system are written $V_{i'}$ such that:

$$\underline{V} = \underline{V}\big|_{(e')} = V_{1'}\,\underline{e}'_{1'} + V_{2'}\underline{e}'_{2'} + V_{3'}\,\underline{e}'_{3'} = V_{i'}\,\underline{e}'_{i'} = V_{k'}\,\underline{e}'_{k'} \tag{2.35}$$

It is noted that there are no primes on the coordinates of $\underline{V}$ but only on the indices. Einstein summation is implied in the two last equalities but it is stressed that the primes cannot be removed from the dummy indices i and k. The motivation for using these notations will appear clear when considering the position vector later in this paragraph. Vectors are first-order tensors and they are thus *invariant* from one basis to the other:

$$\underline{V} = V_i\underline{e}_i = \underline{V}_{i'}\,\underline{e}'_{i'} \tag{2.36}$$

The coordinates are expressed from one frame to the other thanks to the orthogonal transformation matrix $\underline{\underline{L}}$ such that

$$V_{i'} = L_{i'j}V_j, \qquad V_i = L_{j'i}V_{j'} \tag{2.37}$$

which is written using a matricial product as:

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[1]Since the space is Euclidean and the bases are orthonormal there is no need to distinguish the *covariant* coordinates, usually noted V_i, and the *contravariant* coordinates, usually noted V^i.

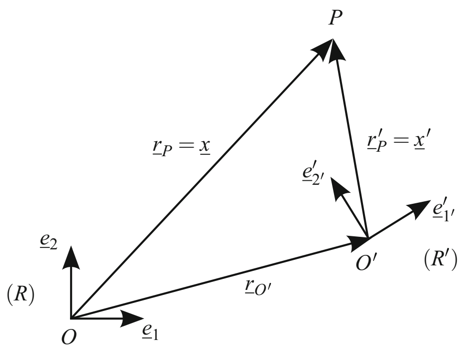

Fig. 2.3 Definition of the position vectors $\underline{r}$ and $\underline{r}'$, and the Eulerian coordinates $\underline{x}$ and $\underline{x}'$. The position vectors mark the trajectory of a given particle P. This position at a given time is expressed by the Eulerian coordinates. Each of these vectors are expressed with respect to the origin of the frame

$$\underline{V}\big|_{(e)} = \underline{\underline{L}} \cdot \underline{V}\big|_{(e')}, \qquad \underline{V}\big|_{(e')} = {}^t\underline{\underline{L}} \cdot \underline{V}\big|_{(e)} \tag{2.38}$$

The *position vector* is defined as the distance between the origin of a frame of reference to a given point in this frame of reference. The position vectors of (R) and (R') are noted $\underline{r} = \underline{OP}$ and $\underline{r}' = \underline{O'P}$ respectively. The position vectors are used to identify the trajectory of a given particle. At a given time, the position of a particle corresponds to the Eulerian positions $\underline{x}$ and $\underline{x}'$ in (R) and (R') respectively. A sketch representing the notations adopted is shown in Fig. 2.3. The *observer transformation* or the *Euclidean transformation* $(\underline{x}, t) \rightarrow (\underline{x}', t)$ is such that:

$$\underline{x}' = \underline{\underline{L}} \cdot \underline{x}\big|_{(e)} + \underline{r}'_O = \underline{\underline{L}} \cdot \underline{x}\big|_{(e)} - \underline{r}_{O'} \tag{2.39}$$

The coordinate transformation is differentiable and non-singular. The inverse transformation is:

$$\underline{x} = {}^t\underline{\underline{L}} \cdot \underline{x}'\big|_{(e')} + \underline{r}_{O'} = {}^t\underline{\underline{L}} \cdot \underline{x}'\big|_{(e')} - \underline{r}'_O \tag{2.40}$$

The vectors $\underline{x}$ and $\underline{x}'$ are expressed in both (e) and (e') as:

$$\underline{x} = x_i \underline{e}_i (= x_{i'} \underline{e}'_{i'}), \qquad \underline{x}' = x'_{i'} \underline{e}'_{i'} (= x'_i \underline{e}_i) \tag{2.41}$$

The parenthesis used in the above equation are present to note that in practice the vector $\underline{x}$ is more easily expressed in (e) and the vector $\underline{x}'$ is more easily expressed in (e'). The coordinate transformations Eqs. 2.39 and 2.40 is such that each Eulerian coordinate in a frame is expressed as a function of the coordinates in the other frames:

$$\text{for } i = 1..3,\ x_i = x_i(x'_{1'}, x'_{2'}, x'_{3'}), \qquad \text{for } i' = 1'..3',\ x'_{i'} = x'_{i'}(x_1, x_2, x_3) \tag{2.42}$$

A quantity Q is said to be *objective* or *frame indifferent* if it is invariant under all observer transformation. The evaluation of the quantity Q in the frame of reference is noted $(Q)_{(R)}$. The fact that the quantity Q is objective is then written:

$$(Q)_{(R)} = (Q)_{(R')} \qquad \text{(objective quantity)} \tag{2.43}$$

The following results are relevant (see e.g. [36, p.107]): mass and temperature are objective quantities; a vector joining two positions is an objective quantity; the velocity and acceleration of a particle are quantities that are not objective; the gradient, curl and divergence of an objective tensor is an objective tensor; the divergence of the velocity field is an objective scalar (since div $\underline{v}_{\text{fix}} = 0$); the viscous stress tensor $\underline{\underline{\tau}}$ is an objective quantity.

Differential of a vector The differential of a vector $\underline{A}$ with respect to time in two different reference frames is considered. The time derivative of the vector in each frame of reference is:

$$\left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_{(R)} \triangleq \left(\frac{\mathrm{d}[A_i\underline{e}_i]}{\mathrm{d}t}\right)_{(R)} = \frac{\mathrm{d}A_i}{\mathrm{d}t}\underline{e}_i, \qquad \left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_{(R')} \triangleq \left(\frac{\mathrm{d}[A_{i'}\underline{e}'_{i'}]}{\mathrm{d}t}\right)_{(R)} = \frac{\mathrm{d}A_{i'}}{\mathrm{d}t}\underline{e}'_{i'} \tag{2.44}$$

since the Cartesian bases are fixed relative to each reference frame, i.e. $\left(\frac{\mathrm{d}\underline{e}_i}{\mathrm{d}t}\right)_{(R)} = \underline{0}$ and $\left(\frac{\mathrm{d}\underline{e}'_{i'}}{\mathrm{d}t}\right)_{(R')} = \underline{0}$. The components of $\underline{A}$ are now written in the basis of (R') and successively differentiated in the reference frame as:

$$\left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}\left[A_{i'}\underline{e}'_{i'}\right]}{\mathrm{d}t}\right)_{(R)} = \frac{\mathrm{d}A_{i'}}{\mathrm{d}t}\underline{e}'_{i'} + A_{i'}\left(\frac{\mathrm{d}\underline{e}'_{i'}}{\mathrm{d}t}\right)_{(R)} \tag{2.45}$$

$$= \left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_{(R')} + A_{i'}\,\underline{\Omega}_{(R'/R)} \times \underline{e}'_{i'} \tag{2.46}$$

where the relation $\left(\frac{\mathrm{d}\underline{e}'_{i'}}{\mathrm{d}t}\right)_{(R)} = \underline{\Omega}_{(R'/R)} \times \underline{e}'_{i'}$ was used. This relation is easily proven when the rotation is directed along a given $\underline{e}'_{j'}$. The time derivatives of the vector in each basis are then related by

$$\left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega}_{(R'/R)} \times \underline{A} \tag{2.47}$$

Equation 2.47 is referred to as *Bour formula* or the *transport theorem*.

Lagrangian particle kinematics The motion of a particle is considered and a Lagrangian formulation is naturally adopted. The following notations are adopted: P is the point where the particle is located, $\underline{r}(t) \equiv \underline{OP}$ is the particle trajectory with respect to O, $\underline{r}'(t) \equiv \underline{O'P}$ is the particle trajectory w.r.t. O', $\underline{r}_{O'}(t) \equiv \underline{OO'}$ is the trajectory of O' w.r.t. O, and $\underline{\Omega}$ is the rotation vector of (R') w.r.t. (R). The position vector in (R) is decomposed as $\underline{r} = \underline{OO'} + \underline{O'P} = \underline{r}_{O'} + \underline{r}'$ and the application of Eq. 2.47 to $\underline{r}'$ and $\underline{v}_{\text{rel}}$ leads to the expression for the speed and acceleration:

$$\underline{r} = \underline{r}_{O'} + \underline{r}' \tag{2.48}$$

$$\underline{v} \triangleq \dot{\underline{r}} = \dot{\underline{r}}_{O'} + \left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega} \times \underline{r}' = \underline{v}_{\text{fix}} + \underline{v}_{\text{rel}} \tag{2.49}$$

$$\underline{a} \triangleq \dot{\underline{v}} = \ddot{\underline{r}}_{O'} + \left(\frac{\mathrm{d}^2\underline{r}'}{\mathrm{d}t^2}\right)_{(R')} + 2\underline{\Omega} \times \underline{v}_{\text{rel}} + \dot{\underline{\Omega}} \times \underline{r}' + \underline{\Omega} \times \underline{\Omega} \times \underline{r}' = \underline{a}_{\text{fix}} + \underline{a}_{\text{cor}} + \underline{a}_{\text{rel}} \tag{2.50}$$

where

$$\underline{v}_{\text{rel}} \triangleq \left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R')}, \qquad \underline{v}_{\text{fix}} \triangleq \dot{\underline{r}}_{O'} + \underline{\Omega} \times \underline{r}' = \underline{v}_{O'} + \underline{\Omega} \times \underline{r}' \tag{2.51}$$

$$\underline{a}_{\text{rel}} \triangleq \left(\frac{\mathrm{d}^2\underline{r}'}{\mathrm{d}t^2}\right)_{(R')}, \qquad \underline{a}_{\text{fix}} \triangleq \ddot{\underline{r}}_{O'} + \dot{\underline{\Omega}} \times \underline{r}' + \underline{\Omega} \times \underline{\Omega} \times \underline{r}', \qquad \underline{a}_{\text{cor}} \triangleq 2\underline{\Omega} \times \underline{v}_{\text{rel}} \tag{2.52}$$

The dot notation was used for the time derivative w.r.t. to (R). The subscript *rel* is used for the *relative* speed and acceleration of the particle w.r.t. (R'). The quantities with the subscript *fix* are values that would be obtained if the particle was fixed in (R'). The value $\underline{a}_{\text{cor}}$ is *Coriolis acceleration*.

Eulerian kinematics Eulerian coordinates are now used to study the flow kinematics. The flow quantities are assumed to be functions of time t and position $\underline{x}$. The Eulerian kinematics require a proper account of the partial derivatives and material derivatives. A given fluid particle P is assumed to occupy at time t the Eulerian position $\underline{x} = \underline{r}(t) \equiv \underline{OP}$ in (R) and the Eulerian position $\underline{x}' = \underline{r}'(t) \equiv \underline{O'P}$ in (R'). By definition of the Eulerian coordinates in (R), the partial derivative with respect to time of $\underline{OP}$ is zero and the material derivative of $\underline{OP}$ is the Eulerian fluid velocity $\underline{u}$, viz.:

$$\left(\left.\frac{\partial \underline{OP}}{\partial t}\right|_{\underline{x}}\right)_{(R)} \equiv \underline{0} \quad \text{i.e.} \quad \left.\frac{\partial x_i}{\partial t}\right|_{\underline{x}} \underline{e}_i = \underline{0} \tag{2.53}$$

$$\left(\frac{\mathrm{d}\underline{OP}}{\mathrm{d}t}\right)_{(R)} \equiv \underline{u} \quad \text{i.e.} \quad \frac{\mathrm{d}x_i}{\mathrm{d}t}\underline{e}_i = \left[\left.\frac{\partial x_i}{\partial t}\right|_{\underline{x}} \underline{e}_i + \underline{u} \cdot \nabla_{\underline{x}} x_i\right] \underline{e}_i = \underline{u} \tag{2.54}$$

The notation $\partial/\partial t|_{\underline{x}}$ means that the position coordinates x_i are kept constant in the time derivation, which is indeed the definition of a partial derivative of a function of time and position. The notation $\nabla_{\underline{x}}$ stands for the gradient relative to the Eulerian coordinates x_i. The vector $\underline{OP}$ is now decomposed as:

$$\underline{OP} = \underline{OO'} + \underline{O'P} = \underline{r}_{O'} + \underline{r}' \tag{2.55}$$

The vector $\underline{O\,O'} \equiv \underline{r}_{O'}$ is not a function of position and hence:

$$\left(\frac{\mathrm{d}\underline{O\,O'}}{\mathrm{d}t}\right)_{(R)} \equiv \left(\left.\frac{\partial \underline{O\,O'}}{\partial t}\right|_{\underline{x}}\right)_{(R)} \triangleq \underline{v}_{O'} \tag{2.56}$$

where $\underline{v}_{O'}$ is the velocity of the origin O' as defined in the previous paragraph using Lagrangian formalism. At the time t, the particle P is at the Eulerian location $\underline{x'} = x'_{i'}\underline{e}'_{i'}$. The vectors of the basis (e') are not a function of position and hence:

$$\text{for } i' = 1', .., 3' \left(\frac{\mathrm{d}\underline{e}'_{i'}}{\mathrm{d}t}\right)_{(R)} \equiv \left(\left.\frac{\partial \underline{e}'_{i'}}{\partial t}\right|_{\underline{x}}\right)_{(R)} = \underline{\Omega} \times \underline{e}'_{i'} \tag{2.57}$$

The time derivatives of $\underline{r'}$ follow from Eq. 2.57:

$$\left(\left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}}\right)_{(R)} = \left(\left.\frac{\partial [x_{i'}\underline{e}'_{i'}]}{\partial t}\right|_{\underline{x}}\right)_{(R)} = \left.\frac{\partial x'_{i'}}{\partial t}\right|_{\underline{x}} \underline{e}'_{i'} + x'_{i'}\underline{\Omega} \times \underline{e}'_{i'} = \left(\left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}}\right)_{(R')} + \underline{\Omega} \times \underline{r}' \tag{2.58}$$

$$\left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}[x'_{i'}\underline{e}'_{i'}]}{\mathrm{d}t}\right)_{(R)} = \frac{\mathrm{d}x'_{i'}}{\mathrm{d}t}\underline{e}'_{i'} + x'_{i'}\underline{\Omega} \times \underline{e}'_{i'} = \left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega} \times \underline{r}' \tag{2.59}$$

These results could have been directly obtained using the transport theorem from Eq. 2.47. Using the decomposition from Eq. 2.55 and the results from above, Eqs. 2.53 and 2.54 become:

$$\left(\left.\frac{\partial \underline{r}}{\partial t}\right|_{\underline{x}}\right)_{(R)} \equiv \underline{0} = \underline{v}_{O'} + \left(\left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}}\right)_{(R')} + \underline{\Omega} \times \underline{r}' = \underline{v}_{\mathrm{fix}} + \left(\left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}}\right)_{(R')} \tag{2.60}$$

$$\left(\frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\right)_{(R)} \equiv \underline{u} = \underline{v}_{O'} + \left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega} \times \underline{r}' = \underline{v}_{\mathrm{fix}} + \left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R')} \tag{2.61}$$

where $\underline{v}_{\mathrm{fix}} = \underline{v}_{O'} + \underline{\Omega} \times \underline{r}'$ was defined in Eq. 2.51. The above equations are rearranged as:

$$\left.\frac{\partial x'_{i'}}{\partial t}\right|_{\underline{x}} \underline{e}'_{i'} \triangleq \left(\left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}}\right)_{(R')} = -\underline{v}_{\mathrm{fix}}, \qquad \frac{\mathrm{d}x'_{i'}}{\mathrm{d}t}\underline{e}'_{i'} \triangleq \left(\frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}\right)_{(R')} = \underline{u} - \underline{v}_{\mathrm{fix}} \triangleq \underline{u}_{\mathrm{rel}} \tag{2.62}$$

The vector $\underline{u}_{\mathrm{rel}}$ is the flow velocity as observed in the frame (R') and its components in this frame are the material derivative of the Eulerian coordinates $x'_{i'}$. The equation on the left states that as the relative frame (R') moves, a fixed point in the frame (R) appears to move at the velocity $-\underline{v}_{\mathrm{fix}}$ in (R'). The converse relation may easily be obtained by considering the partial derivatives w.r.t. to time of $\underline{r}$ when the point $\underline{x'}$ is fixed in (R'):

$$\left(\left.\frac{\partial \underline{r}}{\partial t}\right|_{\underline{x}'}\right)_{(R)} = \underline{v}_{O'} + \left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}'} + \underline{\Omega} \times \underline{r} = \underline{v}_{O'} + \underline{0} + \underline{\Omega} \times \underline{r} = \underline{v}_{\text{fix}} \tag{2.63}$$

The above states that a fixed point in the frame (R') appears to move at the velocity $\underline{v}_{\text{fix}}$ in (R).

Material derivative in inertial and non-inertial frames A flow quantity Q that is a function of time and of the Eulerian position $\underline{x}$ of (R) is considered, i.e. $Q = Q(t, x_1, x_2, x_3)$. The infinitesimal variation of Q is

$$\mathrm{d}Q = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}} \mathrm{d}t + \frac{\partial Q}{\partial x_i}\mathrm{d}x_i = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}} \mathrm{d}t + \underline{\mathrm{d}x} \cdot \nabla_{\underline{x}} Q \tag{2.64}$$

The variation of the quantity during an infinitesimal time $\mathrm{d}t$ and along the fluid trajectory is given by the material derivative:

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{(R)} = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}} + \underline{u} \cdot \nabla_{\underline{x}} Q \tag{2.65}$$

Replacing $\underline{u}$ in Eq. 2.65 by $\underline{u}_{\text{rel}} + \underline{v}_{\text{fix}}$ according to Eq. 2.62 leads to:

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{(R)} = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}} + \underline{v}_{\text{fix}} \cdot \nabla_{\underline{x}} Q + \underline{u}_{\text{rel}} \cdot \nabla_{\underline{x}} Q \tag{2.66}$$

The flow quantity can also be expressed in the frame (R') as a function of time and the Eulerian position $\underline{x}'$ as $Q = Q(t, x'_{1'}, x'_{2'}, x'_{3'})$. The variation of Q during an infinitesimal time $\mathrm{d}t$ and along the fluid trajectory is:

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{(R')} = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}'} + \frac{\partial Q}{\partial x'_{i'}}\frac{\mathrm{d}x'_{i'}}{\mathrm{d}t} = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\text{rel}} \cdot \nabla_{\underline{x}'} Q \tag{2.67}$$

The variation of a scalar quantity Q along a trajectory is an objective quantity (see e.g. [36, p.107]) and hence $(\mathrm{d}Q/\mathrm{d}t)_{(R)} = (\mathrm{d}Q/\mathrm{d}t)_{(R')}$. Also, the gradient of a scalar is a first order tensor (it is the canonical example of a covariant tensor) and is an objective quantity invariant by observer transformation hence $\nabla_{\underline{x}} Q = (\nabla Q)_{(R)} = (\nabla Q)_{(R')} = \nabla_{\underline{x}'} Q$. Equating Eqs. 2.66 and 2.67 leads to the following identification:

$$\left.\frac{\partial Q}{\partial t}\right|_{\underline{x}'} = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}} + \underline{v}_{\text{fix}} \cdot \nabla_{\underline{x}} Q \tag{2.68}$$

Equations 2.65 and 2.67 are rewritten below to summarize the results of this paragraph:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}} + \underline{u} \cdot \nabla_{\underline{x}} Q = \left.\frac{\partial Q}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\mathrm{rel}} \cdot \nabla_{\underline{x}'} Q \tag{2.69}$$

Acceleration The results from the previous paragraphs are applied to derive the acceleration of the fluid. The flow velocity is expanded according to Eq. 2.62 as $\underline{u} = \underline{v}_{\mathrm{fix}} + \underline{u}_{\mathrm{rel}}$. The transport theorem Eq. 2.47 is applied to $\underline{v}_{\mathrm{fix}}$ and the material derivative from Eq. 2.69 is then developed as:

$$\begin{aligned}\left(\frac{\mathrm{d}\underline{v}_{\mathrm{fix}}}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}\underline{v}_{\mathrm{fix}}}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega} \times \underline{v}_{\mathrm{fix}} &= \left.\frac{\partial \underline{v}_{\mathrm{fix}}}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\mathrm{rel}} \cdot \nabla_{\underline{x}'} \underline{v}_{\mathrm{fix}} + \underline{\Omega} \times \underline{v}_{\mathrm{fix}} \\ &= \left.\frac{\partial \underline{v}_{\mathrm{fix}}}{\partial t}\right|_{\underline{x}'} + \underline{\Omega} \times \underline{u}_{\mathrm{rel}} + \underline{\Omega} \times \underline{v}_{\mathrm{fix}}\end{aligned} \tag{2.70}$$

where the relation $\underline{u}_{\mathrm{rel}} \cdot \nabla_{\underline{x}'} \underline{v}_{\mathrm{fix}} = \underline{\Omega} \times \underline{u}_{\mathrm{rel}}$ has been used. It is easily found by inserting the expression of $\underline{v}_{\mathrm{fix}} = \underline{v}_{O'} + \underline{\Omega} \times \underline{r}'$. The similar procedure is applied to $\underline{u}_{\mathrm{rel}}$ and gives:

$$\left(\frac{\mathrm{d}\underline{u}_{\mathrm{rel}}}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}\underline{u}_{\mathrm{rel}}}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega} \times \underline{u}_{\mathrm{rel}} = \left.\frac{\partial \underline{u}_{\mathrm{rel}}}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\mathrm{rel}} \cdot \nabla_{\underline{x}'} \underline{u}_{\mathrm{rel}} + \underline{\Omega} \times \underline{u}_{\mathrm{rel}} \tag{2.71}$$

The total fluid acceleration is given by the sum of Eqs. 2.70 and 2.71:

$$\underline{a} \triangleq \left(\frac{\mathrm{d}\underline{u}}{\mathrm{d}t}\right)_{(R)} = \underline{a}_{\mathrm{fix}} + \underline{a}_{\mathrm{cor}} + \underline{a}_{\mathrm{rel}} \tag{2.72}$$

with

$$\underline{a}_{\mathrm{fix}} \triangleq \left.\frac{\partial \underline{v}_{\mathrm{fix}}}{\partial t}\right|_{\underline{x}'} + \underline{\Omega} \times \underline{v}_{\mathrm{fix}} = \underline{a}_{O'} + \frac{\mathrm{d}\underline{\Omega}}{\mathrm{d}t} \times \underline{r}' + \underline{\Omega} \times \underline{\Omega} \times \underline{r}' \tag{2.73}$$

$$\underline{a}_{\mathrm{cor}} \triangleq 2\underline{\Omega} \times \underline{u}_{\mathrm{rel}} \tag{2.74}$$

$$\underline{a}_{\mathrm{rel}} \triangleq \left(\frac{\mathrm{d}\underline{u}_{\mathrm{rel}}}{\mathrm{d}t}\right)_{(R')} = \left.\frac{\partial \underline{u}_{\mathrm{rel}}}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\mathrm{rel}} \cdot \nabla_{\underline{x}'} \underline{u}_{\mathrm{rel}} \tag{2.75}$$

The expression for $\underline{a}_{\mathrm{fix}}$ in Eq. 2.73 has been expressed using the definition $\underline{v}_{\mathrm{fix}} = \underline{v}_{O'} + \underline{\Omega} \times \underline{r}'$, noting that $\underline{v}_{O'}$ and $\underline{\Omega}$ are functions of time only, using $\left.\frac{\partial \underline{r}'}{\partial t}\right|_{\underline{x}'} = \underline{0}$, using $\underline{a}_{O'} = \left(\frac{\mathrm{d}\underline{v}_{O'}}{\mathrm{d}t}\right)_{(R')} + \underline{\Omega} \times \underline{v}_{O'}$ and $\left(\frac{\mathrm{d}\underline{\Omega}}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}\underline{\Omega}}{\mathrm{d}t}\right)_{(R')}$. An alternative form for $\underline{a}_{\mathrm{fix}}$ is:

$$\underline{a}_{\text{fix}} = \frac{\mathrm{d}\underline{\Omega}}{\mathrm{d}t} \times \underline{r}' + \operatorname{grad}\left[\left(\frac{\mathrm{d}\underline{v}_{O'}}{\mathrm{d}t}\right)_{(R')} \cdot \underline{r}' - \frac{v_{\text{fix}}^2}{2}\right] \tag{2.76}$$

from which it follows that curl $\underline{a}_{\text{fix}} = 2\underline{\Omega}$.

Fluid-mechanic equations in non-inertial frame The derivations from the previous paragraphs are now directly used to express the fluid-mechanics equations in a non-inertial frame. The continuity equation in the frame (R) is given by Eq. 2.26:

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)_{(R)} + \rho \operatorname{div}_{\underline{x}} \underline{u} = 0, \qquad \left.\frac{\partial \rho}{\partial t}\right|_{\underline{x}} + \operatorname{div}_{\underline{x}}(\rho \underline{u}) = 0 \tag{2.77}$$

To go from the left to right equation the term div(ρ) is developed as ρ div $\underline{u} + \underline{u} \cdot \operatorname{grad} \rho$ (see Sect. C.2). Using $\underline{u} = \underline{v}_{\text{fix}} + \underline{u}_{\text{rel}}$, noting that div $\underline{v}_{\text{fix}} = 0$ and using the fact that the divergence of a 1$^{\text{st}}$ order tensor is a 0$^{\text{th}}$-order tensor invariant by observer transformation (i.e. $\operatorname{div}_{\underline{x}} \underline{u} = \operatorname{div}_{\underline{x}'} \underline{u}$) the continuity equation in a non-inertial frame is:

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)_{(R')} + \rho \operatorname{div}_{\underline{x}'} \underline{u}_{\text{rel}} = 0, \qquad \left.\frac{\partial \rho}{\partial t}\right|_{\underline{x}'} + \operatorname{div}_{\underline{x}'}(\rho \underline{u}_{\text{rel}}) = 0 \tag{2.78}$$

Equation 2.68 was used to obtain the equation on the right. It is recalled that Eq. 2.69 holds:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)_{(R)} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)_{(R')} = \left.\frac{\partial \rho}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\text{rel}} \cdot \nabla_{\underline{x}'} \rho \tag{2.79}$$

The momentum equation in the frame (R) is given by Eq. 2.20:

$$\rho \left(\frac{\mathrm{d}\underline{u}}{\mathrm{d}t}\right)_{(R)} = \rho \underline{F} + \operatorname{div}_{\underline{x}} \underline{\underline{\sigma}}, \tag{2.80}$$

Using $\left(\frac{\mathrm{d}\underline{u}}{\mathrm{d}t}\right)_{(R)} = \underline{a}_{\text{fix}} + \underline{a}_{\text{cor}} + \underline{a}_{\text{rel}}$ from Eq. 2.72 and noting that the divergence of a second order tensor is a first order tensor invariant by observer transformation (i.e. $\operatorname{div}_{\underline{x}} \underline{\underline{\sigma}} = \operatorname{div}_{\underline{x}'} \underline{\underline{\sigma}}$), the momentum equation in a non-inertial frame is:

$$\rho \left(\frac{\mathrm{d}\underline{u}_{\text{rel}}}{\mathrm{d}t}\right)_{(R')} = \rho \left(\underline{F} - \underline{a}_{\text{fix}} - \underline{a}_{\text{cor}}\right) + \operatorname{div}_{\underline{x}'} \underline{\underline{\sigma}}, \tag{2.81}$$

where

$$\left(\frac{\mathrm{d}\underline{u}_{\text{rel}}}{\mathrm{d}t}\right)_{(R')} = \left.\frac{\partial \underline{u}_{\text{rel}}}{\partial t}\right|_{\underline{x}'} + \underline{u}_{\text{rel}} \cdot \nabla_{\underline{x}'} \underline{u}_{\text{rel}} \tag{2.82}$$

The stress tensor is expressed in both frames:

$$\begin{aligned}\underline{\underline{\sigma}} &= (-p + \lambda\, \mathrm{div}_{\underline{x}}\, \underline{u})\underline{\underline{\mathbb{1}}} + \mu \left(\nabla_{\underline{x}} \underline{u} + {}^t\nabla_{\underline{x}} \underline{u}\right) \\ &= (-p + \lambda\, \mathrm{div}_{\underline{x'}}\, \underline{u}_{\mathrm{rel}})\underline{\underline{\mathbb{1}}} + \mu \left(\nabla_{\underline{x'}} \underline{u}_{\mathrm{rel}} + {}^t\nabla_{\underline{x'}} \underline{u}_{\mathrm{rel}}\right)\end{aligned}$$

A Newtonian fluid was assumed in the above expressions (see Eq. 2.89). The energy and enthalpy equations in a non-inertial frame are obtained from Eqs. 2.29 and 2.30 in a similar way:

$$\rho\left(\frac{\mathrm{d}e_{t,\mathrm{rel}}}{\mathrm{d}t}\right)_{(R')} = \rho(\underline{F} - \underline{a}_{\mathrm{fix}} - \underline{a}_{\mathrm{cor}})\cdot\underline{u}_{\mathrm{rel}} + \mathrm{div}_{\underline{x'}}(\underline{\underline{\sigma}}\,\underline{u}_{\mathrm{rel}} - \underline{q_s}) \tag{2.83}$$

$$\rho\left(\frac{\mathrm{d}h_{t,\mathrm{rel}}}{\mathrm{d}t}\right)_{(R')} = \rho(\underline{F} - \underline{a}_{\mathrm{fix}} - \underline{a}_{\mathrm{cor}})\cdot\underline{u}_{\mathrm{rel}} + \mathrm{div}_{\underline{x'}}(\underline{\underline{\tau}}\,\underline{u}_{\mathrm{rel}} - \underline{q_s}) + \frac{\partial p}{\partial t} \tag{2.84}$$

with $e_{t,\mathrm{rel}} = e + u_{\mathrm{rel}}^2/2$ and $h_{t,\mathrm{rel}} = e + p/\rho + u_{\mathrm{rel}}^2/2$.

2.1.4 Fluid Mechanics Assumptions

Separation of viscous effects In a fluid in motion the following form for the stress tensor is generally assumed:

$$\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{1}}} + \underline{\underline{\tau}} \tag{2.85}$$

where the first tensor represents non-viscous stresses while the second tensor represents stresses of viscous origin due to the fluid deformation. Equation 2.27 may then be written:

$$\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \underline{F} - \frac{1}{\rho}\,\mathrm{grad}\, p + \frac{1}{\rho}\,\mathrm{div}\,\underline{\underline{\tau}} \tag{2.86}$$

Classical fluid A *classical fluid* is a continuum for which the stress law is of the form:

$$\underline{\underline{\sigma}} = f(\mathrm{grad}\ \underline{u}, \rho, T) \tag{2.87}$$

Stresses are thus directly related to the fact that a velocity field is present within the fluid. Further, stresses do not depend on motions at various locations but only of the stress rate defined by grad $\underline{u}$.

Fourier's law For most isotropic fluid, the heat flux $\underline{q}_s$ satisfies Fourier's law:

$$\underline{q}_s = -k\,\mathrm{grad}\, T \tag{2.88}$$

where k is the thermal conductivity of the fluid. Fourier's law is only presented here to mention the definition of a perfect fluid in the following paragraph.

Newtonian fluid Three properties define a *Newtonian fluid*:

- the stress tensor $\underline{\underline{\sigma}}$ is a linear function of grad $\underline{u}$
- the stress tensor is invariant with respect to any rigid motion applied to the entire domain
- the fluid is isotropic (quantities do not have directional preferences).

For a Newtonian fluid, the viscous stress tensor $\underline{\underline{\tau}}$ and the heat flux $\underline{q}_s$ are written:

$$\underline{\underline{\tau}} = \lambda(\operatorname{div} \underline{u})\underline{\underline{1}} + 2\mu\underline{\underline{\mathscr{D}}} \tag{2.89}$$

$$\underline{q}_s = -k \operatorname{grad} T \tag{2.90}$$

with $\underline{\underline{\mathscr{D}}}$ the deformation tensor (see Sect. 2.2.1, $\underline{\underline{\mathscr{D}}} = \frac{1}{2}(\operatorname{grad} \underline{u} +^t \operatorname{grad} \underline{u})$), λ and μ are *Lamé's viscous coefficients*, and k is the *thermal conductivity* of the fluid. In general, the coefficient λ, μ and k are functions of ρ or T and are not constant. Fluid dynamics equations for Newtonian fluids are known as the *Navier–Stokes equations*.

Perfect fluid In this document a "*perfect fluid*" will correspond to a fluid whose motion may be described without the effect of viscosity and thermal conductivity. For a Newtonian fluid, we will have: $(\lambda, \mu, k) \rightarrow (0, 0, 0)$, which is $\underline{\underline{\tau}} = 0$ and $\underline{q} = 0$. Since this document does not consider heat fluxes, the use of inviscid assumption will be preferred to the "perfect fluid" assumption.

Incompressible fluid For an *incompressible* fluid, the rate of density change of a fluid particle $\mathrm{d}\rho/\mathrm{d}t$ is negligible compared with the component term $\rho\nabla \cdot \underline{u}$. Using Cartesian coordinates, this writes [17, p. 105]:

$$\left|\frac{\mathrm{d}\rho}{\mathrm{d}t}\right| \ll \rho \left|\nabla \cdot \underline{u}\right| \leq \rho \left(\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| + \left|\frac{\partial w}{\partial z}\right|\right) \tag{2.91}$$

By consideration of this condition, the continuity equation given in Eq. 2.26 becomes:

$$\nabla \cdot \underline{u} = 0 \tag{2.92}$$

For an incompressible fluid, the dilatation is then zero, i.e. $\Theta \equiv 0$. Inserting the above into the continuity equation also implies:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \tag{2.93}$$

so that the density is constant on trajectories. It does not imply that the density has the same constant value in the entire domain: different particles can have different densities but their value remains constant. If the fluid is homogeneous at the initial time, then the density remains constant over time and space ($\nabla \rho = 0$): such flow is called a constant-density flow.

By definition of $\underline{\omega} = \text{curl } \underline{u}$, and using "curl curl $=$ grad div $-\nabla^2$", the following relation holds for incompressible flows:

$$\text{curl } \underline{\omega} = -\nabla^2 \underline{u} \tag{2.94}$$

The condition div $\underline{u} = 0$ is also found by considering the assumption that an elementary material volume D_m, followed in its motion, is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_m} \mathrm{d}V = 0 \tag{2.95}$$

The above is using the Green–Ostrogradski theorem from Eq. C.36 [41, § 3.3]:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_m} \mathrm{d}V = \int_{D_m} \frac{\partial 1}{\partial t} \mathrm{d}V + \int_S 1\underline{u} \cdot \underline{n} \, \mathrm{d}S = \int_{D_m} \left(\frac{\partial 1}{\partial t} + \text{div}(1\underline{u}) \right) \mathrm{d}V = \int_{D_m} \text{div } \underline{u} \, \mathrm{d}V = 0 \tag{2.96}$$

Since D_m is an arbitrary material volume, it implies div $\underline{u} = 0$. The incompressible condition from Eq. 2.92 is satisfied identically if (see e.g. [48, p. 10]):

$$\underline{u} = \text{curl } \underline{\psi} \tag{2.97}$$

where $\underline{\psi}$ is a function of position (since "div curl $\equiv 0$").

Baroclinic/non-baroclinic(barotropic) fluid A *baroclinic* fluid satisfies grad $p \times$ grad $\rho \neq 0$. Examples of non-baroclinic fluids, i.e. such that grad $p \times$ grad $\rho \equiv 0$ are barotropic fluids. Barotropic fluids are such that the pressure is a pure function of the density and does not depend on the temperature. In the following the assumption of barotropic fluid will be used, but the condition of non-baroclinic fluid should be enough. Two examples of barotropic fluid are:

1. homogeneous, incompressible fluid
2. perfect fluid in homoentropic flow.

Ideal incompressible fluid This assumption is considered to be satisfied when the Reynolds number is high, the Mach number is low and the ratio of temperatures is small.

Incompressible, Newtonian fluid For an incompressible Newtonian fluid, Eq. 2.89 becomes:

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{\mathscr{D}}} = \mu \left(\text{grad } \underline{u} + {}^t\text{grad } \underline{u} \right) \tag{2.98}$$

2.1.5 Usual Cases - Equations of Euler and Bernoulli

Homogeneous incompressible Newtonian fluid of constant viscosity For an incompressible, homogeneous, Newtonian fluid of constant viscosity, using "div(grad) $= \Delta$" (Appendix C) and Eq. 2.94:

$$\frac{1}{\rho}\,\mathrm{div}\,\underline{\underline{\tau}} = \frac{1}{\rho}\,\mathrm{div}\left(0 + \mu\left(\mathrm{grad}\ \underline{u} + {}^t\mathrm{grad}\ \underline{u}\right)\right) = \nu\Delta u = -\nu\,\mathrm{curl}\ \underline{\omega} \tag{2.99}$$

with $\mu/\rho = \nu$. The conservation of mass and momentum write:

$$\mathrm{div}\,\underline{u} = 0 \tag{2.100}$$

$$\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \underline{F} - \frac{1}{\rho}\,\mathrm{grad}\ p - \nu\,\mathrm{curl}\ \underline{\omega} \tag{2.101}$$

Using $[\mathrm{grad}\,\underline{u}]\cdot\underline{u} = \underline{\omega}\times\underline{u} + \mathrm{grad}(u^2/2)$, Eq. 2.101 writes:

$$\frac{\partial\underline{u}}{\partial t} + \underline{\omega}\times\underline{u} = \underline{F} - \frac{1}{\rho}\,\mathrm{grad}\ p_t - \nu\,\mathrm{curl}\ \underline{\omega} \tag{2.102}$$

where $p_t = p + \frac{1}{2}\rho u^2$ is the total pressure, sometimes written H and referred to as the *Bernoulli constant*. Taking the scalar product of Eq. 2.102 with $\underline{u}$ leads to the following energy equation:

$$\frac{\mathrm{d}p_t}{\mathrm{d}t} = \rho\underline{F}\cdot\underline{u} + \frac{\partial p}{\partial t} - \mu(\mathrm{curl}\ \underline{\omega})\cdot\underline{u} \tag{2.103}$$

Euler's equations *Euler's equations* are obtained under the assumptions of a homogeneous, inviscid, Newtonian fluid under conservative forces. The conservative forces are assumed to derive from a potential V_F such that $\underline{F} = -\,\mathrm{grad}\ V_F$. Using $[\mathrm{grad}\,\underline{u}]\cdot\underline{u} = \underline{\omega}\times\underline{u} + \mathrm{grad}(u^2/2)$, Euler's equations writes:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\mathrm{div}\,\underline{u} = 0 \tag{2.104}$$

$$\frac{\partial\underline{u}}{\partial t} + \underline{\omega}\times\underline{u} = \frac{-1}{\rho}\,\mathrm{grad}\ p - \mathrm{grad}\left(\frac{u^2}{2} + V_F\right) \tag{2.105}$$

$$\rho\frac{\mathrm{d}e}{\mathrm{d}t} = -p\,\mathrm{div}\,\underline{u}, \qquad p = p(\rho, T), e = e(\rho, T) \tag{2.106}$$

The conservation of momentum (Eq. 2.105) in polar coordinates and without external forces writes (see Sect. C.3) is

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2.107}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2.108}$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial z} \tag{2.109}$$

B1: First Bernoulli theorem (weak formulation) The following assumptions are applied: incompressible, steady flow of an inviscid fluid under conservative forces (i.e. deriving from a potential V_F). Under these assumptions, the flow is said to be iso-energetic and satisfy *Bernoulli theorem* in its weak formulation:

$$e + \frac{p}{\rho} + \frac{u^2}{2} + V_F = \text{constant along a streamline} \tag{2.110}$$

This formulation is qualified as weak due to the fact that the constant is in general different from one streamline to another. Bernoulli's equation expresses the conservation of the "total energy" defined here as the sum of the internal energy, the kinetic energy and potential energy associated with volume and pressure forces.

B1: First Bernoulli theorem in a rotating frame The notations and results of the fluid-mechanics equations in a non-inertial frame derived in Sect. 2.1.3 are adopted. A frame rotating with constant angular velocity $\underline{\Omega}$ is considered. The flow velocity in this frame is written $\underline{u}_{\text{rel}}$. The Coriolis and centrifugal fictitious external forces that appear in a rotating system are respectively $-2\underline{\Omega} \times \underline{u}_{\text{rel}}$ and $-\underline{\Omega} \times \underline{\Omega} \times \underline{r}' = \frac{1}{2}\,\text{grad}(\underline{\Omega} \times \underline{r}')^2$. For an incompressible, steady flow (steady in the rotating frame) of an inviscid fluid under the body force $\underline{F}_b$, Eq. 2.103 writes:

$$\frac{\mathrm{d}p_t}{\mathrm{d}t} = \rho\left(\underline{F}_b - 2\underline{\Omega} \times \underline{u}_{\text{rel}} + \frac{1}{2}\,\text{grad}\left[(\underline{\Omega} \times \underline{r}')^2\right]\right) \cdot \underline{u}_{\text{rel}} \tag{2.111}$$

The second term of the RHS is identically 0. Further simplifications are obtained if the body force is 0 or if it is orthogonal to $\underline{u}_{\text{rel}}$.[2] Under this assumption, the first term of the RHS of Eq. 2.111 is 0, leading to:

$$\frac{\mathrm{d}p_t}{\mathrm{d}t} = \text{grad}\left(\frac{1}{2}\rho\underline{\Omega} \times \underline{r}'\right)^2 \cdot \underline{u}_{\text{rel}} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\rho\left(\underline{\Omega} \times \underline{r}'\right)^2\right] \tag{2.112}$$

where the fact that the flow is steady in the rotating frame has been used for the second equality. It follows that

[2] De Vries argued that this is the case for an actuator disk in inviscid flow. Since the flow is inviscid, the actuator disk force is due to a lift force, which is indeed orthogonal to the velocity [15]. For an actuator disk, the lift force is artificially introduced and it is not the result of the pressure field.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p_t - \frac{1}{2}\rho\left(\underline{\Omega}\times\underline{r}'\right)^2\right] = 0 \tag{2.113}$$

and hence:

$$p + \frac{1}{2}\rho u_{\mathrm{rel}}^2 - \frac{1}{2}\rho\left(\underline{\Omega}\times\underline{r}'\right)^2 = \text{constant along a streamline} \tag{2.114}$$

B2: Second Bernoulli theorem (strong formulation) The following assumptions are made: irrotational flow (i.e. curl $\underline{u} = 0$ and hence the velocity is written $\underline{u} = \operatorname{grad}\Phi$), perfect fluid, barotropic fluid (i.e. $p = p(\rho)$ and hence $\frac{1}{\rho}\operatorname{grad} p = \operatorname{grad}\int\frac{\mathrm{d}p}{\rho}$), under conservative forces (i.e. deriving from a potential V_F). Under these assumptions, Bernoulli's strong formulation writes:

$$\frac{\partial\Phi}{\partial t} + \int\frac{\mathrm{d}p}{\rho} + \frac{u^2}{2} + V_F = C(t) \tag{2.115}$$

Unlike the weak formulation, the constant $C(t)$ in the above equation is used for points that do not belong to the same streamline.

The assumption of barotropic fluid may be replaced by the assumption of incompressible fluid. In that case:

$$\frac{\partial\Phi}{\partial t} + \frac{p}{\rho} + \frac{u^2}{2} + V_F = C(t) \tag{2.116}$$

Particular case of B2: homoentropic flow For a irrotational and homoentropic flow of a perfect fluid under conservative forces, Eq. 2.115 becomes:

$$\frac{\partial\Phi}{\partial t} + e + \frac{p}{\rho} + \frac{u^2}{2} + V_F = C(t) \tag{2.117}$$

Indeed, the homoentropic assumption implies $\mathrm{d}h = T\mathrm{d}s + \frac{\mathrm{d}p}{\rho} = \frac{\mathrm{d}p}{\rho}$ and hence $\int\frac{\mathrm{d}p}{\rho} = h$.

Particular case of B1 and B2: homogeneous, incompressible, perfect fluid For a homogeneous, incompressible, perfect fluid under conservative forces, the internal energy is conserved along the streamlines (i.e. $\mathrm{d}e/\mathrm{d}t = 0$), and hence Eq. 2.110 becomes:

$$\frac{p}{\rho} + \frac{u^2}{2} + V_F = \text{constant along a streamline} \tag{2.118}$$

The fluid is barotropic with $\rho = \text{cst}$, so Eq. 2.115 becomes:

$$\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} + \frac{u^2}{2} + V_F = C(t) \tag{2.119}$$

If the flow is also steady, then:

$$\frac{p}{\rho} + \frac{u^2}{2} + V_F = C \tag{2.120}$$

2.2 Flow Kinematics and Vorticity

2.2.1 *Flow Kinematics*

Motion of a material element A *material element* (line, vector, surface, volume) consists of the same fluid particles throughout time. An elementary material vector $\underline{\delta l}(t)$, located at the point $M = \underline{x}$ at the time t is considered. The other vector extremity is written $P(t)$, such that $\underline{\delta l}(t) = P(t) - M(t)$. The material vector evolves according to the trajectory equation $\mathrm{d}\underline{x}/\mathrm{d}t = \underline{u}$. The Taylor expansion of the functions M and P writes:

$$M(t+\delta t) = M(t) + \underline{u}(\underline{x}, t)\delta t \qquad\qquad + O(\delta t^2) \tag{2.121}$$

$$P(t+\delta t) = P(t) \ + \underline{u}(\underline{x} + \underline{\delta l}(t), t)\delta t + O(\delta t^2) \tag{2.122}$$

These equations are gathered by introducing the definition of $\underline{\delta l}$:

$$\underline{\delta l}(t+\delta t) = \underline{\delta l}(t) + \left[\underline{u}\left(\underline{x} + \underline{\delta l}\,(t)\,, t\right) - \underline{u}\left(\underline{x}, t\right)\right]\delta t + O(\delta t^2) \tag{2.123}$$

The Taylor expansion of the velocity field at $\underline{x}$, if defined, writes:

$$\underline{u}(\underline{x} + \underline{h}, t) = \underline{u}(\underline{x}, t) + \text{grad } \underline{u}(\underline{x}, t) \cdot \underline{h} + O(h^2) \tag{2.124}$$

Inserting Eq. 2.124 into 2.123 leads to:

$$\underline{\delta l}(t+\delta t) = \underline{\delta l}(t) + \left[\text{grad } \underline{u}(\underline{x}, t) \cdot \underline{\delta l}(t)\right]\delta t + O(\delta t^2, h^2) \tag{2.125}$$

By evaluation of the limit and using shorter notations, this writes:

$$\frac{\mathrm{d}(\underline{\delta l})}{\mathrm{d}t} = (\text{grad } \underline{u}) \cdot \underline{\delta l} \quad \Leftrightarrow \quad \frac{\mathrm{d}(\underline{\delta l})}{\mathrm{d}t} = \left(\underline{\delta l} \cdot \nabla\right) \underline{u} \tag{2.126}$$

Deformation and rotation matrix At a given point in space where it is defined, the gradient of the velocity field may be decomposed [31, p. 6] into a symmetric part $\mathscr{D}$ and an antisymmetric part Ω as:

$$\text{grad } \underline{u} = \frac{1}{2}\left(\text{grad } \underline{u} +^t \text{grad } \underline{u}\right) + \frac{1}{2}\left(\text{grad } \underline{u} -^t \text{grad } \underline{u}\right) \tag{2.127}$$

$$= \underline{\underline{\mathscr{D}}} + \underline{\underline{\Omega}} \tag{2.128}$$

where $\mathscr{D}$ is the *deformation matrix* or *rate-of-strain matrix*, and Ω is the *rotation matrix*. The trace of the deformation matrix is equal to the divergence of the velocity, i.e. $\text{tr}\,\mathscr{D} = \text{div}\,\underline{u}$. Hence, for incompressible flows the trace of $\mathscr{D}$ is 0. Using the definition of the vorticity (Eq. 2.3) and the definition of Ω, it is shown that:

$$\forall \underline{h} \in \mathbb{R}^3, \quad \underline{\underline{\Omega}} \cdot \underline{h} = \frac{1}{2}\underline{\omega} \times \underline{h} \tag{2.129}$$

This is of relevance for vortex methods, in particular when $\underline{h}$ is chosen as $\underline{\omega}$. More details and physical interpretations are found in standard fluid mechanics text books, e.g. the book of Saffman [43].

Vorticity stretching It will be seen in Sect. 2.3.2 that the "stretching" term $\text{grad } \underline{u} \cdot \underline{\omega} \equiv (\underline{u} \cdot \nabla)\underline{\omega}$ appears in the dynamic equation of vorticity. The development of this term shows that only the deformation tensor contributes to the vorticity stretching:

$$\text{grad } \underline{u} \cdot \underline{\omega} = \left(\underline{\underline{\mathscr{D}}} + \underline{\underline{\Omega}}\right) \cdot \underline{\omega} = \underline{\underline{\mathscr{D}}} \cdot \underline{\omega} + \underline{\underline{\Omega}} \cdot \underline{\omega} = \underline{\underline{\mathscr{D}}} \cdot \underline{\omega} + \frac{1}{2}\underline{\omega} \times \underline{\omega} = \underline{\underline{\mathscr{D}}} \cdot \underline{\omega} \tag{2.130}$$

Using the above, since $\mathscr{D}$ is symmetric, the three following expressions are equal:

$$\text{grad } \underline{u} \cdot \underline{\omega} =^t \text{grad } \underline{u} \cdot \underline{\omega} = \frac{1}{2}\left[{}^t\text{grad } \underline{u} +^t \text{grad } \underline{u}\right] \cdot \underline{\omega} \tag{2.131}$$

The three equalities are written below using the Del notation:

$$(\underline{\omega} \cdot \nabla)\underline{u} = (\underline{\omega} \cdot \nabla^T)\underline{u} = \frac{1}{2}\left[\underline{\omega} \cdot \left(\nabla + \nabla^T\right)\right]\underline{u} \tag{2.132}$$

Alternatively, one can understand these equalities using the following identity: $\left(\nabla\underline{a} - (\nabla\underline{a})^T\right) \cdot \underline{b} = \left(\nabla \times \underline{a}\right) \times \underline{b}$. Applied to $\underline{u}$ and $\underline{\omega}$ this yields: $\left(\nabla\underline{u} - (\nabla\underline{u})^T\right) \cdot \underline{\omega} = \left(\nabla \times \underline{u}\right) \times \underline{\omega} = 0$. It is noted that in general $\nabla\underline{u} \neq (\nabla\underline{u})^T$ and only the multiplication by $\underline{\omega}$ makes this equality true.

2.2.2 *Vorticity and Related Definitions*

Discussion on vortex and vorticity vocabulary The notions of vortex and vorticity are strongly linked. *vorticity* is a measure of the local rotation of a fluid particle (see Eq. 2.128). The definition of a vortex is more ambiguous. For simplicity, a *vortex* is assumed to represent a coherent flow structure which is characterized to some extent by a large scale rotation of the fluid. A vortex has vorticity but a vorticity field does not necessarily represent a vortex: A steady shear layer is a vorticity field with no distinct vortex. The notion of vortex in 3D requires the introduction of a criteria to define/detect a vortex. Most methods are based on the decomposition of the velocity gradient tensor and the extraction of its invariant or eigenvalues. More details on the definition of a vortex is found in the dedicated articles of e.g. Hunt et al. [23], Jeong and Hussain [24], Haller [18] and in the book of Wu [55, pp. 72 and 310].

In many applications, the vorticity is concentrated to small areas. In such case, a vortex denote any finite volume of vorticity immersed in a irrotational fluid. The terms vorticity and vortex are then used without rigorous distinction. For instance, the following terms are indifferently used in this book: vortex sheet/vorticity sheet, vortex methods/vorticity-based method.

Vortex lines In a region where the vorticity does not vanish identically, lines tangent to the local vorticity vector at each point are called *vortex lines* [43, pp. 8–10]. In other words, they are the field lines of the vorticity field. By definition of field lines, a differential element $\underline{\mathrm{d}x}$ tangent to a vorticity line satisfy:

$$\underline{\mathrm{d}x} \times \underline{\omega}(\underline{x}, t) = 0 \tag{2.133}$$

Written differently, vortex lines are solutions of the differential equation:

$$\frac{\mathrm{d}x}{\omega_x} = \frac{\mathrm{d}y}{\omega_y} = \frac{\mathrm{d}z}{\omega_z} \tag{2.134}$$

For any curve L, the continuous set of vorticity lines that pass by this curve form a vorticity surface. If the curve L is a closed path, the vorticity surface forms a *vorticity tube*, referred also as a *vortex tube*. Vortex tubes and lines are studied in details by Saffman [43, pp. 8–10].

Vorticity and rotation - Kinematic interpretation The vorticity is twice the average angular velocity around an infinitesimal circle:

$$\frac{1}{2\pi l} \oint_C \frac{\underline{u}}{l} \cdot \underline{\mathrm{d}l} = \frac{1}{2\pi l^2} \iint_S \underline{\omega} \cdot \mathrm{d}\underline{A} \quad \text{(Stokes's theorem)} \tag{2.135}$$

which tends to $\frac{1}{2}\underline{\omega}$ when $l \to 0$. More details are found in the book of Saffman [43, pp. 6–7].

Vorticity and rotation - Dynamic interpretation The dynamic interpretation of vorticity is obtained by considering the angular momentum about the centroid of a fluid particle rotating as a solid body with angular velocity $\omega/2$. The topic is presented in the book of Saffman [44, p. 7].

Circulation If C design a closed geometric path, then the *circulation* along this path refers to the following curvilinear integral:

$$\Gamma \triangleq \oint_C \underline{u} \cdot \underline{\mathrm{d}s} \qquad [\mathrm{m}^2/\mathrm{s}] \tag{2.136}$$

Using Stokes theorem (Eq. C.50) and $\underline{\omega} = \mathrm{curl}\ \underline{u}$, the circulation is directly related to the flux of vorticity through any surface S that has C as support:

$$\Gamma = \iint_S \underline{\omega} \cdot \underline{n} \mathrm{d}S \tag{2.137}$$

Irrotational flow An irrotational flow is a flow for which the vorticity $\underline{\omega}$ is zero everywhere. An irrotational flow is obtained if the following assumptions hold [22]: - perfect fluid - barotropic fluid (e.g. incompressible homogeneous fluid), - fluid under conservative forces, - initially irrotational: $\forall \underline{x} \in \Omega, \omega(\underline{x}, t = 0) = 0$, - (there is only one streamline exiting an obstacle). The irrotational nature of the flow is a consequence of Lagrange's theorem presented in Sect. 2.6.2.

Solenoidal field - case of the vorticity Because of the vectorial relation "div(curl) $\equiv$ 0", the vorticity field has a divergence of zero. Fields that satisfy this condition are called solenoidal. From the divergence theorem (see Appendix C) which relates a volume integral to a closed surface, it is seen that the flux of vorticity through a closed surface is identically null:

$$\int_{\partial\Omega} \underline{\omega} \cdot \underline{n}\, \mathrm{d}S = \int_{\Omega} \mathrm{div}\, \underline{\omega}\, \mathrm{d}\Omega \equiv 0 \tag{2.138}$$

The vorticity flux through a closed surface is always zero but it is not true in general for an open surface. It does apply for open surfaces that are vorticity surfaces since there the vorticity is orthogonal to the surface normal. Also, this relation should not be confused with Kelvin's theorem (presented in Sect. 2.6.1), which involves the time derivative of the vorticity flux on an open surface under restrictive conditions, which are not present here.

For a differentiable vector field $\underline{f}$, the following relation holds: $\mathrm{div}(x_i \underline{f}) = x_i \,\mathrm{div}\, \underline{f} + \underline{f} \cdot (\mathrm{grad}\, x_i) = x_i \,\mathrm{div}\, \underline{f} + f_i$. It follows that for a solenoidal vector field:

$$\omega_i \equiv \mathrm{div}(x_i \underline{\omega}) \tag{2.139}$$

The application of the divergence theorem leads then to:

$$\int_{\Omega} \omega_i \,\mathrm{d}\Omega = \int_{\Omega} \mathrm{div}(x_i \underline{\omega})\, \mathrm{d}\Omega = \int_{\partial\Omega} x_i \underline{\omega} \cdot \underline{n} \mathrm{d}S \tag{2.140}$$

The application to all components leads to the following vectorial form (see also Eq. C.43):

$$\int_{\Omega} \underline{\omega} \, \mathrm{d}\Omega = \int_{\partial\Omega} \underline{x}(\underline{\omega} \cdot \underline{n}) \mathrm{d}S \tag{2.141}$$

Consequently, if the vorticity is zero outside of a finite region D then the total vorticity in the domain $\Omega \supset D$ is zero. As a side note, the application of Eq. C.47 provides an alternative form for the total vorticity:

$$\int_{\Omega} \underline{\omega} \, \mathrm{d}\Omega = \int_{\partial\Omega} \underline{n} \times \underline{u} \mathrm{d}S \tag{2.142}$$

2.2.3 Helmholtz (First) Law

From Eq. 2.138 it follows that no source or sink of vorticity are present in the volume. The three possible configurations allowed for vorticity lines are then: closed curve, lines of infinite length, and finite lines whose extremities are part of the flow boundaries (walls). Another implication is that in a vorticity tube the vorticity flux and the circulation is conserved along the tube. $\Gamma(t)$ can thus be called the intensity of the vorticity tube.

2.2.4 Helmholtz-(Hodge) Decomposition

The *Helmholtz decomposition* is discussed for instance by: Richardson and Cornish [40], Majda and Bertozzi [31, p. 72], Morino [32], Batchelor [4, p. 84]. A full account of the problem requires a knowledge of the functional space in which the field is defined and a proper account of the boundary conditions (holes, walls, Neumann or Dirichlet conditions). Some elements are given below, but the following treatment is incomplete.

Helmholtz decomposition For most physical applications, a velocity field $\underline{u}$ is written according to the Helmholtz decomposition ([40]):

$$\underline{u} = \underline{u}_0 + \underline{u}_\omega + \underline{u}_\Phi \tag{2.143}$$

where

$$\begin{aligned}
&\mathrm{curl}\ \underline{u}_\Phi \equiv \underline{0}, && \underline{u}_\Phi \text{ is curl free (divergence part)} && (2.144)\\
&\mathrm{div}\, \underline{u}_\omega \equiv 0, && \underline{u}_\omega \text{ is divergence free (rotational part)} && (2.145)\\
&\mathrm{curl}\ \underline{u}_0 \equiv \mathrm{div}\, \underline{u}_0 \equiv 0, && \underline{u}_0 \text{ is divergence free and curl free} && (2.146)
\end{aligned}$$

The decomposition is built from a *scalar potential* Φ and a *vector potential* $\underline{\psi}$ with:

$$\underline{u}_{\Phi} = \nabla \Phi, \qquad \underline{u}_{\omega} = \text{curl } \underline{\psi}, \qquad \underline{u}_0 = \underline{u} - \underline{u}_{\Phi} - \underline{u}_{\omega} \tag{2.147}$$

This decomposition is not unique and depends on the boundary conditions. For incompressible flows, $\underline{u}_{\Phi}$ accounts for boundary conditions. In absence of boundaries and for a flow occupying the entire space $\underline{u}_{\Phi} = 0$. The Helmholtz-Hodge decomposition [11, 37] is chosen such as $\underline{u}_0$ is harmonic, i.e. $\Delta \underline{u}_0 = 0$. In most applications $\underline{u}_0$ is a constant but it is not the only possibility: The example $\underline{u}_0 = \alpha x \underline{e}_x - \alpha y \underline{e}_y$ satisfies Eq. 2.146.

Poisson's equations involved The dilatation and vorticity are defined respectively as $\Theta \triangleq \text{div } \underline{u}$, and $\underline{\omega} \triangleq \text{curl } \underline{u}$ (see Sect. 2.1.1). Choosing $\underline{u}_{\Phi} = \text{grad } \Phi$, $\underline{u}_{\omega} = \text{curl } \underline{\psi}$ and the gauge $\text{div } \underline{\psi} = 0$, the decomposition from Eq. 2.143 leads to

$$\Theta \triangleq \text{div } \underline{u} = \text{div } \underline{u}_{\Phi} = \Delta \Phi \tag{2.148}$$

$$\underline{\omega} \triangleq \text{curl } \underline{u} = \text{curl } \underline{u}_{\omega} = -\Delta \underline{\psi} \qquad (\text{with div } \underline{\psi} = 0) \tag{2.149}$$

For an incompressible flow ($\Theta \equiv 0$), the first equation leads to a *Laplace equation*. For an irrotational flow ($\underline{\omega} \equiv \underline{0}$), the second equation leads to three Laplace equations.

2.2.5 *Bounded and Unbounded Domain - Surface Map - Generalized Helmholtz Decomposition*

Unbounded domain - Biot–Savart law In an unbounded space and in the absence of boundaries, the rotational part of the velocity is retrieved from the vorticity field using the Biot–Savart law (see Sect. 2.6.4):

$$\underline{u}_{\omega}(\underline{x}, t) = \int_D \underline{K}(\underline{x} - \underline{x}') \times \underline{\omega}(\underline{x}', t) \, \mathrm{d}\underline{x}' \tag{2.150}$$

where $\underline{K}$ is the Biot–Savart *kernel* defined in Eq. 2.232 and the integral is taken over $D = D_{\omega}$, with D_{ω} the support of vorticity, possibly infinite.

Reduced domain If one restricts the integral in Eq. 2.150 to a smaller domain $D = D_{\text{in}}$, the contribution from the vorticity outside of the domain is accounted for by means of a *Neumann-to-Dirichlet map* which ensures the continuity of tangential and normal velocity at the domain interface ∂D_{in}. The velocity field is then written: $\underline{u} = \underline{u}_0 + \underline{u}_{\omega} + \underline{u}_{\text{ext}}$ where $\underline{u}_{\text{ext}}$ is the surface integral solution of $\nabla^2 \underline{u} = -\nabla \times \underline{\omega}$ that satisfies the mapping with the external domain. The velocity obtained from the surface map is [35]:

$$\underline{u}_{\text{ext}}(\underline{x}) = \int_{\partial D_{\text{in}}} \left[-\underline{K}(\underline{x} - \underline{x}') \, u_n(\underline{x}') + \underline{K}(\underline{x} - \underline{x}') \times \underline{u}_{\tau}(\underline{x}') \right] \mathrm{d}\underline{x}' \tag{2.151}$$

where u_n is the component of the total velocity field normal to ∂D_{in} such that $u_n = \underline{u} \cdot \underline{n}$, with $\underline{n}$ pointing towards the interior of the domain, $\underline{u}_\tau = \underline{n} \times \underline{u}$ and $\underline{K}$ is the Biot–Savart kernel defined in Eq. 2.232. The gradient $\underline{\nabla u}_{\text{ext}}$ is directly obtained from the expression of $\underline{u}_{\text{ext}}$ as:

$$\underline{\nabla u}_{\text{ext}}(\underline{x}) = \int_{\partial D_{\text{in}}} \left[-\underline{\nabla K}(\underline{x} - \underline{x}')\, u_n(\underline{x}') + \underline{\nabla}\left(\underline{K}(\underline{x} - \underline{x}') \times \underline{u}_\tau\right)\right] \mathrm{d}\underline{x}' \tag{2.152}$$

The addition of the surface integral to the Helmholtz decomposition is sometimes referred to as the *generalized Helmholtz decomposition*.

Interpretation in terms of source and vorticity surfaces Equation 2.151 has the same expression as an integration over a surface distribution of source and vorticity (see Sect. 2.6.4). Introducing the source distribution $\sigma = u_n$ and the vorticity distribution $\underline{\gamma} = \underline{u}_\tau$, Eq. 2.151 is rewritten as:

$$\underline{u}_{\text{ext}}(\underline{x}) = \int_{\partial D_{\text{in}}} \left[-\underline{K}(\underline{x} - \underline{x}')\, \sigma(\underline{x}') + \underline{K}(\underline{x} - \underline{x}') \times \underline{\gamma}(\underline{x}')\right] \mathrm{d}\underline{x}' = \underline{u}_\sigma(\underline{x}) + \underline{u}_\gamma(\underline{x}) \tag{2.153}$$

where $\underline{u}_\sigma$ and $\underline{u}_\gamma$ are the velocities induced by the source and vorticity distributed on the boundary ∂D_{in}.

Applications The expression of $\underline{u}_{\text{ext}}$ is convenient in applications where the support of vorticity is infinite and a smaller computational domain of interest is investigated. This is in particular the case when studying sheared flow in numerical vortex methods. The vorticity associated with the shear profile has a support too large to be handled numerically. The surface map offers a convenient solution to account for this large support of vorticity (see Chap. 30).

The surface map method is also convenient since the outside domain doesn't need to be modelled as long as the velocity on the boundary of the domain is known.

A typical validation case for the numerical implementation of Eq. 2.151 may be a cylindrical domain with constant velocity $\underline{u}$. The surface corresponds then to a vortex cylinder with source terms at its cross-sections and the velocity is indeed constant within the domain. This application is detailed in Sect. 36.1.4.

2.3 Main Dynamics Equations Involving Vorticity

2.3.1 Circulation Equation

General form Applying the definition of the circulation Eq. 2.136 to a material curve, the temporal derivative of the circulation is computed as:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_{L(t)} \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} \cdot \underline{\mathrm{d}M} + \oint_{L(t)} \underline{u} \cdot \underline{\widehat{\mathrm{d}\dot{M}}} \tag{2.154}$$

The first term involves the fluid acceleration which will be expressed using Newton's law Eq. 2.86. The second term involves the material derivative of a material vector, which is given in Eq. 2.126 as $\underline{\widehat{\mathrm{d}\dot{M}}} = [\nabla \underline{u}] \cdot \underline{\mathrm{d}l}$. The second term is then developed as:

$$\oint_{L(t)} \underline{u} \cdot \underline{\widehat{\mathrm{d}\dot{M}}} = \oint_{L(t)} \underline{u} \cdot \left([\nabla \underline{u}] \cdot \underline{\mathrm{d}l}\right) = \oint_{L(t)} \underline{u} \cdot \mathrm{d}\underline{u} = \oint_{L(t)} \mathrm{d}\frac{u^2}{2} = 0 \tag{2.155}$$

As a result of this only the first term of Eq. 2.154 remains. Expressing $\mathrm{d}\underline{u}/\mathrm{d}t$ using the momentum equation from Eq. 2.86 leads to:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_{L(t)} \underline{F} \cdot \underline{\mathrm{d}l} + \int_{\Sigma(t)} \frac{1}{\rho^2} (\nabla\rho \times \nabla p) \cdot \underline{n}\mathrm{d}S + \oint_{L(t)} \frac{1}{\rho} \operatorname{div} \underline{\underline{\tau}} \cdot \underline{\mathrm{d}l} \tag{2.156}$$

In the above equation, the path integral of the pressure term from the Navier–Stokes equation has been replaced by a surface integral using Stokes' theorem (Eq. C.50) and the term curl $(\frac{1}{\rho}$ grad $p)$ has been developed with "curl $(f\,\underline{A}) = f$ curl $\underline{A}$ + grad $f \times \underline{A}$" and "curl (grad f) = 0" (see Sect. C.2).

Sources of circulation Three different sources of circulation are identified from the equation of conservation of circulation (Eq. 2.156): non-conservative forces (e.g. Coriolis force), baroclinicity (i.e. grad $p \times$ grad $\rho \neq 0$), and viscous stresses.

Newtonian fluid of uniform viscosity For a Newtonian fluid of uniform viscosity, $\frac{1}{\rho} \operatorname{div} \underline{\underline{\tau}} = \nu \Delta \underline{u}$ and Eq. 2.156 writes:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_{L(t)} \underline{F} \cdot \underline{\mathrm{d}l} + \int_{\Sigma(t)} \frac{1}{\rho^2} (\nabla\rho \times \nabla p) \cdot \underline{n}\mathrm{d}S + \nu \oint_{L(t)} \Delta\underline{u} \cdot \underline{\mathrm{d}l} \tag{2.157}$$

The integration is performed over a material line $L(t)$, and Σ is an open surface bounded by L.

Homogeneous Incompressible(⇒barotropic) Newtonian fluid of uniform viscosity Recalling $\frac{1}{\rho} \operatorname{div} \underline{\underline{\tau}} = \nu\Delta\underline{u} = -\nu \operatorname{curl} \underline{\omega}$, the integral term on the viscous stresses is rewritten indifferently:

$$\oint_{L(t)} \frac{1}{\rho} \operatorname{div} \underline{\underline{\tau}} \cdot \underline{\mathrm{d}l} = -\nu \oint_{L(t)} \operatorname{curl} \underline{\omega} \cdot \underline{\mathrm{d}l} = \nu \int_{\Sigma(t)} \Delta\omega\underline{\underline{n}}\mathrm{d}S \tag{2.158}$$

where Stokes theorem and "curl curl = grad div $-\Delta$" has been used.

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_{L(t)} \underline{F} \cdot \underline{\mathrm{d}l} - \nu \oint_{L(t)} \operatorname{curl} \underline{\omega} \cdot \underline{\mathrm{d}l} \tag{2.159}$$

The integration takes place over a material line $L(t)$, and Σ is an open surface bounded by L.

2.3.2 Vorticity Equation

Introduction The vorticity equation is obtained by taking the curl of momentum conservation equation given in Eq. 2.86. Information about the divergence part is lost because the divergence and curl operators are orthogonal (see e.g. Chorin [10]). In particular, the pressure term vanishes from the vorticity equation for a barotropic fluid. The pressure is recovered by taking the divergence of Newton's law (see Sect. 2.3.6).

General form The vorticity equation is found by taking the curl of Newton's law Eq. 2.86.

$$\frac{\mathrm{d}\underline{\omega}}{\mathrm{d}t} = \frac{\partial\underline{\omega}}{\partial t} + \underbrace{(\underline{u}\cdot\nabla)\underline{\omega}}_{\text{advection}} = \underbrace{(\underline{\omega}\cdot\nabla)\underline{u}}_{\text{stretching}} - \underbrace{\underline{\omega}\nabla\cdot\underline{u}}_{\text{dilatation}} + \underbrace{\nabla\times\underline{F}}_{\text{n.-c. forces}} + \underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\text{baroclinicity}} + \underbrace{\nabla\times\left(\frac{1}{\rho}\nabla\cdot\underline{\underline{\tau}}\right)}_{\text{v. diffusion}} \tag{2.160}$$

Alternative forms of this equation will be given in Sect. 2.3.4. The vorticity equation is sometimes referred to as *Helmholtz's vorticity equation*.

Sources of vorticity The sources of circulation found in Eq. 2.156 are also present in the vorticity equation, but two additional terms are found: $\underline{\omega}\cdot\operatorname{grad}\underline{u} - \underline{\omega}\operatorname{div}\underline{u}$. These two terms combined are responsible for the stretching and change of direction (dilatation) of the vorticity. This will be studied in more details in Sect. 2.3.3.

2.3.3 Stretching and Dilatation of Vorticity

In this section only the *stretching* and *dilatation* terms are considered. In other words, the fluid considered is an *inviscid*, *barotropic* fluid and *under conservative forces*. The fluid *may be compressible*. For a fluid that satisfies Eq. 2.85, the ideal/inviscid assumption implies $\underline{\underline{\tau}} = 0$.

Rearranging terms The vorticity equation under these conditions writes:

$$\frac{\mathrm{d}\underline{\omega}}{\mathrm{d}t} = \underbrace{(\nabla\underline{u})\cdot\underline{\omega}}_{\text{strain}} + \underbrace{\underline{\omega}\nabla\cdot\underline{u}}_{\text{dilatation}} \tag{2.161}$$

Rearranging the terms and dividing by the density leads to

$$\frac{1}{\rho}\frac{d\underline{\omega}}{dt} - \frac{\underline{\omega}}{\rho^2}\left(\rho\nabla\cdot\underline{u}\right) = \left(\nabla\underline{u}\right)\cdot\frac{\underline{\omega}}{\rho} \tag{2.162}$$

Noting that

$$\frac{d}{dt}\left(\frac{\underline{\omega}}{\rho}\right) = \frac{1}{\rho}\frac{d\underline{\omega}}{dt} - \frac{\underline{\omega}}{\rho^2}\frac{d\rho}{dt} \tag{2.163}$$

and using the continuity equation given by Eq. 2.26, Eq. 2.162 is rewritten:

$$\frac{d}{dt}\left(\frac{\underline{\omega}}{\rho}\right) = (\nabla\underline{u})\cdot\frac{\underline{\omega}}{\rho} \tag{2.164}$$

Comparison with material motion Equation 2.164 may be compared to the kinematic evolution equation of a differential material element $\underline{\delta l}$ as given in Eq. 2.126:

$$\frac{d(\underline{\delta l})}{dt} = \left(\nabla\underline{u}\right)\cdot\underline{\delta l} \tag{2.165}$$

A material vector $\underline{\delta l}(t)$ tangent and along the same direction as $\underline{\omega}$ at a given location and at the time t is now considered. One can write $\underline{\omega} = \frac{\omega}{\delta l}\underline{\delta l}$. Using the kinematic relation $\dot{\widehat{\underline{\delta l}}} = (\nabla\underline{u})\cdot\underline{\delta l}$, Eq. 2.164 is rewritten and integrated as follow:

$$\frac{1}{\omega/\rho}\frac{d}{dt}\left(\frac{\underline{\omega}}{\rho}\right) = \frac{\dot{\widehat{\underline{\delta l}}}}{\delta l} \qquad \rightarrow \qquad \frac{(\underline{\omega}/\rho)(t)}{(\omega/\rho)(t_0)} = \frac{\underline{\delta l}(t)}{\delta l(t_0)} \tag{2.166}$$

Hence if a vorticity line is stretched, the vector $\underline{\omega}/\rho$ will change accordingly. More details are found in the book of Saffman [43, pp. 11–12].

Note for incompressible flows For incompressible flows, Eq. 2.164 is directly obtained from Eq. 2.161.

Relevance for vortex methods The fact that the $\underline{\omega}/\rho$ behave dynamically like a material element is convenient for the Lagrangian tracking of vorticity. The quantity $\underline{\omega}/\rho$ is transported and stretched using the local fluid velocity. This property is used by vortex methods which usually are applied in situations where Eq. 2.161 holds, that is, either for inviscid fluids, or within the context of viscous splitting (see Sect. 41.3). As noted by Voutsinas [50], the observation by Rehbach in 1973 [39] that a concentration of vorticity $\omega\delta V$ follows the same dynamics as a material element in incompressible flows was a key element for the development of vortex particle methods. The result is extended to other "vorticity-dimensions", such as surface vorticity and line vorticity [50]. These other vorticity dimensions are discussed in Sect. 2.4.

Deformation schemes The result mentioned in Sect. 2.2.1 is recalled here. The decomposition of the velocity gradient into a symmetric ($\mathscr{D}$) and anti-symmetric (Ω) part was given in Eq. 2.128. By multiplication with the vorticity, the vorticity stretching term was obtained (Eq. 2.130) as: $(\nabla\underline{u})\cdot\underline{\omega} \equiv (\underline{\omega}\cdot\nabla)\underline{u} = \underline{\underline{\mathscr{D}}}\cdot\underline{\omega}$. Three different forms of this equations were given in Eq. 2.132 as follows:

$$(\underline{\omega} \cdot \nabla)\underline{u} = (\underline{\omega} \cdot \nabla^T)\underline{u} = \frac{1}{2}\left[\underline{\omega} \cdot \left(\nabla + \nabla^T\right)\right]\underline{u} \tag{2.167}$$

The three expressions above lead to different numerical stretching schemes in vortex particle methods (see Sect. 42.2).

2.3.4 Alternative Forms of the Vorticity Equation

The terms of advection and dilatation are gathered into a conservative form using the identity[3] $\operatorname{div}(\underline{\omega} \otimes \underline{u}) = (\underline{u} \cdot \nabla)\underline{\omega} + \underline{\omega} \operatorname{div} \underline{u}$ to give:

$$\frac{\partial \underline{\omega}}{\partial t} + \operatorname{div}(\underline{\omega} \otimes \underline{u}) = (\underline{\omega} \cdot \nabla)\underline{u} - \nabla \times \underline{F} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \underline{\underline{\tau}}\right) \tag{2.168}$$

The stretching term can also be written in the conservative form $\operatorname{div}(\underline{u} \otimes \underline{\omega}) \equiv (\underline{\omega} \cdot \nabla)\underline{u} + \underline{u} \operatorname{div} \underline{\omega}$ since $\operatorname{div} \underline{\omega} \equiv \underline{0}$ to give:

$$\frac{\partial \underline{\omega}}{\partial t} + \operatorname{div}(\underline{\omega} \otimes \underline{u}) = \operatorname{div}(\underline{\omega} \otimes \underline{u}) - \nabla \times \underline{F} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \underline{\underline{\tau}}\right) \tag{2.169}$$

The terms of advection, dilatation and stretching can also be gathered under the term $\operatorname{curl}\left(\underline{\omega} \times \underline{u}\right) = (\underline{u} \cdot \nabla)\underline{\omega} - (\underline{\omega} \cdot \nabla)\underline{u} + \underline{\omega}(\nabla \cdot \underline{u}) - \underline{u}(\nabla \cdot \underline{\omega})$ since $\operatorname{div}\left(\operatorname{curl}(\underline{u})\right) \equiv 0$ (see Appendix C, Sect. C.2). Using this identity, Eq. 2.160 writes:

$$\frac{\partial \underline{\omega}}{\partial t} + \operatorname{curl}(\underline{\omega} \times \underline{u}) = -\nabla \times \underline{F} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \underline{\underline{\tau}}\right) \tag{2.170}$$

The right hand side of this equation is zero within the context of Euler's equation (i.e. ideal homogeneous fluid under conservative forces). Dividing Eq. 2.160 by ρ, and using the continuity equation as in Sect. 2.3.3, the following form is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\underline{\omega}}{\rho}\right) = (\nabla\underline{u}) \cdot \frac{\underline{\omega}}{\rho} + \frac{1}{\rho}\nabla \times \underline{F} + \frac{1}{\rho^3}\nabla\rho \times \nabla p + \frac{1}{\rho}\nabla \times \left(\frac{1}{\rho}\nabla \cdot \underline{\underline{\tau}}\right) \tag{2.171}$$

[3]The divergence of a tensor of order 2 is: $\operatorname{div} \underline{\underline{T}} = \partial_j(T_{ij})\underline{e}_i$. A different convention for the divergence is sometimes found. This is the case for the book of Cottet and Koumoutsakos [13] where the divergence is $\operatorname{div}_2 \underline{\underline{T}} = \partial_i(T_{ij})\underline{e}_j$, and hence the identity becomes: $\operatorname{div}_2(\underline{u} \otimes \underline{\omega}) \equiv (\underline{u} \cdot \nabla)\underline{\omega} + \underline{\omega} \operatorname{div} \underline{u}$. The end result is the same but the divergence definition is different. In the current book, no account is made of covariant and contravariant coordinates.

where the term $\frac{1}{\rho^3}\nabla\rho \times \nabla p$ can also be written $\frac{1}{\rho}\nabla p \times \nabla \frac{1}{\rho}$.

In all the above equations, if the fluid is Newtonian, incompressible and of homogeneous viscosity, the following substitution can be made:

$$\nabla \times \left(\frac{1}{\rho} \nabla \cdot \underline{\underline{\tau}} \right) = \nu \Delta \underline{\omega} \tag{2.172}$$

2.3.5 *Vorticity Equation in Particular Cases*

Ideal (inviscid) barotropic fluid under conservative forces This case was studied in Sect. 2.3.3. For a fluid that satisfies Eq. 2.85, the ideal/inviscid assumption implies $\underline{\underline{\tau}} = 0$. From the direct consequences of all assumptions, Eq. 2.160 becomes:

$$\frac{\mathrm{d}\underline{\omega}}{\mathrm{d}t} = \frac{\partial \underline{\omega}}{\partial t} + \underbrace{(\underline{u} \cdot \nabla)}_{\text{convection}} \underline{\omega} = \underbrace{(\underline{\omega} \cdot \nabla)\underline{u}}_{\text{strain}} + \underbrace{\underline{\omega}\nabla \cdot \underline{u}}_{\text{dilatation}} \tag{2.173}$$

or using the form from Eq. 2.170:

$$\frac{\partial \underline{\omega}}{\partial t} + \text{curl}\,\left(\underline{\omega} \times \underline{u}\right) = 0 \tag{2.174}$$

The fact that the fluid is barotropic may be obtained if the fluid is homoentropic (since it is ideal). The fluid may be compressible.

Incompressible homogeneous Newtonian fluid under conservative forces The incompressibility implies the absence of dilatation. The incompressibility and homogeneity imply that the fluid is barotropic. The condition of incompressibility and the Newtonian nature of the fluid gives curl $(\frac{1}{\rho}\nabla\underline{\underline{\tau}}) = \nu\Delta\underline{\omega}$. In the absence of non-conservative forces and with the previous assumptions, the vorticity equation Eq. 2.160 becomes:

$$\frac{\mathrm{d}\underline{\omega}}{\mathrm{d}t} = \frac{\partial \underline{\omega}}{\partial t} + \underbrace{(\underline{u} \cdot \nabla)}_{\text{convection}} \underline{\omega} = \underbrace{(\underline{\omega} \cdot \nabla)\underline{u}}_{\text{strain}} + \underbrace{\nu\Delta\underline{\omega}}_{\text{diffusion}} \tag{2.175}$$

Incompressible homogeneous inviscid fluid under conservative forces For a fluid that satisfies Eq. 2.85, the inviscid assumption implies $\underline{\underline{\tau}} = 0$. The incompressibility implies the absence of dilatation. The incompressibility and homogeneity implies that the fluid is barotropic. In the absence of non-conservative forces and with the previous assumptions, the vorticity equation Eq. 2.160 becomes:

$$\frac{\mathrm{d}\underline{\omega}}{\mathrm{d}t} = \underbrace{(\underline{\omega} \cdot \nabla)\underline{u}}_{\text{strain}} \tag{2.176}$$

Incompressible homogeneous Newtonian fluid under conservative forces, 2D case In two dimensions, $\underline{\omega}$ reduces to:

$$\omega = \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.177}$$

and the strain term is 0 since $\underline{\omega} \cdot \nabla = \omega_z \partial_z = 0$ (derivatives along z are 0). The two-dimensional form of Eq. 2.175 reduces to the scalar equation

$$\frac{\partial \omega}{\partial t} + \underbrace{(\underline{u} \cdot \nabla)}_{\text{convection}} \omega = \underbrace{\nu \Delta \omega}_{\text{diffusion}} \tag{2.178}$$

Incompressible homogeneous inviscid fluid under conservative forces, 2D case For an inviscid fluid that satisfies Eq. 2.85, then Eq. 2.178 becomes:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0 \tag{2.179}$$

2.3.6 Pressure

In incompressible flows, the pressure is recovered by solving a Poisson equation, see e.g. the book of Cottet and Koumoutsakos [13, p. 6], the book of Saffman [43, p. 18] or the article of Willis et al. [52]. The divergence part of the Navier–Stokes equations gives:

$$\nabla \cdot (\rho \underline{u} \otimes \underline{u}) = \nabla \left(\frac{u^2}{2} \right) + \underline{\omega} \times \underline{u} \Rightarrow -\frac{\nabla^2 p}{\rho} = \nabla \left(\nabla \left(\frac{u^2}{2} \right) + \underline{\omega} \times \underline{u} \right) \tag{2.180}$$

In compressible formulations, the pressure is retrieved using the equation of state of a perfect gas (see e.g. [34])

$$p = (\gamma - 1)\rho \left(E - \frac{|\underline{u}|^2}{2} \right) \tag{2.181}$$

where γ is the adiabatic index and E is the internal energy.

2.3.7 Vortex Force, Image/Generalized/Bound Vorticity, Kutta–Joukowski Relation

The following treatments is detailed in the book of Saffman [43, p. 47].

Vortex force The expression $\rho \underline{u} \times \underline{\omega}$ is referred to as the *vortex force*. For a steady flow of a homogeneous, inviscid, Newtonian fluid, the conservation of momentum given in Eq. 2.102 becomes:

$$\int_V \left[\underline{u} \times \underline{\omega} + \underline{F} - \frac{1}{\rho}\nabla H\right] \mathrm{d}V = 0 \quad \rightarrow \quad \int_V \rho \left[\underline{u} \times \underline{\omega} + \underline{F}\right] \mathrm{d}V - \int_{\partial V} H\, \underline{n} \mathrm{d}S = 0 \tag{2.182}$$

where V is a fixed volume, $\underline{F}$ is a non-conservative external force, H is the Bernoulli constant $H = p + \frac{1}{2}u^2 + V_F$, V_F is the potential associated with the conservative force $\underline{F}_c = \nabla V_F$, and the Green–Gauss identify from Eq. C.45 was used. If the Bernoulli constant H is uniform over ∂V (e.g. a stream-surface), then the surface integral vanishes and it is seen that the external non-conservative forces balances the vortex force: $\int \rho \underline{u} \times \underline{\omega} \mathrm{d}V = -\int \underline{F} \mathrm{d}V$. In other words, the external non-conservative force needed to maintain a steady flow is determined by the vortex force.

Image/generalized/bound vorticity The presence of a body in an incompressible flow is replaced kinematically by a vorticity distribution within the body volume, referred to as *image vorticity* and noted ω_b. In steady motion, the image vorticity is fixed relative to the body. In an unsteady case, the change of flow about the body will result in a change of its generalized vorticity. This change of vorticity, called *shed vorticity* should exit the body. This release occurs at a point of least resistance, typically the trailing edge of an airfoil The extension of the velocity field and vorticity field inside the body is discussed e.g. in the books of Batchelor, Saffman and Lewis [5, 30, 43]. The image vorticity distribution is not unique and depends on the extension method used. The term *generalized vorticity* or *bound vorticity* is also used to refer to the vorticity extended inside the body. The vorticity outside the body is referred to as *free vorticity*. The image vorticity can also be represented using a vortex sheet [43, p. 41]. This representation led to the development of numerical boundary element methods to compute the flow about bodies (see e.g. the book of Katz and Plotkin [25]).

Despite the kinematic extension, the bound vorticity does not follow the same dynamics as the free vorticity. The image vorticity does not in general satisfy the Helmholtz laws. Indeed, a non-conservative force is likely to be present to balance the vortex force $\rho \underline{u}_b \times \underline{\omega}_b$.

Noting $\underline{F}_T$ the total force exerted on the body by the fluid. The total external force applied to the fluid is $-\underline{F}_T$. The application of Eq. 2.182 to a volume which includes the bound vorticity and such that the Bernoulli constant is uniform on its boundary gives:

$$\underline{F}_T = \rho \int_V \underline{u} \times \underline{\omega}\, \mathrm{d}V \tag{2.183}$$

The total vortex force is thus independent of the image distribution used. The integral over the body volume V_b is expressed in term of a surface integral using the Green–Gauss identify from Eq. C.45:

$$\int_{V_b} \underline{u} \times \underline{\omega} \mathrm{d}V = \int_{V_b} \left[\nabla \left(\frac{1}{2} u^2 \right) - \underline{u} \cdot \nabla \underline{u} \right] \mathrm{d}V = \int_{\partial V_b} \left[\frac{1}{2} u^2 \underline{n} - \underline{u} \cdot (\underline{u} \cdot \underline{n}) \right] \mathrm{d}S \tag{2.184}$$

Kutta–Joukowski theorem The *Kutta–Joukowski theorem* is presented in Eq. 3.17. It is demonstrated for a cylinder of circular cross section in Sect. 32.4.1 by integration of the pressure distribution around the cylinder. It can also be obtained by integration of the vortex force. The notations from Sect. 32.4.1 are adopted. The velocity field inside the cylinder is extended based on the exterior field given by Eq. 32.24 and Eq. 32.25 (simply by replacing r with a):

$$u = U_0 + U_0 \left(\frac{y^2}{a^2} - \frac{x^2}{a^2} \right) - \frac{\Gamma y}{2\pi a^2}, \qquad v = -2U_0 \frac{xy}{a^2} + \frac{\Gamma x}{2\pi a^2} \tag{2.185}$$

The resulting vorticity field inside the cylinder is:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\Gamma}{\pi a^2} - 4U_0 \frac{y}{a^2} \tag{2.186}$$

The associated vortex force per length is

$$\underline{F} = \rho \int_{r<a} \underline{u} \times \underline{\omega}\, \mathrm{d}S = \rho \int_{r<a} \left(v\omega_z\, \underline{e}_x - u\omega_z\, \underline{e}_y \right) \mathrm{d}S \tag{2.187}$$

Most terms involved in the integral are zero since they are antisymmetric in x or y. Only the constant term $U_0 \Gamma / \pi a^2$ contributes to the integral. The vortex force is then:

$$\underline{F} = -\rho U_0 \Gamma \underline{e}_y \tag{2.188}$$

The vortex force provides the same expression as the Kutta–Joukowski relation.

2.4 Different Dimensions of Vorticity: Surface, Line and Points

Introduction When the vorticity has two or three dominant components and is confined to thin regions it is convenient to reduce the dimension of the problem by integrating the vorticity over its smallest dimension. This process introduces the concept of *vorticity sheet*, *vorticity line* and *vorticity particle*. The terms *vortex sheet*, *vortex line*, and *vortex particle* are more commonly used in the literature. The terminology is consistent with the notion of the field lines of the vorticity field introduced in Sect. 2.2.2. Vorticity sheets are studied in further details in Sect. 2.8. The reduction process is correct at an infinitesimal level but is unphysical at a larger scale since it effectively introduces discontinuities in the velocity field. The reduction of dimension of vorticity has two main motivations. First, it is a convenient way to simplify some flow situations and derive "vortex models". Second, it is a way to use Lagrangian formulation and benefit from Helmholtz theorem presented in Sect. 2.6.3: this is the approach chosen in "vortex methods" where vorticity is projected onto lower-order vorticity elements that are convected with the local velocity. This second interpretation requires an analysis of the dynamics of vorticity as done in Sect. 2.3 and in particular in Sect. 2.3.3. The dynamics for the different dimensions of vorticity will be briefly mentioned in this section. When reducing the dimension of the problem the direction of vorticity and intensity should be conserved. In fact, as much moments of the vorticity distribution as possible should be conserved.

The reduction of vorticity dimensions is illustrated on Fig. 2.4, adapted from both [16, 49].

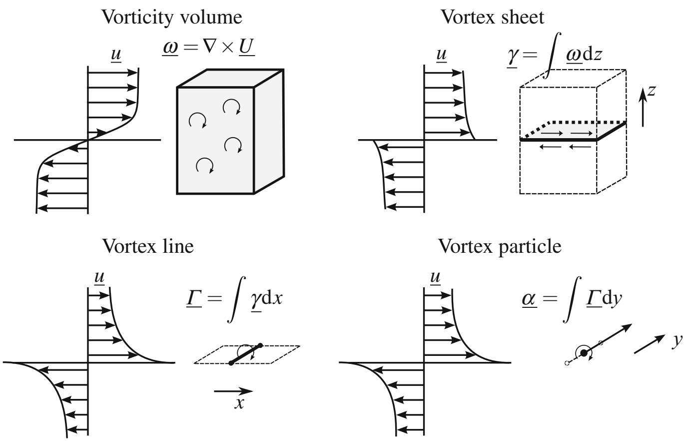

Fig. 2.4 Reduction of vorticity dimensions by integration. The concentration of vorticity introduces singularities in the velocity field close to the vortex elements

Infinitesimal region of vorticity If $\mathrm{d}V$, $\mathrm{d}S$ and $\mathrm{d}l$ are infinitesimal dimensions characterizing the dimensions of a vorticity-volume, vorticity-sheet and vorticity-line, then from a dimensional analysis the respective intensities of these vorticity elements should respect:

$$\underline{\omega}\mathrm{d}V = \underline{\gamma}\mathrm{d}S = \Gamma\underline{\mathrm{d}l} = \underline{\alpha} \tag{2.189}$$

As noted by Hess, there is a kinematic equivalence between a vortex sheet and a continuous dipole distribution μ such that $\underline{\gamma} = \nabla\mu \times \underline{n}$ [20, 50]. The different intensities have the following units:

$$\underline{\omega}\ [\mathrm{s}^{-1}] \qquad \mu\ [\mathrm{s}^{-1}] \qquad \underline{\gamma}\ [\mathrm{m/s}] \qquad \Gamma\ [\mathrm{m}^2/\mathrm{s}] \qquad \underline{\alpha}\ [\mathrm{m}^3/\mathrm{s}] \tag{2.190}$$

The quantity $\Gamma\underline{\mathrm{d}l}$ will also be written $\underline{\Gamma\mathrm{d}l}$. As a general rule the sign and direction of the strength of a vortex filament is such that the direction of the circulation is determined by the right-hand rule.

Dynamics of vorticity The dynamics of vorticity are studied in Sect. 2.3 and in particular in Sect. 2.3.3. For an incompressible, inviscid fluid under conservative force, the evolution of a vortex particle is such that:

$$\frac{\mathrm{d}\underline{\alpha}}{\mathrm{d}t} = \left(\underline{\alpha} \cdot \nabla\right)\underline{u} \tag{2.191}$$

The evolution of the vortex sheet is obtained by consideration of the pressure jump across it [50]:

$$\frac{\partial\mu}{\partial t} + (\underline{u}_{\mathrm{mean}} \cdot \nabla)\mu = 0 \tag{2.192}$$

with

$$\underline{u}_{\mathrm{mean}} = \frac{1}{2}\left(\underline{u}^+ + \underline{u}^-\right) \tag{2.193}$$

Vorticity sheets are studied in further details in Sect. 2.8. More details are found in the article of Voutsinas [50]. The following is quoted from this article:

> It was Rehbach (1973) [39], who first noted that for an incompressible flow, concentrations of vorticity $\underline{\omega}\mathrm{d}V$, as obtained by integrating $\underline{\omega}$ over a small region around a point, will obey exactly the same dynamics [as $\underline{\omega}$]. This is because although the shape of $\mathrm{d}V$ will change, its volume will not ([…] because of this change in shape errors accumulate in time and that is why periodically vorticity has to be redistributed [in vortex methods]). This result can be extended to surface vorticity $\underline{\gamma}\mathrm{d}S$ (Mudry 1982) and to line vorticity $\underline{\mathrm{d}l}$ (Knio and Ghoniem 1990 [27]). […] The material particles on the two sides of a vortex sheet will move differently due to the tangential jump in the velocity. So convection of the vortex sheet in a strict material sense is not possible. In fact, convection of vortex sheets relies on dynamics. Since [the velocity induced by a dipole distribution] has zero divergence, it would

correspond to an incompressible behavior. This means that across the vortex sheet at any point the pressure must be continuous.

Example: Cartesian coordinates Consider a volume, dx × dy × dz, in which the vorticity vector is assumed constant and directed along z. If the vorticity decays rapidly in the x-direction compared to the other dimensions, then the vorticity field is approximated by a vortex sheet of surface dydz. If the vorticity also decays rapidly in the y-direction, the vorticity volume is approximated by a vortex filament. The following relations hold:

$$\underline{\omega}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \underline{\gamma}\mathrm{d}y\mathrm{d}z = \Gamma\mathrm{d}z\underline{e_z} = \underline{\alpha} \tag{2.194}$$

$$\underline{\omega}\mathrm{d}x = \underline{\gamma}\,, \quad \underline{\gamma}\mathrm{d}y = \Gamma\underline{e_z}\,, \quad \underline{\omega}\mathrm{d}S = \Gamma\underline{e_z} \tag{2.195}$$

As an example of application, a vorticity panel of intensity $\underline{\gamma}$ can be discretized into several vorticity filaments (see e.g. [12]). If the filaments are spaced by an interval Δy, the more vortex filaments are used the smaller Δy and hence the smaller the intensity of the vortex filament Γ.

Example: Polar coordinates A thin vorticity layer contained within two cylinders is considered such that the vorticity is decomposed into two components $\underline{\omega_z}$ and $\underline{\omega_\theta}$ using polar coordinates.

$$z\text{-component:} \qquad \underline{\omega_z}r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \underline{\gamma_z}r\mathrm{d}\theta\mathrm{d}z = \Gamma_z\mathrm{d}z\underline{e_z} \tag{2.196}$$

$$\theta\text{-component:} \qquad \underline{\omega_\theta}r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \underline{\gamma_\theta}r\mathrm{d}\theta\mathrm{d}z = \Gamma_\theta r\mathrm{d}\theta\underline{e_\theta} \tag{2.197}$$

$$\underline{\omega_z}\mathrm{d}r = \underline{\gamma_z}\,, \quad \underline{\omega_\theta}\mathrm{d}r = \underline{\gamma_\theta}\,, \quad \underline{\gamma_z}r\mathrm{d}\theta = \Gamma_z\underline{e_z}\,, \quad \underline{\gamma_\theta}\mathrm{d}z = \Gamma_\theta\underline{e_z} \tag{2.198}$$

2.5 Vorticity Moments, Variables and Invariants - Incompressible Flows

Introduction Several flow variables are defined in this section. Most of them are quantities related to the vorticity distribution. Some of these quantities are invariants and they can thus be used to evaluate the accuracy of a numerical methods as time evolves (see Sect. 41.11). Incompressible flows are assumed and the density is omitted in the integrals involved. Definitions including the density ρ are found for instance in the book of Akhmetov [2, p. 12]. Other definitions might also differ by a factor half compared to the definitions given here.

The definitions are given for different conditions: tri-dimensional flows (labelled "3D"), two-dimensional flows ("2D"), Axisymmetric vorticity distribution without swirl ("ring"), 2D flow approximations by vortex points (subscript "h"). Notations for the "ring"-like case are found in the study of axisymmetric flows in Sect. 2.9. The conservation properties of the different variables are briefly mentioned within parenthesis. In general, conservation properties differ in 2D and 3D. Indeed, the extent of the two-dimensional vorticity field (ω_z) is unbounded in the three-dimensional sense and the vorticity cannot be assumed to be zero outside a finite "volume". More details on the conservation properties are found in the review chapter from Winckelmans [53, p. 130] and the book of Cottet and Koumoutsakos [13, p. 50]. More information about the Hamiltonian nature of the system is found in the book of Chorin [11]. The relation between Kinetic energy and the Hamiltonian is discussed in the book of Akhmetov [2, p. 13].

The vector $\underline{\psi}$ represents the vector potential such that $\Delta\underline{\psi} = -\underline{\omega}$ in the context of Helmholtz decomposition (see Sect. 2.2.4). In two dimensions, the scalar stream function is written Ψ.

Total vorticity (Conserved in 3D unbounded flows) (Conserved in 2D unbounded flows with non-zero circulation)

$$\underline{\Omega}_{3D} = \int_V \underline{\omega}\,\mathrm{d}V \qquad\qquad \Gamma = \int_S \omega_z\,\mathrm{d}S \tag{2.199}$$

$$\Gamma_{\mathrm{Ring}} = \int_S \omega_\phi(r,\theta) r\,\mathrm{d}r\mathrm{d}\theta \qquad\qquad I_0^h = \sum_p \alpha_p \tag{2.200}$$

For 3D unbounded flows in which the vorticity vanishes outside some finite region, the total vorticity is zero according to Eq. 2.142. The vortex centroid is thus ill-defined. More discussions on the topic are found in the book of Saffman [43, p. 61].

Linear Momentum (In general not well defined in an infinite region) (Conserved in 3D unbounded flows) (Conserved in 2D unbounded flows with zero circulation) (Not defined in 2D unbounded flows with $\Gamma \neq 0$)

$$\underline{I}_{u,3D} = \int \underline{u}\,\mathrm{d}\underline{x} \qquad\qquad \underline{I}_{u,2D} = \int \underline{u}\,\mathrm{d}\underline{x} \tag{2.201}$$

Using Eq. C.48 with $\underline{f} = \underline{u}$, one obtains $\underline{I}_{u,3D} = \frac{1}{2}\int \underline{x}\times\underline{\omega}\,\mathrm{d}\underline{x} + \frac{1}{2}\int_S \underline{x}\times\underline{u}\times\underline{n}\,\mathrm{d}S$. For an unbounded flow, it is shown that the surface integral is $-\frac{1}{3}I_{3D}$ and thus $\underline{I}_{u,3D} = \frac{2}{3}\underline{I}_{3D}$. The demonstration is done by see Saffman [43, p. 51].

Linear Impulse (Conserved in 3D unbounded flows) (Conserved in 2D unbounded flows with non-zero circulation) (First vorticity moment, similar to a dipole strength) (The value of I is independent of the choice of the origin when $\underline{\Omega}$ (or Γ in 2D) is zero)

$$\underline{I}_{3D} = \frac{1}{2}\int \underline{x}\times\underline{\omega}\,\mathrm{d}\underline{x} \qquad\qquad \underline{I}_{2D} = \int \underline{x}\times\omega_z\underline{e}_z\,\mathrm{d}\underline{x} \tag{2.202}$$

$$\underline{I}_{\mathrm{Ring}} = \pi\int_S \omega_\phi(\rho,z)\rho^2\mathrm{d}\rho\mathrm{d}z\,\underline{e}_z \qquad\qquad \underline{I}_{1,2D} = \int \underline{x}\omega_z\,\mathrm{d}\underline{x} \tag{2.203}$$

$$\underline{I}_{1,2D}^h = \sum_p \underline{x}_p\alpha_p \tag{2.204}$$

The difference of factor between the 1D and 2D case is due to the fact that the vortex lines are not being closed in 2D (see Saffman [43, p. 65]).

Angular momentum (Conserved in 3D unbounded flows) (Conserved in 2D unbounded flows with zero circulation) (Not defined in 2D unbounded with $\Gamma \neq 0$)

$$\underline{A}_{u,3D} = \int \underline{x} \times \underline{u}\, \mathrm{d}\underline{x} \qquad (A_u \underline{e_z}) = \int \underline{x} \times \underline{u}\, \mathrm{d}\underline{x} \tag{2.205}$$

$$I_{2,2D} = \int |\underline{x}|^2\, \omega_z\, \mathrm{d}\underline{x} \tag{2.206}$$

$$I^h_{2,2D} = \sum_p |\underline{x}_p|^2 \alpha_p \tag{2.207}$$

Angular impulse (Conserved in 3D unbounded flows) (Conserved in 2D unbounded flows with zero circulation) (Conserved in 2D unbounded inviscid flows with non-zero circulation)

$$\underline{A}_{3D} = \frac{1}{3}\int \underline{x} \times (\underline{x} \times \underline{\omega})\, \mathrm{d}\underline{x} = \frac{-1}{2}\int r^2 \underline{\omega}\, \mathrm{d}\underline{x} \qquad (A\underline{e_z}) = \frac{-1}{2}\int r^2 \omega_z \underline{e}_z\, \mathrm{d}\underline{x} \tag{2.208}$$

Kinetic Energy (Conserved in 3D unbounded inviscid flows) (Conserved in 2D unbounded flows with zero circulation) (Not defined in 2D unbounded flows with non-zero circulation)

$$E_{u,3D} = \frac{1}{2}\int \underline{u} \cdot \underline{u}\, \mathrm{d}\underline{x} \qquad E_{u,2D} = \frac{1}{2}\int \underline{u} \cdot \underline{u}\, \mathrm{d}\underline{x} = \frac{1}{2}\int |\nabla \Psi|^2\, \mathrm{d}\underline{x} \tag{2.209}$$

Hamiltonian (In some cases $E_u = E$, see e.g. the book of Akhmetov [2, p. 13]) (Conserved in 3D unbounded inviscid flows) (Conserved in 2D unbounded flows with zero circulation) (Conserved in 2D unbounded inviscid flows with zero circulation)

$$E_{3D} = \frac{1}{2}\int \underline{\omega} \cdot \underline{\psi}\, \mathrm{d}\underline{x} \qquad E_{2D} = \frac{1}{2}\int \omega_z \Psi\, \mathrm{d}\underline{x} = \frac{1}{2}\iint G(\underline{x} - \underline{x}')\omega_z(\underline{x})\omega_z(\underline{x}')\mathrm{d}\underline{x}\mathrm{d}\underline{x}' \tag{2.210}$$

$$E_{\mathrm{Ring}} = \pi \int_S \omega_\phi \psi_\phi\, \rho \mathrm{d}\rho \mathrm{d}z \qquad E^h_{2D} = \sum_{p,p'} \alpha_p \alpha'_p G_\varepsilon \left(\underline{x}_p - \underline{x}_{p'}\right) \tag{2.211}$$

Helicity (Related to the degree of knottedness of the vortex lines) (Conserved in 3D unbounded inviscid flows)

$$\mathscr{J}_{3D} = \frac{1}{2}\int \underline{\omega} \cdot \underline{u}\, \mathrm{d}\underline{x} \qquad \mathscr{J}_{2D} = \frac{1}{2}\int \underline{\omega} \cdot \underline{u}\, \mathrm{d}\underline{x} = 0 \tag{2.212}$$

Vortical Helicity

$$\mathscr{Q}_{3D} = \frac{1}{2}\int \underline{\omega}\cdot\underline{(\nabla\times\underline{\omega})}\,\mathrm{d}\underline{x} \qquad \mathscr{Q}_{2D} = \frac{1}{2}\int \underline{\omega}\cdot\underline{(\nabla\times\underline{\omega})}\,\mathrm{d}\underline{x} = 0 \tag{2.213}$$

Enstrophy (Conserved in 2D unbounded flows with zero circulation) (Conserved in 2D unbounded inviscid flows with zero circulation)

$$\mathscr{E}_{3D} = \frac{1}{2}\int \underline{\omega}\cdot\underline{\omega}\,\mathrm{d}\underline{x} \qquad \mathscr{E}_{2D} = \frac{1}{2}\int \omega_z\omega_z\,\mathrm{d}\underline{x} = \frac{1}{2}\int (\Delta\Psi)^2\,\mathrm{d}\underline{x} \tag{2.214}$$

$$\mathscr{E}^h = \frac{1}{2}\sum_p \mathscr{V}_p|\omega_p^h|^2 \tag{2.215}$$

Palinstrophy (Means "again rotation" see [14, p. 571])

$$\mathscr{P}_{3D} = \frac{1}{2}\int |\nabla\times\underline{\omega}|^2\,\mathrm{d}\underline{x} \qquad \mathscr{P}_{2D} = \frac{1}{2}\int (\nabla\underline{\omega})\cdot(\nabla\underline{\omega})\,\mathrm{d}\underline{x} = \frac{1}{2}\int |\nabla\Delta\Psi|^2\,\mathrm{d}\underline{x} \tag{2.216}$$

2.6 Main Theorems Involving Vorticity

2.6.1 Kelvin's Theorem

The three sources of production of circulation are identified in the right hand side of the equation of conservation of circulation (Eq. 2.156). *Kelvin's theorem* applies in a context where these sources are zero, that is under the assumption of an *ideal, barotropic fluid under conservative volume forces*. Under these assumptions the circulation around any closed material curve is conserved when followed in its motion. Similarly, using Stoke's theorem, the flux of vorticity through a material surface $S_m(t)$ is conserved in its motion. Kelvin's theorem is thus written:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma(t)} \underline{\omega}\cdot\underline{n}\,\mathrm{d}S = 0 \tag{2.217}$$

2.6.2 Lagrange's Theorem

Lagrange's theorem applies under the same assumptions of Kelvin's theorem with the addition of the following condition: *the flow is irrotational at a given time* t_0 (i.e. $\forall\underline{x},\ \underline{\omega}(\underline{x}, t_0) = 0$). As a consequence of Kelvin's theorem, the flow stays irrotational for all successive times. Lagrange's theorem is written formally as:

$$\forall t \geq t_0, \quad \forall \underline{x}, \quad \underline{\omega}(\underline{x}, t) = 0 \tag{2.218}$$

For a barotropic and perfect fluid, Lagrange's theorem is used for instance in the case of a fluid which is at rest at the initial time, or for a steady flow with uniform velocity far-upstream. From Lagrange's theorem, it is seen that such flow will remain irrotational.

2.6.3 Helmholtz Theorem

Helmholtz laws were presented when studying the kinematics of vorticity in Sect. 2.2.3. The following theorem is a dynamic theorem which is a direct consequence of Kelvin's theorem, it is hence applied under the same assumptions (ideal, barotropic fluid under conservative forces). *Helmholtz theorem* applies to vorticity tubes, surface and lines, which are in general not of the same nature as material surfaces or lines (see Sect. 2.2.1).

Helmholtz theorem is stated as follows: "For an ideal and barotropic fluid under conservative volume forces, vorticity tubes, surfaces and lines are *material* surfaces or lines. Hence, they follow the fluid's motion." This is shown by considering a vorticity surface S_0 at a given time t_0. A vorticity surface is such that on any point of S_0, the tangent vector to the surface is oriented along the vorticity vector. Consequently, for any *geometrical* surface Σ_0 included in the vorticity surface S_0, the flux of vorticity through this surface will be zero:

$$\int_{\Sigma_0} \underline{\omega} \cdot \underline{n} \mathrm{d}\Sigma_0 = 0 \tag{2.219}$$

If one considers the particles belonging to the geometrical surface Σ_0 at t_0, then Σ_0 is seen as a material surface that will be written $\Sigma(t_0)$. Following the motion of these particles, at a given time $t > t_0$, the particles forms the material surface $\Sigma(t)$. In the same way, all the particles that belonged to S_0 at t_0 form a surface denoted $S(t)$ at t. Kelvin's theorem is applied to the material surface $\Sigma(t)$ between the instant t_0 and t:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma(t)} \underline{\omega} \cdot \underline{n} \, \mathrm{d}\Sigma(t) = 0 \quad \Rightarrow \quad \int_{\Sigma(t_0)} \underline{\omega} \cdot \underline{n} \, \mathrm{d}\Sigma(t_0) = \int_{\Sigma(t)} \underline{\omega} \cdot \underline{n} \, \mathrm{d}\Sigma(t) \tag{2.220}$$

From Eq. 2.219 it follows that $\int_{\Sigma(t)} \underline{\omega} \cdot \underline{n} \, \mathrm{d}\Sigma(t) = 0$. It is concluded that the vorticity vector is tangent to the surface $\Sigma(t)$. The procedure is valid for any surface Σ_0, and hence the surface $S(t)$ is a vorticity surface. From this, it is seen that the vorticity surface S_0 was transported with the fluid, which is Helmholtz's theorem. The intensity Γ of a vortex tube is conserved through its motion. This is found by applying Kelvin's theorem to a path surrounding the vortex tube.

2.6.4 Biot–Savart Law

Introduction The *Biot–Savart law* is named after the two scientists who developed it in 1820 to solve Poisson's equation. The Helmholtz decomposition presented in Sect. 2.2.4 led to two different Poisson's equation (see Eq. 2.149). The second one is seen as the inversion of the vorticity definition $\underline{\omega} = \text{curl } \underline{u}$ in order to obtain $\underline{u} = f(\omega)$. Stating the conditions for which this inversion is possible and unique is beyond the scope of this section. The reader is referred e.g. to the book of Saffman [44]. As an example, the inversion is possible and unique if the following conditions are met: the fluid is incompressible, the vorticity field is compact, no solid boundaries are present (implying also that the domain is unbounded) and the velocity vanishes at infinity.

The Biot–Savart law is obtained from the resolution of Poisson's equation by convolution with the Green function. General derivations involving the solution of Poisson's screened equation may be found in Section B.1. Green's function evaluation for Poisson's equation is found in Section B.1.4. The derivation of the Biot–Savart law from a distribution of vorticity is found in Section B.2.1. Further derivations re found in the work of Wu and Thomson [54], and Walther [51, p. 20])

The Biot–Savart law introduces a causal link between vorticity and velocity so that one refers to the velocity *induced* by the vorticity. Yet, as mentioned by Morino [32, p. 69] this link is more of a mathematical nature than of a physical nature. It is indeed an artifact of incompressible flow to consider that the vorticity has an instantaneous impact on the entire domain.

Biot–Savart law in three dimensions The velocity field induced at a point $M(x, y, z)$ by a vorticity distribution $\underline{\omega}$ in a domain Ω is given by the Biot–Savart law as:

$$\underline{u}(M) = \frac{1}{4\pi} \int_{\Omega} \frac{\underline{\omega}(M') \times \underline{M'M}}{\|\underline{M'M}\|^3} \, \mathrm{d}v(M') \tag{2.221}$$

This velocity field originates from a vector potential $\underline{\psi}$ such that $\underline{u} = \nabla \times \underline{\psi}$, with

$$\underline{\psi}(\underline{r}) = G_{\Delta} * \underline{\omega} = \frac{1}{4\pi} \int_{\Omega} \frac{\underline{\omega}(\underline{r}')}{\left|\underline{r} - \underline{r}'\right|} \mathrm{d}v(\underline{r}') \tag{2.222}$$

For a volume V with distributed vorticity, the Biot–Savart law is:

$$\underline{u}(\underline{x}) = \frac{-1}{4\pi} \iiint_V \frac{(\underline{x} - \underline{x}')}{\|\underline{x} - \underline{x}'\|^3} \times \underline{\omega}(\underline{x}') \mathrm{d}v(\underline{x}') = \frac{1}{4\pi} \iiint_V \text{grad}_x \left[\frac{1}{|\underline{x}' - \underline{x}|} \right] \times \underline{\omega}(\underline{x}') \mathrm{d}v(\underline{x}') \tag{2.223}$$

The notation grad_x is introduced to stress that the differentiation is according to $\underline{x}$ and not $\underline{x}'$. For a vorticity surface S, the Biot–Savart law is:

$$\underline{u}(\underline{x}) = \frac{-1}{4\pi}\iint_S \frac{(\underline{x}-\underline{x}')}{\|\underline{x}-\underline{x}'\|^3} \times \underline{\gamma}(\underline{x}')\mathrm{d}S(\underline{x}') = \frac{1}{4\pi}\iint_S \mathrm{grad}_x\left[\frac{1}{|\underline{x}'-\underline{x}|}\right] \times \underline{\gamma}(\underline{x}')\mathrm{d}S(\underline{x}') \tag{2.224}$$

For a curvilinear vorticity segment, the Biot–Savart law is:

$$\underline{u}(\underline{x}) = \frac{-1}{4\pi}\int_{\underline{x}_1}^{\underline{x}_2} \frac{(\underline{x}-\underline{x}')}{\|\underline{x}-\underline{x}'^3\|} \times \Gamma \mathrm{d}\underline{x}' = \frac{1}{4\pi}\int_{\underline{x}_1}^{\underline{x}_2} \mathrm{grad}_x\left[\frac{1}{|\underline{x}'-\underline{x}|}\right] \times \Gamma \mathrm{d}\underline{x}' \tag{2.225}$$

Biot–Savart law in two dimensions The two-dimensional plane is here assumed to be orthogonal to $\underline{e}_z$. The Biot–Savart law in two dimensions is obtained by resolution of the Poisson equation but it can also be obtained by integration of the 3D relations along the z axis. Writing $\underline{x}$ the projection of the point $\underline{X}$ onto the two-dimensional plane, the integral along z give:

$$\int_z \frac{(\underline{X}-\underline{X}')}{\|\underline{X}-\underline{X}'\|^3}\,\mathrm{d}z = 2\frac{(\underline{x}-\underline{x}')}{\|\underline{x}-\underline{x}'\|^2} \tag{2.226}$$

Writing $(\underline{x}-\underline{x}') = (r_x, r_y)$, the Biot–Savart law in this two-dimensional plane for a given vorticity distribution $\underline{\omega} = \omega_z \underline{e}_z$ is:

$$\underline{u}(\underline{x}) = \frac{-1}{2\pi}\iint_S \frac{(\underline{x}-\underline{x}')}{\|\underline{x}-\underline{x}'\|^2} \times \omega_z(\underline{x}')\underline{e}_z \mathrm{d}S(\underline{x}') = \frac{-1}{2\pi}\iint_S \frac{(r_y, -r_x)\omega_z(\underline{x}')}{\|\underline{x}'-\underline{x}\|^2}\mathrm{d}S(\underline{x}') \tag{2.227}$$

$$\psi_z(\underline{x}) = \frac{1}{2\pi}\int_S \log\left[\frac{L}{|\underline{x}-\underline{x}'|}\right]\omega_z(\underline{x}')\mathrm{d}S(\underline{x}') \tag{2.228}$$

with L any reference length since the vector potential is defined up to a constant.

Unification of notations for 2D and 3D It is common to introduce notations that unify the 2D and 3D cases. The number of dimension will further be written n. Adopting the notations of [6], the Biot–Savart law is written in the following forms:

$$\underline{u}(\underline{z}, t) = \nabla \times (G_{\Delta^-} * \underline{\omega}) = \left(\underline{K} * \underline{\omega}\right)(\underline{z}, t) = \int \underline{K}(\underline{z}'-\underline{z})\underline{\omega}(\underline{z}', t)\,\mathrm{d}z' \tag{2.229}$$

with $\underline{K} \equiv \underline{K_n} : \mathbb{R}^n \to \mathbb{R}^n$ is an integral matrix-valued *kernel* referred to as the *Biot–Savart kernel*, $G \equiv G_{\Delta^-_n} : \mathbb{R}^n \to \mathbb{R}$ is the Green function (with the "minus sign convention" of Eq. B.2) associated with the Laplace operator in dimension n, and the convolution of two vector-functions is performed component per component. $\underline{K}$ is seen as the rotational counterpart of the Green function associated with the Laplace operator. The meaning of $\underline{K}$ is thus different in two and three dimensions.

The notation $\underline{K} = \nabla \times G$ is sometimes used but should be considered with care since G is scalar. To stress out this fact, the notation $\underline{K} = [\nabla\times]G$ will be used. More generally, the notation $[\underline{r}\times]$ is introduced with different meaning depending on the dimension:

$$[\underline{r}\times] = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \text{ (in 3D) }, \qquad [\underline{r}\times] = \begin{bmatrix} r_y \\ -r_x \end{bmatrix} \text{ (in 2D)} \tag{2.230}$$

The Green functions associated to the Laplace operator in 2 and 3 dimensions are (see e.g. [19] and Section B.1.4):

$$G_{\Delta_2^-}(\underline{r}) = -\frac{1}{2\pi}\log\|\underline{r}\|, \qquad G_{\Delta_3^-}(\underline{r}) = \frac{1}{4\pi\,\|\underline{r}\|} \tag{2.231}$$

In the above the "minus sign convention" of Eq. B.2 has been used. The opposite functions are obtained with the opposite convention. In three dimensions, the kernel is written indifferently (by identification with Eq. 2.223):

$$\underline{K}_3(\underline{r}) = -\frac{1}{4\pi\,\|\underline{r}\|^3}[\underline{r}\times] = \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} G_{\Delta_3^-}(\underline{r}) = [\nabla\times]G_{\Delta_3^-} \tag{2.232}$$

Further, using the radial symmetry $G(\underline{r}) = G(r)$ one has $\partial_x = x/r\partial_r$, $\partial_y = y/r\partial_r$, $\partial_z = z/r\partial_r$. Using the above definition of $[\underline{r}\times]$, the last equality of Eq. 2.232 becomes:

$$\underline{K}_3(\underline{r}) = \frac{1}{r}\left(\partial_r G_{\Delta_3^-}(r)\right)\,[\underline{r}\times] \tag{2.233}$$

For $\underline{r} = (r_x, r_y, 0)$, the in-plane component of the cross product $[\underline{r}\times]\underline{v}$ receives only contribution from the out of plane component of $\underline{v}$. The vorticity component is in any-case only directed along z in 2D, so that $\underline{\omega} = \omega_z\underline{e}_z$ is seen as a scalar in Eq. 2.229. The kernel in two-dimensions may thus be written indifferently as:

$$\underline{K}_2(\underline{r}) = \frac{-1}{2\pi\,\|\underline{r}\|^2}(\underline{r}\times) = \frac{-1}{2\pi\,\|\underline{r}\|^2}\begin{bmatrix} r_y \\ -r_x \end{bmatrix} = \begin{bmatrix} \partial_y \\ -\partial_x \end{bmatrix}\frac{-1}{2\pi}\log\|\underline{r}\| = \begin{bmatrix} \partial_y \\ -\partial_x \end{bmatrix} G_{\Delta_2^-}(\underline{r}) \tag{2.234}$$

Further, using the radial symmetry $G(\underline{r}) = G(r)$ one has:

$$\underline{K}_2(\underline{r}) = \frac{1}{r}\left(\partial_r G_{\Delta_2^-}(r)\right)\,[\underline{r}\times] \tag{2.235}$$

The following relations unify the 2D and 3D case:

$$\underline{K}(\underline{r}) = [\nabla\times]G(r) = \frac{1}{r}\left(\partial_r G(r)\right)\,[\underline{r}\times] \tag{2.236}$$

A regularization of the kernel is usually applied to avoid its singularity as r approaches zero. Examples of regularization are found in Sect. 41.8.

2.7 Vortices in Viscous and Inviscid Fluid - Results and Classical Flows

As noted by Sarpkaya [45, p. 7] "the solution of real fluid flow problems with vortex models often forces one to think (at times to defend) simultaneously the behavior of vortices in terms of viscous and inviscid concepts.". The properties of vortices in these two distinct cases are briefly discussed below and examples of analytical viscous and inviscid vortices are given.

2.7.1 Vortex in Inviscid Fluid

Within the context of Kelvin's theorem given in Sect. 2.6.1, the circulation around any closed material curve is invariant with time. Under the same assumptions, Helmholtz theorem imply that vorticity is transported by convection of the fluid. The circulation intensity of a vortex tube is conserved throughout its motion and may be determined by any circulation path surrounding this sole vortex tube. In potential flows, the linearity of Poisson's equation gives rise to the principle of superposition.

Inviscid vorticity patches Analytical of inviscid vorticity patches are given in a dedicated section: Sect. 33.1.

Rayleigh stability criterion *Rayleigh stability criterion* for an inviscid flow consisting of concentric circular streamlines is stated as: "A circulation always increasing outwards ensures stability" Details of the derivation is given in the work of Brenner [9].

2.7.2 Vortex in Viscous Fluid - Standard Solutions

Introduction In a viscous fluid, the circulation around a vortex tube depends on the contour of integration and is not time-invariant since diffusion occurs. In this regard, Taylor discussed the contour of integration around a viscous 2D airfoil (see Sect. 3.1.5 and [47]).

For a viscous fluid, the principle of superposition of vortex fields does not apply due to the non-linearity of the Navier–Stokes equation. It will be mentioned that the Lamb-Oseen vortex is a canonical vortex solution of the Navier–Stokes equation. Yet a superposition of several of these vortices do not form an exact solution of the Navier–Stokes equation [45, p. 8].

Note: The solution given below are solutions to the diffusion equation, so that the convective part of the equation has been discarded. This can either be justified in the case of low Reynolds number flow, or by noticing that the solutions are axisymmetric such that the convection does not change the vorticity distribution.

Rankine vortex The *Rankine vortex* is a crude but simple model of a viscous vortex. It is not a solution of the Navier–Stokes equations. The vortex rotates as a solid body in its core and follows a potential flow outside of it. The tangential velocity field possess an artificial discontinuity at a radius corresponding to the core radius r_c (see e.g. [45, p. 8]):

$$u_\theta = \begin{cases} \frac{\Gamma}{2\pi r} & r > r_c \\ \frac{\Gamma}{2\pi}\frac{r}{r_c^2} & r < r_c \end{cases} \tag{2.237}$$

Lamb-Oseen Vortex The *Lamb-Oseen vortex i*s an exact solution of the 2D Navier–Stokes equations. It corresponds to an axisymmetric viscous vortex in an unbounded incompressible domain [29]. The vorticity equation in polar coordinates is given in Eq. 2.265. Using $\omega = \omega(r)$ and $u_r \equiv 0$, leads to:

$$\frac{\partial \omega}{\partial t} = \nu \left[\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right] \tag{2.238}$$

The solution is a Gaussian vorticity distribution:

$$\omega(r,t) = \frac{\Gamma}{4\pi \nu t}\exp\left(-\frac{r^2}{4\nu t}\right) \tag{2.239}$$

for which the velocity field solution is:

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right] \tag{2.240}$$

The standard deviation of the Gaussian vorticity distribution is $\sigma_\omega = \sqrt{2\nu t}$. The radius of maximum tangential velocity is $r_m = 2.24\sqrt{\nu t}$ [45, p. 8]. Since the velocity field consists of concentric streamlines, the convection and the diffusion are decoupled. The Lamb-Oseen vortex is thus a good candidate for studying different viscous schemes [3].

"Wavelet" vortex For the validation of viscous implementation, Cottet and Koumoutsakos [13, p. 136] use a one dimensional solution of the diffusion equation for which the initial vorticity distribution is:

$$\omega(x,0) = x e^{-x^2} \tag{2.241}$$

The distribution corresponds to the first derivative of the Gaussian function without scaling. This distribution will be further referred to as the *wavelet vortex* in this document. The solution of the diffusion equation with initial value Eq. 2.241 is:

$$\omega(x,t) = \int_{-\infty}^{\infty} \mathscr{G}(x-y,\nu t) y e^{-y^2} \mathrm{d}y = \frac{x e^{-x^2/(1+4\nu t)}}{(1+4\nu t)^{3/2}} \tag{2.242}$$

where $\mathscr{G}(x,\sigma)$ defines a Gaussian [13, p. 136].

Viscous diffusion of a vortex sheet The diffusion of an infinite vortex sheet of constant intensity possesses an analytical solution (see Batchelor [4, p. 187], Lewis [30, p. 375]). The normal to the vortex sheet is taken as the y axis and the strength is assumed to be $\gamma = 2U$. The solution of the diffusion equation $\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$ is

$$\omega(y,t) = \frac{U}{\sqrt{\pi \nu t}} e^{-y^2/4\nu t} \tag{2.243}$$

The velocity u is directly obtained by integration of the vorticity $\omega = \partial u/\partial y$ as:

$$u(y,t) = \frac{U}{\sqrt{\pi \nu t}} \int_0^y e^{-y'^2/4\nu t} \mathrm{d}y' = U \operatorname{erf}\left(y/\sqrt{4\nu t}\right) \tag{2.244}$$

where the characteristic length of the shear layer is proportional to $\sqrt{\nu t}$. Lewis [30, p. 375] studied the diffusion about a vortex sheet (numerically finite) using the random walk method and also uses a superposition of Lamb-Oseen solutions [30, p. 484]. Though convenient for numerical investigations, it is yet to be noted that the superposition of Lamb-Oseen vortices is not a solution to the (non-linear) Navier–Stokes equations.

2.7.3 Life of a Vortex - Vortex Decay, Collapse and Stability

Only references on the topic are given in this paragraph. Spalart studied the decay of airplane trailed vortices in 1998 [46]. In the work of Okulov and Sørensen [33] the stability of helical vortex systems is studied.

2.8 Surface Representations - Vortex Sheets

2.8.1 Introduction

The current section focuses on few properties of *vortex sheets*. More information is found for instance in the book of Batchelor [4, p. 96]. The interpretation of a vortex sheet in terms of doublet distribution is discussed by Hess [20]. The convection of a doublet distribution is found in the article of Voutsinas [50].

2.8.2 Vortex Sheets Kinematics

Induced velocity From the Biot–Savart law, Eq. 2.224, the velocity induced by a vortex sheet S at a point $P = \underline{x}$ not located on S is:

$$\underline{u}(\underline{x}) = \frac{-1}{4\pi}\iint_S \frac{(\underline{x}' - \underline{x})}{\|\underline{x}' - \underline{x}\|^3} \times \underline{\gamma}(\underline{x}')\mathrm{d}S(\underline{x}') = \frac{1}{4\pi}\iint_S \mathrm{grad}_x\left[\frac{1}{\underline{x}' - \underline{x}}\right] \times \underline{\gamma}(\underline{x}')\mathrm{d}S(\underline{x}') \tag{2.245}$$

When the point P passes through the surface S, the velocity $\underline{u}$ is subjected to a discontinuity such that:

$$[\![\underline{u}]\!] \triangleq \underline{u}_+ - \underline{u}_- = \underline{\gamma} \times \underline{n} \tag{2.246}$$

and conversely

$$\underline{\gamma} = [\![\underline{u}]\!] \times \underline{n} \tag{2.247}$$

where $\underline{n}$ is the unit vector normal to S at P. For a point located on S on the side $(+)$ or $(-)$, one has thus [39]:

$$\underline{u}_\pm(\underline{x}) = \frac{1}{4\pi}\,\mathrm{PV}\!\iint_S \mathrm{grad}_x\left[\frac{1}{\underline{x}' - \underline{x}}\right] \times \underline{\gamma}(\underline{x}')\mathrm{d}S(\underline{x}') \mp \frac{1}{2}\underline{n}(\underline{x}) \times \underline{\gamma}(\underline{x}) \tag{2.248}$$

Convection velocity To account for the possible presence of a free-stream velocity $\underline{U}_\infty$, the total velocity is further written $\underline{U} = \underline{u} + \underline{U}_\infty$. The convection velocity of a vortex sheet is taken as the mean between the upper and lower velocity:

$$\underline{U}_m = \frac{1}{2}\left(\underline{U}_+ + \underline{U}_-\right) \tag{2.249}$$

If there is pressure continuity through S, then $|\underline{U}_+| = |\underline{U}_-|$. This is in particular the case for wing wakes modelled as vortex sheets. In the specific case where $|\underline{U}_+| =$

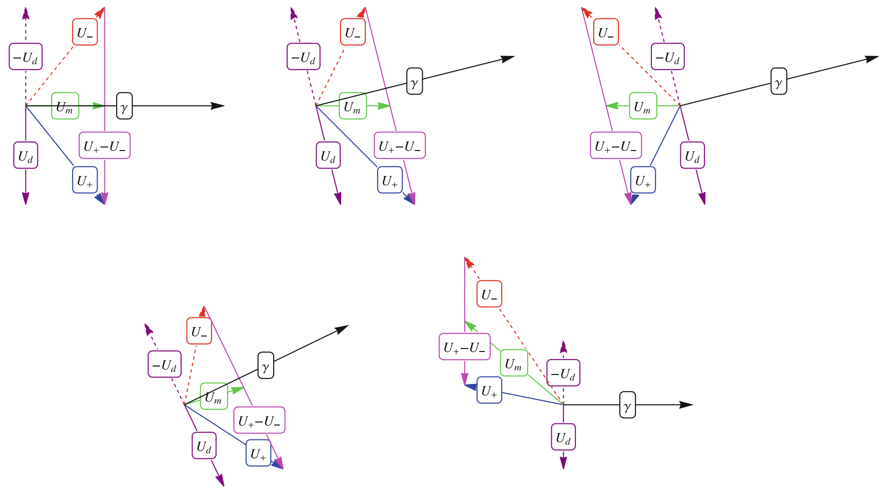

Fig. 2.5 Sketch of velocities and vortex sheet intensity in the plane tangent to the vortex sheet. Five cases are shown for illustration purposes. In the top-left figure, the velocities about the sheet have the same norm and the mean velocity $\underline{U}_m$ is thus parallel to the sheet intensity $\underline{\gamma}$

$|\underline{U}\_|$, then from Eq. 2.246 it follows that $\underline{\gamma}$ is parallel to the average velocity $\underline{U}_m$ about the sheet.

Following the work of Kerwin [26], the half difference of velocity across the sheet is introduced as:

$$\underline{U}_d = \frac{\underline{U}_+ - \underline{U}_-}{2} \tag{2.250}$$

It follows immediately that:

$$\underline{U}_+ = \underline{U}_m + \underline{U}_d, \qquad \underline{U}_- = \underline{U}_m - \underline{U}_d, \qquad \underline{\gamma} = 2[\underline{n} \times \underline{V}_d] \tag{2.251}$$

Illustration of the different components are illustrated in Fig. 2.5. The figure is shown in a plane tangent to the vortex sheet. This representation is uncommon but fruitful.

2.8.3 *Vortex Sheets Dynamics*

The force exerted by the fluid on a vortex sheet at a point P is:

$$\underline{F} = -\rho \iint_S \underline{\gamma}(\underline{x}) \times \underline{U}_m(\underline{x}) \mathrm{d}S(\underline{x}) \tag{2.252}$$

As mentioned in Sect. 2.8.2, the pressure continuity through S implies $|\underline{U}_+| = |\underline{U}_-|$ and then from Eq. 2.246 it follows that $\underline{\gamma}$ is parallel to the average velocity $\underline{U}_m$ about the sheet and hence $\underline{F} = \underline{0}$. This should in particular be the case for wing wakes modelled as vortex sheets. This is why wakes are said to be "force-free". The equilibrium position of such wake sheet is thus defined by the condition of constant pressure through the sheet.

2.8.4 Vortex Sheet Convection and Stability

The convection and stability of vortex sheets is discussed e.g. by Lewis [30, p. 326] and Batchelor [4]. The Kelvin Helmholtz instability is discussed in these references. The roll-up of a vortex sheet behind a square wing was studied by Rehbach in 1973 [39]. The convection of a doublet distribution is found in the article of Voutsinas [50]. The following references are also relevant for the topic: [7, 8, 28, 42].

2.8.5 Vortex Surfaces in 2D

The induced velocity equations of a vortex sheet are obtained by integration of the Biot–Savart law. A vortex sheet may be thought as a continuous distribution of point vortices since the point vortex corresponds to the kernel of the Biot–Savart integral (see Eq. 32.9).

Flat panel of constant strength In a Cartesian coordinate system attached to the sheet, the induced velocity is:

$$u(x, y) = \frac{-1}{2\pi}\int_{x_1}^{x_2} \frac{\gamma(\xi)y}{(x-\xi)^2 + y^2} d\xi = \frac{\gamma}{2\pi}\left[\operatorname{atan}\frac{x-\xi}{y}\right]_{x_1}^{x_2} \tag{2.253}$$

$$v(x, y) = \frac{1}{2\pi}\int_{x_1}^{x_2} \frac{\gamma(\xi)(x-\xi)}{(x-\xi)^2 + y^2} d\xi = \frac{-\gamma}{4\pi}\left[\log\left((x-\xi)^2 + y^2\right)\right]_{x_1}^{x_2} \tag{2.254}$$

The stream function is:

$$\begin{aligned}\Psi(x, y) &= \frac{\gamma}{2\pi}\int_{x_1}^{x_2} \log\left[\frac{1}{\sqrt{(x-\xi)^2 + y^2}}\right] d\xi \\ &= \frac{\gamma}{4\pi}\Bigg[(x - x_1)\log\left[(x-x_1)^2 + y^2\right] + 2y \operatorname{atan}\left(\frac{x-x_1}{y}\right) \\ &\quad + (x_2 - x)\log\left[(x-x_2)^2 + y^2\right] - 2y \operatorname{atan}\left(\frac{x-x_2}{y}\right) + 2(x_1 - x_2)\Bigg]\end{aligned} \tag{2.255}$$

The velocity potential is:

$$\Phi(x, y) = -\frac{\gamma}{2\pi}\int_{x_1}^{x_2} \text{atan}\left(\frac{y}{x-\xi}\right) d\xi \tag{2.256}$$
$$= -\frac{\gamma}{2\pi}\left[\frac{1}{2} y \log\left[\frac{(x-x_1)^2 + y^2}{(x-x_2)^2 + y^2}\right] + (x - x_1)\,\text{atan}\left[\frac{y}{x-x_1}\right] - (x-x_2)\,\text{atan}\left[\frac{y}{x-x_2}\right]\right]$$

Flat infinite vortex sheet of constant strength In a Cartesian coordinate system attached to the sheet, the induced velocity is:

$$u(x, y) = \frac{-1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma(\xi) y}{(x-\xi)^2 + y^2} d\xi = \frac{\gamma}{2\pi}\left[\text{atan}\,\frac{x-\xi}{y}\right]_{-\infty}^{+\infty} = \begin{cases} \frac{-\gamma}{2} & , y > 0 \\ \frac{\gamma}{2} & , y < 0 \end{cases} \tag{2.257}$$

$$v(x, y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma(\xi)(x-\xi)}{(x-\xi)^2 + y^2} d\xi = 0 \tag{2.258}$$

2.9 Incompressible Flow Equations in Polar Coordinates - 2D and 3D Flows - Axisymmetric Flows

Introduction In this section, polar coordinates are understood as cylindrical polars and spherical polars. A cylindrical coordinate system (r, θ, z) is adopted instead of the ISO standard 80000-2 notations: (ρ, ϕ, z). The standard notations (r, θ, ϕ) are used for the spherical coordinate system. Care should be used when compare equations obtained using both coordinate systems.

The definitions of the various operators in these coordinate systems are given in Sect. C.3. The notations from the Helmholtz decomposition introduced in Sect. 2.2.4 are used: $\underline{u} = \underline{u}_0 + \underline{u}_\Phi + \underline{u}_\omega$. The rotational part of the velocity derives from a vector potential $\underline{\psi}$ such that $\underline{u}_\omega = \nabla \times \underline{\psi}$. Only the rotational part of the velocity contributes to the vorticity and hence $\underline{\omega} \triangleq \nabla \times \underline{u} \equiv \nabla \times \underline{u}_\omega = -\nabla^2 \underline{\psi}$. The divergence part of the velocity derives from a scalar potential Φ such that $\underline{u}_\Phi = \text{grad}\,\Phi$. The divergence of $\underline{u}_\omega$ is zero by construction. The incompressible assumption implies that $\text{div}\,\underline{u} \equiv \text{div}\,\underline{u}_\Phi = \nabla^2\Phi = 0 \equiv \text{div}\,\underline{u}_\omega$.

The equations are presented for arbitrary 2D and 3D flows and axisymmetrical flows around the z-axis with and without swirl. In axisymmetrical flows without swirl, the flow is effectively two-dimensional but is still of a 3D nature. The velocity may be defined by a stream function which is different than the one obtained in 2D potential flows. It is referred to as the *Stokes stream function* and is noted Ψ. It is stressed that the vector potential $\underline{\psi}$ is not a stream function in 3D: it is not constant along streamlines. The *stream function* in 2D corresponds to the vector potential:

$$\Psi \equiv \psi_z \qquad (2D) \tag{2.259}$$

Stokes stream function in 3D cylindrical coordinates is:

$$\Psi \triangleq r\,\psi_\theta \qquad \text{(Cylindrical coordinates } (r,\theta,z),\ r=\sqrt{x^2+y^2}\text{)} \tag{2.260}$$

Stokes stream function in 3D spherical coordinates is:

$$\Psi \triangleq \psi_\phi\, r\sin\theta \qquad \text{(Spherical coordinates } (r,\theta,\phi),\ r=\sqrt{x^2+y^2+z^2}\text{)} \tag{2.261}$$

2.9.1 2D Arbitrary Flow (Cylindrical Coordinates)

The cylindrical coordinate system (r, θ, z) is adopted. The 2D assumption implies $\partial_z \equiv 0$. For a 2D flow, the rotational part of the velocity $\underline{u}_\omega$ is:

$$\underline{u}_\omega = \nabla \times \underline{\psi} = \frac{1}{r}\frac{\partial \psi_z}{\partial \theta}\underline{e}_r - \frac{\partial \psi_z}{\partial r}\underline{e}_\theta \tag{2.262}$$

The vorticity is

$$\omega_z = (\nabla \times \underline{u})\cdot\underline{e}_z \equiv (\nabla \times \underline{u}_\omega)\cdot\underline{e}_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right) \tag{2.263}$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_z}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \psi_z}{\partial \theta^2} = -\nabla^2\psi_z \tag{2.264}$$

The stretching term $(\underline{\omega}\cdot\nabla)\underline{u}$ is identically 0 in two dimensions. The vorticity equation given by Eq. 2.175 reduces then to:

$$\frac{\partial \omega_z}{\partial t} + (\underline{u}\cdot\nabla)\omega_z = \nu\Delta\omega_z$$

$$\frac{\partial \omega_z}{\partial t} + \left[u_r\frac{\partial \omega_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial \omega_z}{\partial \theta}\right] = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \omega_z}{\partial \theta^2}\right] \tag{2.265}$$

The continuity equation writes:

$$\nabla\cdot\underline{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 = \nabla^2\Phi, \qquad \left(\nabla\cdot\underline{u}_\omega = \frac{\partial^2\psi_z}{\partial r\partial\theta} - \frac{\partial^2\psi_z}{\partial\theta\partial r} \equiv 0\right) \tag{2.266}$$

2.9.2 3D Arbitrary Flow (Cylindrical Coordinates)

The cylindrical coordinate system (r, θ, z) is adopted. The rotational part of the velocity and the vorticity are:

$$\underline{u}_\omega = \nabla \times \underline{\psi} = \left(\frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z}\right)\underline{e}_r + \left(\frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r\psi_\theta) - \frac{\partial \psi_r}{\partial \theta}\right)\underline{e}_z \tag{2.267}$$

$$\underline{\omega} = \nabla \times \underline{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\underline{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right)\underline{e}_z \tag{2.268}$$

The vorticity also satisfies the Laplace equation $\omega = -\nabla^2 \underline{\psi}$. For an incompressible flow, the continuity equation writes:

$$\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial (ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 = \nabla^2 \Phi \tag{2.269}$$

The vorticity dynamics are governed by Eq. 2.175 as:

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\omega} = (\underline{\omega} \cdot \nabla)\underline{u} + \nu \Delta \underline{\omega} \tag{2.270}$$

Each term is written in cylindrical coordinates using the following:

$$\begin{aligned}(\underline{A} \cdot \nabla)\,\underline{B} = &\left(A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r}\right)\underline{e}_r \\ &+ \left(A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + A_z \frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r}\right)\underline{e}_\theta \\ &+ \left(A_r \frac{\partial B_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} + A_z \frac{\partial B_z}{\partial z}\right)\underline{e}_z\end{aligned}$$

$$\begin{aligned}\nu \Delta \underline{\omega} = &\,\nu\left(\Delta \omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial \omega_\theta}{\partial \theta}\right)\underline{e}_r \\ &+ \nu\left(\Delta \omega_\theta - \frac{\omega_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \omega_r}{\partial \theta}\right)\underline{e}_\theta \\ &+ \nu \Delta \omega_z \underline{e}_z\end{aligned}$$

2.9.3 3D Axisymmetric Flows with Swirl (Cylindrical Coordinates)

The cylindrical coordinate system (r, θ, z) is adopted. The axisymmetry implies that $\partial_\theta \equiv 0$.

Conservation laws - Flow equations The continuity equation is:

$$\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0 = \nabla^2 \Phi \tag{2.271}$$

The vorticity equation from Eq. 2.175 writes:

$$\begin{aligned}
\frac{\partial \omega_r}{\partial t} + \left(u_r \frac{\partial \omega_r}{\partial r} + u_z \frac{\partial \omega_r}{\partial z}\right) &= \left(\omega_r \frac{\partial u_r}{\partial r} + \omega_z \frac{\partial u_r}{\partial z}\right) + \nu \left(\Delta \omega_r - \frac{\omega_r}{r^2}\right) \\
\frac{\partial \omega_\theta}{\partial t} + \left(u_r \frac{\partial \omega_\theta}{\partial r} + u_z \frac{\partial \omega_\theta}{\partial z} + \frac{u_\theta \omega_r}{r}\right) &= \left(\omega_r \frac{\partial u_\theta}{\partial r} + \omega_z \frac{\partial u_\theta}{\partial z} + \frac{\omega_\theta u_r}{r}\right) + \nu \left(\Delta \omega_\theta - \frac{\omega_\theta}{r^2}\right) \\
\frac{\partial \omega_z}{\partial t} + \left(u_r \frac{\partial \omega_z}{\partial r} + u_z \frac{\partial \omega_z}{\partial z}\right) &= \left(\omega_r \frac{\partial u_z}{\partial r} + \omega_z \frac{\partial u_z}{\partial z}\right) + \nu \Delta \omega_z
\end{aligned} \tag{2.272}$$

Kinematics: Vorticity, vector potential and Stokes stream function For an incompressible axisymmetric flow, the vorticity $\underline{\omega} = \text{curl } \underline{u} = \text{curl } \underline{u}_\omega$ is

$$\underline{\omega} = -\frac{\partial u_\theta}{\partial z}\underline{e}_r + \left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right]\underline{e}_\theta + \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}\underline{e}_z \tag{2.273}$$

The Laplace equation $\nabla^2 \underline{\psi} = -\underline{\omega}$ also gives

$$-\underline{\omega} = \nabla^2 \underline{\psi} = \left(\Delta \psi_r - \frac{\psi_r}{r^2}\right)\underline{e}_r + \left(\Delta \psi_\theta - \frac{\psi_\theta}{r^2}\right)\underline{e}_\theta + \Delta \psi_z \underline{e}_z \tag{2.274}$$

The relation $\underline{u}_\omega = \text{curl } \underline{\psi}$ is

$$\underline{u}_\omega = -\frac{\partial \psi_\theta}{\partial z}\underline{e}_r + \left[\frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}\right]\underline{e}_\theta + \frac{1}{r}\frac{\partial (r \psi_\theta)}{\partial r}\underline{e}_z \tag{2.275}$$

$$u_{\omega r} = -\frac{\partial \psi_\theta}{\partial z}, \qquad u_{\omega \theta} = \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}, \qquad u_{\omega z} = \frac{1}{r}\frac{\partial\, (r \psi_\theta)}{\partial r} \tag{2.276}$$

Stokes stream function is defined as:

$$\Psi \triangleq r\, \psi_\theta \qquad (\text{Cylindrical coordinates } (r, \theta, z), r = \sqrt{x^2 + y^2}) \tag{2.277}$$

Introducing this notation into Eq. 2.276 leads to:

$$u_{\omega r} = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad u_{\omega z} = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \qquad \left(u_{\omega \theta} = \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}\right) \tag{2.278}$$

This notation is useful for "vortex-rings"-like flows where $u_\theta = 0$ (see Sect. 2.9.4). The function Ψ is then seen to act like a stream function. Indeed, in such flows ψ_θ is not constant along streamlines and inversely proportional to r (see discussion around Eq. 2.282).

2.9.4 3D Axisymmetric Flows Without Swirl (Cylindrical Coordinates)

Integrations of the Biot–Savart law in this case is found in the book of Pozrikidis [38, p. 586].

Kinematics From Eq. 2.273, it is readily seen that if there is no *swirl* ($u_\theta \equiv 0$), then the vorticity is purely along $\underline{e}_\theta$. Writing, $\underline{\omega} = \omega \underline{e}_\theta$, the vorticity definition from Eq. 2.273, and the velocity from Eq. 2.278 writes:

$$\omega = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \qquad u_{\omega r} = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad u_{\omega z} = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \tag{2.279}$$

The vorticity is expressed in term of the vector potential, or the stream function, from the Laplace equation $\nabla^2 \underline{\psi} = -\underline{\omega}$ or Eq. 2.279 as:

$$\omega = \frac{\partial u_{\omega r}}{\partial z} - \frac{\partial u_{\omega z}}{\partial r} = -\frac{1}{r}\left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2}\right] + \frac{1}{r^2}\frac{\partial \Psi}{\partial r} \tag{2.280}$$

$$= -\nabla^2 \psi_\theta + \frac{\psi_\theta}{r^2} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\psi_\theta}{\partial r}\right) - \frac{\partial^2 \psi_\theta}{\partial z^2} + \frac{\psi_\theta}{r^2} = -\frac{1}{r}\left[\frac{\partial^2 r\psi_\theta}{\partial r^2} + \frac{\partial^2 r\psi_\theta}{\partial z^2}\right] + \frac{1}{r^2}\frac{\partial r\psi_\theta}{\partial r}$$

The above expressions are relevant for vortex rings, the Hill's vortex, or irrotational flows ($\omega = 0$) where analytical solution in terms of Ψ are sought (see e.g. 34). It is recalled that the variation of a variable along a streamline is given in the material derivative by the term $(\underline{u} \cdot \nabla)\bullet$. Using Eq. 2.279, it is immediately seen that the value of the Stokes stream function is constant along a streamline:

$$\left[(\underline{u} \cdot \nabla)\Psi \underline{e}_\theta\right] \cdot \underline{e}_\theta = u_r \frac{\partial \Psi}{\partial r} + u_z \frac{\partial \Psi}{\partial z} = 0 \tag{2.281}$$

On the other hand, the vector potential is not constant along a streamline, viz.:

$$\left[(\underline{u} \cdot \nabla)\psi_\theta \underline{e}_\theta\right] \cdot \underline{e}_\theta = u_r \frac{\partial \psi_\theta}{\partial r} + u_z \frac{\partial \psi_\theta}{\partial z} = \frac{\psi_\theta}{r} \tag{2.282}$$

Equations for the total circulation, the Linear impulse and Hamiltonian are found respectively in Eqs. 2.199, 2.202 and 2.210.

Flow equations The continuity equation from Eq. 2.271 remains unchanged:

$$\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \tag{2.283}$$

The tangential component of vorticity dynamics equation Eq. 2.272 reduces to

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \omega}{\partial t} + \left(u_r \frac{\partial \omega}{\partial r} + u_z \frac{\partial \omega}{\partial z}\right) = \frac{\omega u_r}{r} + \nu \left[\frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} - \frac{\omega}{r^2}\right] \tag{2.284}$$

Inviscid steady flows When $\nu = 0$ and $\partial/\partial t = 0$, Eq. 2.284 reduces to:

$$u_r \frac{\partial \omega}{\partial r} + u_z \frac{\partial \omega}{\partial z} = \frac{\omega u_r}{r} \tag{2.285}$$

Using the above equation, the following is obtained:

$$\frac{\mathrm{d}(\omega/r)}{\mathrm{d}t} = u_r \frac{\partial(\omega/r)}{\partial r} + u_z \frac{\partial(\omega/r)}{\partial z} = \frac{1}{r}\left[u_r \frac{\partial \omega}{\partial r} + u_z \frac{\partial \omega}{\partial z}\right] - \frac{\omega u_r}{r^2} = 0 \tag{2.286}$$

As noted in the book of Akhmetov [2, p. 9], it follows that in steady axisymmetric ring flows the quantity ω/r is constant along a streamline and depends only on the value of the stream function Ψ. The condition of steadiness is written in the form:

$$\frac{\omega}{r} = f(\Psi) \tag{2.287}$$

where f is an arbitrary function of Ψ. Combining the steady condition with Eq. 2.280 leads to

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = -r^2 f(\Psi) \tag{2.288}$$

The analogy of this equation with plasma physics is discussed in the book of Akhmetov [2, p. 21]. A solution to this equation was given by Hill for $f(\Psi) = \mathrm{cst}$ (see Sect. 34.2, [21]).

Flow deriving only from a vector potential If $\underline{u} = \underline{u}_{\omega}$, the velocity is directly determined by the Stokes stream function Ψ according to Eq. 2.278.

2.9.5 3D Arbitrary Flow (Spherical Coordinates)

A spherical coordinate system (r, θ, ϕ) is adopted in this section. Care should be used when comparing the equations with the ones from the cylindrical coordinates sections since the variables r and θ are different. The vorticity and the rotational part of the velocity are:

$$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r \sin\theta}\left(\frac{\partial}{\partial\theta}(u_\phi \sin\theta) - \frac{\partial u_\theta}{\partial\phi}\right)\underline{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r u_\phi)\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right)\underline{e}_\phi \tag{2.289}$$

$$\underline{u}_\omega = \nabla \times \underline{\psi} = \frac{1}{r \sin\theta}\left(\frac{\partial}{\partial\theta}(\psi_\phi \sin\theta) - \frac{\partial \psi_\theta}{\partial\phi}\right)\underline{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial \psi_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r \psi_\phi)\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r \psi_\theta) - \frac{\partial \psi_r}{\partial\theta}\right)\underline{e}_\phi \tag{2.290}$$

For an incompressible flow, the continuity equation writes

$$\nabla \cdot \underline{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0 \tag{2.291}$$

The vorticity equation is lengthy to develop but it can easily be obtained using the definitions of the various operators in spherical coordinates given in Sect. C.3.

2.9.6 3D Axisymmetric Flows with Swirl (Spherical Coordinates)

A spherical coordinate system (r, θ, ϕ) is adopted in this section. The axisymmetry implies $\partial_\phi \equiv 0$. The velocity and vorticity are:

$$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\phi \sin\theta)\underline{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r u_\phi)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right)\underline{e}_\phi \tag{2.292}$$

$$\underline{u}_\omega = \nabla \times \underline{\psi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi_\phi \sin\theta)\underline{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r \psi_\phi)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r \psi_\theta) - \frac{\partial \psi_r}{\partial\theta}\right)\underline{e}_\phi \tag{2.293}$$

The vorticity equation is lengthy to develop and it is thus not presented in this section.

2.9.7 3D Axisymmetric Flows Without Swirl (Spherical Coordinates)

A spherical coordinate system (r, θ, ϕ) is adopted in this section. Care should be used when comparing the equations with the ones from the cylindrical coordinates sections since the variables r and θ are different. The axisymmetry implies $\partial_\phi \equiv 0$

and the absence of swirl implies $u_\phi \equiv 0$. The following assumes that potential part is zero, i.e. $\underline{u}_\Phi \equiv 0$ and hence $\underline{u} \equiv \underline{u}_\omega$.

Kinematics From the conditions stated above, the vorticity consists of a component along $\underline{e}_\phi$, that will be further written $\underline{\omega} = \omega \underline{e}_\phi$:

$$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\underline{e}_\phi \tag{2.294}$$

From Eq. 2.293, the fact that $\underline{u}_\phi = 0$ implies that ψ_r and ψ_θ do not play a role in the velocity field and can thus be chosen as zero. This leads to:

$$\underline{u} = \nabla \times \underline{\psi} = \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\psi_\phi \sin\theta)\underline{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r\psi_\phi)\underline{e}_\theta \tag{2.295}$$

$$u_r = \frac{1}{r\sin\theta}\frac{\partial(\psi_\phi \sin\theta)}{\partial\theta}, \qquad u_\theta = -\frac{1}{r}\frac{\partial(r\psi_\phi)}{\partial r} \tag{2.296}$$

Introducing Stokes stream function in spherical coordinates

$$\Psi = \psi_\phi \, r \sin\theta \qquad \text{(Spherical coordinates } (r, \theta, \phi), r = \sqrt{x^2 + y^2 + z^2}) \tag{2.297}$$

Then the velocity $\underline{u} = \nabla \times (\psi_\phi \underline{e}_\phi) = \nabla \times (\frac{\Psi}{r\sin\theta}\underline{e}_\phi)$ is:

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r} \tag{2.298}$$

The vorticity is expressed in terms of the Stokes stream function by inserting these expressions into Eq. 2.294:

$$\omega = -\frac{1}{r\sin\theta}\left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right] \tag{2.299}$$

Irrotational flows should thus be such that the above equation is 0 (see e.g. the flow around a sphere in Chap. 34). As noted in the book of Acheson [1, p. 173], and as obtained in the cylindrical coordinate case (see Eq. 2.281), the Stokes stream function is constant along streamlines in such axisymmetric flow. This is shown using Eq. 2.298 as:

$$\left[(\underline{u}\cdot\nabla)\Psi\underline{e}_\phi\right]\cdot\underline{e}_\phi = u_r\frac{\partial\Psi}{\partial r} + \frac{u_\theta}{r}\frac{\partial\Psi}{\partial\theta} = 0 \tag{2.300}$$

Conservation laws The continuity, with, and without swirl is the same equation. From Eq. 2.291 it writes:

$$\nabla\cdot\underline{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) = 0 \tag{2.301}$$

Inviscid steady flows For steady axisymmetric flows without swirl, the vorticity equation was seen to reduce to Eq. 2.286 in cylindrical coordinates. This equation is readily transformed to spherical coordinates as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega}{r\sin\theta}\right) = (\underline{u}\cdot\nabla)\left(\frac{\omega}{r\sin\theta}\right) = 0 \tag{2.302}$$

where ω is the ϕ-component of $\underline{\omega}$. Since $\omega/(r\sin\theta)$ is constant along streamlines, the condition for steady flow writes (as was done in cylindrical coordinates, Eq. 2.287)

$$\frac{\omega}{r\sin\theta} = f(\Psi) \tag{2.303}$$

Expressing the vorticity using Eq. 2.299 leads to an equation for Ψ (similar to the one in cylindrical coordinates given in Eq. 2.288):

$$-f(\Psi)\,r^2\sin^2\theta = \frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) \tag{2.304}$$

$$= \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) \tag{2.305}$$

2.10 2D Potential Flows

The *potential flow* assumption corresponds to the irrotational flow of an ideal, homogeneous, incompressible fluid under conservative volume forces. The potential flow assumptions do not apply in places dominated by viscous effects like boundary layers and trailed vortex sheet. In this section the main equations of 2D potential flows are presented. Examples of potential flow elements/solutions will be presented in Chap. 32. Since 2D flows are considered, derivatives along the z direction are identically zero. The vector potential is written $\underline{\psi} = (0, 0, \Psi)$, where $\psi_z = \Psi$ is the *stream function*. As a consequence of the continuity equation, the velocity field is expressed in the two following forms:

$$\underline{u} = \text{grad}\,\Phi \tag{2.306}$$

$$\underline{u} = \text{curl}\,\underline{\psi} = \text{grad}\,\Psi \times \underline{e}_z \tag{2.307}$$

Using Cartesian (x, y) or polar coordinates (r, θ), this writes:

$$u = \frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \quad u_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} \tag{2.308}$$

$$v = \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x} u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r} \tag{2.309}$$

Poisson equations involved The irrationality condition $\omega_z \equiv 0$ implies:

$$\omega_z = (\nabla \times \underline{u})\cdot\underline{e}_z = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right) = -\nabla^2 \Psi = 0 \quad (2.310)$$

The incompressibility conditions $\nabla\cdot\underline{u} \equiv 0$ implies:

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \nabla^2 \Phi = 0 \quad (2.311)$$

Flux across a surface and stream function Flux across a surface delimited by a curve A-B in the $x - y$ plane and of unitary length in the z direction, (see e.g. [48, p. 10]) is

$$F = \int_A^B u\mathrm{d}y - v\mathrm{d}x = \int_A^B \frac{\partial \Psi}{\partial y}\mathrm{d}y + \frac{\partial \Psi}{\partial x}\mathrm{d}x = \int_A^B \mathrm{d}\Psi = [\Psi]_A^B \quad (2.312)$$

The flux is thus zero for iso-Ψ value lines. Since the flux across a streamline is zero, these lines Ψcst are streamlines, which justifies the naming of stream function.

Boundary conditions The velocity field is computed from the stream function or the potential which both satisfy the Laplace equation. Nevertheless, the boundary conditions are different. Boundary conditions for the potential are:

$$\underline{n} \cdot \mathrm{grad}\, \Phi = 0 \quad \text{on } S_{\mathrm{body}} \quad (2.313)$$

$$\Phi \to U_0\, y \quad \text{when } |x| \to \infty \quad (2.314)$$

Boundary conditions for the stream function are:

$$\Psi(x, y) = \text{constant} \quad \text{on } S_{\mathrm{body}} \quad (2.315)$$

$$\Psi \to U_0\, y \quad \text{when } |x| \to \infty \quad (2.316)$$

Ψ and Φ are harmonic functions since they satisfy Laplace equations and are assumed twice continuously differentiable. Further, from the definition of the velocity potential and the stream function, Φ and Ψ satisfy the *Cauchy-Riemann conditions*:

$$(u =) \quad \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \qquad (v =) \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \quad (2.317)$$

The streamlines $\Psi(x, y) = \text{cst}$ and the iso-potential lines $\Phi(x, y) = \text{cst}$ form a network of orthogonal curves. It results that these two functions are conjugate harmonic functions and it invites us to define a complex potential as:

$$f(z) = \Phi(x, y) + i\Psi(x, y) \quad (2.318)$$

where $z = x + iy$. The complex velocity is defined as:

$$w(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = u(x, y) - iv(x, y) = (u_r(r, \theta) - iu_\theta(r, \theta))\, e^{-i\theta} \tag{2.319}$$

2.11 Conformal Map Solutions

The mathematical technique of conformal mapping is convenient to solve 2D potential flows analytically or numerically. The analytical solutions are in particular useful to validate 2D airfoil panel codes.

2.11.1 Conformal Mapping - Definitions and Properties

Two complex planes of complex variable $z = x + iy$ and $Z = X + iY$ are here considered.

Conformal map The transformation $Z = h(z)$ which maps each point m of the z plane to a point M of the Z-plane is a *conformal map* if it conserves angles magnitude and orientation.

Holomorphic functions If the function $Z = h(z)$ is *holomorphic*[4] around z_0 and such that $h'(z_0) \neq 0$, it has an inverse function $z = H(Z)$ in the neighborhood of z_0, and the set $\{h, H\}$ defines a conformal mapping of the z-plane into the Z-plane in the same neighborhood. To verify the above, the differential of each plane is written: $\mathrm{d}Z = |h'(z)|e^{i \arg h'(z)}\mathrm{d}z$. An elementary vector around z_0 is hence elongated with a factor $|h'(z)|$ and rotated by an angle $\arg h'(z)$ by the mapping. The angles are conserved by such transformation, hence the above property.

Singular points A point z_0 such that $h'(z_0) = 0$ for the map $Z = h(z)$ is called a singular point. The map h is not conformal in the neighborhood of a singular point.

Properties - In the neighborhood of a zero of order n for h', the map $Z = h(z)$ multiply angles by $n + 1$.
- If $f(z)$ corresponds to the complex potential of a flow in the z-plane, the flow transformed in the Z-plane by the conformal mapping $Z = h(z)$ has the following complex potential and speed:

$$\mathscr{F}(Z) = f\,(H(Z)) \tag{2.320}$$

$$W(Z) \triangleq \frac{\mathrm{d}\mathscr{F}}{\mathrm{d}Z} = w\,(H(Z))\;H'(Z) \tag{2.321}$$

[4]The term analytic can also be used since complex analytic functions coincides with holomorphic functions.

The following properties follows:

- The orthogonal grid of equipotential and stream-lines of the z-planes are mapped to the equipotential and stream-lines of the Z planes unless z, with exception for the singular points of $h(z)$.
- Circulation and flow rate is conserved through a conformal map.
- Sources and vortices are mapped to sources and vortices of the same intensity.
- A doublet of intensity $\mu e^{i\alpha}$ located at z_0 is mapped to a doublet of intensity $h'(z_0)\mu e^{i\alpha}$ located at $Z_0 = h(z_0)$.
- A cyclic flow of the z-plane is mapped to a cyclic flow in the Z-plane with same cyclic constant.

2.11.2 Reference Airfoil Flow: Flow Around a Cylinder and Kutta Condition

The cyclic flow around a lifting cylinder is a classical 2D potential flow result. It is presented in Sect. 32.4.3. This flow is presented here since several conformal maps transform this flow to a flow around an airfoil of a given geometry. The notations used are introduced in Fig. 2.6. The tangential velocity around a lifting cylinder is:

$$u_\theta = -2\underline{U}_0 \sin(\theta - \alpha) + \frac{\Gamma}{2\pi r_c} \tag{2.322}$$

where U_0 is the free stream velocity of incidence α and Γ is the circulation around the cylinder. There are potentially two solutions θ_s that satisfies the stagnation point equation $\underline{u}_\theta = 0$. If β stands for the angle to the rear stagnation point, and r_c the radius of the circle (see Fig. 2.6), then the Kutta condition requirement (no flow around the rear stagnation point, see Sect. 3.1.4) leads to:

$$\Gamma = 2\pi r_c \sin(\beta - \alpha) \tag{2.323}$$

where from geometrical considerations:

$$r_c = \sqrt{(a - X_c)^2 + Y_c^2}, \qquad \beta = \text{atan}\,\frac{-Y_c}{a - X_c} = \text{asin}\,\frac{Y_c}{r_c} \tag{2.324}$$

2.11.3 Joukowski's Conformal Map

A family of airfoil shapes are obtained using *Joukowski's conformal map*. The transformation of the velocity field around a circle given in Sect. 2.11.2 provides the velocity field around the transformed airfoil. For a a real constant, Joukowski's conformal map $Z \mapsto z$ is defined as:

$$Z \mapsto z : \quad z = H(Z) = Z + \frac{a^2}{Z^2} \tag{2.325}$$

The inverse transformation is given as:

$$Z(z) = -a\left[\left(\frac{z-2}{z+2}\right)^{\frac{1}{2}} + 1\right]\left[\left(\frac{z-2}{z+2}\right)^{\frac{1}{2}} - 1\right]^{-1} = \frac{a}{4}(z+2)\left[\left(\frac{z-2}{z+2}\right)^{\frac{1}{2}} - 1\right]^{2} \tag{2.326}$$

Properties

- Joukowski's map $Z \mapsto z$ has two singular points in $Z = \pm a$. The segment $|x| < a$ may be removed from the z-plane to make $h : z \mapsto Z$ well defined.
- The image of the circle $|Z| = a$ (i.e. $X_c = Y_c = 0$) is the segment $|x| = 2a$. There is a bijection between the outside of the circle and the cut z-plane.
- The singular points $Z = \pm a$ are first order zeros of $H'(Z)$. Angles are locally multiplied by two through the map $Z \mapsto z$. Any curve passing by $Z = \pm a$ will have a corner.
- The point $Z = \infty$ is the only fixed point of the transformation.

Relation to airfoil properties In practice, Joukowski's conformal map is used to transform a circle which is located around the point Z_c and which intersect the real axis at $x = a$. Realistic airfoil shape in the z plane are then produced by this transformation. The following notations are introduced (see Fig. 2.6):

$$Z_c = me^{i\delta}, \qquad z = Re^{i\theta} \tag{2.327}$$

$$r_c^2 = m^2 + c^2 - 2ma\cos\delta, \qquad r_c^2 = m^2 + R^2 - 2mR\cos(\delta - \theta) \tag{2.328}$$

$$R = a\left[1 + \varepsilon\left(-\cos\delta + \cos(\delta - \theta)\right)\right] + o(\varepsilon^2), \qquad \varepsilon = m/a \tag{2.329}$$

Using these notations, the thickness and camber are determined by the position of the center as:

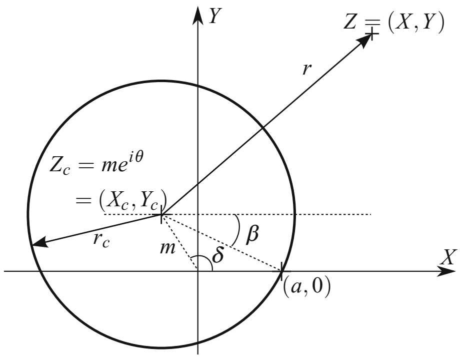

Fig. 2.6 Notations used for the circle in the Z-plane

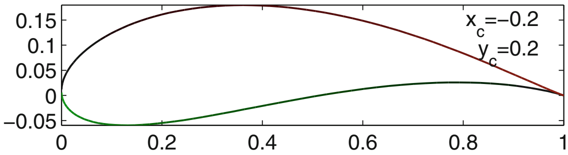

Fig. 2.7 Example of Joukowski airfoil obtained for $X_c = -0.2$ and $Y_c = 0.2$. The airfoil coordinates have been normalized to a unit chord

$$\frac{t}{l} = \frac{3\sqrt{2}}{4}\varepsilon \quad \text{(thickness)} \qquad\qquad \frac{h}{l} = \frac{1}{2}\varepsilon \sin\delta \quad \text{(camber)} \tag{2.330}$$

where $l = 4a$ is the airfoil chord. The trailing edge angle τ is zero for the Joukowski airfoils. The trailing edge is said to be a cusped trailing edge.An example of airfoil shape obtained using the transformation is shown in Fig. 2.7.

2.11.4 Karman-Trefftz Conformal Map

The *Karman-Trefftz conformal map* is an extension of Joukowski's conformal map. For (a, λ) two real constants, the Karman-Trefftz conformal map $Z \mapsto z$ is defined as:

$$Z \mapsto z : \quad z = H(Z) = \lambda a \frac{(Z+a)^\lambda + (Z-a)^\lambda}{(Z+a)^\lambda - (Z-a)^\lambda} \tag{2.331}$$

which is also written:

$$\frac{z - \lambda a}{z + \lambda a} = \frac{(Z-a)^\lambda}{(Z+a)^\lambda} \tag{2.332}$$

The derivative of the mapping function is then:

$$\frac{\mathrm{d}z}{\mathrm{d}Z} = 4\lambda^2 a^2 \frac{(Z+a)^{\lambda-1}(Z-a)^{\lambda-1}}{\left[(Z+a)^\lambda - (Z-a)^\lambda\right]^2} \tag{2.333}$$

The inverse transformation is given as:

$$Z(z) = -a\left[\left(\frac{z-\lambda}{z+\lambda}\right)^{\frac{1}{\lambda}} + 1\right]\left[\left(\frac{z-\lambda}{z+\lambda}\right)^{\frac{1}{\lambda}} - 1\right]^{-1} \tag{2.334}$$

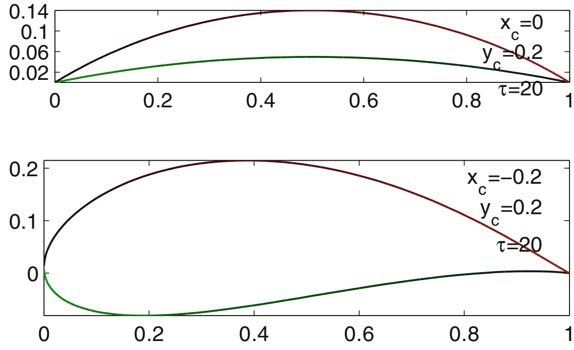

Fig. 2.8 Examples of Karman-Trefftz airfoils obtained for $Y_c = 0.2$, $\tau = 20°$ and $X_c = 0$ (top) $X_c = -0.2$ (bottom). The airfoil coordinates have been normalized to a unit chord

Properties

- The parameter λ is related to the trailing edge angle as:

$$\tau = \pi(2 - \lambda) \tag{2.335}$$

- For $\lambda = 2$, Karman-Trefftz's transform gives Joukowski's map, which has a cusped trailing edge ($\tau = 0$).

Example of airfoil shapes obtained using the transformation is shown in Fig. 2.8.

2.11.5 Van de Vooren Conformal Map

With $(a, \lambda, l, \varepsilon)$ four real constants, the *Van de Vooren conformal map* $Z \mapsto z$ is defined as:

$$Z \mapsto z: \quad z = H(Z) = \frac{(Z - a)^{\lambda}}{(Z - a\varepsilon)^{\lambda-1}} + l \tag{2.336}$$

Relation to airfoil properties In practice this map is applied to a circle of radius a centered in the origin of the Z plane to provide a realistic airfoil shape in the z plane. In such case, the points $Z = a$ and $Z = -a$ will correspond respectively to the trailing and leading edge of the airfoil shape in the z plane. When the chord is chosen to be $c = 2l$, the parameter a is then found to be:

$$a = c(1 + \varepsilon)^{\lambda-1} 2^{-\lambda} \tag{2.337}$$

Coordinates in the z plane of such airfoil are found in [25, p. 163]. The advantage of this transformation is that its parameters can directly be related to common airfoil parameters: l is related to the chord, ε to the thickness and λ to the trailing edge angle. The parameter λ is related to the trailing edge angle (with the same relation

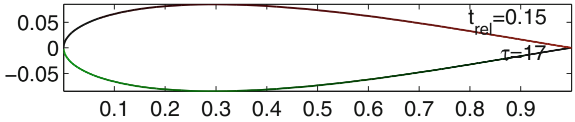

Fig. 2.9 Examples of Van de Vooren airfoil obtained for $t_{rel} = 0.15$ and $\tau = 17°$. The airfoil coordinates have been normalized to a unit chord

as the Karman-Trefftz transform):

$$\tau = \pi(2 - \lambda) \tag{2.338}$$

An example of airfoil shape obtained using the transformation is shown in Fig. 2.9.

2.11.6 Matlab Source Code

A Matlab source code to compute the Karman-Trefftz map is given below. The code computes the airfoil geometry, the pressure distribution on the profile surface and the velocity field about the airfoil.

```
%% Initialization
clearvars; close all; clc;
%% Parameters for Karman Trefftz airfoil
U0        = 1    ; % Free Stream Velocity
alpha_deg = 05   ; % Angle of Attack [deg]
xc        = -0.2; % Circle Center Location (<0)
yc        = 0.1  ; % Circle Center
tau       = 10   ; % Tail Angle
% ---  Display parameters
n=100; % Number of points for pressure distribution
nx=200; ny=200; %  grid points in cart. coordinates (in Z-plane)
nStreamlines = 21; %Number of Streamlines to plot
XLIM=[-3.5 3.5]; YLIM=[-3 3];
%% Main function call (with grid for contour plots, but not
    necessary)
[Xg,Yg]=meshgrid(linspace(XLIM(1),XLIM(2),nx),linspace(YLIM(1),
    YLIM(2),ny));
[X_p,Y_p,Cp,Ug,Vg,CP] = fProfileKarmanTrefftz(xc,yc,tau,n,U0,
    alpha_deg,Xg,Yg);
%% Plotting pressure distribution about the airfoil
figure, hold on, grid on, box on;
plot((X_p-min(X_p))./(max(X_p)-min(X_p)), Cp, 'k.-')
xlabel('x/c [-]'); ylabel('C_p [-]');
title(sprintf('Karman-Trefftz C_p \alpha = %.1f deg.',alpha_deg));
xlim([0 1]); axis ij
%% Plotting airfoil, streamlines and pressure distribution around
    the foil
figure,grid on,hold on;axis equal; box on;
contourf(Xg,Yg,CP,15); fill(X_p,Y_p,'w');
Y0=linspace(YLIM(1),YLIM(2),nStreamlines); X0=Y0*0+XLIM(1);
hlines=streamline(stream2(Xg,Yg,Ug,Vg,X0,Y0));set(hlines,'Color','
    k')
ylim(YLIM);xlim(XLIM);title('Karman-Trefftz Cp and streamlines');
```

```
function [X_profile,Y_profile,Cp,Ug,Vg,CP,Gamma] =
    fProfileKarmanTrefftz( xc,yc,tau,n,varargin)
% returns Karman-Trefftz profiles, and optionally pressure
    distribution and velocity field
% AUTHOR: E. Branlard
% INPUT:
%     xc,yc    % Circle Center Location
%     tau      % Tail Angle in deg
%     n        % Number of points along mapped foil surface
% OUTPUT:
%    X and Y coordinates of airfoil
%    Cp : pressure coeff at foil surface
%    Xg,Yg,Ug,Vg,CP : grid points, velocity field and  pressure
    coeff
% EXAMPLES:
%    [X_p,Y_p]     = fProfileKarmanTrefftz(xc,yc,tau,n);
%    [X_p,Y_p,Cp] = fProfileKarmanTrefftz(xc,yc,tau,n,U0,alpha_deg);
%    [X_p,Y_p,Cp,Ug,Vg,CP] = fProfileKarmanTrefftz(xc,yc,tau,n,U0,
    alpha_deg,Xg,Yg);
%% --- Optional arguments
bComputeGridVelocity=0; bComputeAero=0;
if nargin>4
  bComputeAero=1; U0=varargin{1}; alpha=varargin{2}; % deg!!!
  if nargin> velocity computation on polar Grid
    bComputeGridVelocity=1; Xg=varargin{3}; Yg=varargin{4};
  end
end
Cp=[]; U=[]; V=[]; CP =[]; Gamma=[];
%% Main parameters
a      = 1.0                    ; % x intersectoin
rc     = sqrt((a-xc)^2 + yc^2); % radius of circle
beta   = asin(-yc/(rc))         ; % Angle to rear stagnation point
lambda = 2-tau/180              ;
%% Coordinates of profile (using transform of the circle)
vtheta_circ = 0:2*pi/n:2*pi-pi/n;
z0          = (xc + i*yc)       ; % center of circle
z_circ      = z0 + rc*exp(i*vtheta_circ);
% --- Karman-Trefftz Conformal map - Profile Shape
[Z_profile,dZdz] = fConformalMapKarmanTrefftz(z_circ,a,lambda);
X_profile=real(Z_profile(:)); Y_profile=imag(Z_profile(:));
%% Aero computation if required
if bComputeAero
  %% ---Pressure distribution on the airfoil
  Gamma = 4*pi*rc*U0*sin(beta-alpha*pi/180); % from Kutta condition
  % Velocity at circle sruface
  [ u_circ, v_circ ] = fUi_Cylinder2D(real(z_circ),imag(z_circ),xc,
      yc,rc,U0,alpha,Gamma);
  % Velocities, -Cp on surface
  W_circ  = (u_circ-i*v_circ)./dZdz   ; % [u-iv]_Z = [u-iv]_z/DZ/Dz
  U_circ = real(W_circ); V_circ = -imag(W_circ); % velocity in Zeta
      -plane
  Q      = sqrt(U_circ.^2 + V_circ.^2); % velocity Magnitude
  Cp     = 1-(Q./U0).^2               ; % pressure coefficient
  %% Pressure distribution on a grid (using direct transform and a
      polar grid)
  if bComputeGridVelocity
     % Inverse Karman Trefftz Conformal map
     Zg = Xg + 1i*Yg;
     [z,dzdZ] = fConformalMapKarmanTrefftz(Zg,a,lambda,true );
     x = real(z); y = imag(z);
```

```
      [u,v] = fUi_Cylinder2D(x,y, xc,yc,rc, U0, alpha, Gamma);
      W = (u-1i.*v).*dzdZ;
      Ug = real(W); Vg = -imag(W);
      Q  = sqrt(Ug.^2 + Vg.^2) ;  % velocity magnitude
      CP = 1-(Q./U0).^2         ;  % pressure coefficient
  end
end
%% Standardizing coordinates
% [P PS SS TE chord IPin IPout]=fProfileStandardize(X_profile,
    Y_profile,[]);

function [z_out,dz_out] = fConformalMapKarmanTrefftz(z_in,a,l,
    bFlagReverse);
% Karman Trefftz conformal map (and inverse map)
% AUTHOR: E. Branlard
if ~exist('bFlagReverse','var'); bFlagReverse=false; end;
if ~bFlagReverse  % from z to Z plane
z=z_in;
Z   =l*a*((z+a).^l + (z-a).^l)./((z+a).^l - (z-a).^l);
dZdz=(4*(l*a)^2)*(((z-a).^(l-1)).*((z+a).^(l-1)))./((((z+a).^l)-((z
    -a).^l)).^2);
z_out= Z ; dz_out = dZdz;
else % from Z to z plane
Z=z_in;
z   =-a.*(   ( ((Z-l)./(Z+l)).^(1/l) )+1   ) ./ (  (((Z-l)./(Z+l))
    .^(1/l))-1);
dzdZ=1./((4*(l*a)^2)*(((z-a).^(l-1)).*((z+a).^(l-1)))./((((z+a).^l)
    -((z-a).^l)).^2));
z_out=z; dz_out=dzdZ;
end

function [U,V] = fUi_Cylinder2D(X,Y,xc,yc,rc,U0,alpha,G)
% Flow about a 2D cylinder, free stream U0, circulation G. NOTE:
    alpha[deg]
r          = sqrt((X-xc).^2 + (Y-yc).^2);
theta      = atan2(Y-yc,X-xc)            ;
alph = alpha*pi/180                ;
U = U0*cos(alph)-(U0*((rc./r).^2).*cos(2*theta-alph))-G.*sin(theta)
    ./(2*pi.*r);
V = U0*sin(alph)-(U0*((rc./r).^2).*sin(2*theta-alph))+G.*cos(theta)
    ./(2*pi.*r);
U(r<rc-10*eps)=0; V(r<rc-10*eps)=0;
```

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