Fractional Differentiation: Leibniz Meets Hölder

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Abstract We discuss how to estimate the fractional derivative of the product of two functions, not in the pointwise sense, but on Lebesgue spaces whose indices satisfy Hölder's inequality

Keywords Kato-Ponce inequality • bilinear operators • Riesz and Bessel potentials

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1 Introduction

We recall Leibniz's product rule of differentiation

$$(fg)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k)} \tag{1}$$

which is valid for $\mathcal{C}^m$ functions f, g on the real line. Here $g^{(k)}$ denotes the kth derivative of the function g on the line. This rule can be extended to functions of n variables. For a given multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Z}^+ \cup \{0\})^n$ we set

$$\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} f\,.$$

The nth dimensional extension of the Leibniz rule (2) is

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$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)\, (\partial^\beta g) \tag{2}$$

where $\beta \le \alpha$ means $\beta_j \le \alpha_j$ for all $j = 1, \dots, n$, and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} = \prod_{j=1}^{n} \frac{\alpha_j !}{\beta_j !(\alpha_j - \beta_j)!} \,. \tag{3}$$

Identity (3) can be used to control the Lebesgue norm of $\partial^\alpha (fg)$ in terms of Lebesgue norms of partial derivatives of f and g via Hölder's inequality:

$$\|FG\|_{L^r(\mathbf{R}^n)} \le \|F\|_{L^p(\mathbf{R}^n)} \|G\|_{L^q(\mathbf{R}^n)}$$

where $0 < p, q, r \le \infty$ and $1/r = 1/p + 1/q$.

Unlike convolution, which captures the smoothness of its smoother input, multiplication inherits the smoothness of the rougher function. In this note we study the smoothness of the product of two functions of equal smoothness. The results we prove are quantitative and we measure smoothness in terms of Sobolev spaces. We focus on a version of (3) in which the multiindex α is replaced by a non-integer positive number, for instance a fractional power. Fractional powers are defined in terms of the Fourier transform.

We denote by $\mathcal{S}(\mathbf{R}^n)$ the space of all rapidly decreasing functions on $\mathbf{R}^n$, called Schwartz functions. The Fourier transform of an integrable function f on $\mathbf{R}^n$ (in particular of a Schwartz function) is defined by

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

and its inverse Fourier transform is defined by

$$f^\vee(\xi) = \widehat{f}(-\xi)$$

for all $\xi \in \mathbf{R}^n$. The Laplacian of a $\mathcal{C}^2$ function f on $\mathbf{R}^n$ is defined by

$$\Delta f = \sum_{j=1}^{n} \partial_j^2 f$$

and this can be expressed in terms of the Fourier transform as follows:

$$\Delta f = \big(-4\pi^2 |\xi|^2 \widehat{f}(\xi) \big)^\vee \,. \tag{4}$$

Identity (4) can be used to define fractional derivatives of f as follows: given $s > 0$ define

$$\Delta^{s/2}f = \big((2\pi|\xi|)^s\widehat{f}(\xi)\big)^{\vee}$$

and

$$J^s(f) = (1-\Delta)^{s/2}(f) = \big((1+4\pi^2|\xi|^2)^{s/2}\widehat{f}(\xi)\big)^{\vee}.$$

The operators $\Delta^{s/2}$ and $J^s = (1-\Delta)^{s/2}$ are called the Riesz potential and Bessel potential on $\mathbf{R}^n$, respectively. Heuristically speaking, $\Delta^{s/2}f$ is the "sth derivative" of f, while $J^s(f)$ "captures" all derivatives of all f of orders up to and including s.

Concerning J^s, in [14], Kato and Ponce obtained the commutator estimate

$$\|J^s(fg) - f(J^s g)\|_{L^p(\mathbf{R}^n)} \le C\Big[\|\nabla f\|_{L^\infty(\mathbf{R}^n)}\big\|J^{s-1}g\big\|_{L^p(\mathbf{R}^n)} + \|J^s f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^\infty(\mathbf{R}^n)}\Big]$$

for $1 < p < \infty$ and $s > 0$, where ∇ is the n-dimensional gradient, f, g are Schwartz functions, and C is a constant depending on n, p, and s. This estimate was motivated by a question stated in Remark 4:1 in Kato's work [13].

Using the Riesz potential $D^s = (-\Delta)^{s/2}$, Kenig, Ponce, and Vega [16] obtained the related estimate

$$\|D^s[fg] - fD^s g - gD^s f\|_{L^r} \le C(s, s_1, s_2, r, p, q)\|D^{s_1}f\|_{L^p}\|D^{s_2}g\|_{L^q},$$

where $s = s_1 + s_2$ for $s, s_1, s_2 \in (0,1)$, and $1 < p, q, r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{r}$.

Instead of the original statement given by Kato and Ponce, the following variant is known in the literature as the Kato-Ponce inequality (also *fractional Leibniz rule*)

$$\|J^s(fg)\|_{L^r(\mathbf{R}^n)} \le C\Big[\|f\|_{L^{p_1}(\mathbf{R}^n)}\|J^s g\|_{L^{q_1}(\mathbf{R}^n)} + \|J^s f\|_{L^{p_2}(\mathbf{R}^n)}\|g\|_{L^{q_2}(\mathbf{R}^n)}\Big] \tag{5}$$

where $s > 0$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ for $1 < r < \infty$, $1 < p_1, q_2 \le \infty$, $1 < p_2, q_1 < \infty$ and $C = C(s, n, r, p_1, p_2, q_1, q_2)$. It is important to note that in the preceding formulation the L^∞ norm does not fall on the terms with the Bessel potential. There is an analogous Kato-Ponce version of (5) in which the Bessel potential is replaced by the Riesz potential

$$\|D^s(fg)\|_{L^r(\mathbf{R}^n)} \le C\Big[\|f\|_{L^{p_1}(\mathbf{R}^n)}\|D^s g\|_{L^{q_1}(\mathbf{R}^n)} + \|D^s f\|_{L^{p_2}(\mathbf{R}^n)}\|g\|_{L^{q_2}(\mathbf{R}^n)}\Big] \tag{6}$$

and the indices are as before. In this note we study (5) and (6) focusing on (6).

There are further generalizations of the aforementioned Kato-Ponce inequalities. For instance, Muscalu, Pipher, Tao, and Thiele [18] extended this inequality to allow for partial fractional derivatives in $\mathbf{R}^2$. Bernicot, Maldonado, Moen, and Naibo [1] proved the Kato-Ponce inequality in weighted Lebesgue spaces under certain restrictions on the weights. The last authors also extended the Kato-Ponce inequality to indices $r < 1$ under the assumption $s > n$. Additional work on the Kato-Ponce inequality was done by Christ and Weinstein [3], Gulisashvili and Kon [12], and

Cordero and Zucco [6], present a way to obtain the homogeneous inequality from the inhomogeneous via a limiting process. There is also a discussion about the Kato-Ponce inequality in [8].

When $r \geq 1$ inequality (5) is valid for all $s \geq 0$ but when $r < 1$ the author and Oh [9] showed that there is a restriction $s > n/r - n$. Moreover, there is an example that inequality (5) fails for $s \leq n/r - n$. This restriction in the case $r < 1$ was independently obtained by Muscalu and Schlag [17].

2 The counterexample

In this section we provide an example to determine the range of $r < 1$ for which $D^s(fg)$ can lie in $L^r(\mathbf{R}^n)$ and in particular (6) can hold.

We set

$$f(x) = g(x) = e^{-\pi|x|^2/2}\,.$$

Then

$$fg(x) = e^{-\pi|x|^2}$$

and for $s > 0$ we have

$$(-\Delta)^{s/2}(fg)(x) = (2\pi)^s \int_{\mathbf{R}^n} \big(e^{-\pi|x|^2}\big)^\wedge(\xi)|\xi|^s e^{2\pi i x\xi}\,dx = (2\pi)^s \int_{\mathbf{R}^n} e^{-\pi|\xi|^2}|\xi|^s e^{2\pi i x\xi}\,dx.$$

We now consider two cases: (a) s is an even integer. In this case we have that $(-\Delta)^{s/2}(fg)$ is a Schwartz function and thus it has fast decay at infinity. (b) s is not an even integer. In the second case $(-\Delta)^{s/2}(fg)$ is given by multiplication on the Fourier transform side by $c\,|\xi|^s$ and thus it is given by convolving $e^{-\pi|x|^2}$ with the distribution

$$W_s = c\,\frac{\pi^{-\frac{s}{2}}}{\Gamma(-\frac{s}{2})}\,\frac{\Gamma(\frac{s+n}{2})}{\pi^{\frac{s+n}{2}}}|x|^{-n-s}\,.$$

Notice that $e^{-\pi|\cdot|^2} * W_s$ is the convolution of a Schwartz function with a tempered distribution and thus it is a smooth function with at most polynomial growth at infinity [7]. Thus $e^{-\pi|\cdot|^2} * W_s$ is a smooth and bounded function on any compact set.

Next we study the decay of this function as $|x| \to \infty$. To do so, we introduce a nonnegative function ϕ with support contained in $|x| \leq 2$ and equal to 1 on the ball $|x| \leq 1$. Then we have

$$e^{-\pi|\cdot|^2} * W_s = e^{-\pi|\cdot|^2} * \phi W_s + e^{-\pi|\cdot|^2} * (1-\phi)W_s\,.$$

First we notice that for $|x| \geq 10$ we have

$$\begin{aligned}
\big|(1-\phi)W_s * e^{-\pi|\cdot|^2}(x)\big| &= \left|c_{s,n}\int_{\mathbf{R}^n}(1-\phi(y))|y|^{-n-s}e^{-\pi|x-y|^2}dy\right| \\
&\geq |c_{s,n}|\int_{|y|\geq 1}|y|^{-n-s}e^{-\pi|x-y|^2}dy \\
&\geq |c_{s,n}|\int_{\substack{|y|\geq 1\\ |x-y|\leq 1}}|y|^{-n-s}e^{-\pi|x-y|^2}dy \\
&\geq |c_{s,n}|\, e^{-\pi}\int_{\substack{|y|\geq 1\\ |x-y|\leq 1}}|y|^{-n-s}dy \\
&\geq c'\,|x|^{-n-s}.
\end{aligned}$$

As for the other term, for $|x| \geq 10$ and for $N = [s]+1$ we have

$$\begin{aligned}
&\int_{|y|\leq 2} e^{-\pi|x-y|^2}\phi(y)\frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\,dy \\
&= \int_{|y|\leq 2}\left[e^{-\pi|x-y|^2} - \sum_{|\gamma|\leq N}\frac{\partial^\gamma(e^{-\pi|\cdot|^2})}{\gamma!}(x)y^\gamma\right]\phi(y)\frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\,dy \\
&\qquad + \sum_{|\gamma|\leq N}\frac{\partial^\gamma(e^{-\pi|\cdot|^2})}{\gamma!}(x)\int_{|y|\leq 2}\phi(y)y^\gamma\frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\,dy \\
&= \sum_{|\gamma|=N+1}\int_{|y|\leq 2}\frac{\partial^\gamma(e^{-\pi|\cdot|^2})}{\gamma!}(x-\theta_y y)\ \phi(y)\frac{y^\gamma\ |y|^{-s-n}}{\Gamma(-\frac{s}{2})}\,dy \\
&\qquad + \sum_{|\gamma|\leq N}\frac{\partial^\gamma(e^{-\pi|\cdot|^2})}{\gamma!}(x)\int_{|y|\leq 2}\phi(y)y^\gamma\frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\,dy\,,
\end{aligned}$$

where $\theta_y \in [0,1]$. Notice that

$$\partial^\gamma(e^{-\pi|\cdot|^2})(x) = P_\gamma(x)e^{-\pi|x|^2},$$

where P_γ is a polynomial of n variables depending on γ. Also $|x-\theta_y y| \geq \frac{1}{2}|x|$, hence $\partial^\gamma(e^{-\pi|\cdot|^2})(x-\theta_y y)$ has exponential decay at infinity, while the integral

$$\int_{|y|\leq 2}\phi(y)y^\gamma\frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\,dy$$

is absolutely convergent when $|\gamma| = N+1 = [s]+2 > s+1$ which is equivalent to $N-s-n > -n$. Moreover, for $|\gamma| \le [s]+1$, the quantities

$$\int_{|y|\le 2} \phi(y) y^\gamma \frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\, dy = \int_{|y|\le 1} y^\gamma \frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\, dy + \int_{1\le |y|\le 2} \phi(y) y^\gamma \frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\, dy$$

are finite constants in view of the following observation: If $|\gamma|$ is odd, then the displayed expression below is zero, while if $|\gamma| = 2m$ is an even integer, we have

$$\int_{|y|\le 1} y^\gamma \frac{|y|^{-s-n}}{\Gamma(-\frac{s}{2})}\, dy = \Big(\int_{\mathbf{S}^{n-1}} \varphi^\gamma\, d\varphi\Big) \frac{\int_0^1 r^{|\gamma|-s-1} dr}{\Gamma(-\frac{s}{2})} = \Big(\int_{\mathbf{S}^{n-1}} \varphi^\gamma\, d\varphi\Big) \frac{1/2}{(m-\frac{s}{2})\Gamma(-\frac{s}{2})}$$

and this is a well-defined constant, since the entire function $\Gamma(-w)^{-1}$ has simple zeros at the positive integers and thus $(m-w)^{-1}\Gamma(-w)^{-1}$ is also entire in w for any $m < |w|$. An important observation here is that $w = s/2$ is not an integer, since s is not an even integer, hence m can never be equal to $w = s/2$.

The outcome of this discussion is that $e^{-\pi|\cdot|^2} * \phi W_s$ is a smooth function that decays exponentially as $|x| \to \infty$. Combining this result with the corresponding obtained for $e^{-\pi|\cdot|^2} * (1-\phi) W_s$, we deduce that

$$\big|(e^{-\pi|\cdot|^2} * W_s)(x)\big| \ge c''|x|^{-n-s}$$

as $|x| \to \infty$, provided s is not an even integer.

Finally, this assertion is not valid if s is an even integer, since in this case we have that

$$(-\Delta)^{s/2}(e^{-\pi|\cdot|^2}) = \underbrace{\Big(\sum_{j=1}^n \partial_j^2\Big) \cdots \Big(\sum_{j=1}^n \partial_j^2\Big)}_{s/2 \ \text{ times}} (e^{-\pi|\cdot|^2})$$

has obviously exponential decay at infinity, as obtained by a direct differentiation.

The preceding calculation imposes a restriction on the p for which $(-\Delta)^{s/2}(fg)$ lies in $L^p(\mathbf{R}^n)$. In fact the simple example $f(x) = g(x) = e^{-\pi|x|^2/2}$ introduced at the beginning of this section provides a situation in which f, g, $(-\Delta)^{s/2}(f)$, and $(-\Delta)^{s/2}(g)$ lie in all the L^{p_j} spaces for $p_j > 1$ when $s > 0$, but $(-\Delta)^{s/2}(fg)$ lies in $L^r(\mathbf{R}^n)$ only if

$$(-s-n)r < -n,$$

that is, when $r > \frac{n}{n+s}$. Thus, when $1 \le p_1, p_2, q_1, q_2 \le \infty$ and

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$$

the inequality (6) fails when

$$r \le \frac{n}{n+s}.$$

Obviously, since $r > 1/2$, so this restriction is relevant only when $0 < s \le n$. Recall that Bernicot, Maldonado, Moen, and Naibo [1] showed that (6) holds when $s > n$, so this work fills in the gap $0 < s < n$.

3 The sharp Kato-Ponce inequalities and preliminaries

The following theorem is contained in the joint article of the author with Seungly Oh [9]. In this section we discuss some preliminary facts needed to prove the first inequality below.

Theorem 1 *Let $\frac{1}{2} < r < \infty$, $1 < p_1, p_2, q_1, q_2 \le \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Given $s > \max\left(0, \frac{n}{r} - n\right)$ or $s \in 2\mathbf{N}$, there exists $C = C(n, s, r, p_1, q_1, p_2, q_2) < \infty$ such that for all $f, g \in \mathcal{S}(\mathbf{R}^n)$ we have*

$$\|D^s(fg)\|_{L^r(\mathbf{R}^n)} \le C\left[\|D^s f\|_{L^{p_1}(\mathbf{R}^n)}\|g\|_{L^{q_1}(\mathbf{R}^n)} + \|f\|_{L^{p_2}(\mathbf{R}^n)}\|D^s g\|_{L^{q_2}(\mathbf{R}^n)}\right], \tag{7}$$

$$\|J^s(fg)\|_{L^r(\mathbf{R}^n)} \le C\left[\|f\|_{L^{p_1}(\mathbf{R}^n)}\|J^s g\|_{L^{q_1}(\mathbf{R}^n)} + \|J^s f\|_{L^{p_2}(\mathbf{R}^n)}\|g\|_{L^{q_2}(\mathbf{R}^n)}\right]. \tag{8}$$

Moreover if $r < 1$ and any one of the indices p_1, p_2, q_1, q_2 is equal to 1, then (7) and (8) hold when the $L^r(\mathbf{R}^n)$ norms on the left-hand side of the inequalities are replaced by the $L^{r,\infty}(\mathbf{R}^n)$ quasi-norm.

We remark that the statement above does not include the endpoints $L^1 \times L^\infty \to L^{1,\infty}$, $L^\infty \times L^1 \to L^{1,\infty}$, and $L^\infty \times L^\infty \to L^\infty$. However, the endpoint case $p_1 = p_2 = q_1 = q_2 = r = \infty$ was completed by Bourgain and Li [2]. An earlier version of this endpoint inequality was obtained by Grafakos, Maldonado, and Naibo [11].

As a consequence of (8) we obtain Hölder's inequality for Sobolev spaces. We have

Corollary 1 *Let $s > 0$, $\frac{n}{s+n} < r < \infty$, $1 < p, q < \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then there exists $C = C(n, s, p, q) < \infty$ such that for all $f, g \in \mathcal{S}(\mathbf{R}^n)$ we have*

$$\|J^s(fg)\|_{L^r(\mathbf{R}^n)} \le C\|J^s f\|_{L^p(\mathbf{R}^n)}\|J^s g\|_{L^q(\mathbf{R}^n)}. \tag{9}$$

We note that (9) is an easy consequence of (8), since

$$\|f\|_{L^p} \le C\|J^s f\|_{L^p}$$

for $1 < p < \infty$.

In the rest of this section we discuss some background material needed to prove Theorem 1. First we recall the classical multiplier result of Coifman and Meyer [4] (see also [5]) for the case $r > 1$ and its extension to the case $r \le 1$ by Grafakos and Torres [10] and independently by Kenig and Stein [15].

Theorem A *Let $m \in L^\infty(\mathbf{R}^{2n})$ be smooth away from the origin. Suppose that there exists a constant $A > 0$ satisfying*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \le A\,(|\xi| + |\eta|)^{-|\alpha|-|\beta|} \tag{10}$$

for all $\xi, \eta \in \mathbf{R}^n$ with $|\xi| + |\eta| \neq 0$ and $\alpha, \beta \in \mathbf{Z}^n$ multi-indices with $|\alpha|, |\beta| \le 2n+1$. Then the bilinear operator

$$T_m(f,g)(x) = \int_{\mathbf{R}^{2n}} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x}\,d\xi\,d\eta\,,$$

defined for all $f, g \in \mathcal{S}(\mathbf{R}^n)$, satisfies

$$\|T_m(f,g)\|_{L^r(\mathbf{R}^n)} \le C(p,q,r,A)\,\|f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^q(\mathbf{R}^n)}$$

where $\frac{1}{2} < r < \infty$, $1 < p, q \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Furthermore, when either p or q is equal to 1, then the $L^r(\mathbf{R}^n)$ norm on left-hand side can be replaced by the $L^{r,\infty}(\mathbf{R}^n)$ norm.

In the sequel we will use the notation $\Psi_t(x) = t^{-n}\Psi(x/t)$ when $t > 0$ and $x \in \mathbf{R}^n$. The following result will be of use to us.

Theorem 2 *Let $m \in \mathbf{Z}^n \setminus \{0\}$ and $\Psi(x) = \psi(x+m)$ for some Schwartz function ψ whose Fourier transform is supported in the annulus $1/2 \le |\xi| \le 2$. Let $\Delta_j(f) = \Psi_{2^{-j}} * f$. Then there is a constant C_n such that for $1 < p < \infty$ we have*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n \ln(1+|m|)\max\big(p,(p-1)^{-1}\big)\big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{11}$$

There also exists $C_n < \infty$ such that for all $f \in L^1(\mathbf{R}^n)$,

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n \ln(1+|m|)\big\|f\big\|_{L^1(\mathbf{R}^n)}. \tag{12}$$

Proof We recall the following form of the Littlewood-Paley theorem: Suppose that Ψ is an integrable function on $\mathbf{R}^n$ that satisfies

$$\sum_{j\in\mathbf{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 \le B^2 \tag{13}$$

and

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \sum_{j\in\mathbf{Z}} \int_{|x|\ge 2|y|} \left|\Psi_{2^{-j}}(x-y) - \Psi_{2^{-j}}(x)\right| dx \le B \tag{14}$$

Then there exists a constant $C_n < \infty$ such that for all $1 < p < \infty$ and all f in $L^p(\mathbf{R}^n)$,

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R}^n)} \le C_n B \max\big(p, (p-1)^{-1}\big) \|f\|_{L^p(\mathbf{R}^n)} \tag{15}$$

where $\Delta_j(f) = \Psi_{2^{-j}} * f$. There also exists a $C_n' < \infty$ such that for all f in $L^1(\mathbf{R}^n)$,

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n' B \|f\|_{L^1(\mathbf{R}^n)}. \tag{16}$$

We make a few remarks about the proof. Clearly the required estimate holds when $p = 2$ in view of (13). To obtain estimate (16) and thus the case $p \ne 2$, we define an operator $\vec{T}$ acting on functions on $\mathbf{R}^n$ as follows:

$$\vec{T}(f)(x) = \{\Delta_j(f)(x)\}_j\,.$$

The inequalities (15) and (16) we wish to prove say simply that $\vec{T}$ is a bounded operator from $L^p(\mathbf{R}^n, \mathbf{C})$ to $L^p(\mathbf{R}^n, \ell^2)$ and from $L^1(\mathbf{R}^n, \mathbf{C})$ to $L^{1,\infty}(\mathbf{R}^n, \ell^2)$. We indicated that this statement is true when $p = 2$, and therefore the first hypothesis of Theorem 4.6.1 in [7] is satisfied. We now observe that the operator $\vec{T}$ can be written in the form

$$\vec{T}(f)(x) = \left\{ \int_{\mathbf{R}^n} \Psi_{2^{-j}}(x-y) f(y)\, dy \right\}_j = \int_{\mathbf{R}^n} \vec{K}(x-y)(f(y))\, dy,$$

where for each $x \in \mathbf{R}^n$, $\vec{K}(x)$ is a bounded linear operator from $\mathbf{C}$ to ℓ^2 given by

$$\vec{K}(x)(a) = \{\Psi_{2^{-j}}(x)a\}_j. \tag{17}$$

We clearly have that $\big\|\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2} = \big(\sum_j |\Psi_{2^{-j}}(x)|^2\big)^{\frac{1}{2}}$, and to be able to apply Theorem 4.6.1 in [7] we need to know that

$$\int_{|x|\ge 2|y|} \big\|\vec{K}(x-y) - \vec{K}(x)\big\|_{\mathbf{C}\to\ell^2}\, dx \le C_n B, \qquad y \ne 0. \tag{18}$$

We clearly have

$$\begin{aligned}\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2} &= \Big(\sum_{j\in\mathbf{Z}}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|^2\Big)^{\frac{1}{2}}\\ &\le \sum_{j\in\mathbf{Z}}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|\end{aligned}$$

and so condition (14) implies (18).

Note

$$\widehat{\Psi}(\xi)=\widehat{\psi}(\xi)e^{2\pi i m\cdot\xi}.$$

The fact that $\widehat{\psi}$ is supported in the annulus $1/2\le|\xi|\le 2$ implies condition (13) for $\widehat{\Psi}$. We now focus on condition (14) for Ψ.

We fix a nonzero y in $\mathbf{R}^n$ and $j\in\mathbf{Z}$. We look at

$$\int_{|x|\ge 2|y|}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|dx=\int_{|x|\ge 2|y|}2^{jn}\big|\psi(2^jx-2^jy+m)-\psi(2^jx+m)\big|dx$$

Changing variables we can write the above as

$$I_j=\int_{|x|\ge 2|y|}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|dx=\int_{|x-m|\ge 2^{j+1}|y|}\big|\psi(x-2^jy)-\psi(x)\big|dx$$

Case 1: $2^j\ge 2\,|m|\,|y|^{-1}$. In this case we estimate I_j by

$$\begin{aligned}&\int_{|x-m|\ge 2^{j+1}|y|}\frac{c}{(1+|x-2^jy|)^{n+2}}dx+\int_{|x-m|\ge 2^{j+1}|y|}\frac{c}{(1+|x|)^{n+2}}\,dx\\ &=\int_{|x+2^jy-m|\ge 2^{j+1}|y|}\frac{c}{(1+|x|)^{n+2}}dx+\int_{|x-m|\ge 2^{j+1}|y|}\frac{c}{(1+|x|)^{n+2}}\,dx\end{aligned}$$

Suppose that x lies in the domain of integration of the first integral. Then

$$|x|\ge|x+2^jy-m|-2^j|y|-|m|\ge 2^{j+1}|y|-2^j|y|-\frac{1}{2}\,2^j|y|=\frac{1}{2}\,2^j|y|.$$

If x lies in the domain of integration of the second integral, then

$$|x|\ge|x-m|-|m|\ge 2^{j+1}|y|-|m|\ge 2^{j+1}|y|-\frac{1}{2}\,2^j|y|=\frac{3}{2}\,2^j|y|.$$

In both cases we have

$$I_j \le 2 \int_{|x| \ge \frac{1}{2} 2^j |y|} \frac{c}{(1+|x|)^{n+2}}\, dx \le \frac{C}{2^j |y|} \int_{\mathbf{R}^n} \frac{1}{(1+|x|)^{n+1}}\, dx \le \frac{C_n}{2^j |y|},$$

and clearly

$$\sum_{j:\ 2^j |y| \ge 2|m|} I_j \le \sum_{j:\ 2^j |y| \ge 2} I_j \le C_n.$$

Case 2: $|y|^{-1} \le 2^j \le 2\,|m|\,|y|^{-1}$. The number of j's in this case are $O(\ln|m|)$. Thus, uniformly bounding I_j by a constant, we obtain

$$\sum_{j: 1 \le 2^j |y| \le 2|m|} I_j \le C_n (1 + \ln|m|).$$

Case 3. $2^j \le |y|^{-1}$. In this case we have

$$\left|\psi(x - 2^j y) - \psi(x)\right| = \left| \int_0^1 2^j \nabla \psi(x - 2^j t y) \cdot y\, dt \right| \le 2^j |y| \int_0^1 \frac{c}{(1 + |x - 2^j t y|)^{n+1}}\, dt\,.$$

Integrating over $x \in \mathbf{R}^n$ gives the bound $I_j \le C_n\, 2^j |y|$. Thus, we obtain

$$\sum_{j: 2^j |y| \le 1} I_j \le C_n.$$

Overall, we obtain the bound $C_n \ln(1 + |m|)$ for (14), which yields the desired statement using the Littlewood-Paley theorem.

4 The proof of the homogeneous inequality (7)

In this section we prove the homogeneous inequality (7) of Theorem 1.

Proof We fix a function $\widehat{\Phi} \in \mathcal{S}(\mathbf{R}^n)$ such that $\widehat{\Phi}(\xi) \equiv 1$ on $|\xi| \le 1$ and which is supported in $|\xi| \le 2$. We define another function

$$\widehat{\Psi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$$

and we note that $\widehat{\Psi}$ is supported on the annulus $\{\xi : 1/2 < |\xi| < 2\}$ and satisfies

$$\sum_{k \in \mathbf{Z}} \widehat{\Psi}(2^{-k}\xi) = 1$$

for all $\xi \neq 0$.

Given $f, g \in \mathcal{S}(\mathbf{R}^n)$, we decompose $D^s[fg]$ as follows:

$$
\begin{aligned}
D^s[fg](x) &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta \\
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \left(\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)\right)\left(\sum_{k\in\mathbf{Z}} \widehat{\Psi}(2^{-k}\eta)\widehat{g}(\eta)\right) e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta \\
&= \Pi_1[f,g](x) + \Pi_2[f,g](x) + \Pi_3[f,g](x)\,,
\end{aligned}
$$

where

$$
\Pi_1[f,g](x) = \sum_{j\in\mathbf{Z}}\sum_{k:k\le j-2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta
$$

$$
\Pi_2[f,g](x) = \sum_{k\in\mathbf{Z}}\sum_{j:j\le k-2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta
$$

$$
\Pi_3[f,g](x) = \sum_{k\in\mathbf{Z}}\sum_{j:|j-k|\le 1}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta\,.
$$

For Π_1, we can write

$$
\Pi_1[f,g](x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \left\{\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j+2}\eta)\frac{|\xi+\eta|^s}{|\xi|^s}\right\} \widehat{D^s f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta.
$$

In Π_1 the variable ξ dominates η and so the ratio $\frac{|\xi+\eta|^s}{|\xi|^s}$ vanishes only at the origin in $\mathbf{R}^{2n}$. It is easy to verify that the expression in the square bracket above is a bilinear Coifman-Meyer multiplier, hence Theorem A implies that $\Pi_1[f,g]$ satisfies the inequality

$$
\|\Pi_1[f,g]\|_{L^r} \le C\|D^s f\|_{L^{p_1}}\|g\|_{L^{q_1}}
$$

and thus (7) holds for this term. The argument for Π_2 is identical under the apparent symmetry and one obtains

$$
\|\Pi_2[f,g]\|_{L^r} \le C\|f\|_{L^{p_2}}\|D^s g\|_{L^{q_2}}\,.
$$

For $\Pi_3[f,g]$, note that the summation in j is finite, we may only focus on one term, say $j=k$ and in this case it suffices to show estimate (7) for the term

$$
\left\|\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \widehat{\Psi}(2^{-k}\xi)\widehat{f}(\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta\right\|_{L^r(\mathbf{R}^n)}. \tag{19}
$$

When $s \in 2\mathbf{N}$, (19) can be written as

$$\left\| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left\{ \sum_{k \in \mathbf{Z}} \frac{|\xi + \eta|^s}{|\eta|^s} \widehat{\Psi}(2^{-k}\xi) \widehat{\Psi}(2^{-k}\eta) \right\} \widehat{f}(\xi) \widehat{D^s g}(\eta) e^{2\pi i (\xi+\eta)\cdot x} \, d\xi \, d\eta \right\|_{L^r(\mathbf{R}^n)} .$$

The expression in the bracket above belongs to Coifman-Meyer class, i.e., it satisfies (10), so the claimed inequality is a consequence of Theorem A in this case. When s is not an even integer we argue differently. In this case, the estimate for Π_3 requires a more careful analysis. We consider the following cases:

Case 1: $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$ or $\frac{1}{2} \le r < 1$, $1 \le p, q < \infty$.

In this case, we may have the strong L^r norm on the left-hand side of (7) when $p, q > 1$ or the weak L^r norm instead when either p or q is equal to 1. In view of Theorem A and Theorem 2, the strategy for the proof in both of these subcases will be identical. For simplicity, we will only prove the estimate with a strong L^r norm on the left-hand side.

Notice that when $|\xi|, |\eta| \le 2 \cdot 2^k$, then $|\xi + \eta| \le 2^{k+2}$ and thus

$$\widehat{\Phi}(2^{-k-2}(\xi + \eta)) = 1$$

on the support of the integral giving Π_3. In view of this we may write

$$\begin{aligned}
&\Pi_3[f, g](x) \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \sum_{k \in \mathbf{Z}} |\xi + \eta|^s \widehat{\Psi}(2^{-k}\xi) \widehat{f}(\xi) \widehat{\Psi}(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i (\xi+\eta)\cdot x} \, d\xi \, d\eta \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \sum_{k \in \mathbf{Z}} |\xi + \eta|^s \widehat{\Phi}(2^{-k-2}(\xi + \eta)) \widehat{\Psi}(2^{-k}\xi) \widehat{f}(\xi) \widehat{\Psi}(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i (\xi+\eta)\cdot x} \, d\xi \, d\eta \\
&= 2^{2s} \sum_{k \in \mathbf{Z}} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \widehat{\Phi_s}(2^{-k-2}(\xi + \eta)) \widehat{\Psi}(2^{-k}\xi) \widehat{f}(\xi) \widetilde{\Psi}(2^{-k}\eta) \widehat{D^s g}(\eta) e^{2\pi i (\xi+\eta)\cdot x} \, d\xi \, d\eta \\
&= 2^{2s} \sum_{k \in \mathbf{Z}} 2^{2nk} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \widehat{\Phi_s}(2^{-2}(\xi + \eta)) \widehat{\Psi}(\xi) \widehat{f}(2^k\xi) \widehat{\widetilde{\Psi}}(\eta) \widehat{D^s g}(2^k\eta) e^{2\pi i 2^k (\xi+\eta)\cdot x} \, d\xi \, d\eta,
\end{aligned}$$

where

$$\widehat{\widetilde{\Psi}}(\xi) = |\xi|^{-s} \widehat{\Psi}(\xi)$$

$$\widehat{\Phi_s}(\xi) = |\xi|^s \widehat{\Phi}(\xi)$$

Now the function $\xi \mapsto \widehat{\Phi_s}(2^{-2}\xi)$ is supported in $[-8, 8]^n$ and can be expressed in terms of its Fourier series multiplied by the characteristic function of the set $[-8, 8]^n$, denoted $\chi_{[-8,8]^n}$.

$$\widehat{\Phi_s}(2^{-2}(\xi+\eta)) = \sum_{m\in\mathbf{Z}^n} c_m^s e^{\frac{2\pi i}{16}(\xi+\eta)\cdot m}\chi_{[-8,8]^n}(\xi+\eta),$$

where

$$c_m^s = \frac{1}{16^n}\int_{[-8,8]^n} |y|^s\widehat{\Phi}(2^{-2}y)e^{-\frac{2\pi i}{16}y\cdot m}\,dy\,.$$

It is an easy calculation that

$$c_m^s = O((1+|m|)^{-s-n}) \tag{20}$$

as $|m|\to\infty$ and c_m^s is uniformly bounded for all $m\in\mathbf{Z}$. This calculation is similar to the one in Section 2.

Due to the support of $\widehat{\Psi}$ and $\widehat{\widehat{\Psi}}$, we also have

$$\chi_{[-8,8]^n}(\xi+\eta)\widehat{\Psi}(\xi)\widehat{\widehat{\Psi}}(\eta) = \widehat{\Psi}(\xi)\widehat{\widehat{\Psi}}(\eta),$$

so that the characteristic function may be omitted from the integrand. Using this identity, we write $\Pi_3[f,g](x)$ as

$$\begin{aligned}
&= 2^{2s}\sum_{k\in\mathbf{Z}} 2^{2nk}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\sum_{m\in\mathbf{Z}^n} c_m^s e^{\frac{2\pi i}{16}(\xi+\eta)\cdot m}\widehat{\Psi}(\xi)\widehat{f}(2^k\xi)\widehat{\widehat{\Psi}}(\eta)\widehat{D^s g}(2^k\eta)e^{2\pi i 2^k(\xi+\eta)\cdot x}\,d\xi\,d\eta\\
&= 2^{2s}\sum_{m\in\mathbf{Z}^n} c_m^s\sum_{k\in\mathbf{Z}}\Delta_k^m(f)(x)\widetilde{\Delta_k^m}(D^s g)(x),
\end{aligned}$$

where Δ_k^m is the Littlewood-Paley operator given by multiplication on the Fourier transform side by $e^{2\pi i 2^{-k}\xi\cdot\frac{m}{16}}\widehat{\Psi}(2^{-k}\xi)$, while $\widetilde{\Delta_k^m}$ is the Littlewood-Paley operator given by multiplication on the Fourier side by $e^{2\pi i 2^{-k}\xi\cdot\frac{m}{16}}\widehat{\widehat{\Psi}}(2^{-k}\xi)$. Both Littlewood-Paley operators have the form:

$$\int_{\mathbf{R}^n} 2^{nk}\Theta(2^k(x-y)+\tfrac{1}{16}m)f(y)\,dy$$

for some Schwartz function Θ whose Fourier transform is supported in some annulus centered at zero.

Let $r_* = \min(r,1)$. Taking the L^r norm of the right-hand side above, we obtain

$$\|D^s[fg]\|_{L^r}^{r_*} \le \sum_{m\in\mathbf{Z}^n}|c_m^s|^{r_*}\left\|\sum_{k\in\mathbf{Z}}\Delta_k^m(f)(x)\widetilde{\Delta_k^m}D^s(g)(x)\right\|_{L^r(\mathbf{R}^n)}^{r_*}$$

$$\leq \sum_{m\in\mathbf{Z}^n} |c_m^s|^{r_*} \left\| \sqrt{\sum_{k\in\mathbf{Z}} |\Delta_k^m(f)|^2} \right\|_{L^p(\mathbf{R}^n)}^{r_*} \left\| \sqrt{\sum_{k\in\mathbf{Z}} |\widetilde{\Delta_k^m}(D^s g)|^2} \right\|_{L^q(\mathbf{R}^n)}^{r_*}$$

whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. By Theorem 2, the preceding expression is bounded by a constant multiple of

$$\sum_{m\in\mathbf{Z}^n} |c_m^s|^{r_*} [\ln(2+|m|)]^{2r_*} \|f\|_{L^p}^{r_*} \|D^s g\|_{L^q}^{r_*}$$

if $1 < p, q < \infty$ and the preceding series converges in view of (20) and of the assumption $r_*(n+s) > n$. This concludes Case 1.

Case 2: $1 < r < \infty$, $(p,q) \in \{(r,\infty), (\infty, r)\}$

In this case we use an argument inspired by Christ and Weinstein [3]. We have

$$\|\Pi_3[f,g]\|_{L^r(\mathbf{R}^n)} \leq C(r,n) \left\| \left(\sum_{j\in\mathbf{Z}} |\Delta_j(\Pi_3[f,g])|^2 \right)^{\frac{1}{2}} \right\|_{L^r(\mathbf{R}^n)} .$$

The summand in j above can be estimated as follows:

$$\begin{aligned}
&\Delta_j \Pi_3[f,g](x) \\
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\xi+\eta|^s \widehat{\Psi}(2^{-j}(\xi+\eta)) \sum_{k\geq j-2} \widehat{\Psi}(2^{-k}\xi)\widehat{f}(\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{g}(\eta)\, e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta \\
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} 2^{js}\widehat{\Psi_s}(2^{-j}(\xi+\eta))\widehat{\Psi}(2^{-k}\xi)\widehat{f}(\xi) \sum_{k\geq j-2} 2^{-ks}\widehat{\Psi_{-s}}(2^{-k}\eta)\widehat{D^s g}(\eta)\, e^{2\pi i(\xi+\eta)\cdot x}\, d\xi\, d\eta \\
&= 2^{js} \sum_{k\geq j-2} 2^{-ks} \widetilde{\Delta_j^s}\Big[\Delta_k(f)\widetilde{\Delta_k^{-s}}(D^s g)\Big](x) \\
&\leq 2^{js} \left(\sum_{k\geq j-2} 2^{-2ks}\right)^{\frac{1}{2}} \left(\sum_{k\geq j-2} \left|\widetilde{\Delta_j^s}\left[\Delta_k(f)\widetilde{\Delta_k^{-s}}(D^s g)\right](x)\right|^2\right)^{\frac{1}{2}} \\
&\leq C(s) \left(\sum_{k\geq j-2} \left|\widetilde{\Delta_j^s}\Big[\Delta_k(f)\widetilde{\Delta_k^{-s}}(D^s g)\Big](x)\right|^2\right)^{\frac{1}{2}} ,
\end{aligned}$$

where

$$\widehat{\Psi_s}(\xi) := |\xi|^s \widehat{\Psi}(\xi)$$

and

$$\widetilde{\Delta_k^s}(f)^\wedge(\xi) := \widehat{\widetilde{\Psi_s}}(2^{-k}\xi)\widehat{f}(\xi).$$

Thus we have

$$\|\Pi_3[f,g]\|_{L^r} \le C(r,n,s)\left\| \left(\sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \left| \widetilde{\Delta_j^s}[\Delta_k(f)\, \widetilde{\Delta_k^{-s}}(D^s g)] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r}.$$

We apply [7, Proposition 4.6.4] to extend $\{\widetilde{\Delta_k^s}\}_{k\in\mathbf{Z}}$ from $L^r \to L^r\ell^2$ to $L^r\ell^2 \to L^r\ell^2\ell^2$ for $1 < r < \infty$. This gives

$$\begin{aligned}
\|\Pi_3[f,g]\|_{L^r} &\le C(r,n,s)\left\| \left(\sum_{k\in\mathbf{Z}} \left| \Delta_k(f)\, \widetilde{\Delta_k}(D^s g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \\
&\le C(r,n,s)\left\| \sup_{k\in\mathbf{Z}} \widetilde{\Delta_k}(D^s g) \right\|_{L^\infty} \left\| \left(\sum_{k\in\mathbf{Z}} |\Delta_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \\
&\le C(r,n,s) \sup_{k\in\mathbf{Z}} \left\| \widetilde{\Delta_k}(D^s g) \right\|_{L^\infty} \|f\|_{L^r} \\
&\le C(r,n,s) \left\| \widetilde{\Psi_{-s}} \right\|_{L^1} \|D^s g\|_{L^\infty} \|f\|_{L^r}.
\end{aligned}$$

This proves the case when $(p,q) = (r,\infty)$ while the case $(p,q) = (\infty, r)$ follows by symmetry. □

5 Final remarks

We make a few comments about the inhomogeneous Kato-Ponce inequality (7). It can be obtained from the corresponding homogeneous inequality (8) via the following observation

$$\|J^s f\|_{L^p} = \|(I-\Delta)^{s/2} f\|_{L^p} \approx \|f\|_{L^p} + \|(-\Delta)^{s/2} f\|_{L^p} \tag{21}$$

which is valid for $1 < p < \infty$ and $s > 0$.

Then in the case where $r > 1$ we may use (21) and Hölder's inequality

$$\|fg\|_{L^r} \le \|f\|_{L^{p_j}} \|g\|_{L^{q_j}}$$

for $j = 1, 2$ to obtain (8). In the case $r \le 1$ another argument is needed, similar to that given in Section 4.

Inequalities (7) and (8) are also valid for complex values of s with nonnegative real part. This is an easy consequence of fact that the functions

$$|\xi|^{it}, \qquad (1+|\xi|^2)^{it/2}$$

for t real are L^p Fourier multipliers for $1 < p < \infty$. Finally, it is worth noting that the inequality (7) fails when $s < 0$; see [9].

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