Chapter 2
Measurement Errors

2.1 Errors of Measurements

All experimental measurements are inaccurate to some degree. By this statement we mean that the measured quantity (e.g. length, weight, mass, time interval, speed, temperature etc.) *has* a *real value*, which we wish to determine, but any measurement we perform of this magnitude, direct or indirect, does not have as its result this real value but some value which differs from it by an unknown amount. We call the difference between the numerical result of the measurement and the real value of the magnitude being measured, *error of the measurement*.

Measurement errors are classified in two categories: *accidental* and *systematic*. There is no clear definition of the exact difference between them. Neither do they obey any simple law. It is difficult to distinguish them; the error of a measurement is usually a combination of errors of both kinds.

2.1.1 Accidental or Random Errors

Accidental or *random errors* are due to many unpredictable factors and their presence may be revealed by repetitions of the measurement. The main characteristics of accidental errors are that they have no regularity in successive measurements of the same magnitude and that their sign is equally probable to be positive or negative. The basic property of accidental errors to be positive or negative with equal probabilities, as well as the fact that small deviations from the real value are more probable than large

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ones, make possible the determination of an estimate of the uncertainty with which the measured magnitude is known. This is achieved by repeating the measurement, under identical experimental conditions, many times, so that, on average, accidental errors mutually cancel out to a certain degree.

Classical examples of random errors are those due to *thermal noise*. The inevitable thermal noise affects all systems, mechanical and electrical. The indications of a torsion balance or a galvanometer, for example, are always non-zero and vary with time (*Brownian motion*). This fact is due to their thermal interaction with their surroundings through their collisions with air molecules or even the photons of the ambient electromagnetic radiation. The noise in electronic instruments is of the same (thermal) origin. This noise (known as *Johnson* or *Nyquist* noise) is due to the random thermal motion of the electrons in the components of the instrument, such as resistors, which leads to the appearance of small potential differences across them. These signals, after being processed by the instrument, appear at its output as random variations of its reading. There exist, of course, means of minimizing thermal noise (suitable filtering of electric signal, lowering the temperature since the noise depends on it etc.). It is, however, both practically and theoretically impossible to eliminate this noise completely. The noise is added to the signal being measured changing its value. The often-used term *signal to noise ratio* describes precisely this situation.

2.1.2 Systematic Errors

Systematic errors may be due to imperfections of the instruments or the method used, and to the observer. They are the most difficult to deal with, as the repetition of the measurement does not reveal their existence. Some examples of sources of systematic errors will help clarify this statement.

Zero error is one of the commonest of systematic errors. If, for example, the pointer of an instrument (for those instruments that still have pointers!) has been shifted relative to its scale, in such a way that for zero input signal the instrument shows a non-zero output signal x_z, then all the instrument's readings will differ by x_z from what they should be. In this example, the systematic error is constant.

If a ruler was marked so that a length of 999 mm was subdivided into 1000 equal parts, which are supposed to have a length of one mm, then each measurement of length using this ruler will give results which are systematically larger by 0.1%. In this example, the systematic error is equal to a constant proportion of the measured quantity. In addition, if the subdivision of the length into equal parts was not performed with the necessary precision, this will add more systematic errors.

A mercury thermometer, whose column does not have a constant cross section, will give systematically and irregularly wrong values for the temperature. This, of course, is true assuming that the scale of the thermometer was drawn using the usual method of establishing the points corresponding to 0 and 100 °C and subdividing the distance between them into 100 equal intervals, each corresponding to 1 °C.

An instrument which needs a certain time to reach a state of equilibrium (e.g. thermal) before it can function normally, will systematically give erroneous readings during its transition period. In this case the systematic error will be a function of time.

The use of a wrong numerical value in the processing of experimental results, or an approximate theoretical relation involving the measured magnitudes, will lead to systematic errors. It is of course arguable whether these errors may be considered to be experimental errors. A classical example is the case of the measurement of the electronic charge by Millikan. The value found by Millikan was $e = 1.591 \times 10^{-19}$ C, with an estimated probable uncertainty of the order of 0.002×10^{-19} C, or about 0.1%. The accepted value today is $e = 1.602 \times 10^{-19}$ C, accurate to the significant digits given. It is seen that the real error of Millikan's value was greater than 0.7%, which is five times that given by Millikan. The problem arose from the fact that the value for viscosity of air available to Millikan was wrong. Due to this error, all values of the atomic constants, such as Planck's constant and Avogadro's constant, the determination of which depends on the value of e, were wrong by errors larger than 0.7% until 1930.

The avoidance of systematic errors depends mainly on the observer's experience. Systematic errors are difficult to detect and usually are the most important errors present in measurements. The most common way of detecting systematic errors is the *calibration* of the measuring instrument, by comparing it with another instrument which is known to have greater precision and negligible systematic errors. Another way of testing for systematic errors in an instrument or procedure is to use it in the measurement of a *standard*. A balance, for example, may be tested by weighing a standard of known weight. A voltmeter may be calibrated by the measurement of a standard of *emf*. A radioactive source of well known activity may be used in the calibration of an arrangement for the measurement of radioactivity.

Figure 2.1 illustrates the relationship between random and systematic errors.

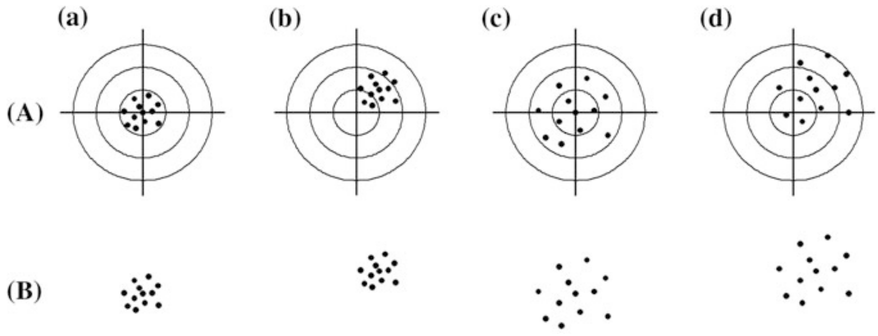

Fig. 2.1 A schematic illustration of random and systematic errors: ***a*** Random errors only. ***b*** Random and systematic errors. ***c*** Only random errors, but larger than those of (***a***). ***d*** Random errors larger than those of (***a***) and systematic errors. In line **A** the real value of the measured quantity is also shown (*center of the circles*) and the distinction between random and systematic errors is possible. In practice, however, the real value is not known [line **B**] and the detection of systematic errors is difficult

2.1.3 Personal Errors

The habits of experimentalists differ and it has been adequately documented, mainly from astronomical observations, that, in measurements where subjective judgment is important, some systematic errors are characteristic of the observer or the instrument-observer combination. Bessel had examined the positions of stars, as these were determined by leading astronomers of his age by measuring the times of passage across the meridians of various observatories and found systematic differences between them. Another good example refers to the estimation of the sunspot activity. This is measured by a number R, which was proposed by R. Wolf of the Zurich Observatory and is defined as:

$$R = k[10 \times (\text{number of visible groups of sunspots}) + (\text{number of all sunspots})].$$

The coefficient k in the definition of this so-called *Wolf number*, depends on the combination of the observer and the telescope used. It is determined by comparison with some such combination which is used as standard for $k = 1$. It is found that it has values which differ, in some cases, by up to 20% from unity. The k coefficients are known for many combinations of telescope-observer, a fact that makes possible the coordinated observation of solar activity by many observatories simultaneously.

2.1.4 Occasional Errors

In some experimental arrangements it is possible for signals to be detected and measured which occur very rarely and cannot be considered to be a permanent source of noise. For example, in a system which measures pulses resulting from some process under investigation, a false event may be recorded, such as one due to the relatively rare high-energy cosmic ray showers. It must be stressed here that some of science's greatest discoveries were made possible when an experienced observer realized that such a signal was not a spurious noise signal but was caused by an unknown effect.

More pedestrian causes may lead to similar mistakes; an example from real life is that of the postgraduate student who was tormented for days trying to explain a small peak that appeared in the curve of light intensity versus time he was recording in his studies, before he realized that the source of the signal was the building's elevator arriving at the fourth floor where his laboratory was situated! Apart from working in a hut outside the main building, the student learned the advantages of good shielding of his apparatus from external electrical signals.

The first author had a similar, very striking experience during his participation in a project to detect gravitational radiation back in the 70s. The detector, a cylinder of mass over 600 kg, was so sensitive that it could detect minute sound signals before the chamber in which it was situated was evacuated to a low enough pressure.

This resulted in pulses appearing at the output, which one could either interpret as gravitational radiation pulses emitted when a star fell into a black hole at the center of the Galaxy or to the sound waves created when someone was walking along the corridor of the floor above. The final site for the detector was actually a farm belonging to the university, outside the city, where the farm's cows could not cause such pulses.

The problem with occasional errors is mainly that they are so rare that we cannot predict the number of expected events during our experiment. This number is small, in any case. Occasional errors, however, may be very significant and cause a modification of the results of a sensitive experiment when they occur. We might call them *parasitic*, although this term fits any unwanted signal in general.

2.1.5 The Errors in Reading the Indications of Instruments

Despite the fact that in modern instruments the participation of the observer in the taking of readings becomes more and more rare, there are still many instances in which the subjectivity factor and the habits of the observer play an important role in the reading of the indications of instruments. Most uncertainties which are due to the observer may be minimized as the observer's experience increases, both in general and due to the fact that a procedure is repeated many times. In all cases, however, it is necessary for the observer to be fully conscious of both his capabilities and of the instruments he is using. Overestimation of these capabilities may lead to problems.

For example, if we measure a length 5 times using a ruler and record the results

$$17 \quad 17 \quad 17 \quad 17 \quad 17 \text{ mm}$$

are we justified in stating that the measured length is equal to 17 mm *exactly*? Obviously not. If the length was actually 17.01 mm would we be able to measure this? The answer is 'no', because the method we used did not allow us to detect the difference between 17.00 and 17.01 mm. The *accuracy with which we read the scale* of the instrument we are using is a basic quantity that we must always have in mind. In a well designed experiment, the accuracy of the method should manifest itself in the differences among the numerical results of repeated measurements. By this we mean that the accuracy with which the instrument's scale can be read should be such that the random errors of the measurements become apparent. Such a set of measurements would be, for example,

$$17.2 \quad 17.0 \quad 16.8 \quad 17.1 \quad 16.9 \text{ mm.}$$

This would mean, however, that we have the ability to take readings with an accuracy of 0.1 mm, which is not the case when we measure lengths with a ruler and the naked eye. The belief that we have this ability is over-ambitious in this particular case. The solution is obvious:

In those cases in which the random errors are not apparent in our measurements, the uncertainty in our result should be based on the accuracy with which we can read the indications of the instrument used.

Examples are given below:

Let us assume that we are measuring the length of an object using a ruler and the naked eye. We naturally assume that the ends of the object are well defined, so that it makes sense to talk of its *exact length*. If the smallest subdivisions on the ruler correspond to mm, then the procedure would be to place one end of the object next to an incision on the ruler and then see with which incision of the ruler the other end coincides. The difference of the two readings on the ruler will be the length of the object. The positioning of the one end of the object next to an incision on the ruler may usually be achieved with an accuracy of 1/5 mm. The reading of the position of the other end has the same uncertainty. The *reading error* for the length of the object will, therefore, be of the order of $2 \times 1/5 = 0.4$ mm. A more realistic estimate of the reading error for measurements with a common ruler would usually be 1 mm. So, for the 5 measurements we had above, all of which gave the result of 17 mm, it follows that:

The length measured is 17 mm, with a possible error of the order of 1 mm.

What we have said above also apply to the case when we have only one measurement of a quantity. In this case no estimate of the random error can be made and the reading error should be considered to be a lower estimate of the error of the measurement.

Another example is that of the measurement of time. We assume that we use a chronometer of the traditional kind, with a circular scale and subdivisions of 1/5 of a second. If we measure the time that passes between two events and we read an indication of, say, 15.8 s, just how sure are we of this result? We assume that the two events are well defined in time so that it makes sense to talk of accuracy in the timing of their occurrence equal to 1/5 s. Our reaction time in pressing the chronometer's knob at the right time is not zero. It might be that a reaction time of 1/10 s is possible for some people. However, do the mechanical parts of the chronometer's knob react at such a speed? Something else that must be taken into account is that the chronometer's pointer does not change position continuously but in steps of 1/5 s. In the best of cases, therefore, the accuracy of our measurements cannot be smaller than about 1/5 s, assuming of course that the chronometer is that well manufactured.

The problem is also present in measurements performed with instruments having digital indications. If the instrument has a 4-digit display, we must assume that the error corresponds to one unit in the last digit. Most digital instruments do not round to the nearest previous digit but simply reject all digits beyond those shown in the display. The *round-off error* must thus be taken to be equal to one unit in the last digit of the display. So, if the indication of a digital instrument is, for example, 1.245, the reading error must be taken as being equal to 0.001 units.

Good quality instruments are usually accompanied by instructions on how to estimate the highest possible systematic error, something that is determined by the manufacturer by calibrating the instrument. A voltmeter may, for example be accompanied by a certificate stating that:

The maximum possible error in the measured voltage is equal to:

0.005 × (*maximum indication of the scale used*) + *0.010* × (*indication of the instrument*).

If, for example we are using the scale of 0–3 mV and the indication of the instrument is 2.45 mV, the maximum possible systematic error due to the instrument is 0.005 × 3 + 0.010 × 2.45 = 0.015 + 0.025 = 0.04 mV.

While it is certain that an accurate value of the magnitude being measured is desirable, it is not always necessary for us to do *everything we can* in order to lower the error as much as possible; in most cases, a *reasonable* error is tolerable. However, what is certainly needed is for us to have a good estimate of the possible error in our measurement. The use of our measurements so as to derive from them the most accurate estimate for the real value of the magnitude being measured, as well as of the possible error in this value, is the main purpose of the first part of this book. The theory to be used in the mathematical analysis is valid only for random errors. We must never forget, therefore, that our measurements may contain systematic errors which are much larger than the random errors and which will be definitive for the usefulness of our results.

2.2 Errors in Compound Quantities

We will now examine the methods of evaluating the error in a compound quantity $Q = Q(x, y, \ldots)$ which is a function of the quantities x, y, …, which we have measured. If x_0, y_0, … are the *real values* of the quantities x, y, … and x_{m}, y_{m}, … the numerical values that resulted from their measurement, then, the *errors* in these magnitudes are defined as

$$e_x \equiv x_{\mathrm{m}} - x_0, \quad e_y \equiv y_{\mathrm{m}} - y_0, \ldots. \tag{2.1}$$

The *reduced or fractional error* in the quantity x is defined as

$$f_x \equiv \frac{e_x}{x_0}. \tag{2.2}$$

Obviously, it is

$$x_{\mathrm{m}} = x_0 + e_x = x_0(1 + f_x). \tag{2.3}$$

The *percentage error* is also defined as $100e_x/x_0\%$.

If the fractional error in x is small compared to unity ($f_x \ll 1$), the following approximate relations hold

$$\frac{x_0}{x_{\mathrm{m}}} = \frac{1}{1 + f_x} \approx 1 - f_x \tag{2.4}$$

and

$$\frac{e_x}{x_0} = \frac{e_x}{x_{\mathrm{m}}}\frac{x_{\mathrm{m}}}{x_0} = \frac{e_x}{x_{\mathrm{m}}}(1 + f_x) \approx \frac{e_x}{x_{\mathrm{m}}}. \tag{2.5}$$

The errors e_x, e_y, … are unknown to us, since we do not know the real values x_0, y_0, … of the magnitudes x, y, …. As a consequence, the error in the compound quantity Q will also be unknown to us. We could say that the numerical results of the examples to follow would be known only to somebody who knew, apart for our experimental results, the real values of the quantities being measured as well. We will, however, examine the way in which the errors in x, y, … affect the estimated value Q_{m} of the compound quantity Q, because this will help us understand the concept of *propagation of errors*, i.e. the evaluation of the deviation of the value Q_{m} from the real value Q_0, due to the errors in x_{m}, y_{m},….

2.2.1 Error in a Sum or a Difference

If it is $Q = x + y$, then $Q_0 = x_0 + y_0$ and if the measurements of x and y gave the results x_{m} and y_{m}, it will be

$$Q_{\mathrm{m}} = x_{\mathrm{m}} + y_{\mathrm{m}} = x_0 + y_0 + e_x + e_y = Q_0 + e_Q, \tag{2.6}$$

where e_Q is the error in Q_{m}. The fractional error in $Q = x + y$ is, therefore,

$$f_Q = \frac{e_Q}{Q_0} = \frac{e_x + e_y}{x_0 + y_0} = \frac{x_0 f_x + y_0 f_y}{x_0 + y_0}. \tag{2.7}$$

The relation also holds when either x_0 or y_0 is negative, in which case Eq. (2.7) gives the error in the difference of the two quantities.

We observe that, if f_x and f_y are comparable and $x_0 \gg \mathrm{y}_0$, then $f_Q \approx f_x$, while, if $y_0 \gg \mathrm{x}_0$, then $f_Q \approx f_y$.

The result may be generalized to give the fractional error in $Q_0 = x_0 + y_0 + z_0 + \ldots$ as

$$f_Q = \frac{e_Q}{Q_0} = \frac{e_x + e_y + e_z + \ldots}{x_0 + y_0 + z_0 + \ldots} = \frac{x_0 f_x + y_0 f_y + z_0 f_z + \ldots}{x_0 + y_0 + z_0 + \ldots}. \tag{2.8}$$

In particular, if it is $Q_0 = kx_0$ where k is an integer, putting $x_0 = y_0 = z_0 = \ldots$ (k terms) in Eq. (2.8) we have

$$f_Q = \frac{e_Q}{Q_0} = \frac{ke_x}{kx_0} = \frac{e_x}{x_0} = f_x \quad \text{or} \quad f_Q = f_x. \tag{2.9}$$

The result is true for every k: Since $Q_0 = kx_0$, $Q_\text{m} = kx_\text{m}$ and $x_\text{m} = x_0(1 + f_x)$, we have

$$Q_m = kx_0(1 + f_x) = Q_0(1 + f_x),$$

and, because $Q_m = Q_0(1 + f_Q)$, it follows that $f_Q = f_x$.

Example 2.1

The measurements of x and y gave the results $x_\text{m} = 6.2$ cm and $y_\text{m} = 3.6$ cm. The real values of these quantities are $x_0 = 6.1$ cm and $y_0 = 3.4$ cm. What is the error in the sum $Q = x + y$?

Obviously, the real value of Q is $Q_0 = x_0 + y_0 = 6.1 + 3.4 = 9.5$ cm. The value determined by the measurements is $Q_\text{m} = x_\text{m} + y_\text{m} = 6.2 + 3.6 = 9.8$ cm. It is immediately seen that the error in Q is equal to $e_Q = 9.8 - 9.5 = 0.3$ cm and the fractional error is $f_Q = e_Q/Q_0 = 0.3/9.5 = 0.03$, or 3%.

Using Eq. (2.7), we find again

$$f_Q = \frac{x_0 f_x + y_0 f_y}{x_0 + y_0} = \frac{(x_\text{m} - x_0) + (y_\text{m} - y_0)}{x_0 + y_0} = \frac{x_\text{m} + y_\text{m}}{x_0 + y_0} - 1 = 0.03.$$

Example 2.2

The quantities x and y were measured with fractional errors $f_x = 0.01$ and $f_y = 0.02$. If the real values of these quantities are $x_0 = 15$ m and $y_0 = 5$ m, what is the fractional error in the sum $Q = x + y$?

From Eq. (2.7) we have

$$f_Q = \frac{x_0 f_x + y_0 f_y}{x_0 + y_0} = \frac{15 \times 0.01 + 5 \times 0.02}{15 + 5} = \frac{0.15 + 0.10}{20} = 0.0125, \text{ or } 1.25\%.$$

2.2.2 Error in a Product

If $Q = xy$, then $Q_0 = x_0 y_0$ and if the measurements of x and y gave the results x_{m} and y_{m}, it will be

$$\begin{aligned} Q_{\mathrm{m}} &= x_{\mathrm{m}} y_{\mathrm{m}} = (x_0 + e_x) \times (y_0 + e_y) = x_0(1 + f_x) \times y_0(1 + f_y) \\ &= x_0 y_0 (1 + f_x)(1 + f_y). \end{aligned} \tag{2.10}$$

Since for small f_x and f_y it is $(1 + f_x)(1 + f_y) \approx 1 + f_x + f_y$, it follows that

$$Q_{\mathrm{m}} \approx Q_0(1 + f_x + f_y) \tag{2.11}$$

and the fractional error in Q is

$$f_Q = \frac{Q_{\mathrm{m}} - Q_0}{Q_0} = f_x + f_y, \tag{2.12}$$

i.e., *the fractional error in $Q = xy$ is equal to the sum of the fractional errors in x and y.*

The result may be generalized and in the case of $Q = xyz\ldots$. We have

$$Q_{\mathrm{m}} = x_{\mathrm{m}} y_{\mathrm{m}} z_{\mathrm{m}} \ldots = x_0 y_0 z_0 \ldots (1 + f_x)(1 + f_y)(1 + f_z) \ldots \approx Q_0(1 + f_x + f_y + f_z + \ldots) \tag{2.13}$$

and, therefore,

$$f_Q = \frac{Q_{\mathrm{m}} - Q_0}{Q_0} = f_x + f_y + f_z + \ldots, \tag{2.14}$$

i.e., *the fractional error in $Q = xyz\ldots$ is equal to the algebraic sum of the fractional errors in x, y, z, ….*

Example 2.3

The measurements of the quantities x and y gave results with fractional errors $f_x = 0.01$ and $f_y = 0.02$, respectively. Which is the fractional error in the product $Q = xy$?

Equation (2.12) gives $f_Q = f_x + f_y = 0.01 + 0.02 = 0.03$ or 3%.

2.2.3 Error in a Power

For the special case of $Q = x^n$ where n is a positive integer, Eq. (2.14) gives $f_Q = nf_x$ or that *the fractional error of the power x^n is equal to n times the fractional error in x.*

Generally, let $Q = kx^n$, where n is any real number and k a constant. Since it is $Q_\mathrm{m} = Q_0(1+f_Q)$, $x_\mathrm{m} = x_0(1+f_x)$, $Q_0 = kx_0^n$ and $Q_\mathrm{m} = kx_\mathrm{m}^n$, we have

$$Q_\mathrm{m} = Q_0(1+f_Q) = kx_\mathrm{m}^n = kx_0^n(1+f_x)^n \approx Q_0(1+nf_x) \qquad (2.15)$$

for $f_x \ll 1$. From the equality of the first term and the last term, it follows that

$$f_Q = nf_x \qquad (2.16)$$

or that *the fractional error in any multiple of the n-th power of x is equal to n times the fractional error in x.*

Special cases: If $Q = x^2$ it is $f_Q = 2f_x$ and if $Q = \sqrt{x}$ it is $f_Q = \frac{1}{2}f_x$.

Example 2.4

If the result of the measurement of x has a fractional error $f_x = 0.005$, what is the fractional error in the quantity $Q = 7x^{3/2}$?

From Eq. (2.16), $f_Q = nf_x = \frac{3}{2} \times 0.005 = 0.0075 \approx 0.008$.

2.2.4 Error in a Quotient

If it is $Q = x/y$ and the measurements of x and y gave the results x_m and y_m, then

$$Q_\mathrm{m} = \frac{x_\mathrm{m}}{y_\mathrm{m}} = \frac{x_0 + e_x}{y_0 + e_y} = \frac{x_0(1+f_x)}{y_0(1+f_y)} \approx \frac{x_0}{y_0}(1+f_x)(1-f_y) \approx Q_0(1+f_x-f_y) \qquad (2.17)$$

for small f_x and f_y, and the fractional error in Q is

$$f_Q = \frac{Q_\mathrm{m} - Q_0}{Q_0} = f_x - f_y, \qquad (2.18)$$

i.e. *the fractional error in $Q = x/y$ is equal to the difference of the fractional errors of x and y.*

The result may be generalized and in the case when it is $Q = \frac{x'y'z'\ldots}{x''y''z''\ldots}$ we have

$$\begin{aligned} Q_\mathrm{m} &= \frac{x'_\mathrm{m}y'_\mathrm{m}z'_\mathrm{m}\cdots}{x''_\mathrm{m}y''_\mathrm{m}z''_\mathrm{m}\cdots} = \frac{x'_0y'_0z'_0\cdots}{x''_0y''_0z''_0\cdots} \times \frac{(1+f_{x'})(1+f_{y'})(1+f_{z'})\ldots}{(1+f_{x''})(1+f_{y''})(1+f_{z''})\ldots} \\ &\approx Q_0(1+f_{x'}+f_{y'}+f_{z'}+\ldots-f_{x''}-f_{y''}-f_{z''}-\ldots) \end{aligned} \qquad (2.19)$$

and

$$f_Q = \frac{Q_\mathrm{m} - Q_0}{Q_0} = (f_{x'}+f_{y'}+f_{z'}+\ldots) - (f_{x''}+f_{y''}+f_{z''}+\ldots), \qquad (2.20)$$

i.e. *the fractional error in Q is equal to the algebraic sum of the fractional errors in $x', y', z', \ldots$, minus the algebraic sum of the fractional errors in $x'', y'', z'', \ldots$.*

Example 2.5

If the quantities x and y were measured with fractional errors $f_x = -0.015$ and $f_y = 0.02$, respectively, what will the fractional error in $Q = x^2/y$ be?

We initially evaluate the fractional error in x^2. Equation (2.16) gives $f_{x^2} = 2f_x = -0.03$. Then Eq. (2.18) gives $f_Q = f_{x^2} - f_y = -0.03 - 0.02 = -0.05$.

2.2.5 *The Use of Differentials*

2.2.5.1 Functions of One Variable

If $Q(x)$ is a function of one variable, x, its derivative is $\frac{\mathrm{d}Q}{\mathrm{d}x}$. From the definition of the derivative $\lim\limits_{\delta x \to 0} \frac{\delta Q}{\delta x} = \frac{\mathrm{d}Q}{\mathrm{d}x}$, it follows that for small δx it is, approximately,

$$\delta Q \approx \frac{\mathrm{d}Q}{\mathrm{d}x}\delta x. \tag{2.21}$$

Equation (2.21) gives the change δQ in Q due to a small change δx in x. If now it is $x_\mathrm{m} = x_0 + e_x$ and $Q_\mathrm{m} = Q_0 + e_Q$, and we put $\delta x \equiv x_\mathrm{m} - x_0 = e_x$ and $\delta Q \equiv Q_\mathrm{m} - Q_0 = e_Q$ in Eq. (2.21), we will have, to a good approximation for small e_x,

$$e_Q = \frac{\mathrm{d}Q}{\mathrm{d}x}e_x, \tag{2.22}$$

a relationship which correlates the error in Q to the error in x.

The geometrical interpretation of the relations (2.21) and (2.22) is given in Fig. 2.2. Assuming a linear relationship between δQ and δx, which is shown in the figure by the tangent to the curve $Q(x)$ at x, we evaluate the error in δQ. The dashed line gives a better value for δQ, because it takes into account the non-linearity of $Q(x)$. We would also have a better value for δQ by taking point x at the center of δx. These, however, are second-order corrections, which are not important for small values of δx.

For example, if $Q = x^2$, then $\frac{\mathrm{d}Q}{\mathrm{d}x} = 2x$ and, therefore, $e_Q = 2xe_x$. Dividing on the left by Q and on the right by x^2, we find that $\frac{e_Q}{Q} = 2\frac{e_x}{x}$, or $f_Q = 2f_x$, as we found above. In the same way we may verify that, if it is $Q = \sqrt{x}$, then $f_Q = \frac{1}{2}f_x$.

Strictly speaking, the derivative should be evaluated at $x = x_0$. However, since this value is not known to us and because we assume that x_m does not differ by much from x_0, we can do nothing else but evaluate the derivative at the point $x = x_\mathrm{m}$.

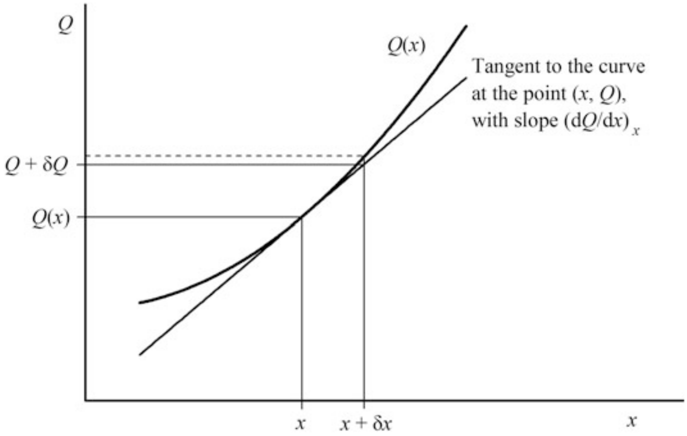

Fig. 2.2 The relationship between the error δx in the variable x and the corresponding error δQ in the function $Q(x)$

Warning: When angles are involved in Eqs. (2.21) and (2.22), we must bear in mind that the relations $\delta(\sin\theta) \approx \frac{\mathrm{d}(\sin\theta)}{\mathrm{d}\theta}\delta\theta = \cos\theta\,\delta\theta$, $\delta(\cos\theta) \approx \frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\theta}\delta\theta = -\sin\theta\,\delta\theta$, as well as other similar trigonometric relations, are valid only if the error in the angle, $\delta\theta$, is given in radians. This is a common source of errors when one meets such problems for the first time.

Example 2.6

On measuring the radius of a sphere, the value $r_{\mathrm{m}} = 10.1$ mm was found, instead of the real value $r_0 = 10$ mm. What will the error be in the volume of the sphere, if this is evaluated using the value r_{m}?

Since the volume of the sphere is given by $V = \frac{4}{3}\pi r^3$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$, it will be $e_V = \delta V = 4\pi r^2 \delta r = 4\pi r^2 e_r$, where $e_r = 10.1 - 10 = 0.1$ mm. If in the evaluation of $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$ we use the real value $r_0 = 10$ mm, we find that

$$e_V = \delta V = 4\pi r_0^2 e_r = 4\pi \times (10)^2 \times 0.1 = 126\,\mathrm{mm}^3.$$

The real volume of the sphere is $V_0 = \frac{4}{3}\pi r_0^3 = 4189$ mm^3.

Using the value $r_{\mathrm{m}} = 10.1$ mm, we find $V_{\mathrm{m}} = \frac{4}{3}\pi r_{\mathrm{m}}^3 = 4316$ mm^3, which is larger than V_0 by 127 mm^3.

The fractional error in the volume is equal to 127/4189 = 0.030 (or 3%), which is three times the fractional error 0.1/10 = 0.010 (or 1%) in the radius. This is expected, since it is $V \propto r^3$.

2.2.5.2 Functions of Many Variables

If $Q(x, y, z, \ldots)$ is a function of the variables x, y, z,…, then it is known from differential calculus that the *differential* of the function is

$$\mathrm{d}Q = \frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z + \ldots, \tag{2.23}$$

where $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \ldots$ are the *partial derivatives* of the function $Q(x, y, z, \ldots)$ with respect to the variables x, y, z, …, respectively. (The concept of the partial derivative is simple: the partial derivative $\frac{\partial Q}{\partial x}$ of Q with respect to x is found by differentiating $Q(x, y, z, \ldots)$ with respect to x, keeping all the other variables, y, z, … constant. $\frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \ldots$ are found in a similar way.)

For small changes δx, δy, $\delta z, \ldots$ in x, y, z, …, the change in $Q(x, y, z, ..)$ is given by

$$\delta Q = \frac{\partial Q}{\partial x}\delta x + \frac{\partial Q}{\partial y}\delta y + \frac{\partial Q}{\partial z}\delta z + \ldots, \tag{2.24}$$

which contains an infinite number of terms of higher order in δx, δy, $\delta z, \ldots$, of the form $\frac{\partial^2 Q}{\partial x^2}(\delta x)^2$, $\frac{\partial^2 Q}{\partial y^2}(\delta y)^2$, $\frac{\partial^2 Q}{\partial y \partial x}(\delta x)(\delta y)$ etc., which have been omitted as negligible.

If $\delta x \equiv x_\mathrm{m} - x_0 = e_x$, $\delta y \equiv y_\mathrm{m} - y_0 = e_y$, $\delta z \equiv z_\mathrm{m} - z_0 = e_z, \ldots$ are the errors in the values of x, y, z, …, then

$$e_Q = \frac{\partial Q}{\partial x}e_x + \frac{\partial Q}{\partial y}e_y + \frac{\partial Q}{\partial z}e_z + \ldots \tag{2.25}$$

is the error in Q.

Strictly speaking, the evaluation of the partial derivatives $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \ldots$ should be done using the real values of $x, y, z, \ldots$, which are not known. However, if the fractional errors in these, $f_x, f_y, f_z, \ldots$, are small enough, the measured values $x_\mathrm{m}, y_\mathrm{m}, z_\mathrm{m}, \ldots$ may be used without introducing a significant error in the calculations. This method is adopted in most of the examples that follow.

The relations (2.24) and (2.25) are two equivalent formulations of the *principle of superposition of errors*. Its physical interpretation is evident if we consider the term $\frac{\partial Q}{\partial x}e_x$ to be the error in Q which is due to the error e_x of x etc. Of course, in the evaluation of the error in Q, due to the propagation of errors, we will find that we can only make statistical predictions for $e_Q = \delta Q$, since we can only make statistical predictions concerning the values of $e_x = \delta x$, $e_y = \delta y$, $e_z = \delta z, \ldots$. So, if δx, δy, $\delta z, \ldots$ are the *probable errors* in x, y, z, …, we will prove that the probable error in Q is

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial x}\delta x\right)^2 + \left(\frac{\partial Q}{\partial y}\delta y\right)^2 + \left(\frac{\partial Q}{\partial z}\delta z\right)^2 + \ldots}, \tag{2.26}$$

i.e. we will prove that *the probable error in Q is equal to the square root of the sum of the squares of the contributions of the probable errors in x, y, z, ... to the error in Q.*

Example 2.7

The real values of the quantities x, y and z are $x_0 = 1$, $y_0 = 2$ and $z_0 = 3$. They were measured with fractional errors $f_x = 0.01$, $f_y = -0.02$ and $f_z = 0.01$, respectively. What is the fractional error in the function $Q = 2x^2y + 5z/y$?

The real value of Q is $Q_0 = 2x_0^2y_0 + 5z_0/y_0 = 2 \times 1^2 \times 2 + 5 \times 3/2 = 11.5$. Also, the errors in x, y and z are

$$\delta x = 1 \times 0.01 = 0.01, \quad \delta y = 2 \times (-0.02) = -0.04, \quad \delta z = 3 \times 0.01 = 0.03.$$

The partial derivatives of $Q = 2x^2y + 5z/y$ are $\frac{\partial Q}{\partial x} = 4xy, \frac{\partial Q}{\partial y} = 2x^2 - \frac{5z}{y^2}, \frac{\partial Q}{\partial z} = \frac{5}{y}$. From Eq. (2.24), it is $\delta Q = \frac{\partial Q}{\partial x}\delta x + \frac{\partial Q}{\partial y}\delta y + \frac{\partial Q}{\partial z}\delta z$ and we have

$$\delta Q = 4xy\delta x + \left(2x^2 - \frac{5z}{y^2}\right)\delta y + \frac{5}{y}\delta z.$$

Substituting, we find $\delta Q = 4 \times 1 \times 2 \times 0.01 + \left(2 \times 1^2 - 5 \times \frac{3}{2^2}\right) \times (-0.04) + \frac{5}{2} \times 0.03 = 0.225$ and $f_Q = \delta Q/Q_0 = 0.225/11.5 = 0.0196 = 0.02$ or 2%.

Example 2.8

The hypotenuse of a right-angled triangle was measured to be equal to $a = 10.3$ m, with fractional error $f_a = 0.01$, and one of the triangle's angles was measured to be $B = 56.3°$, with fractional error $f_B = 0.02$. Using the results for a and B, find the values and the fractional errors of the other elements of the triangle (angle C and sides b and c).

The other elements of the triangle are given by $C = 180° - A - B$, $b = a\sin B$, $c = a\cos B$.

(a) *The other acute angle*: $C = 180° - A - B$.

We take angle A of the triangle to be a right angle and, therefore, $A = 90°$ exactly. Therefore, $C = 90° - B = 90° - 56.3° = 33.7°$
which has an error equal to $e_C = e_{90°} - e_B = 0° - 56.3° \times 0.02 = -1.1°$
and a fractional error $f_C = e_C/C_0 \approx e_C/C = -1.1°/33.7° = -0.033$ or –3.3%.

(b) *The opposite side*: $b = a\sin B$.

Since $B = 56.3°$ and $a = 10.3$ m, we have $b = a\sin B = 10.3 \times \sin 56.3° = 8.57$ m.
Here, $\frac{\partial b}{\partial a} = \sin B$ and $\frac{\partial b}{\partial B} = a\cos B$ and, therefore, $\delta b = \frac{\partial b}{\partial a}\delta a + \frac{\partial b}{\partial B}\delta B = \sin B\delta a + a\cos B\delta B$.
The error in a is $\delta a \approx 10.3 \times 0.01 = 0.10$ m.

Also, $\delta B = 56.3° \times 0.02 = 1.13°$ and therefore $\delta B = \frac{2\pi}{360°} \times 1.13° = 0.0197 = 0.020$ rad.

So, $\delta b = \sin 56.3° \times 0.103 + 10.3 \times \cos\, 56.3° \times 0.020 = 0.0857 + 0.1143 = 0.20$ m.

(c) *The adjacent side*: $c = a \cos B$.

The length of the side is $c = 10.3 \cos 56.3° = 5.71$ m.

Here $\frac{\partial c}{\partial a} = \cos B$ and $\frac{\partial c}{\partial B} = -a \sin B$ and, therefore, $\delta c = \frac{\partial c}{\partial a}\delta a + \frac{\partial c}{\partial B}\delta B = \cos B \delta a - a \sin B \delta B$.

So, $\delta c = \cos 56.3° \times 0.103 - 10.3 \times \sin 56.3° \times 0.020 = 0.0571 - 0.1714 = -0.11$ m.

Example 2.9

The acceleration of gravity g may be determined by measuring the period T of a pendulum of length l and using the relation $g = 4\pi^2 l/T^2$. In one such experiment, with a pendulum of length l = 1.000 m, whose fractional error is $f_l = -0.005$, a period of T = 2.01 s was measured, with a fractional error $f_T = 0.01$. Find the value of g and its fractional error f_g.

The value of g found by using the measured values of l and T is equal to $g = 4\pi^2 l/T^2 = 4 \times (3.1416)^2 \times 1.000/(2.01)^2 = 9.76$ m/s^2.

Because it is $\delta g = \frac{\partial g}{\partial l}\delta l + \frac{\partial g}{\partial T}\delta T = 4\pi^2\left(\frac{1}{T^2}\delta l - 2\frac{l}{T^3}\delta T\right) = 4\pi^2 \frac{l}{T^2}\left(\frac{\delta l}{l} - 2\frac{\delta T}{T}\right) = g\left(\frac{\delta l}{l} - 2\frac{\delta T}{T}\right)$, the fractional error in g is equal to

$$f_g = \frac{\delta g}{g} = \frac{\delta l}{l} - 2\frac{\delta T}{T} = f_l - 2f_T = -0.005 - 2 \times 0.01 = -0.025.$$

This is equivalent to an error in g equal to $\delta g = f_g g = -0.025 \times 9.76 = -0.24$ m/s^2.

It is worth examining the following question: What portion of the error in g is due to the fact that we have used the approximate value $\pi \approx 3.14$ instead of the exact value?

If we consider π to be a variable with error $\delta\pi = 3.14 - \pi = 3.14 - 3.14159\ldots = -0.0016$, then the contribution of $\delta\pi$ to the error δg in g will be equal to

$$\delta g_\pi = \frac{\partial g}{\partial \pi}\delta\pi = \frac{8\pi l}{T^2}\delta\pi = 2g\frac{\delta\pi}{\pi} = 2 \times 9.76 \times \frac{(-0.0016)}{3.14159} = -0.00994$$
$$= -0.01\,\text{m/s}^2,$$

which is negligible compared to the error $\delta g = -0.24$ m/s^2 due to the errors in l and T. This should have been expected, since the fractional error in π is only $f_\pi = \delta\pi/\pi = -0.0016/3.14159 = -0.0005$ or -0.05%, while the fractional errors in l and T are -0.5% and 1%, respectively.

Example 2.10

Find the relationship between the errors in the variables $x, y, z, \ldots$ and the error in the function $Q(x, y, x, \ldots) = Ax^{\alpha}y^{\beta}z^{\gamma}\ldots$, where A is a constant.

The natural logarithm of the function is $\ln Q = \ln A + \alpha \ln x + \beta \ln y + \gamma \ln z + \ldots$. From this relation, by taking differentials, we get $\frac{dQ}{Q} = \alpha\frac{dx}{x} + \beta\frac{dy}{y} + \gamma\frac{dz}{z} + \ldots$. Therefore, for small δx, δy, $\delta z, \ldots$ the approximation $\frac{\delta Q}{Q} = \alpha\frac{\delta x}{x} + \beta\frac{\delta y}{y} + \gamma\frac{\delta z}{z} + \ldots$ holds.

In Example 2.9, the function was $g = 4\pi^2 lT^{-2}$. Therefore, the last relation gives $\frac{\delta g}{g} = \frac{\delta l}{l} - 2\frac{\delta T}{T}$. Assuming that π is also a variable, it is $\frac{\delta g}{g} = 2\frac{\delta\pi}{\pi} + \frac{\delta l}{l} - 2\frac{\delta T}{T}$, in agreement with the last results.

Example 2.11

If it is difficult to differentiate an expression, the error of which we require, it is possible to use numerical methods for this purpose. We will apply this technique to the expression $F = x^{y^z}$.

Let $x = 2.00$, $y = 1.50$, $z = 1.20$ and $\delta x = 0.10$, $\delta y = -0.15$, $\delta z = 0.20$. We need to find the corresponding δF.

Given F, it is true that:

$$\frac{\partial F}{\partial x}\delta x \approx F(x+\delta x, y, z) - F(x, y, z) = (x+\delta x)^{y^z} - x^{y^z} = 2.10^{1.50^{1.20}} - 2.00^{1.50^{1.20}} = 0.2535$$

$$\frac{\partial F}{\partial y}\delta y \approx F(x, y+\delta y, z) - F(x, y, z) = x^{(y+\delta y)^z} - x^{y^z} = 2.00^{1.35^{1.20}} - 2.00^{1.50^{1.20}} = -0.3870$$

$$\frac{\partial F}{\partial z}\delta z \approx F(x, y, z+\delta z) - F(x, y, z) = x^{y^{z+\delta z}} - x^{y^z} = 2.00^{1.50^{1.40}} - 2.00^{1.50^{1.20}} = 0.3086$$

Summing, we have $\delta F = 0.175$.

Completing Sect. 2.2, we repeat that most of what were mentioned are usually of no use in arithmetical applications, since the errors we referred to are not known. The topics we examined, however, are of great theoretical importance and constitute the background for the understanding of the theory of errors, something which will become obvious in the following chapters.

Problems

2.1 When you stand on the bathroom scales, its reading is 70.5 kg. When you get off it, it shows –1.5 kg. How much do you weigh?

2.2 The fractional errors in the lengths of the sides a and b of a rectangle are –2 and 3%, respectively. Find the fractional error in its area.

2.3 The lengths of the edges of a rectangular parallelepiped have real values $a = 1$ m, $b = 2$ m and $c = 3$ m. The lengths of these edges were measured and found to be $a_m = 1.02$ m, $b_m = 1.99$ m and $c_m = 3.05$ m. Find:

(a) the volume of the parallelepiped using first a, b and c, and then a_m, b_m and c_m.
(b) the fractional error in the volume of the parallelepiped using the results of (a).
(c) the fractional errors in a, b and c.
(d) the fractional error in the volume of the parallelepiped using the formula for the evaluation of the fractional error of a compound quantity in terms of the fractional errors of the variables on which it depends. Compare with the result of (b).

2.4 Find the fractional error in $Q = x^2 y z^{-2}$ in terms of the fractional errors in the variables.
2.5 Find the fractional error in $Q = x^2 (y + 2) z^{-2}$ in terms of the fractional errors in the variables.
2.6 For the determination of the focal length f of a lens, the distances $s = 0.53$ m and $s' = 0.32$ m of the object and the image from the lens are measured and the formula $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ is used. If the errors in s and s' are $\delta s = 0.01$ m and $\delta s' = 0.02$ m, find the fractional error in the value of f calculated.
2.7 The rate of flow, $\phi = dV/dt$, of a fluid with viscosity η through a cylindrical pipe of length l and radius r is $\phi = \frac{\pi p r^4}{8 l \eta}$, where p is the pressure difference between the two ends of the pipe (Poiseuille's formula). (a) Find the fractional error in ϕ in terms of the fractional errors in η, l, r and p. (b) Which quantity must be measured with the greatest accuracy if we want to have a small error in ϕ?
2.8 The relativistic mass of a body moving with speed V, is given by the relation $m = \frac{m_0}{\sqrt{1 - V^2/c^2}}$, where m_0 is a constant of the body, known as its rest mass. If the ratio V/c is very much smaller than unity, find the fractional error in m, in terms of the fractional error in V.
2.9 The displacement x of a simple harmonic oscillator as a function of time t is given by the relation $x = a \sin(\omega t)$, where a and ω are constants. If measurements of a and ω gave the results a_m and ω_m, which have fractional errors δa and $\delta \omega$, respectively, find the fractional error in x as a function of time.

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