Chapter 3
Permanents

Introduced in 1812 by Binet and Cauchy, permanents are of interest to combinatorics, as they enumerate perfect matchings in bipartite graphs, to physics as they compute certain integrals and to computer science as they occupy a special place in the computational complexity hierarchy. This is our first example of a partition function and we demonstrate in detail how various approaches work. Connections with $\mathbb{H}$-stable polynomials lead, in particular, to an elegant proof of the van der Waerden lower bound for the permanent of a doubly stochastic matrix. Combining it with the Bregman - Minc upper bound, we show that permanents of doubly stochastic matrices are strongly concentrated. Via matrix scaling, this leads to an efficient approximation of the permanent of non-negative matrices by a function with many convenient properties: it is easily computable, log-concave and generally amenable to analysis. As an application of the interpolation method, we show how to approximate permanents of a reasonably wide class of complex matrices and also obtain approximations of logarithms of permanents of positive matrices by low degree polynomials.

3.1 Permanents

3.1.1 Permanent. Let $A = \left(a_{ij}\right)$ be an $n \times n$ real or complex matrix. The *permanent* of A is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{3.1.1.1}$$

where S_n is the symmetric group of all $n!$ permutations of the set $\{1, \ldots, n\}$.

One can see that the permanent does not change when the rows or columns of the matrix are permuted and that per A is linear in each row and each column of A. Moreover, if $n > 1$, then denoting by A_j the $(n-1) \times (n-1)$ matrix obtained from

A. Barvinok, *Combinatorics and Complexity of Partition Functions*,
Algorithms and Combinatorics 30, DOI 10.1007/978-3-319-51829-9_3

A by crossing out the first row and the j-th column, we obtain the row expansion

$$\operatorname{per} A = \sum_{j=1}^{n} a_{1j} \operatorname{per} A_j. \tag{3.1.1.2}$$

3.1.2 Permanents and perfect matchings. If A is a real matrix and $a_{ij} \in \{0, 1\}$ for all i, j then per A has a combinatorial interpretation as the number of perfect matchings in a *bipartite graph* G with *biadjacency matrix* A. Namely, the vertices of G are $1L, 2L \ldots, nL$ and $1R, 2R, \ldots, nR$ ("L" is for "left" and "R" is for "right"), whereas the edges of G are all unordered pairs $\{iL, jR\}$ for which $a_{ij} = 1$. A *perfect matching* in a graph G is a collection of edges which contain every vertex of G exactly once, see Fig. 3.1.

In this case, per A is the number of perfect matchings in G, since every perfect matching in G corresponds to a unique permutation σ such that $a_{i\sigma(i)} = 1$ for all $i = 1, \ldots, n$. For example, Fig. 3.1 pictures a graph encoded by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.1.2.1}$$

and a perfect matching corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \tag{3.1.2.2}$$

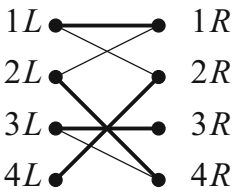


Fig. 3.1 A bipartite graph and a perfect matching (thick edges)

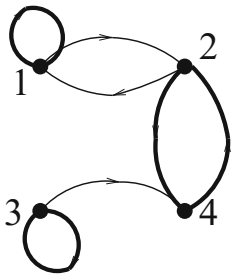


Fig. 3.2 A graph and a cycle cover (thick edges)

3.1.3 Permanents and cycle covers. A different interpretation of the permanent of a 0–1 matrix A arises if we interpret A as the *adjacency matrix* of a directed graph G. In this case, the vertices of G are $1, \ldots, n$ whereas the edges of G are all ordered pairs (i, j) such that $a_{ij} = 1$ (in particular, we allow loops). A *cycle cover* of G is a collection of edges which contain every vertex of G exactly once as the beginning point of an edge and exactly once as an endpoint of an edge, see Fig. 3.2.

In this case, per A is the number of cycle covers of G, since every cycle cover of G corresponds to a unique permutation σ such that $a_{i\sigma(i)} = 1$ for all $i = 1, \ldots, n$. For example, Fig. 3.2 pictures a graph encoded by the matrix (3.1.2.1) and a cycle cover corresponding to the permutation (3.1.2.2).

Interpretations of Sects. 3.1.2 and 3.1.3 explain why permanents are of interest to combinatorics, see [LP09] for more.

3.1.4 Permanents as integrals. Let μ_n be the Gaussian probability measure on the complex vector space $\mathbb{C}^n$ with density

$$\frac{1}{\pi^n} e^{-\|z\|^2} \quad \text{where} \quad \|z\|^2 = |z_1|^2 + \ldots + |z_n|^2 \quad \text{for} \quad z = (z_1, \ldots, z_n)\,.$$

The measure μ_n is normalized in such a way that

$$\mathbf{E}\,|z_i|^2 = 1 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \mathbf{E}\, z_i \overline{z_j} = 0 \quad \text{for} \quad i \neq j.$$

Let $f_1, \ldots, f_n; g_1, \ldots, g_n : \mathbb{C}^n \longrightarrow \mathbb{C}$ be linear forms and let us define an $n \times n$ matrix $A = \big(a_{ij}\big)$ by

$$a_{ij} = \mathbf{E}\, f_i \overline{g_j} = \int_{\mathbb{C}^n} f_i(z) \overline{g_j(z)}\, d\mu_n \quad \text{for all} \quad i, j.$$

Then

$$\mathbf{E}\ (f_1 \cdots f_n \overline{g_1 \cdots g_n}) = \operatorname{per} A. \tag{3.1.4.1}$$

Formula (3.1.4.1) is known as (a version of) *Wick's formula*, see for example, [Zv97] and [Gu04]. To prove it, we note that both sides of (3.1.4.1) are linear in each f_i and antilinear in each g_j. Namely, denoting the left hand side of (3.1.4.1) by $L\,(f_1, \ldots, f_n; g_1, \ldots, g_n)$ and the right hand side by $R\,(f_1, \ldots, f_n; g_1, \ldots, g_n)$, we observe that

$$\begin{aligned} &L\left(f_1, \ldots, f_{i-1}, \alpha_1 f_i' + \alpha_2 f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \\ &\quad = \alpha_1 L\left(f_1, \ldots, f_{i-1}, f_i', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \\ &\qquad\quad + \alpha_2 L\left(f_1, \ldots, f_{i-1}, f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \\ &\qquad \text{and} \\ &R\left(f_1, \ldots, f_{i-1}, \alpha_1 f_i' + \alpha_2 f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \end{aligned}$$

$$= \alpha_1 R\left(f_1, \ldots, f_{i-1}, f_i', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) + \alpha_2 R\left(f_1, \ldots, f_{i-1}, f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right)$$

as well as

$$\begin{aligned} &L\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, \alpha_1 g_i' + \alpha_2 g_i'', g_{i+1} \ldots, g_n\right) \\ &\quad = \overline{\alpha_1} L\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i', g_{i+1}, \ldots, g_n\right) \\ &\qquad + \overline{\alpha_2} L\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i'', g_{i+1}, \ldots, g_n\right) \\ &\qquad \text{and} \\ &R\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, \alpha_1 g_i' + \alpha_2 g_i'', g_{i+1}, \ldots, g_n\right) \\ &\quad = \overline{\alpha_1} R\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i', g_{i+1}, \ldots, g_n\right) \\ &\qquad + \overline{\alpha_2} R\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i'', g_{i+1}, \ldots, g_n\right). \end{aligned}$$

Hence it suffices to check (3.1.4.1) when each f_i and g_j is a coordinate function. Suppose therefore that

$$(f_1, \ldots, f_n) = \left(\underbrace{z_1, \ldots, z_1}_{m_1 \text{ times}}, \ldots, \underbrace{z_n, \ldots, z_n}_{m_n \text{ times}}\right) \quad \text{and}$$

$$(g_1, \ldots, g_n) = \left(\underbrace{z_1, \ldots, z_1}_{k_1 \text{ times}}, \ldots, \underbrace{z_n, \ldots, z_n}_{k_n \text{ times}}\right),$$

where $m_1, \ldots, m_n$ and $k_1, \ldots, k_n$ are non-negative integers such that

$$m_1 + \ldots + m_n = k_1 + \ldots + k_n = n.$$

If we have $m_i \neq k_i$ for some i then the left hand side of (3.1.4.1) is 0 since

$$\mathbf{E}\, z_i^{m_i} \overline{z_i^{k_i}} = 0 \quad \text{provided} \quad m_i \neq k_i.$$

On the other hand, the right hand side of (3.1.4.1) is also 0. Indeed, without loss of generality, we may assume that $m_i > k_i$. The matrix A contains an $m_i \times (n - k_i)$ block of 0s and if $m_i > k_i$ each of the $n!$ terms of (3.1.1.1) contains and least one entry from that block and hence is 0. Thus it remains to prove (3.1.4.1) in the case when $m_i = k_i$ for all $i = 1, \ldots, n$. Since

$$\mathbf{E}\, z_i^{m_i} \overline{z_i^{m_i}} = m_i!,$$

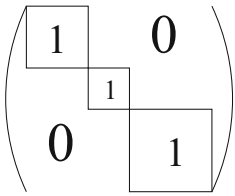


Fig. 3.3 The structure of matrix A

we conclude that the left hand side of (3.1.4.1) is $m_1! \cdots m_n!$. The matrix A in this case consists of the diagonal blocks filled by 1 s of sizes $m_1, \ldots, m_n$, see Fig. 3.3, and hence the right hand side of (3.1.4.1) is also $m_1! \cdots m_n!$. □

One immediate corollary of (3.1.4.1) is that

$$\text{per } A \ \geq \ 0 \quad \text{provided } A \text{ is Hermitian positive semidefinite.} \tag{3.1.4.2}$$

Indeed, any such $A = \big(a_{ij}\big)$ can be written as

$$a_{ij} = \mathbf{E}\left(f_i \overline{f_j}\right) \quad \text{for all} \quad i, j$$

and some linear forms $f_1, \ldots, f_n$, in which case by (3.1.4.1) we have

$$\text{per } A = \mathbf{E}\left(f_1 \cdots f_n \overline{f_1} \cdots \overline{f_n}\right) = \mathbf{E}\left(|f_1|^2 \cdots |f_n|^2\right) \geq 0.$$

The identity of Sect. 3.1.4 has some relevance to statistics of bosons in quantum physics, see, for example, [AA13] and [Ka16].

3.1.5 Permanents in computational complexity. Permanents occupy a special place in the theory of computational complexity. Valiant [Va79] proved that computing permanents of 0–1 matrices exactly (that is, counting perfect matchings in bipartite graphs exactly) is an example of a **#P**-complete problem, that is, counting perfect matchings in bipartite graphs in polynomial time exactly would lead to a polynomial time counting of the number of acceptable computations of a general non-deterministic polynomial time Turing machine, see also [AB09] and [Go08]. This is especially striking since finding whether there exists a perfect matching in a given bipartite graph is a famous problem solvable in polynomial time, see for example, [LP09]. Exact computation of permanents of 0–1 matrices leads by interpolation to exact computation of permanents of matrices with 0 and ± 1 entries and those turn out to be sufficient to encode rather involved computations. In the algebraic complexity theory, permanents stand out as universal polynomials, see Part 5 of [B+97].

Permanents also stand out as an example of the problem where randomized algorithms so far substantially outperform deterministic algorithms. The Monte Carlo Markov Chain algorithm of Jerrum, Sinclair and Vigoda [J+04] approximates permanents of non-negative matrices in polynomial time and none of the deterministic algorithms could achieve that so far, see also Sects. 3.7 and 3.9 below.

3.2 Permanents of Non-negative Matrices and $\mathbb{H}$-Stable Polynomials

3.2.1 Permanents and products of linear forms. Let $A = (a_{ij})$ be an $n \times n$ matrix and let $z_1, \ldots, z_n$ be complex variables. The following simple formula has many important consequences:

$$\operatorname{per} A = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right). \tag{3.2.1.1}$$

In other words, per A is the coefficient of $z_1 \cdots z_n$ in the product (3.2.1.1) of linear forms.

We note that if $A = (a_{ij})$ is a non-negative real matrix with non-zero rows, then the polynomial

$$f(z_1, \ldots, z_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right)$$

is $\mathbb{H}$-stable, see Sect. 2.4, since

$$\Im \left(\sum_{j=1}^{n} a_{ij} z_j \right) > 0 \quad \text{provided} \quad \Im z_j > 0 \quad \text{for} \quad j = 1, \ldots, n.$$

More generally, let $a_1, \ldots, a_n$ be the columns of A, so that $A = [a_1, \ldots, a_n]$. Given a non-negative integer vector $m = (m_1, \ldots, m_n)$ such that $m_1 + \ldots + m_n = n$, let

$$A_m = \left[\underbrace{a_1, \ldots, a_1}_{m_1 \text{ times}}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k \text{ times}}, \ldots, \underbrace{a_n, \ldots, a_n}_{m_n \text{ times}} \right]$$

be the $n \times n$ matrix with columns consisting of m_k copies of a_k for $k = 1, \ldots, n$. Then

$$\frac{\partial^n}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right) = \operatorname{per} A_m \tag{3.2.1.2}$$

(if $m_k = 0$ for some k then the corresponding partial derivative is missing and so are the copies of a_k in A_m). Indeed, the left hand side of (3.2.1.2) is the coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in the product of linear forms

$$f_i(z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{ij} z_j,$$

multiplied by $m_1! \cdots m_n!$. Hence the left hand side of (3.2.1.2) can be written as

$$\int_{\mathbb{C}^n} f_1 \cdots f_n \overline{z_1^{m_1} \cdots z_n^{m_n}}\, d\mu_n,$$

for the Gaussian measure μ_n of Sect. 3.1.4, and (3.2.1.2) follows by (3.1.4.1).

3.2.2 Alexandrov - Fenchel inequalities. One immediate application of (3.2.1.1) and (3.2.1.2) is an inequality for permanents of non-negative matrices, which is a particular case of the Alexandrov - Fenchel inequality for mixed volumes of convex bodies, see, for example, [Sa93].

Let $[a_1, \ldots, a_n]$ denote the $n \times n$ matrix with non-negative real columns $a_1, \ldots, a_n$. Then

$$\mathrm{per}^2[a_1, \ldots, a_n] \;\geq\; \mathrm{per}[a_1, a_1, a_3, \ldots, a_n]\,\mathrm{per}[a_2, a_2, a_3, \ldots, a_n]. \qquad (3.2.2.1)$$

By continuity, it suffices to prove (3.2.2.1) assuming that the coordinates of $a_1, \ldots, a_n$ are strictly positive. Let $a_{ij} > 0$ be the i-th coordinate of a_j. Then, from Sect. 3.2.1, the polynomial

$$f(z_1, \ldots, z_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right)$$

is $\mathbb{H}$-stable. Let

$$g(z_1, z_2) = \frac{\partial^{n-2}}{\partial z_3 \cdots \partial z_n} f = u z_1^2 + 2 v z_1 z_2 + w z_2^2.$$

Using (3.2.1.2) we observe that

$$u = \frac{1}{2}\,\mathrm{per}[a_1, a_1, a_3, \ldots, a_n], \quad v = \frac{1}{2}\,\mathrm{per}[a_1, \ldots, a_n] \quad \text{and}$$
$$w = \frac{1}{2}\,\mathrm{per}[a_2, a_2, a_3, \ldots, a_n].$$

By the repeated application of Part (3) of Lemma 2.4.2, the quadratic polynomial q is $\mathbb{H}$-stable, which implies that $v^2 \geq uw$ and we get (3.2.2.1). Indeed, if $v^2 < uw$ then the univariate polynomial $t \longmapsto u + 2vt + wt^2$ has a pair of complex conjugate roots $\alpha \pm \beta i$ for some $\beta > 0$. Then, for any $\epsilon > 0$, the point $z_1 = 1 + i\epsilon, z_2 = (\alpha + \beta i)(1 + i\epsilon)$ is a root of $q(z_1, z_2)$ and if $\epsilon > 0$ is sufficiently small, we have $\Im z_2 = \alpha\epsilon + \beta > 0$, which contradicts the $\mathbb{H}$-stability of q.

The connection of (3.2.2.1) to the Alexandrov - Fenchel inequality for mixed volumes is as follows. Let $K_1, \ldots, K_n \subset \mathbb{R}^n$ be convex bodies and let $\lambda_1, \ldots, \lambda_n$ be positive real numbers. We consider a combination $\lambda_1 K_1 + \ldots + \lambda_n K_n$, where

$$\lambda K = \{\lambda x : x \in K\}$$

is the dilation/contraction by a factor of λ and "+" stands for the Minkowski sum of convex bodies:

$$A + B = \{x + y : x \in A, y \in B\}.$$

As is known, the volume vol $(\lambda_1 K_1 + \ldots + \lambda_n K_n)$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ and its coefficient

$$V(K_1, \ldots, K_n) = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \text{vol}\,(\lambda_1 K_1 + \ldots + \lambda_n K_n)$$

is called the *mixed volume* of $K_1, \ldots, K_n$. The Alexandrov - Fenchel inequality asserts that

$$V^2(K_1, \ldots, K_n) \geq V(K_1, K_1, K_3, \ldots, K_n) V(K_2, K_2, K_3, \ldots, K_n). \quad (3.2.2.2)$$

We obtain (3.2.2.1), if we choose K_j to be the parallelepiped, that is the direct product of axis-parallel intervals:

$$K_j = [0,\ a_{1j}] \times \ldots \times [0,\ a_{nj}].$$

In this case $\lambda_1 K_1 + \ldots + \lambda_n K_n$ is the parallelepiped

$$\left[0,\ \sum_{j=1}^{n} a_{1j}\lambda_j\right] \times \ldots \times \left[0,\ \sum_{j=1}^{n} a_{nj}\lambda_j\right],$$

cf. Fig. 3.4,
so that

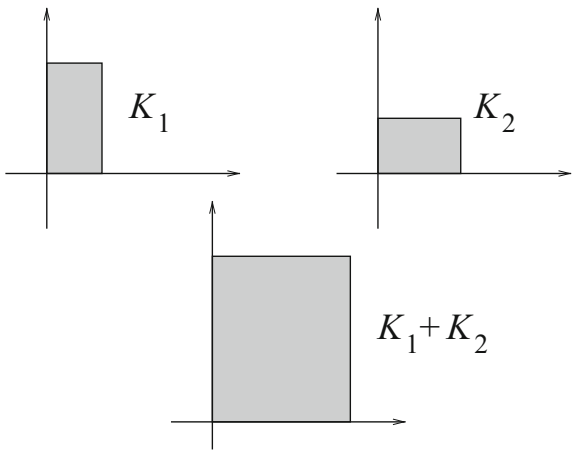


Fig. 3.4 Parallelepipeds K_1, K_2 and their Minkowski sum $K_1 + K_2$

$$\operatorname{vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \lambda_j \right)$$

and

$$V(K_1, \ldots, K_n) = \operatorname{per} A \quad \text{where} \quad A = \left(a_{ij}\right),$$

We note that for general convex bodies $K_1, \ldots, K_n$, the polynomial $\operatorname{vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$ does not have to be $\mathbb{H}$-stable, cf. [Kh84].

3.3 The van der Waerden Inequality and Its Extensions

3.3.1 Doubly stochastic matrices. A real $n \times n$ matrix $A = \left(a_{ij}\right)$ is called *doubly stochastic* if

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for} \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} a_{ij} = 1 \quad \text{for} \quad j = 1, \ldots, n$$

and

$$a_{ij} \geq 0 \quad \text{for all} \quad i, j.$$

In words: a matrix is doubly stochastic if it is non-negative real with all row and column sums equal 1.

Clearly, permutation matrices (matrices, containing in each row and column exactly one non-zero entry equal to 1) are doubly stochastic, as well as the matrix

$$\frac{1}{n} J_n,$$

where J_n is the $n \times n$ matrix of all 1s.

The main goal of this section is to prove the following result, known as the van der Waerden conjecture.

3.3.2 Theorem. *Let A be an $n \times n$ doubly stochastic matrix. Then*

$$\operatorname{per} A \ \geq \ \frac{n!}{n^n}.$$

Moreover, the equality is attained if and only if $A = \dfrac{1}{n} J_n$.

Theorem 3.3.2 was first proved by Falikman [Fa81] and Egorychev [Eg81] (earlier Friedland [Fr79] proved a slightly weaker bound $\operatorname{per} A \ \geq \ e^{-n}$). Our exposition

follows Gurvits' paper [Gu08] with some simplifications introduced in [Wa11] and [LS10]. We use the notion of capacity, see Sect. 2.1.5, Theorem 2.4.3 and Corollary 2.4.6.

3.3.3 Lemma. *Let $A = \left(a_{ij}\right)$ be an $n \times n$ doubly stochastic matrix and let*

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right).$$

Then

$$\inf_{x_1, \ldots, x_n > 0} \frac{p\,(x_1, \ldots, x_n)}{x_1 \cdots x_n} = 1.$$

Proof. Clearly, $p(1, \ldots, 1) = 1$ and hence the infimum does not exceed 1. On the other hand, using the arithmetic-geometric mean inequality, see Sect. 2.1.1.1, we conclude that for $x_1, \ldots, x_n > 0$ we get

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \geq \prod_{i=1}^{n} \left(\prod_{j=1}^{n} x_j^{a_{ij}} \right) = \prod_{j=1}^{n} \left(\prod_{i=1}^{n} x_j^{a_{ij}} \right) = \prod_{j=1}^{n} \left(x_j^{\sum_{i=1}^{n} a_{ij}} \right) = \prod_{j=1}^{n} x_j$$

and hence the infimum is at least 1. □

To prove the van der Waerden inequality, we use $\mathbb{H}$-stability, see Sect. 3.2.

3.3.4 Proof of Theorem 3.3.2. As in Sect. 3.2.1, we define a polynomial $p = p_A$ in n variables $x_1, \ldots, x_n$:

$$p\,(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right).$$

As we discussed in Sect. 3.2.1, the polynomial p is $\mathbb{H}$-stable and hence by Corollary 2.4.6, we have

$$\frac{\partial^n p}{\partial x_1 \cdots \partial x_n} \geq \frac{n!}{n^n} \inf_{x_1, \ldots, x_n > 0} \frac{p\,(x_1, \ldots, x_n)}{x_1 \cdots x_n}. \tag{3.3.4.1}$$

By (3.2.1.1), the left hand side of (3.3.4.1) is per A, while by Lemma 3.3.3, the infimum in the right hand side of (3.3.4.1) is 1.

In the uniqueness proof, we follow [LS10]. Suppose now that A is a doubly stochastic matrix such that $\operatorname{per} A = n!/n^n$. Then inequality (3.3.4.1) is, in fact, equation. Analyzing the proof of Theorem 2.4.3 in Sect. 2.4.5, we conclude that for

$$q(x_1, \ldots, x_{n-1}) = \frac{\partial}{\partial x_n} \left(\prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \right) \Bigg|_{x_n = 0} = \sum_{k=1}^{n} a_{kn} \prod_{i : i \neq k} \left(\sum_{j=1}^{n-1} a_{ij} x_j \right),$$

we must have

$$\inf_{x_1,\ldots,x_{n-1}>0} \frac{q\,(x_1,\ldots,x_{n-1})}{x_1\cdots x_{n-1}} = \left(\frac{n-1}{n}\right)^{n-1}. \tag{3.3.4.2}$$

Applying the arithmetic-geometric mean inequality, see Sect. 2.1.1.1, we conclude that for all $x_1 > 0, \ldots, x_{n-1} > 0$, we get

$$\begin{aligned} q(x_1,\ldots,x_{n-1}) &\geq \prod_{k=1}^{n}\prod_{i:i\neq k}\left(\sum_{j=1}^{n-1} a_{ij}x_j\right)^{a_{kn}} = \prod_{i=1}^{n}\prod_{k:k\neq i}\left(\sum_{j=1}^{n-1} a_{ij}x_j\right)^{a_{kn}} \\ &= \prod_{i=1}^{n}\left(\sum_{j=1}^{n-1} a_{ij}x_j\right)^{1-a_{in}}. \end{aligned}$$

Using the arithmetic-geometric mean inequality again, we conclude that for all $x_1 > 0, \ldots, x_{n-1} > 0$, we have

$$\begin{aligned} q(x_1,\ldots,x_{n-1}) &\geq \prod_{i=1}^{n}\left((1-a_{in})\sum_{j=1}^{n-1}\frac{a_{ij}}{1-a_{in}}x_j\right)^{1-a_{in}} \\ &\geq \prod_{i=1}^{n}\left((1-a_{in})^{1-a_{in}}\prod_{j=1}^{n-1}x_j^{a_{ij}}\right) \\ &= \left(\prod_{i=1}^{n}(1-a_{in})^{1-a_{in}}\right)\left(\prod_{j=1}^{n-1}x_j\right). \end{aligned}$$

Therefore,

$$\inf_{x_1,\ldots,x_{n-1}>0} \frac{q\,(x_1,\ldots x_{n-1})}{x_1\ldots x_{n-1}} \geq \prod_{i=1}^{n}(1-a_{in})^{1-a_{in}}.$$

By (3.3.4.2), we must have

$$\prod_{i=1}^{n}(1-a_{in})^{1-a_{in}} \leq \left(\frac{n-1}{n}\right)^{n-1}. \tag{3.3.4.3}$$

Now, since the function $t \longmapsto t\ln t$ is strictly convex for $t > 0$, see Sect. 2.1.1.2, we conclude that

$$\frac{1}{n}\sum_{i=1}^{n} t_i \ln t_i \geq \frac{t_1+\ldots+t_n}{n}\ln\frac{t_1+\ldots+t_n}{n}$$

for all $t_1, \dots, t_n$ with equality if and only if $t_1 = \dots = t_n$. Applying it with $t_i = 1 - a_{in}$, we get

$$\frac{1}{n}\sum_{i=1}^{n}(1-a_{in})\ln(1-a_{in}) \;\geq\; \frac{n-1}{n}\ln\frac{n-1}{n}$$

with equality if and only if $a_{in} = 1/n$ for $i = 1, \dots, n$. In other words,

$$\prod_{i=1}^{n}(1-a_{in})^{1-a_{in}} \;\geq\; \left(\frac{n-1}{n}\right)^{n-1}$$

with equality if and only if $a_{in} = 1/n$ for $i = 1, \dots, n$. Comparing this with (3.3.4.3), we conclude that if per $A = n!/n$, we must have $a_{in} = 1/n$ for $i = 1, \dots, n$. Since the matrix obtained from a doubly stochastic matrix by a permutation of columns remains doubly stochastic with the same permanent, we conclude that $a_{ij} = 1/n$ for all i and j as desired. □

3.3.5 Sharpening. Suppose that A is a doubly stochastic matrix and that, additionally, the j-th column of A contains not more than k_j non-zero entries for some $1 \leq k_j \leq n$ and $j = 1, \dots, n$. Using Theorem 2.4.3, we obtain

$$\operatorname{per} A = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p \;\geq\; \prod_{j=1}^{n}\left(\frac{k_j-1}{k_j}\right)^{k_j-1} \tag{3.3.5.1}$$

or, even sharply,

$$\operatorname{per} A \;\geq\; \prod_{j=1}^{n}\left(\frac{\min\{j,k_j\}-1}{\min\{j,k_j\}}\right)^{\min\{j,k_j\}-1}, \tag{3.3.5.2}$$

where the corresponding factor is 1 if $\min\{j, k_j\} = 1$. Inequalities (3.3.5.1) and (3.3.5.2) are also due to Gurvits [Gu08]. In the case when all $k_j = 3$ for all j, the inequality (3.3.5.2) was obtained by Voorhoeve [Vo79] and in the case when all k_j are equal, the inequality (3.3.5.1) was obtained by Schrijver [Sc98]. In the case of all k_j equal, we will give a different proof of (3.3.5.1) in the particular case when the non-zero entries of A are $1/k$ in Theorem 5.3.6, where we also show, following Csikvári [Cs14], that asymptotically, as n grows, the bound is logarithmically exact.

3.4 The Bregman–Minc Inequality and Its Corollaries

The following inequality was conjectured by Minc, cf. [Mi78], and proved by Bregman [Br73]. We follow the approach of Radhakrishnan [Ra97], only using the language of partitions instead that of random variables.

3.4.1 Theorem. *Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} \in \{0, 1\}$ for all i, j. Let*

$$r_i = \sum_{j=1}^{n} a_{ij}$$

be the number of 1 s in the ith row of A. Then

$$\operatorname{per} A \ \leq \ \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Let us define

$$\Omega = \{\sigma \in S_n : \ a_{i\sigma(i)} = 1 \quad \text{for} \quad i = 1, \ldots, n\}.$$

Hence

$$\operatorname{per} A = |\Omega|.$$

Without loss of generality, we assume that $\Omega \neq \emptyset$, in which case we consider Ω as a probability space with uniform measure.

We start with a probabilistic argument.

3.4.2 Lemma. *Let us fix a permutation $\sigma \in \Omega$ and an index $1 \leq i \leq n$. Let us choose a permutation $\tau \in S_n$ uniformly at random, find k such that $\tau(k) = i$ and cross out from A the columns indexed by $\sigma(\tau(1)), \ldots, \sigma(\tau(k-1))$. Let x be the number of 1 s remaining in the ith row of A after the columns are crossed out. Then*

$$\mathbf{Pr}\,(x = a) = \frac{1}{r_i} \quad \textit{for} \quad a = 1, \ldots, r_i.$$

Proof. Let J be the set of indices of columns where the ith row of A contains 1 and let $I = \sigma^{-1}(J)$. Then $i \in I$ and x is the number of indices in $\tau^{-1}(I)$ that are greater than or equal to $k = \tau^{-1}(i)$. Since $\tau \in S_n$ is chosen uniformly at random, $\tau^{-1}(i)$ is equally probable to be the largest, second largest, etc. element of $\tau^{-1}(I)$. □

3.4.3 Proof of Theorem 3.4.1

For a permutation $\tau \in S_n$ we construct a family of partitions

$$\mathcal{F}_{\tau,0} \preceq \mathcal{F}_{\tau,1} \preceq \ldots \preceq \mathcal{F}_{\tau,n}$$

of Ω as follows. We let $\mathcal{F}_{\tau,0} = \{\Omega\}$. The partition $\mathcal{F}_{\tau,1}$ consists of the events

$$F_i = \big\{\sigma \in \Omega : \ \sigma(\tau(1)) = i\big\} \quad \text{for} \quad i = 1, \ldots, n$$

(note that not more than $r_{\tau(1)}$ of the events F_i are non-empty). Generally, the partition $\mathcal{F}_{\tau,k}$ consists of the events

$$F_{i_1,\dots,i_k} = \{\sigma \in \Omega : \ \sigma(\tau(1)) = i_1, \dots, \sigma(\tau(k)) = i_k\}$$
$$\text{for distinct} \quad 1 \le i_1, \dots, i_k \le n$$

(again, some of the events can be empty). In particular, the non-empty events in $\mathcal{F}_{\tau,n}$ are singletons. From (2.1.2.4), using that $H(\{\Omega\}) = 0$ and $H(\{\mathcal{F}_{\tau,n}\}) = \ln|\Omega|$, we obtain

$$\ln|\Omega| = \sum_{k=1}^{n} H(\mathcal{F}_{\tau,k}|\mathcal{F}_{\tau,k-1}).$$

Averaging over all $\tau \in S_n$, we obtain

$$\ln|\Omega| = \frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H(\mathcal{F}_{\tau,k}|\mathcal{F}_{\tau,k-1}). \tag{3.4.3.1}$$

For a permutation $\sigma \in \Omega$, let $F_{\tau,k-1}(\sigma)$ be the block of $\mathcal{F}_{\tau,k-1}$ that contains σ. We consider $F_{\tau,k-1}(\sigma)$ as a probability space with conditional probability measure and let $\mathcal{F}_{\tau,k-1}(\sigma)$ be the partition of that space by the events of $\mathcal{F}_{\tau,k}$. Then

$$H(\mathcal{F}_{\tau,k}|\mathcal{F}_{\tau,k-1}) = \sum_{\sigma\in\Omega} \mathbf{Pr}(\sigma) H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big),$$

cf. (2.1.2.3), and by (3.4.3.1) we have

$$\begin{aligned}\ln|\Omega| &= \frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n}\sum_{\sigma\in\Omega} \mathbf{Pr}(\sigma) H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big)\\ &= \sum_{\sigma\in\Omega} \mathbf{Pr}(\sigma)\frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big).\end{aligned} \tag{3.4.3.2}$$

We fix an arbitrary $\sigma \in \Omega$ and consider the sum

$$\frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big). \tag{3.4.3.3}$$

Recall that $\mathcal{F}_{\tau,k-1}(\sigma)$ is the partition of the probability space Ω consisting of all permutations $\pi \in \Omega$ such that $\pi(\tau(1)) = \sigma(\tau(1)), \dots, \pi(\tau(k-1)) = \sigma(\tau(k-1))$ into the events defined by the choice of $\pi(\tau(k))$. We rearrange (3.4.3.3) in accordance with the value of $i = \tau(k)$:

$$\frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big) = \sum_{i=1}^{n}\frac{1}{n!}\sum_{\tau\in S_n} H\big(\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)\big) \tag{3.4.3.4}$$

and consider each term

$$\frac{1}{n!}\sum_{\tau\in S_n} H\left(\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)\right) \tag{3.4.3.5}$$

separately.

Now, the partition $\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)$ looks as follows. We fixed $\sigma\in\Omega$ and $1\le i\le n$. For the permutation τ, we find k such that $\tau(k)=i$, consider the probability space of all permutations $\pi\in\Omega$ such that $\pi(\tau(1))=\sigma(\tau(1)),\dots,\pi(\tau(k-1))=\sigma(\tau(k-1))$ endowed with uniform probability measure and partition it according to the value of $\pi(i)$. By (2.1.2.2),

$$H\left(\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)\right) \le \ln a \quad \text{provided } \mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma) \text{ contains } a \text{ events.}$$

By Lemma 3.4.2, the value of (3.4.3.5) does not exceed

$$\frac{1}{r_i}\sum_{a=1}^{r_i}\ln a = \frac{1}{r_i}\ln(r_i!).$$

Then by (3.4.3.4), the value of (3.4.3.3) does not exceed

$$\sum_{i=1}^{n}\frac{1}{r_i}\ln(r_i!).$$

By (3.4.3.2), we get

$$\ln|\Omega| \le \sum_{i=1}^{n}\frac{1}{r_i}\ln(r_i!),$$

and the proof follows. □

3.4.4 Remark. Let J_r be the $r\times r$ matrix filled with 1s. If A is a block-diagonal matrix with blocks $J_{r_1},\dots,J_{r_m}$, then

$$\operatorname{per} A = \prod_{i=1}^{m} r_i!,$$

from which it follows that the bound of Theorem 3.4.1 is sharp.

Theorem 3.4.1 allows us to bound permanents of stochastic matrices.

3.4.5 Corollary. *Suppose that $A=\left(a_{ij}\right)$ is an $n\times n$ stochastic matrix, that is, $a_{ij}\ge 0$ for all i, j and*

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \textit{for all} \quad i = 1, \ldots, n. \tag{3.4.5.1}$$

Suppose that

$$a_{ij} \;\le\; \frac{1}{b_i} \quad \textit{for all} \quad i, j \tag{3.4.5.2}$$

and some positive integers $b_1, \ldots, b_n$. *Then*

$$\operatorname{per} A \;\le\; \prod_{i=1}^{n} \frac{(b_i!)^{1/b_i}}{b_i}.$$

Proof. Let us fix all but the i-th row of an $n \times n$ matrix A and allow the ith row vary. Then per A is a linear function in the i-th row $a_i = (a_{i1}, \ldots, a_{in})$. Let us consider the polytope P_i of all n-vectors $a_i = (a_{i1}, \ldots, a_{in})$ such that all entries a_{ij} are non-negative and the conditions (3.4.5.1) and (3.4.5.2) are met. By linearity, the maximum value of per A on P_i is attained at a vertex of P_i, in which case we necessarily have $a_{ij} \in \{0, 1/b_{ij}\}$ for $j = 1, \ldots, n$. Indeed, if $0 < a_{ij_1} < 1/b_i$ for some j_1 then there is another $j_2 \neq j_1$ such that $0 < a_{ij_2} < 1/b_i$ (recall that b_i is an integer). In that case, we can write $a_i = \left(a_i^1 + a_i^2\right)/2$, where a_i^1 is obtained from a_i by the perturbation $a_{ij_1} := a_{ij_1} + \epsilon$, $a_{ij_2} := a_{ij_2} - \epsilon$ and a_i^2 is obtained from a_i by the perturbation $a_{ij_1} := a_{ij_1} - \epsilon$, $a_{ij_2} := a_{ij_2} + \epsilon$ for a sufficiently small $\epsilon > 0$, which implies that a_i is not a vertex of P_i.

Hence we conclude that the maximum of per A on the set of $n \times n$ non-negative matrices $A = \left(a_{ij}\right)$ satisfying (3.4.5.1) and (3.4.5.2) is attained when $a_{ij} \in \{0, 1/b_{ij}\}$ for all i, j. Let B be the matrix obtained from such a matrix A by multiplying the i-th row by b_i. Then

$$\operatorname{per} B = \left(\prod_{i=1}^{n} \frac{1}{b_i}\right) \operatorname{per} A \quad \text{and} \quad \operatorname{per} B \;\le\; \prod_{i=1}^{n} (b_i!)^{1/b_i}$$

by Theorem 3.4.1. □

The author learned Corollary 3.4.5 and its proof from A. Samorodnitsky [Sa01], see also [So03] for a somewhat more general statement with b_i not required to be integer.

3.4.6 Concentration of the permanent of doubly stochastic matrices. The van der Waerden bound (Theorem 3.3.2) together with the Bregman - Minc bound (Corollary 3.4.5) implies that per A does not vary much if A is a doubly stochastic matrix with small entries. Indeed, suppose that A is an $n \times n$ doubly stochastic matrix. Then, by Theorem 3.3.2,

$$\operatorname{per} A \;\ge\; \frac{n!}{n^n} \;\ge\; e^{-n}.$$

Let us fix an $\alpha \ge 1$ and suppose that, additionally,

$$a_{ij} \leq \frac{\alpha}{n} \quad \text{for all} \quad i, j.$$

Let

$$b = \left\lfloor \frac{n}{\alpha} \right\rfloor,$$

so that

$$a_{ij} \leq \frac{1}{b} \quad \text{for all} \quad i, j$$

and by Corollary 3.4.5,

$$\text{per } A \leq \left(\frac{(b!)^{1/b}}{b} \right)^n = e^{-n} n^{O(\alpha)}.$$

Hence if the entries of an $n \times n$ doubly stochastic matrix are within a constant factor of each other, the permanent of the matrix varies within a polynomial in n factor.

In fact,

$$\prod_{i,j=1}^{n} \left(1 - a_{ij}\right)^{1-a_{ij}} \leq \text{per } A \leq 2^n \prod_{i,j=1}^{n} \left(1 - a_{ij}\right)^{1-a_{ij}} \tag{3.4.6.1}$$

for any $n \times n$ doubly stochastic matrix A (if $a_{ij} = 1$ the corresponding factor is 1), where the lower bound is due to Schrijver [Sc98] and the upper bound was recently established by Gurvits and Samorodnitsky [GS14], who also conjectured that the upper bound holds with 2^n replaced by $2^{n/2}$.

The following useful inequality was conjectured by Vontobel [Vo13] and deduced by Gurvits [Gu11] from the lower bound in (3.4.6.1)

Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive real matrix and let $B = \left(b_{ij}\right)$ be an $n \times n$ doubly stochastic matrix.Then

$$\ln \text{per } A \geq \sum_{i,j=1}^{n} b_{ij} \ln \frac{a_{ij}}{b_{ij}} + \sum_{i,j=1}^{n} \left(1 - b_{ij}\right) \ln \left(1 - b_{ij}\right).$$

We prove the inequality in Theorem 5.4.2 following the approach of Lelarge [Le15]. Note that if A is doubly stochastic, by choosing $B = A$ we recover the lower bound in (3.4.6.1).

3.5 Matrix Scaling

Results of Sects. 3.3 and 3.4 provide us with some rather useful estimates of permanents of doubly stochastic matrices. It turns out that computing the permanent of any *positive* real matrix can be easily reduced to computing the permanent of a doubly stochastic matrix.

3.5.1 Matrix scaling. Let $A = (a_{ij})$ be an $n \times n$ matrix. We say that A is obtained by *scaling* from an $n \times n$ matrix $B = (b_{ij})$ if

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \text{for all} \quad i, j$$

and some numbers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$.

We note that in this case

$$\operatorname{per} A = \left(\prod_{i=1}^{n} \lambda_i \right) \left(\prod_{j=1}^{n} \mu_j \right) \operatorname{per} B. \tag{3.5.1.1}$$

3.5.2 Theorem. *For any $n \times n$ matrix $A = (a_{ij})$ such that*

$$a_{ij} > 0 \quad \textit{for all} \quad i, j,$$

there exists a unique $n \times n$ doubly stochastic matrix $B = (b_{ij})$ and positive $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ such that

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \textit{for all} \quad i, j. \tag{3.5.2.1}$$

The numbers λ_i and μ_j are unique up to a rescaling

$$\lambda_i \longmapsto \lambda_i \tau, \quad \mu_j \longmapsto \mu_j \tau^{-1}$$

for some $\tau > 0$.

Proof. Without loss of generality, we may assume that $n \geq 2$. Let Ω_n be the polytope of all $n \times n$ doubly stochastic matrices $X = (x_{ij})$ and let us consider a function $f : \Omega_n \longrightarrow \mathbb{R}$ defined by

$$f(X) = \sum_{i,j=1}^{n} x_{ij} \ln \frac{x_{ij}}{a_{ij}}.$$

Then f is a strictly convex function, cf. Sect. 2.1.1.2, and hence it attains its unique minimum, say $B = (b_{ij})$, on Ω_n.

First, we establish that $b_{ij} > 0$ for all i, j. Indeed,

$$\frac{\partial}{\partial x_{ij}} f(X) = \ln \frac{x_{ij}}{a_{ij}} + 1. \tag{3.5.2.2}$$

If $x_{ij} = 0$ we consider the right derivative and conclude that it is equal to $-\infty$, while for any $x_{ij} > 0$ the derivative is finite. Let $\frac{1}{n} J_n \in \Omega_n$ be the matrix with all entries equal to $1/n$ and let $B(t) = (1-t)B + t\frac{1}{n}J_n$, so that $B(0) = B$ and $B(1) = \frac{1}{n}J_n$. If $b_{ij} = 0$ for some i, j then for all sufficiently small $t > 0$ we have

$$f(B_t) < f(B),$$

which contradicts the definition of B as the minimum point of f.

Thus B is a positive matrix and therefore lies in the relative interior of Ω_n. It follows from (3.5.2.2) by the Lagrange multiplier conditions that there are numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that

$$\ln \frac{b_{ij}}{a_{ij}} = \alpha_i + \beta_j \quad \text{for all} \quad i, j.$$

Letting

$$\lambda_i = e^{-\alpha_i} \quad \text{and} \quad \mu_j = e^{-\beta_j},$$

we obtain (3.5.2.1).

On the other hand, if a doubly stochastic matrix $B = \left(b_{ij}\right)$ satisfies (3.5.2.1) then necessarily $b_{ij} > 0$ for all i, j and B is a critical point of f on Ω_n. Since f is strictly convex, B must be the unique minimum point of f on Ω_n, which proves the uniqueness of B.

From (3.5.2.1) and the uniqueness of B, we obtain the uniqueness of λ_i and μ_j up to a rescaling. □

Scaling can be obtained by solving a different optimization problem.

3.5.3 Lemma. *Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive matrix. Let us define a function $g_A : \mathbb{R}^n \oplus \mathbb{R}^n \longrightarrow \mathbb{R}$ by*

$$g_A(x, y) = \sum_{i,j=1}^{n} a_{ij} e^{x_i + y_j} \quad \textit{where} \quad x = (x_1, \ldots, x_n) \quad \textit{and} \quad y = (y_1, \ldots, y_n)$$

and let $\mathcal{L} \subset \mathbb{R}^n \oplus \mathbb{R}^n$ be the subspace defined by the equations

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{n} y_j = 0.$$

Then g attains its minimum on $\mathcal{L}$ at some point (x^*, y^*) where $x^* = (\xi_1, \ldots, \xi_n)$ and $y^* = (\eta_1, \ldots, \eta_n)$. Let

$$\lambda_i = e^{-\xi_i}\sqrt{\frac{g_A(x^*, y^*)}{n}} \quad \text{and} \quad \mu_j = e^{-\eta_j}\sqrt{\frac{g_A(x^*, y^*)}{n}}$$

for all i, j and let us define an $n \times n$ matrix $B = \left(b_{ij}\right)$ by

$$b_{ij} = \lambda_i^{-1}\mu_j^{-1}a_{ij} \quad \text{for all} \quad i, j.$$

Then B is a doubly stochastic matrix.

Proof. First, we claim that the minimum of g_A on $\mathcal{L}$ is indeed attained at some point. Let

$$\delta = \min_{ij} a_{ij} > 0.$$

Since for all $(x, y) \in \mathcal{L}$, we have $x_i \geq 0$ and $y_j \geq 0$ for some i and j, we have

$$g_A(x, y) > g_A(0, 0) \quad \text{if} \quad x_i > \ln\frac{g_A(0, 0)}{\delta} \quad \text{or} \quad y_j > \ln\frac{g_A(0, 0)}{\delta}$$

for some i, j. On the other hand, if for some $(x, y) \in \mathcal{L}$ we have $x_i < -t$ for some $t > 0$ then $x_j > t/n$ for some j and, similarly, if $y_i < -t$ for some $t > 0$ then $y_j > t/n$ for some j. Therefore, the minimum of g_A on $\mathcal{L}$ is attained on the compact subset

$$|x_i|, \left|y_j\right| \leq n \ln\frac{g_A(0, 0)}{\delta} \quad \text{for all} \quad i, j.$$

At the minimum point, the gradient of $g_A(x, y)$ is orthogonal to $\mathcal{L}$, so for some α and β we have

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij}e^{\xi_i+\eta_j} = \alpha \quad \text{for} \quad i = 1, \ldots, n \\ &\qquad \text{and} \\ &\sum_{i=1}^{n} a_{ij}e^{\xi_i+\eta_j} = \beta \quad \text{for} \quad j = 1, \ldots, n. \end{aligned} \tag{3.5.3.1}$$

Summing the first set of equations over $i = 1, \ldots, n$ and the second set of equations over $j = 1, \ldots, n$, we conclude that

$$\sum_{i,j=1}^{n} a_{ij}e^{\xi_i+\eta_j} = n\alpha = n\beta,$$

so

$$\alpha = \beta = \frac{1}{n} g_A\left(x^*, y^*\right)$$

and the proof follows from (3.5.3.1). □

3.5.4 Remark. Theorem 3.5.2 was proved by Sinkhorn [Si64], who used a different approach. He showed that, given a positive matrix A, the repeated row and column scaling (first, scale all rows to row sum 1, then scale all columns to column sum 1, then again rows, then again columns, etc.) converges to the desired doubly stochastic matrix B. An approach to scaling via a solution of an appropriate optimization problem (similar to our Lemma 3.5.3) was used in [MO68] and several other papers since then.

Clearly, not every non-negative matrix can be scaled to doubly stochastic (for example, the matrix of all zeros cannot). Some non-negative matrices can be scaled arbitrarily close to doubly stochastic, but cannot be scaled exactly, for example the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Indeed, multiplying the first column by $\epsilon > 0$ and the first row by ϵ^{-1}, we obtain the matrix

$$B = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}$$

with row and column sums arbitrarily close to 1, but never exactly 1. It is shown in [L+00] that a non-negative matrix A can be scaled arbitrarily close to a doubly stochastic matrix if and only if $\operatorname{per} A > 0$ and that it can be scaled exactly to a doubly stochastic matrix, if, in addition, whenever for a set $I \subset \{1, \ldots, n\}$ of rows and for a set $J \subset \{1, \ldots, n\}$ of columns such that $|I| + |J| = n$ we have $a_{ij} = 0$ for $i \in I$ and $j \in J$, we must also have $a_{ij} = 0$ for all $i \notin I$ and $j \notin J$. The conditions for approximate and exact scaling can be efficiently (in polynomial time) verified. Also [L+00] contains the fastest known algorithm for matrix scaling.

As is observed in [L+00], formula (3.5.1.1) together with the inequality

$$\frac{n!}{n^n} \leq \operatorname{per} B \leq 1$$

for the permanent of a doubly stochastic matrix B allows one to estimate the permanent of any $n \times n$ non-negative matrix A within a multiplicative factor of roughly e^n and the inequality (3.4.6.1) improves the factor further to 2^n (and, conjecturally, to $2^{n/2}$). Computationally, matrix scaling is very efficient and in view of Sect. 3.4.6 it is natural to ask for which matrices A their doubly stochastic scaling B will not have large entries, so that a better upper bound on $\operatorname{per} B$ can be used.

3.5.5 Definition. Let $A = (a_{ij})$ be an $n \times n$ positive matrix. For $\alpha \geq 1$ we say that A is α-*conditioned* if

$$a_{ij_1} \leq \alpha a_{ij_2} \quad \text{for any} \quad 1 \leq i, j_1, j_2 \leq n$$

and

$$a_{i_1 j} \leq \alpha a_{i_2 j} \quad \text{for any} \quad 1 \leq i_1, i_2, j \leq n.$$

In words: an $n \times n$ positive matrix is α-conditioned if the ratio of any two entries of A in the same row and the ratio of any two entries of A in the same column do not exceed α.

3.5.6 Lemma. *Let A be an $n \times n$ matrix which is α-conditioned for some $\alpha \geq 1$. Let $B = (b_{ij})$ be the doubly stochastic matrix obtained from A by scaling. Then B is α^2-conditioned. In particular,*

$$b_{ij} \leq \frac{\alpha^2}{n} \quad \textit{for all} \quad i, j.$$

Proof. Let $A = (a_{ij})$ and let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be positive real such that

$$b_{ij} = \lambda_i \mu_j a_{ij} \quad \text{for all} \quad i, j.$$

Then

$$\frac{b_{ij_1}}{b_{ij_2}} = \frac{\mu_{j_1}}{\mu_{j_2}} \frac{a_{ij_1}}{a_{ij_2}} \leq \frac{\mu_{j_1}}{\mu_{j_2}} \alpha \quad \text{for all} \quad 1 \leq j_1, j_2 \leq n. \tag{3.5.6.1}$$

Since

$$\sum_{i=1}^{n} b_{ij_1} = \sum_{i=1}^{n} b_{ij_2} = 1,$$

we conclude that

$$\frac{\mu_{j_1}}{\mu_{j_2}} \geq \frac{1}{\alpha} \quad \text{for all} \quad j_1, j_2.$$

On the other hand, since

$$\frac{a_{ij_1}}{a_{ij_2}} \geq \frac{1}{\alpha} \quad \text{for all} \quad j_1, j_2,$$

from (3.5.6.1) we conclude that

$$\frac{b_{ij_1}}{b_{ij_2}} \geq \frac{1}{\alpha^2} \quad \text{for all} \quad j_1, j_2. \tag{3.5.6.2}$$

Similarly, we prove that

$$\frac{b_{i_1 j}}{b_{i_1 j}} \geq \frac{1}{\alpha^2} \quad \text{for all} \quad i_1, i_2$$

and hence B is α^2-conditioned.

Since

$$\sum_{j=1}^{n} b_{ij} = 1 \quad \text{for all} \quad i = 1, \ldots, n,$$

we have

$$b_{ij} \geq \frac{1}{n} \quad \text{for every} \quad i \quad \text{and some} \quad j$$

and the proof follows by (3.5.6.2). □

Lemma 3.5.6 together the observation of Sect. 3.4.6 and formula (3.5.1.1) allows us, given an $n \times n$ positive matrix A whose entries are within a constant factor of each other, to compute per A by scaling within a polynomial in n factor.

Although the scaling factors $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are not uniquely defined by the matrix, Theorem 3.5.2 implies that their product $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$ is a function of the matrix. It has some interesting convex properties.

3.5.7 Lemma. *For an $n \times n$ positive matrix $A = (a_{ij})$, let us define a number $f(A)$ as follows: Let $B = (b_{ij})$ be a doubly stochastic matrix and let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be positive numbers such that*

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \textit{for all} \quad i, j.$$

Let

$$f(A) = \left(\prod_{i=1}^{n} \lambda_i \right) \left(\prod_{j=1}^{n} \mu_j \right).$$

Then f is well-defined and satisfies the following properties:

(1) Function f is homogeneous of degree n:

$$f(\alpha A) = \alpha^n f(A) \quad \textit{for all} \quad \alpha > 0$$

and all positive $n \times n$ matrices A;

(2) Function f is monotone:

$$f(C) \leq f(A)$$

for any positive $n \times n$ matrices $A = (a_{ij})$ and $C = (c_{ij})$ such that

$$c_{ij} \leq a_{ij} \quad \textit{for all} \quad i, j;$$

(3) Function $f^{1/n}$ is concave:

$$f^{1/n}(\alpha_1 A_1 + \alpha_2 A_2) \geq \alpha_1 f^{1/n}(A_1) + \alpha_2 f^{1/n}(A_2)$$

for any positive $n \times n$ matrices A_1 and A_2 and any $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$.

Proof. Theorem 3.5.2 implies that f is well-defined and Part (1) is straightforward. As in Lemma 3.5.3, let us define

$$g_A(x, y) = \sum_{i,j=1}^{n} a_{ij} e^{x_i + y_j}$$

and let $\mathcal{L} \subset \mathbb{R}^n \oplus \mathbb{R}^n$ be the subspace defined by the equations $x_1 + \ldots + x_n = 0$ and $y_1 + \ldots + y_n = 0$. Then, by Lemma 3.5.3,

$$f(A) = \frac{1}{n^n} \min_{(x,y)\in\mathcal{L}} g_A^n(x, y).$$

Since $g_C(x, y) \leq g_A(x, y)$ for all $(x, y) \in \mathcal{L}$ provided $c_{ij} \leq a_{ij}$ for all i, j, the proof of Part (2) follows.

We have

$$f^{1/n}(A) = \frac{1}{n} \min_{(x,y)\in\mathcal{L}} g_A(x, y)$$

and hence for $A = \alpha_1 A_1 + \alpha_2 A_2$ we have

$$\begin{aligned} f^{1/n}(A) =& \frac{1}{n} \min_{(x,y)\in\mathcal{L}} g_A(x, y) = \frac{1}{n} \min_{(x,y)\in\mathcal{L}} \alpha_1 g_{A_1}(x, y) + \alpha_2 g_2 A_2(x, y) \\ \geq& \frac{\alpha_1}{n} \min_{(x,y)\in\mathcal{L}} g_{A_1}(x, y) + \frac{\alpha_2}{n} \min_{(x,y)\in\mathcal{L}} g_{A_2}(x, y) = \alpha_1 f^{1/n}(A_1) + \alpha_2 f^{1/n}(A_2), \end{aligned}$$

which completes the proof of Part (3). □

It is not hard to see that the function f of Lemma 3.5.7 is the capacity

$$\inf_{x_1,\ldots,x_n>0} \frac{p(x_1, \ldots, x_n)}{x_1 \cdots x_n}$$

of the polynomial

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right),$$

cf. Sect. 2.1.5 and Lemma 3.3.3.

We state the scaling theorem in the most general form (we will use it later in Chap. 8).

3.5.8 Theorem. *Let* $r = (r_1, \dots, r_m)$ *and* $c = (c_1, \dots, c_n)$ *be positive integer vectors such that*

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j = N.$$

Then for any positive $m \times n$ *matrix* $A = \big(a_{ij}\big)$ *there exists an* $m \times n$ *positive matrix* $B = \big(b_{ij}\big)$ *with row sums* $r_1, \dots, r_m$ *and column sums* $c_1, \dots, c_n$ *and positive real* $\lambda_1, \dots, \lambda_m$ *and* $\mu_1, \dots, \mu_n$ *such that*

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \textit{for all} \quad i, j.$$

Moreover, given r, c *and* A, *the matrix* B *is unique and can be found as the minimum point of the function*

$$f = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} x_{ij} \ln \frac{x_{ij}}{a_{ij}}$$

on the polytope $\Omega_{r,c}$ *of non-negative* $m \times n$ *matrices with row sums* r *and column sums* c. *The numbers* λ_i *and* μ_j *are unique up to a rescaling*

$$\lambda_i \longmapsto \lambda_i \tau, \quad \mu_j \longmapsto \mu_j \tau^{-1}$$

for some $\tau > 0$ *and can be found as follows:*

Let us define $g_A : \mathbb{R}^m \oplus \mathbb{R}^n \longrightarrow \mathbb{R}$ *by*

$$g_A(x, y) = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} e^{x_i + y_j} \quad \textit{for} \quad x = (x_1, \dots, x_m) \quad \textit{and} \quad y = (y_1, \dots, y_n)$$

and let $\mathcal{L}_{r,c} \subset \mathbb{R}^m \oplus \mathbb{R}^n$ *be the subspace defined by the equations*

$$\sum_{i=1}^{m} r_i x_i = 0 \quad \textit{and} \quad \sum_{j=1}^{n} c_j y_j = 0.$$

Then the minimum of g_A *on* $\mathcal{L}_{r,c}$ *is attained at some point* $x^* = (\xi_1, \dots, \xi_m)$ *and* $y^* = (\eta_1, \dots, \eta_n)$ *and we may let*

$$\lambda_i = e^{-\xi_i} \sqrt{\frac{g_A(x^*, y^*)}{N}} \quad \textit{and} \quad \mu_j = e^{-\eta_j} \sqrt{\frac{g_A(x^*, y^*)}{N}}$$

for all i, j. □

The proof is very similar to those of Theorem 3.5.2 and Lemma 3.5.3 and therefore omitted.

3.6 Permanents of Complex Matrices

In this section, we take a look at the permanents of matrices with complex entries. Such permanents are of interest in physics, see, for example, [AA13] and [Ka16]. First, we prove that the permanents of matrices sufficiently close to the $n \times n$ matrix J_n of all 1s is not 0.

3.6.1 Theorem. *There exists an absolute constant $\delta_0 > 0$ (one can choose $\delta_0 = 0.5$) such that for any $n \times n$ matrix $A = (a_{ij})$ with complex entries satisfying*

$$|1 - a_{ij}| \leq \delta_0 \quad \text{for all} \quad i, j$$

we have

$$\operatorname{per} A \neq 0.$$

Geometrically, the ℓ^∞ distance from the matrix J_n to the hypersurface per $Z = 0$ in the space $\mathbb{C}^{n \times n}$ of $n \times n$ complex matrices is bounded below by a positive constant, independent on n. Later, in Theorem 5.5.3, we prove that per $A \neq 0$ if the ℓ^1 distance of every row and column of an $n \times n$ complex matrix A to the vector of all 1 s does not exceed γn for some absolute constant $\gamma > 0$ (one can choose $\gamma = 0.0696$).

In view of Theorem 3.6.1, we can choose a branch of $\ln \operatorname{per} A$ for all matrices $A = (a_{ij})$ satisfying $|1 - a_{ij}| \leq \delta_0$ such that $\ln \operatorname{per} J_n$ is a real number, where J_n is the $n \times n$ matrix of all 1s.

3.6.2 Theorem. *Let us fix some $0 < \delta < \delta_0$, where δ_0 is the constant in Theorem 3.6.1. Then there exists $\gamma = \gamma(\delta) > 0$ and for any $\epsilon > 0$ and positive integer n there exists a polynomial $p = p_{n,\delta,\epsilon}$ in the entries of an $n \times n$ complex matrix $A = (a_{ij})$ satisfying*

$$\deg p \leq \gamma(\ln n - \ln \epsilon)$$

and

$$|\ln \operatorname{per} A - p(A)| \leq \epsilon$$

provided

$$|1 - a_{ij}| \leq \delta \quad \text{for all} \quad i, j.$$

As we will see, the polynomial $p(A)$ can be efficiently computed. The gist of Theorem 3.6.2 is that $\ln \operatorname{per} A$ can be efficiently approximated by a low-degree polynomial in the vicinity of the matrix J_n of all 1s, and, in particular, per A can be approximated there within a relative error of ϵ in quasi-polynomial $n^{O(\ln n - \ln \epsilon)}$ time.

Theorems 3.6.1 and 3.6.2 were first proved in [B16b] with a worse constant $\delta_0 = 0.195$. Following [B16+], we give a much simplified proof achieving a better constant.

First we prove Theorem 3.6.1 and then deduce Theorem 3.6.2 from it. We identify $\mathbb{C} = \mathbb{R}^2$ and measure angles between complex numbers as vectors in the plane.

3.6.3 Lemma. *Let $u_1, \ldots, u_n \in \mathbb{R}^2$ be non-zero vectors and suppose that the angle between any two vectors u_i and u_j does not exceed α for some $0 \le \alpha < 2\pi/3$. Let $u = u_1 + \ldots + u_n$. Then*

$$|u| \ \ge \ \left(\cos\frac{\alpha}{2}\right)\sum_{i=1}^{n}|u_i|.$$

Proof. First, we note that 0 cannot lie in the convex hull of the vectors $u_1, \ldots, u_n$, since otherwise by the Carathéodory Theorem it would have lied in the convex hull of some three vectors u_i, u_j, u_k and then the angle between some two of these three vectors would have been at least $2\pi/3$, see Fig. 3.5.

Hence the vectors $u_1, \ldots, u_n$ lie in an angle measuring at most α. Let us consider the orthogonal projections of $u_1, \ldots, u_n$ onto the bisector of the angle, see Fig. 3.6.

Then the length of the projection of u_i is at least $|u_i|\cos(\alpha/2)$ and the length of the projection of u is at least $(|u_1| + \ldots + |u_n|)\cos(\alpha/2)$. Since the length of u is at least as large as the length of its orthogonal projection, the result follows. □

In [B16b] a weaker bound with $\sqrt{\cos\alpha}$ instead of $\cos(\alpha/2)$ is used (assuming that $\alpha < \pi/2$). The current enhancement is due to Bukh [Bu15].

3.6.4 Lemma. *Let $u_1, \ldots, u_n \in \mathbb{C}$ be non-zero complex numbers, such that the angle between any two vectors u_i and u_j does not exceed α for some $0 \le \alpha < 2\pi/3$ and let $0 \le \delta < \cos(\alpha/2)$ be a real number. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers such that*

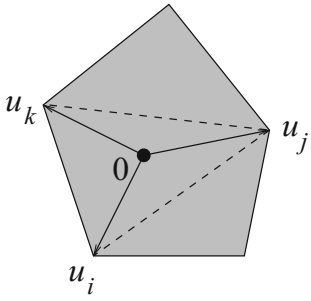


Fig. 3.5 If the origin lies in the convex hull of the vectors then the angle between some two vectors is at least $2\pi/3$

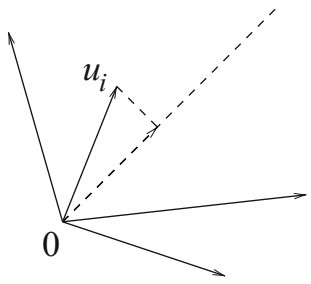


Fig. 3.6 Projecting vectors onto the bisector of the angle

$$\left|1 - a_j\right| \le \delta \quad \textit{and} \quad \left|1 - b_j\right| \le \delta \quad \textit{for} \quad j = 1, \ldots, n.$$

Let

$$v = \sum_{j=1}^{n} a_j u_j \quad \textit{and} \quad w = \sum_{j=1}^{n} b_j u_j.$$

Then $v \neq 0$, $w \neq 0$ *and the angle between* v *and* w *does not exceed*

$$2 \arcsin \frac{\delta}{\cos(\alpha/2)}.$$

Proof. Let $u = u_1 + \ldots + u_n$. Then, by Lemma 3.6.3, $u \neq 0$ and

$$|u| \ge \cos\left(\frac{\alpha}{2}\right) \sum_{j=1}^{n} |u_j|.$$

By the triangle inequality, we have

$$|v - u| \le \sum_{j=1}^{n} \left|1 - a_j\right| |u_j| \le \delta \sum_{j=1}^{n} |u_j|.$$

Therefore, the angle between $v = (v - u) + u$ and u does not exceed

$$\theta = \arcsin \frac{|v - u|}{|u|} \le \arcsin \frac{\delta}{\cos(\alpha/2)},$$

see Fig. 3.7.

Similarly, the angle between w and u does not exceed θ and hence the angle between v and w does not exceed 2θ. □

3.6.5 Proof of Theorem 3.6.1. Let us choose

$$\delta_0 = 0.5 \quad \text{and} \quad \alpha = \frac{\pi}{2}.$$

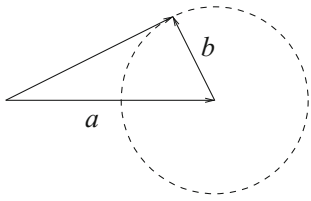


Fig. 3.7 The angle between a and $a + b$ does not exceed $\arcsin \frac{|b|}{|a|}$ provided $|b| < |a|$

We denote by $\mathcal{U}_n$ the closed polydisc $\mathcal{U}_n \subset \mathbb{C}^{n\times n}$ consisting of the $n \times n$ complex matrices $A = (a_{ij})$ such that

$$|1 - a_{ij}| \leq \delta_0 \quad \text{for all} \quad i, j.$$

We prove by induction on n the following statement.

For every matrix $Z \in \mathcal{U}_n$ we have per $Z \neq 0$ and, moreover, if $A, B \in \mathcal{U}_n$ are two matrices that differ in one row (one column) only, then the angle between non-zero complex numbers per A and per B does not exceed α.

If $n = 1$ then any $a \in \mathcal{U}_1$ is necessarily non-zero, since $\delta_0 < 1$. Moreover, the angle between any two $a, b \in \mathcal{U}_1$ does not exceed $2 \arcsin \delta_0 = \pi/3 < \alpha$, cf. Fig. 3.7.

Suppose that $n \geq 2$ and assume that the above statement holds for matrices from $\mathcal{U}_{n-1}$. Let $A, B \in \mathcal{U}_n$ be two matrices that differ in one row or in one column only. Without loss of generality, we assume that the matrix B is obtained from A by replacing the entries a_{1j} in the first row by some complex numbers b_{1j}, where $j = 1, \ldots, n$. Using the row expansion (3.1.1.2), we obtain

$$\operatorname{per} A = \sum_{j=1}^{n} a_{1j} \operatorname{per} A_j \quad \text{and} \quad \operatorname{per} B = \sum_{j=1}^{n} b_{1j} \operatorname{per} A_j,$$

where A_j is the $(n-1) \times (n-1)$ matrix obtained from A by crossing out the first row and the j-th column. We have $A_j \in \mathcal{U}_{n-1}$ and, moreover, up to a permutation of columns, any two matrices A_{j_1} and A_{j_2} differ in at most one column. Therefore, by the induction hypothesis per $A_j \neq 0$ for $j = 1, \ldots, n$ and the angle between any two non-zero complex numbers per A_{j_1} and per A_{j_2} does not exceed α.

We apply Lemma 3.6.4 with $u_j = \operatorname{per} A_j$, $a_j = a_{1j}$ and $b_j = b_{1j}$ for $j = 1, \ldots, n$. Since $\delta_0 < \cos(\alpha/2)$, by Lemma 3.6.4 we have per $A \neq 0$ and per $B \neq 0$ and the angle between per A and per B does not exceed

$$2 \arcsin \frac{\delta_0}{\cos(\alpha/2)} = 2 \arcsin \frac{0.5}{\cos(\pi/4)} = 2 \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{2} = \alpha,$$

which completes the proof. □

The value of $\delta_0 = 0.5$ is the largest value of δ for which the equation

$$\alpha = 2 \arcsin \frac{\delta}{\cos(\alpha/2)}$$

has a solution α. Indeed, the above equation can be written as

$$\left(\sin \frac{\alpha}{2}\right)\left(\cos \frac{\alpha}{2}\right) = \delta, \quad \text{that is,} \quad \sin \alpha = 2\delta.$$

3.6.6 The optimal value of δ_0. What is the optimal value of δ_0 in Theorem 3.6.1? To be more precise, since it is not even clear whether the optimal value δ_0 exists, what is the supremum of all possible values of δ_0 in Theorem 3.6.1? Since

$$\operatorname{per}\begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} = 0$$

we must have

$$\delta_0 < \frac{\sqrt{2}}{2} \approx 0.7071067810.$$

Moreover, Bukh [Bu15] showed that for

$$a = \frac{1+i}{2} \quad \text{and} \quad b = \frac{1-i}{2}$$

we have

$$\operatorname{per} \underbrace{\begin{pmatrix} a & b & a & b & \dots & a & b \\ b & a & b & a & \dots & b & a \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a & b & a & b & \dots & a & b \\ b & a & b & a & \dots & b & a \end{pmatrix}}_{n \equiv 2 \mod 4} = 0$$

and hence there is no hope that the value of δ_0 might improve as n grows.

Now we deduce Theorem 3.6.2 from 3.6.1.

3.6.7 Proof of Theorem 3.6.2. Let $A = (a_{ij})$ be an $n \times n$ complex matrix satisfying $|a_{ij} - 1| \le \delta$ for all i, j and let $J = J_n$ be the $n \times n$ matrix of all 1s. We define a univariate polynomial

$$g(z) = \operatorname{per}\big(J + z(A - J_n)\big)$$

with $\deg g \le n$. Let

$$\beta = \frac{\delta_0}{\delta} > 1.$$

By Theorem 3.6.1,

$$g(z) \neq 0 \quad \text{provided} \quad |z| \le \beta.$$

Let

$$f(z) = \ln g(z) \quad \text{for} \quad |z| \le 1,$$

where we choose the branch of the logarithm that is real for $z = 0$. We note that by Theorem 3.6.1 the function f is well defined and we have

$$f(0) = \ln n! \quad \text{and} \quad f(1) = \ln \operatorname{per} A.$$

We consider the Taylor polynomial of f at $z = 0$:

$$p_m(z) = f(0) + \sum_{k=1}^{m} \frac{z^k}{k!} \frac{d^k}{dz^k} f(z)\Big|_{z=0.} \tag{3.6.7.1}$$

By Lemma 2.2.1, we have

$$|p_m(1) - \ln \operatorname{per} A| = |p_m(A) - f(1)| \;\le\; \frac{n}{(m+1)\beta^m(\beta-1)}$$

In particular, to approximate $\ln \operatorname{per} A$ within an additive error of $\epsilon > 0$, we can choose $m \le \gamma(\ln n - \ln \epsilon)$ in (3.6.7.1) for some $\gamma = \gamma(\delta) > 0$.

It remains to show that $p_m(1)$ is a polynomial of degree m in the matrix entries a_{ij} of A. Our first observation is that the k-th derivative $g^{(k)}(0)$ is a polynomial of degree k in the entries of the matrix A, which can be computed in $n^{O(k)}$ time. Indeed,

$$\begin{aligned}\frac{d^k}{dz^k} g(z)\Big|_{z=0} &= \frac{d^k}{dz^k} \sum_{\sigma\in S_n} \prod_{i=1}^{n} \left(1 + z\left(a_{i\sigma(i)} - 1\right)\right)\Big|_{z=0} \\ &= \sum_{\sigma\in S_n} \sum_{(i_1,\dots,i_k)} \left(a_{i_1\sigma(i_1)} - 1\right)\cdots\left(a_{i_k\sigma(i_k)} - 1\right),\end{aligned}$$

where the last sum is taken over all ordered k-subsets $(i_1, \dots, i_k)$ of indices $1 \le i_j \le n$. Since there are $(n-k)!$ permutations $\sigma \in S_n$ that map a given ordered k-subset $(i_1, \dots, i_k)$ into a given ordered k-subset $(j_1, \dots, j_k)$, we can write

$$g^{(k)}(0) = (n-k)! \sum_{\substack{(i_1,\dots,i_k)\\(j_1,\dots,j_k)}} \left(a_{i_1 j_1} - 1\right)\cdots\left(a_{i_k j_k} - 1\right), \tag{3.6.7.2}$$

where the last sum is taken over all pairs of ordered k-subsets $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ of indices between 1 and n. As follows from Sect. 2.2.2, the derivatives $f^{(k)}(0)$ for $k = 1, \dots, m$ can be found in $O(m^2)$ time as linear combinations of the derivatives $g^{(k)}(0)$ for $k = 1, \dots, m$ with coefficients depending on k only, which completes the proof. □

Kontorovich and Wu [KW16] implemented the algorithm of Sect. 3.6.7 for computing the polynomial $p(A)$ and performed numerical experiments. Computing $g^{(k)}(0)$ reduces to computing the sum of permanents of $k \times k$ submatrices of $A - J_n$ and Kontorovich and Wu used for that purpose an efficient algorithm of [FG06]. It turned out that for $n \times n$ matrices $A = \left(a_{ij}\right)$ satisfying $|1 - a_{ij}| \le 0.5$ and $n \le 20$ (so that the exact value of per A can be computed for comparison), polynomials p of degree 3 already provide reasonable approximations (they approximate $\ln \operatorname{per} A$ within an about 1% error). On the other hand, polynomials p of degree 3 can be easily computed for 100×100 matrices.

Let A be an $n \times n$ complex matrix such that $\operatorname{per} A \neq 0$ and suppose that the ℓ^∞-distance from A to the complex hypersurface $\operatorname{per} Z = 0$ is at least δ_0 for some $\delta_0 > 0$. It follows from the proof of Sect. 3.6.7 that for any $0 < \delta < \delta_0$ there is a constant $\gamma = \gamma(\delta) > 0$ and for any $0 < \epsilon < 1$ there is a polynomial $p = p_{A,\delta,\epsilon}$ in the entries of an $n \times n$ matrix $B = \left(b_{ij}\right)$ such that $\deg p \leq \gamma(\ln n - \ln \epsilon)$ and

$$\left|\ln \operatorname{per} B - p_{A,\delta,\epsilon}(B)\right| \leq \epsilon \quad \text{provided} \quad \left|a_{ij} - b_{ij}\right| \leq \delta \quad \text{for all} \quad i, j.$$

Of course, depending on A, the polynomial p might be hard to compute (it is easy when $A = J_n$, the matrix of all 1s).

3.6.8 Remark. If the entries of an $n \times n$ real matrix $A = \left(a_{ij}\right)$ are (weakly) decreasing down each column, that is, if $a_{ij} \geq a_{(i+1)j}$ for all i, j then the roots of the polynomial $p(z) = \operatorname{per}(J_n + zA)$ are real. Moreover, the n-variate polynomial

$$p(z_1, \ldots, z_n) = \operatorname{per}(J_n D(z_1, \ldots, z_n) + A),$$

where $D(z_1, \ldots, z_n)$ is the diagonal matrix having $z_1, \ldots, z_n$ on the diagonal, is $\mathbb{H}$-stable [B+11].

A different approach to approximation of permanents by Taylor polynomial expansions around J_n is described in [Mc14].

3.7 Approximating Permanents of Positive Matrices

As follows from Sect. 3.5, for any $\alpha \geq 1$, fixed in advance, the permanent of an α-conditioned $n \times n$ positive matrix A can be approximated in polynomial time within an $n^{O(\alpha^2)}$ factor. Understanding permanents of complex matrices allows us to approximate permanents of such matrices better: we show that we can approximate the permanent within arbitrarily small relative error in quasi-polynomial time. More precisely, we prove the following result.

3.7.1 Theorem. *For any $0 \leq \delta < 1$, there exists $\gamma = \gamma(\delta) > 0$ such that for any positive integer n and any real $0 < \epsilon \leq 1$ there exists a polynomial $p = p_{n,\delta,\epsilon}$ with $\deg p \leq \gamma\left(\ln n - \ln \epsilon\right)$ in the entries a_{ij} of an $n \times n$ real matrix $A = \left(a_{ij}\right)$ such that*

$$|\ln \operatorname{per} A - p(A)| \leq \epsilon$$

provided

$$\left|1 - a_{ij}\right| \leq \delta \quad \text{for all} \quad i, j.$$

We show that the polynomial $p_{n,\delta,\epsilon}$ can be computed in $n^{O(\ln n - \ln \epsilon)}$ time, where the implicit constant in the "O" notation depends on δ alone.

We deduce Theorem 3.7.1 from the following result.

3.7.2 Theorem. *Let us fix a real* $0 \le \delta < 1$ *and le*

$$\tau = (1-\delta)\sin\left(\frac{\pi}{4} - \arctan\delta\right) > 0.$$

Let $Z = \left(z_{ij}\right)$ *be an* $n \times n$ *complex matrix such that*

$$\left|1 - \Re\, z_{ij}\right| \le \delta \quad \textit{and} \quad \left|\Im\, z_{ij}\right| \le \tau \quad \textit{for all} \quad 1 \le i, j \le n.$$

Then

$$\operatorname{per} Z \neq 0.$$

We note that

$$(1-\delta)\sin\left(\frac{\pi}{4} - \arctan\delta\right) \ge \frac{(1-\delta)^2}{2} \quad \text{for all} \quad 0 \le \delta \le 1$$

and so

$$\tau = \frac{(1-\delta)^2}{2}$$

satisfies the condition of Theorem 3.7.2.

We prove Theorem 3.7.2 first and then deduce Theorem 3.7.1 from it.

As in Sect. 3.6, we identify $\mathbb{C} = \mathbb{R}^2$ and measure angles between non-zero complex numbers as between non-zero vectors in the plane. We start with a simple geometric lemma.

3.7.3 Lemma. *Let* $u_1, \ldots, u_n \in \mathbb{C}$ *be non-zero complex numbers such that the angle between any two* u_i, u_j *does not exceed* $\pi/2$.

(1) Let

$$v = \sum_{j=1}^{n} \alpha_j u_j \quad \textit{and} \quad w = \sum_{j=1}^{n} \beta_j u_j$$

where $\alpha_1, \ldots, \alpha_n$ *are non-negative real and* $\beta_1, \ldots, \beta_n$ *are real such that*

$$|\beta_j| \le \alpha_j \quad \textit{for} \quad j = 1, \ldots, n.$$

Then

$$|w| \le |v|;$$

(2) Let

$$v = \sum_{j=1}^{n} \alpha_j u_j \quad \textit{and} \quad w = \sum_{j=1}^{n} \beta_j u_j$$

where $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *are real such that*

$$|1-\alpha_j| \le \delta \quad \textit{and} \quad |1-\beta_j| \le \delta \quad \textit{for} \quad j=1,\dots,n$$

and some $0 \le \delta < 1$. *Then* $v \ne 0$, $w \ne 0$ *and the angle between* v *and* w *does not exceed*

$$2\arctan\delta.$$

(3) Let

$$v=\sum_{j=1}^{n}\alpha_j u_j \quad \textit{and} \quad w=\sum_{j=1}^{n}\beta_j u_j$$

where

$$|1-\Re\,\alpha_j| \le \delta, \quad |1-\Re\,\beta_j| \le \delta \quad \textit{and}$$
$$|\Im\,\alpha_j| \le \tau, \quad |\Im\,\beta_j| \le \tau \quad \textit{for} \quad j=1,\dots,n$$

and some $0 \le \delta < 1$ *and* $0 \le \tau < 1-\delta$. *Then* $v \ne 0$, $w \ne 0$ *and the angle between* v *and* w *does not exceed*

$$2\arctan\delta + 2\arcsin\frac{\tau}{1-\delta}.$$

Proof. We consider the standard inner product in $\mathbb{R}^2=\mathbb{C}$, so

$$\langle a,b\rangle = \Re\, a\overline{b}.$$

Hence

$$\langle u_i,u_j\rangle \ge 0 \quad \text{for all} \quad i,j.$$

We have

$$|w|^2=\sum_{1\le i,j\le n}\beta_i\beta_j\langle u_i,u_j\rangle \le \sum_{1\le i,j\le n}\alpha_i\alpha_j\langle u_i,u_j\rangle = |v|^2$$

and the proof of Part (1) follows.

To prove Part (2), let

$$u=\sum_{j=1}^{n}\left(\frac{\alpha_j+\beta_j}{2}\right)u_j \quad \text{and} \quad x=\sum_{j=1}^{n}\left(\frac{\alpha_j-\beta_j}{2}\right)u_j,$$

so that $v=u+x$ and $w=u-x$, see Fig. 3.8. Clearly, $|u|>0$.

Now, if $|1-\alpha| \le \delta$ and $|1-\beta| \le \delta$ for some $0 \le \delta < 1$ and $\alpha \ge \beta$ we have

$$\frac{\alpha}{\beta} \le \frac{1+\delta}{1-\delta} \quad \text{and hence} \quad \alpha(1-\delta) \le \beta(1+\delta)$$

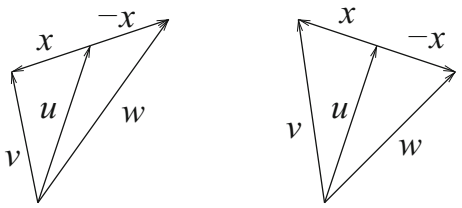


Fig. 3.8 Given $|u|$ and $|x|$, the angle between $v = u + x$ and $w = u - x$ is the largest when u is orthogonal to x

and

$$\frac{\alpha - \beta}{\alpha + \beta} - \delta = \frac{\alpha - \beta - \delta(\alpha + \beta)}{\alpha + \beta} = \frac{\alpha(1 - \delta) - \beta(1 + \delta)}{\alpha + \beta} \leq 0.$$

Therefore for all α and β such that $|1 - \alpha| \leq \delta$ and $|1 - \beta| \leq \delta$ for some $0 \leq \delta < 1$ we have

$$\frac{|\alpha - \beta|}{\alpha + \beta} \leq \delta.$$

Therefore, by Part (1),

$$|x| \leq \delta |u|.$$

The angle between v and w is

$$\arccos \frac{\langle v, w \rangle}{|v||w|},$$

where

$$\langle v, w \rangle = |u|^2 - |x|^2.$$

We have

$$|v|^2 + |w|^2 = 2|u|^2 + 2|x|^2$$

and hence

$$|v||w| \leq |u|^2 + |x|^2$$

with equality attained when $|v|^2 = |w|^2 = |u|^2 + |x|^2$, that is, when x is orthogonal to u. Therefore, the angle between v and w does not exceed

$$\arccos \frac{|u|^2 - |x|^2}{|u|^2 + |x|^2}$$

with equality attained when x is orthogonal to u and the angle is

$$2 \arctan \frac{|x|}{|u|} \leq 2 \arctan \delta,$$

see Fig. 3.8. The proof of Part (2) now follows.

In Part (3), let

$$v' = \sum_{j=1}^{n} \left(\Re\, \alpha_j\right) u_j, \quad v'' = \sum_{j=1}^{n} \left(\Im\, \alpha_j\right) u_j, \quad w' = \sum_{j=1}^{n} \left(\Re\, \beta_j\right) u_j$$
$$\text{and} \quad w'' = \sum_{j=1}^{n} \left(\Im\, \beta_j\right) u_j.$$

By Part (2), the angle between non-zero vectors v' and w' does not exceed $2 \arctan \delta$. By Part (1), we have

$$|v''| \le \frac{\tau}{1-\delta}|v'| \quad \text{and} \quad |w''| \le \frac{\tau}{1-\delta}|w'|.$$

Hence $v = v' + i\,v'' \neq 0$ and $w = w' + i\,w'' \neq 0$ and the angle between v and v' and the angle between w and w' do not exceed

$$\arcsin \frac{\tau}{1-\delta},$$

see Fig. 3.7. The proof of Part (3) now follows. □

Now we are ready to prove Theorem 3.7.2.

3.7.4 Proof of Theorem 3.7.2. For a positive integer n, let $\mathcal{U}_n = \mathcal{U}_n(\delta, \tau)$ be the set of $n \times n$ complex matrices $Z = \left(z_{ij}\right)$ such that

$$\left|1 - \Re\, z_{ij}\right| \le \delta \quad \text{and} \quad \left|\Im\, z_{ij}\right| \le \tau \quad \text{for all} \quad i, j.$$

We prove by induction on n a stronger statement:

For any $Z \in \mathcal{U}_n$ we have $\operatorname{per} Z \neq 0$ and, moreover, if $A, B \in \mathcal{U}_n$ are two matrices that differ in one row (or in one column) only, then the angle between the non-zero complex numbers $\operatorname{per} A$ and $\operatorname{per} B$ does not exceed $\pi/2$.

Since $\tau < 1 - \delta$, the statement holds for $n = 1$. Assuming that the statement holds for matrices in $\mathcal{U}_{n-1}$, let us consider two matrices $A, B \in \mathcal{U}_n$ that differ in one row or in one column only. Without loss of generality, we assume that B is obtained from A by replacing the entries a_{1j} in the first row with complex numbers b_{1j} for $j = 1, \ldots, n$. Let A_j be the $(n-1) \times (n-1)$ matrix obtained from A by crossing out the first row and the j-th column. Applying the row expansion (3.1.1.2), we get

$$\operatorname{per} A = \sum_{j=1}^{n} a_{1j} \operatorname{per} A_j \quad \text{and} \quad \operatorname{per} B = \sum_{j=1}^{n} b_{1j} \operatorname{per} A_j.$$

We have $A_j \in \mathcal{U}_{n-1}$ for all $j = 1, \ldots, n$, and, moreover any two matrices A_{j_1} and A_{j_2} differ, up to a permutation of columns, in one column only. Therefore, by the

induction hypothesis, we have per $A_j \neq 0$ for $j = 1, \ldots, n$ and the angle between any two non-zero complex numbers A_{j_1} and A_{j_2} does not exceed $\pi/2$. Applying Part (3) of Lemma 3.7.3 with

$$u_j = \operatorname{per} A_j, \quad \alpha_j = a_{1j} \quad \text{and} \quad \beta_j = b_{1j} \quad \text{for} \quad j = 1, \ldots, n,$$

we conclude that per $A \neq 0$, per $B \neq 0$ and the angle between per A and per B does not exceed

$$2 \arctan \delta + 2 \arcsin \frac{\tau}{1-\delta} = \frac{\pi}{2}.$$

□

3.7.5 Proof of Theorem 3.7.1. Let $A = \big(a_{ij}\big)$ be an $n \times n$ real matrix such that

$$\big|1 - a_{ij}\big| \ \leq \ \delta \quad \text{for all} \quad i, j,$$

let $J_n = J$ be the $n \times n$ matrix filled with 1 s and let us define a univariate polynomial

$$r(z) = \operatorname{per}\big(J + z(A - J)\big) \quad \text{for} \quad z \in \mathbb{C}.$$

Hence

$$r(0) = \operatorname{per} J = n!, \quad r(1) = \operatorname{per} A \quad \text{and} \quad \deg r \leq n.$$

First, we observe that as long as $-\alpha \ \leq \ \Re z \ \leq \ 1 + \alpha$ for some $\alpha > 0$, the real part of each entry of the matrix $J + z(A - J)$ lies in the interval

$$[1 - \delta(1 + \alpha), \ 1 + \delta(1 + \alpha)].$$

Similarly, as long as $|\Im z| \leq \rho$ for some $\rho > 0$, the imaginary part of each entry of the matrix $J + z(A - J)$ does not exceed $\rho\delta$ in the absolute value. Let us choose an $\alpha = \alpha(\delta) > 0$ such that $\delta' = \delta(1 + \alpha) < 1$ and choose

$$\rho = \rho(\delta) = \frac{1 - \delta'}{\delta} \sin\left(\frac{\pi}{4} - \arctan \delta'\right) > 0.$$

It follows from Theorem 3.7.2 that

$$r(z) \neq 0 \quad \text{provided} \quad -\alpha \ \leq \ \Re z \ \leq \ 1 + \alpha \quad \text{and} \quad |\Im z| \ \leq \ \rho. \tag{3.7.5.1}$$

Let $\phi(z) = \phi_\delta(z)$ be the univariate polynomial constructed in Lemma 2.2.3, such that

$$\phi(0) = 0, \quad \phi(1) = 1$$

and

$$-\alpha \ \leq \ \Re \phi(z) \ \leq \ 1 + \alpha \quad \text{and} \quad |\Im \phi(z)| \ \leq \ \rho$$

provided

$$|z| \leq \beta \quad \text{for some} \quad \beta = \beta(\delta) > 1.$$

The degree of $\phi(z)$ is bounded by a constant depending on δ alone.

Let us define

$$g(z) = r(\phi(z)).$$

Then $g(z)$ is a univariate polynomial and $\deg g = (\deg r)(\deg \phi) = O(n)$ where the implicit constant in the "O" notation depends only on δ. We have

$$g(0) = r(0) = n!, \quad g(1) = r(1) = \operatorname{per} A$$

and from (3.7.5.1) it follows that

$$g(z) \neq 0 \quad \text{provided} \quad |z| \leq \beta.$$

Let us choose a branch of $f(z) = \ln g(z)$ in the disc $|z| \leq 1$ so that

$$f(0) = \ln n! \quad \text{and} \quad f(1) = \ln \operatorname{per} A$$

and let p_m be the Taylor polynomial of degree m of $f(z)$ computed at $z = 0$, so

$$p_m(z) = f(0) + \sum_{k=1}^{m} \left(\frac{d^k}{dz^k} f(z) \Big|_{z=0} \right) \frac{z^k}{k!}.$$

By Lemma 2.2.1, we have

$$|f(1) - p_m(1)| \leq \frac{\deg g}{(m+1)\beta^m(\beta - 1)}.$$

Hence one can choose $m \leq \gamma\big(\ln n - \ln \epsilon\big)$ for some constant $\gamma = \gamma(\delta) > 0$ such that

$$|\ln \operatorname{per} A - p_m(1)| \leq \epsilon.$$

It remains to show that

$$p_m(1) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!}$$

is a polynomial of degree at most m in the entries a_{ij} of the matrix A that can be computed in $n^{O(m)}$ time.

As follows from Sect. 2.2.2, the derivatives $f^{(k)}(0)$ for $k = 1, \ldots, m$ can be found in $O(m^2)$ time as linear combinations of the derivatives $g^{(k)}(0)$ for $k = 1, \ldots, m$ with coefficients depending on k only.

For a univariate polynomial $q(z)$ and a positive integer m, let $q_{[m]}(z)$ be the truncated polynomial obtained from q by erasing all monomials of degree higher than m.

Since $\phi(0) = 0$, the constant term of $\phi(z)$ is 0 and to compute $g_{[m]}(z)$, we compute the truncated polynomials $\phi_{[m]}(z)$, $r_{[m]}(z)$ and then truncate the composition $r_{[m]}(\phi_{[m]}(z))$ by discarding all terms of degree higher than m. As in Sect. 3.6.7, we observe that the k-th derivative $r^{(k)}(0)$ is a polynomial of degree k in the entries of the matrix A, which can be computed in $n^{O(k)}$ time. Hence $g^{(k)}(0)$ and thus $f^{(k)}(0)$ are polynomials of degree at most k in the entries a_{ij} of the matrix $A = \left(a_{ij}\right)$. The proof now follows. □

3.8 Permanents of α-Conditioned Matrices and Permutations with Few Cycles

Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive matrix which is α-conditioned for some $\alpha \geq 1$, cf. Definition 3.5.5. Let us fix α and let n grow. It turns out that the bulk of the permanent of A is carried by permutations with a small (logarithmic) number of cycles. We interpret permanents as sums over cycle covers, see Sect. 3.1.3.

The following result was proved in [Ba15].

3.8.1 Theorem. *Let $c(\sigma)$ denote the number of cycles of a permutation $\sigma \in S_n$. For an α-conditioned $n \times n$ matrix $A = \left(a_{ij}\right)$, we have*

$$\sum_{\substack{\sigma \in S_n: \\ c(\sigma) < 3\alpha^2 \ln n + 6}} \prod_{i=1}^{n} a_{i\sigma(i)} \ \geq \ \frac{1}{2} \operatorname{per} A.$$

Given a positive matrix $A = \left(a_{ij}\right)$, we consider the symmetric group S_n as a probability space, where

$$\mathbf{Pr}\,(\sigma) = (\operatorname{per} A)^{-1} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right) \quad \text{for} \quad \sigma \in S_n.$$

3.8.2 Lemma. *Let us define random variables*

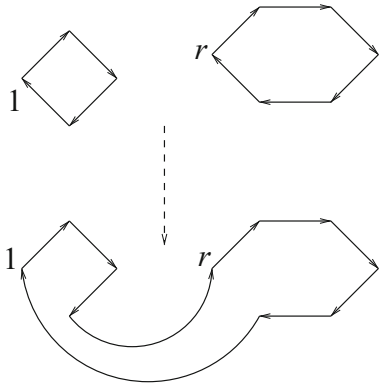
$$l_i : \ S_n \longrightarrow \mathbb{R} \quad \textit{for} \quad i = 1, \ldots, n,$$

where $l_i(\sigma)$ is the length of the cycle of permutation σ that contains i. Assuming that A is α-conditioned, we have

$$\mathbf{Pr}\left(\sigma \in S_n : \ l_i(\sigma) = m\right) \ \leq \ \frac{\alpha^2}{n-m} \quad \textit{for} \quad i = 1, \ldots, n$$

and $m = 1, \ldots, n-1$.

Fig. 3.9 Merging two cycles



Proof. Without loss of generality, we assume that $i = 1$. Let $X \subset S_n$ be the set of permutations $\sigma \in S_n$ such that $l_1(\sigma) = m$. We construct a set $Y \subset S_n$ as follows. Each permutation $\sigma \in X$ contributes $n - m$ permutations into Y: we write the cycle of σ containing 1 as

$$1 = j_1 \to j_2 \to \ldots \to j_m \to 1, \tag{3.8.2.1}$$

pick an element r of the $n - m$ elements not in the cycle, write the cycle of σ containing r as

$$r = j_{m+1} \to j_{m+2} \to \ldots \to j_{m+k} \to r \tag{3.8.2.2}$$

and produce a permutation $\tau \in Y$ by merging the two cycles together:

$$1 = j_1 \to j_2 \to \ldots \to j_m \to r = j_{m+1} \to j_{m+2} \to \ldots \to j_{m+k} \to 1, \tag{3.8.2.3}$$

see Fig. 3.9.

Since A is α-conditioned, we have

$$\mathbf{Pr}\,(\sigma) \;\leq\; \alpha^2 \mathbf{Pr}\,(\tau). \tag{3.8.2.4}$$

Next, we observe that each permutation $\tau \in Y$ is obtained from a unique permutation $\sigma \in X$. To reconstruct σ from τ, we find the cycle of σ containing 1, write it as in (3.8.2.3) and cut into the cycles (3.8.2.1) and (3.8.2.2), see Fig. 3.10

Using (3.8.2.4), we conclude that

$$\mathbf{Pr}\,(X) \;\leq\; \frac{\alpha^2}{n-m} \mathbf{Pr}\,(Y) \;\leq\; \frac{\alpha^2}{n-m}.$$

□

3.8.3 Proof of Theorem 3.8.1. Let l_i be the random variables of Lemma 3.8.2. Using Lemma 3.8.2, we estimate

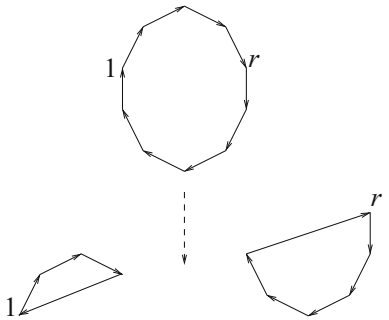


Fig. 3.10 Cutting a cycle into two

$$\begin{aligned}
\mathbf{E}\left(l_i^{-1}\right) &= \sum_{m=1}^{n} \frac{1}{m}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right)\\
&= \sum_{1\le m\le n/3} \frac{1}{m}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right) + \sum_{n/3<m\le n} \frac{1}{m}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right)\\
&\le \frac{3\alpha^2}{2n}\sum_{1\le m\le n/3}\frac{1}{m} + \frac{3}{n}\sum_{n/3<m\le n}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right)\\
&\le \frac{3\alpha^2\ln n}{2n} + \frac{3}{n}.
\end{aligned}$$

Next, we note that

$$c(\sigma) = \sum_{i=1}^{n} l_i^{-1}(\sigma),$$

since the sum of $l_i^{-1}(\sigma)$ for all i in a cycle of σ is 1. Therefore,

$$\mathbf{E}\,c(\sigma) = \sum_{i=1}^{n}\mathbf{E}\left(l_i^{-1}(\sigma)\right) \le \frac{3\alpha^2\ln n}{2} + 3.$$

Applying the Markov inequality, we conclude that

$$\mathbf{Pr}\left(\sigma:\ c(\sigma) \ge 3\alpha^2\ln n + 6\right) \le \frac{1}{2},$$

and the proof follows. □

As is shown in [Ba15], one immediate corollary of Theorem 3.8.1 is that on α-conditioned matrices, the permanent of A and the *Hamiltonian permanent* of A,

$$\operatorname{ham} A = \sum_{\substack{\sigma \in S_n: \\ c(\sigma)=1}} \prod_{i=1}^{n} a_{i\sigma(i)}$$

differ by a factor of $n^{O(\alpha^2 \ln n)}$ (permutations consisting of a single cycle are called *Hamiltonian cycles*). Similarly to the proof of Lemma 3.8.2, the result is obtained by patching a permutation with $O\left(\alpha^2 \ln n\right)$ cycles into a single cycle. Consequently, for α fixed in advance, using the scaling algorithm of Sect. 3.5, we obtain a polynomial time algorithm for computing $\operatorname{ham} A$ within a factor of $n^{O(\alpha^2 \ln n)}$. As is discussed in [Ba15], this allows one to distinguish in polynomial time directed graphs on n vertices that contain many Hamiltonian cycles (at least $\epsilon^n(n-1)!$ for some fixed $\epsilon > 0$) from graphs that are sufficiently far from having a Hamiltonian cycle (need at least ϵn new edges added to acquire one). The algorithm is obtained by approximating $\operatorname{per} A$ and hence $\operatorname{ham} A$ for a "soft" version $A = \left(a_{ij}\right)$ of the adjacency matrix of the graph,

$$a_{ij} = \begin{cases} 1 & \text{if } i \to j \text{ is an edge} \\ \delta & \text{otherwise} \end{cases}$$

for a sufficiently small $\delta = \delta(\epsilon) > 0$.

Vishnoi [Vi12] used the van der Waerden bound for the permanent (see Sect. 3.3) to prove the existence of long cycles (and of an efficient algorithm to find such cycles) in regular graphs.

3.9 Concluding Remarks

3.9.1 Permanents and determinants. It is tempting to compare the permanent

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

with the syntactically similar determinant

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

and try exploit the similarity. Godsil and Gutman [GG78] suggested the following construction.

Suppose that $A = \left(a_{ij}\right)$ is an $n \times n$ non-negative real matrix. Let ξ_{ij} be real-valued independent random variables such that

$$\mathbf{E}\, \xi_{ij} = 0 \quad \text{and} \quad \mathbf{var}\, \xi_{ij} = 1 \quad \text{for all} \quad i, j = 1, \ldots, n$$

and let us define a random $n \times n$ matrix $B = \left(b_{ij}\right)$ by

$$b_{ij} = \xi_{ij}\sqrt{a_{ij}} \quad \text{for all} \quad i, j = 1, \ldots, n.$$

It is not hard to show that

$$\mathbf{E}\ (\det B)^2 = \operatorname{per} A$$

and one can ask how well $\det^2 B$ is likely to approximate per A, see also Chap. 8 of [LP09]. Since $\det^2 B$ is non-negative, the Markov inequality implies that $\det^2 B$ is unlikely to overestimate per A by a lot (for example, the probability that $\det^2 B > 10 \operatorname{per} A$ does not exceed $1/10$). However, it may happen that $\det^2 B$ grossly underestimates per A. For example, if $n = 2m$ and A is a block-diagonal matrix consisting of m blocks $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then $\operatorname{per} A = 2^m$. If we choose ξ_{ij} to be random signs, so that

$$\mathbf{Pr}\ (\xi_{ij} = 1) = \frac{1}{2} \quad \text{and} \quad \mathbf{Pr}\ (\xi_{ij} = -1) = \frac{1}{2}$$

then $\det B = 0$ with probability $1 - 2^{-m}$. This effect can be mitigated if ξ_{ij} are continuous random variables. In [Ba99] it is shown that if ξ_{ij} are standard Gaussian with density

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

then with probability approaching 1 as n grows, we have

$$(\det B)^2 \ \geq\ (0.28)^n \operatorname{per} A \tag{3.9.1.1}$$

(the worst-case scenario is when $A = I_n$, the $n \times n$ identity matrix). It is also shown that if ξ_{ij} are complex Gaussian with density

$$\frac{1}{\pi} e^{-|z|^2} \quad \text{for} \quad z \in \mathbb{C},$$

in which case $\mathbf{E}\ |\det B|^2 = \operatorname{per} A$ then with probability approaching 1 as n grows, we have

$$|\det B|^2 \ \geq\ (0.56)^n \operatorname{per} A \tag{3.9.1.2}$$

(again, the worst case scenario is when $A = I_n$).

Finally, let us choose ξ_{ij} to be quarternionic Gaussian with density

$$\frac{4}{\pi^2} e^{-|h|^2} \quad \text{for} \quad h \in \mathbb{H}$$

(so that $\mathbf{E}\ |h|^2 = 1$, here $\mathbb{H}$ denotes the skew field of quaternions and not the upper half-plane of $\mathbb{C}$ as elsewhere in the book). Then B is an $n \times n$ quaternionic matrix

which we write as

$$B = R + \mathbf{i}\, S + \mathbf{j}\, T + \mathbf{k}\, U,$$

where R, S, T and U are $n \times n$ real matrices. Let $B_{\mathbb{C}}$ denote the $2n \times 2n$ complex matrix

$$B_{\mathbb{C}} = \begin{pmatrix} R + iS & T + iU \\ -T + iU & R - iS \end{pmatrix}.$$

It is show in [Ba99] that $\det B_{\mathbb{C}}$ is a non-negative real number such that $\mathbf{E}\ \det B_{\mathbb{C}} = \operatorname{per} A$ and that

$$\det B_{\mathbb{C}} \ \geq\ (0.76)^n \operatorname{per} A \tag{3.9.1.3}$$

with probability approaching 1 as n grows (again, the worst-case scenario is when $A = I_n$).

The idea behind the inequalities of (3.9.1.1)–(3.9.1.3) is roughly as follows. We note that $\det B$ is linear in every row of B. We consider $\det B$ as a function of n independent Gaussian n-vectors $x_i = (\xi_{i1}, \ldots, \xi_{in})$. In the real case $(\det B)^2$ is a quadratic form in each x_i, once the values of the remaining vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are fixed. In the complex case, $|\det B|^2$ is a Hermitian form in each x_i, once the values of the remaining vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are fixed. In the quaternionic case, $\det B_{\mathbb{C}}$ is a quaternionic Hermitian form in each x_i, once the values of the remaining vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are fixed.

We deduce (3.9.1.1) from the following: if $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a positive semidefinite quadratic form on the space $\mathbb{R}^n$ equipped with the standard Gaussian measure and such that $\mathbf{E}\, q = 1$ then

$$\mathbf{E}\ \ln q \ \geq\ -\ln 2 - \gamma, \tag{3.9.1.4}$$

where $\gamma \approx 0.5772156649$ is the Euler constant and the bound (3.9.1.4) is attained if q is a form of rank 1, for example,

$$q\,(x_1, \ldots, x_n) = x_1^2 \quad \text{where} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Since every positive semidefinite quadratic form is a convex combination of positive semidefinite forms of rank 1, by Jensen's inequality the minimum in (3.9.1.4) is indeed attained on forms of rank 1. The constant in (3.9.1.1) is $e^{-\ln 2 - \gamma} \approx 0.28$.

We deduce (3.9.1.2) from the following: if $q : \mathbb{C}^n \longrightarrow \mathbb{R}$ is a positive semidefinite Hermitian form on the space $\mathbb{C}^n$ equipped with the standard Gaussian measure and such that $\mathbf{E}\, q = 1$ then

$$\mathbf{E}\ \ln q \ \geq\ -\gamma, \tag{3.9.1.5}$$

and the bound in (3.9.1.5) is attained if q is a form of rank 1, for example,

$$q\,(z_1, \ldots, z_n) = |z_1|^2 \quad \text{where} \quad (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Similarly to the real case, since every positive semidefinite Hermitian form is a convex combination of positive semidefinite Hermitian forms of rank 1, by Jensen's inequality the minimum in (3.9.1.5) is indeed attained on forms of rank 1. We get a better bound than in the real case, because a complex Hermitian form of rank 1 can be viewed as a real quadratic form of rank 2. The constant in (3.9.1.2) is $e^{-\gamma} \approx 0.56$.

We deduce (3.9.1.3) from the following: if $q : \mathbb{H}^n \longrightarrow \mathbb{R}$ is a positive semidefinite Hermitian form on the space $\mathbb{H}^n$ equipped with the standard Gaussian measure and such that $\mathbf{E}\, q = 1$ then

$$\mathbf{E}\, \ln q \;\geq\; 1 - \gamma - \ln 2 \tag{3.9.1.6}$$

and the bound in (3.9.1.6) is attained if q is a form of rank 1, for example,

$$q\,(h_1, \ldots, h_n) = |h_1|^2 \quad \text{where} \quad (h_1, \ldots, h_n) \in \mathbb{H}^n.$$

The constant in (3.9.1.3) is $e^{1-\gamma-\ln 2} \approx 0.76$.

For various special classes of matrices, a subexponential approximation factor is achieved by (real) Gaussian [F+04], [RZ16] and some non-Gaussian [CV09] random variables ξ_{ij}.

3.9.2 Algorithms for computing permanents. For a general $n \times n$ real or complex matrix A, the most efficient method known of computing per A exactly, is, apparently, Ryser's method and its modifications, see Chap. 7 of [Mi78], which achieves $O(n^2 2^n)$ complexity. Essentially, it uses the formula

$$\operatorname{per} A = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x_1, \ldots, x_n) \quad \text{where} \quad p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right),$$

and computes the derivative as

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x_1, \ldots, x_n) = \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} p(x_I), \tag{3.9.2.1}$$

where x_I is the 0–1 vector with 0s in positions I and 1s elsewhere (as is easy to see, formula (3.9.2.1) holds for any homogeneous polynomial p of degree n in $x_1, \ldots, x_n$). The exact computation of the permanent is a **#P**-hard problem already for 0–1 matrices [Va79], which makes a polynomial time algorithm rather unlikely. Efficient (polynomial time) algorithms for computing permanents exactly are known for some rather restricted classes of matrices, for example, for matrices of a small (fixed in advance) rank [Ba96] and for 0–1 matrices with small (fixed in advance) permanents [GK87].

Given an $n \times n$ matrix $A = \big(a_{ij}\big)$, let $G(A)$ be the bipartite graph with $2n$ vertices $1_L, \ldots n_L$ and $1_R, \ldots, n_R$, where vertices i_L and j_R are connected by an edge if and only if $a_{ij} \neq 0$, see Sect. 3.1.2. Cifuentes and Parillo found a polynomial time

algorithm to compute per A exactly provided the treewidth of $G(A)$ is bounded by a constant, fixed in advance [CP16]. The algorithm is applicable to matrices over any commutative ring. One can obtain graphs $G(A)$ of a small treewidth provided A is sufficiently sparse, that is, contains relatively few non-zeros. This is the case, for example, if A has a band structure, that is, $a_{ij} = 0$ provided $|i - j| \geq \omega$ for some ω, fixed in advance.

The greatest success in approximation algorithms is achieved by Jerrum, Sinclair and Vigoda [J+04] who constructed a Markov Chain Monte Carlo based fully polynomial time randomized approximation scheme for computing permanents of non-negative matrices. A scaling based deterministic polynomial time algorithm approximating permanents of $n \times n$ non-negative matrices within a factor of e^n is constructed in [L+00], see also Remark 3.5.4. The approximation factor was improved to 2^n [GS14] and it is conjectured that the same algorithm actually achieves a $2^{n/2}$ approximation factor, cf. (3.4.6.1). Using the "correlation decay" idea from statistical physics, Gamarnik and Katz obtained a $(1 + \epsilon)^n$ approximation factor for any $\epsilon > 0$, fixed in advance, when A is a 0–1 matrix of a constant degree expander graph [GK10].

Less is known about approximation algorithms for not necessarily non-negative matrices (but see Sects. 3.6, 5.5 and also [Mc14]). Gurvits [Gu05] presented a randomized algorithm, which, given an $n \times n$ complex matrix A approximates per A in $O(n^2/\epsilon^2)$ time within an additive error of $\epsilon \|A\|^n$, where $\|A\|$ is the operator norm of A, see also [AA13] for an exposition. The idea of the algorithm is to use the formula

$$\operatorname{per} A = \mathbf{E}\, x_1 \cdots x_n \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right),$$

where $x_i = \pm 1$ are independent Bernoulli random variables and replace the expectation by the sample average.

http://www.springer.com/978-3-319-51828-2

Combinatorics and Complexity of Partition Functions

Barvinok, A.

2016, VI, 303 p. 51 illus., 42 illus. in color., Hardcover
ISBN: 978-3-319-51828-2